Filomat 40:6 (2026), 2107–2120
https://doi.org/10.2298/FIL2606107R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Cohomology and deformation for dendriform algebra with a derivation

Jingru Ren[a]

[a] *School of Humanities and Fundamental Sciences, Shenzhen University of Information Technology, Shenzhen 518172, P.R.China*

Abstract. Dendriform algebras can be regarded as associative algebras whose product is decomposed into two operations satisfying some laws of dendriform algebra, which together form the associative law of the undecomposed associative algebra. We introduce a cohomology theory for dendriform algebras equipped with derivations and show the deformation theory is controlled by the cohomology and the collection of equivalence classes of abelian extensions is given by the second cohomology.

1. Introduction

A dendriform algebra is a vector space together with two operations whose sum gives an associative algebra structure of the underlying space. Dendriform algebra is defined by Loday in [10] as well as its (co)homology theory. According to Loday [10], the original motivation for dendriform algebra is that it appears as the Koszul dual of the dialgebra operad, which is another type of algebra introduced in that same paper for the purpose of studying periodicity phenomena in algebraic K-theory.

A derivation for an algebra is a generalization of the usual derivatives for functions. To be precise, a linear map $d : A \to A$ is called a derivation for an associative algebra A if d satisfies $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. Derivations for different algebraic structures are useful in many ways. For example, in constructing homotopy Lie algebras [14], deformation formulas [1], differential Galois theory [12]. Lie algebras with derivations (called LieDer pairs) are recently explored in [13]. In particular, the authors define and investigate LieDer pairs cohomology theory, as well as how its relationship with deformation theory and (central) extensions of LieDer pairs. Later, similar cohomology theory and deformation theory/(central) extensions for other types of algebras (including 3-Lie algebras, associative algebras, Lie triple systems, Leibniz algebras, Leibniz triple systems, Hom-Lie algebras, Hom-Leibniz algebras, etc.) equipped with derivations are discussed in [7], [4], [6], [2], [15], [8], [9].

In this paper, we introduce a cohomology theory for dendriform algebras with derivations, study their formal one-parameter deformation theory and abelian extensions. Similar relations between Hochschild cohomology, algebraic deformation theory and abelian extensions of associative algebras/LieDer pairs/LeibDer pairs are obtained. It is shown that the infinitesimal of any deformation is a 2-cocycle, the rigidity of the underlying dendriform algebra (that is, all deformation is equivalent to the undeformed deformation) is

2020 *Mathematics Subject Classification*. Primary 16E40; Secondary 16S80, 16W10.
Keywords. dendriform algebra; derivation; cohomology; deformation; abelian extension.
Received: 30 June 2025; Revised: 24 December 2025; Accepted: 02 January 2026
Communicated by Dijana Mosić
Research supported by Guangdong Basic and Applied Basic Research Foundation (Grants: 2022A1515012176, 2018A030313581), and Shenzhen University of Information Technology (Grant: 2023djpjgyb05)
Email address: jingrur124@outlook.com (Jingru Ren)
ORCID iD: https://orcid.org/0000-0002-1584-4367 (Jingru Ren)

controlled by the vanishing of the second cohomology, the extension of a finite order deformation to the next order is possible if the third cohomology vanishes and the collection of equivalence classes of abelian extensions is given by the second cohomology of the underlying dendriform algebra with a derivation.

Although we only mention dendriform algebras with derivations, the method used in this paper can be generalized to any Loday algebra with a derivation (the word Loday here refers to the algebras appeared in [16] [17]) as the constructions used in our approach mainly relies on the fact that the sequence of cochain modules of these algebras forms an operad with a multiplication (See Theorem 1.5 [16]). However, it is still necessary to workout the specifics for different types of algebras because the difficulties have shifted for concrete algebras. For dendriform algebras with derivations, for example, it boils down to finding out how to put everything together using the functions of the dendriform pre-operadic system (See Section three) and this does not come for free from the abstract operad setting. There are many details in this paper because some of them are not found in the literature, and it was our goal to make the paper as self-contained as possible.

Convention All vector spaces considered in the this paper are over a field k with characteristic zero.

2. Preliminaries

Our reference for operad theory is [11]. Recall that a nonsymmetric operad consists of a sequence of vector spaces $O = \{O(n)\}_{n\geqslant 1}$ and for each integers $k, n_1, \cdots, n_k$ a composition

$$\gamma : O(k) \otimes O(n_1) \otimes \cdots \otimes O(n_k) \to O(n_1 + \cdots + n_k)$$

and an element $id \in O(1)$ satisfying the associativity and identity properties. All operads considered in this paper are nonsymmetric so we simply call them operads below.

An equivalent way to describe an operad is a sequence of vector spaces endowed with the so-called partial compositions

$$\circ_i : O(m) \otimes O(n) \to O(m + n - 1)$$

for $1 \leqslant i \leqslant m$ and $n \geqslant 0$ and an element $id \in O(1)$ satisfying associativity, equivariance and unitality requirements. The two compositions are related by

$$f \circ_i g = \gamma(f; \overbrace{id, \cdots, id, \underbrace{g}_{i^{th}}, id, \cdots, id}^{m})$$

$$\gamma(f; g_1, \cdots, g_m) = (\cdots((f \circ_m g_m) \circ_{m-1} g_{m-1}) \cdots) \circ_1 g_1)$$

for $f \in O(m)$. For any operad (O, γ, id), one can define certain brace operations

$$\{f\}\{g_1, \cdots, g_n\} := \sum (-1)^{\epsilon} \gamma(f; id, \cdots, g_1, \cdots, g_n, \cdots, id)$$

where $\epsilon = \sum_{p=1}^{n} |g_p| i_p$ (recall that $|g| = m - 1$ if $g \in O(m)$ and i_p is the total number of inputs in front of g_p) and the summation is given by all possible substitutions of the g_i's into f in the prescribed order. Further, there is a circle product $\circ : O(m) \otimes O(n) \to O(m + n - 1)$ defined by

$$f \circ g := \{f\}\{g\} = \sum_{i=1}^{m} (-1)^{(i-1)|g|} f \circ_i g$$

and the bracket $[f, g] := f \circ g - (-1)^{|f||g|} g \circ f$ is a degree -1 graded Lie bracket on $\bigoplus_{n\geqslant 1} O(n)$.

Suppose O has a multiplication, i.e., an element $\pi \in O(2)$ such that $\pi \circ \pi = 0$, then π defines a homotopy G-algebra structure on $\bigoplus_{n\geqslant 1} O(n)$ [5]. This is in particular a dg Lie algebra with the above bracket and differential $d_\pi(-) := [\pi, -]$

Definition 2.1. *A **dendriform algebra** is a k vector space D with two binary operations* $\prec, \succ: D \otimes D \to D$ *such that*

$$(x \prec y) \prec z = x \prec (y \prec z + y \succ z)$$
$$(x \succ y) \prec z = x \succ (y \prec z)$$
$$(x \prec y + x \succ y) \succ z = x \succ (y \succ z)$$

for any $x, y, z \in D$.

Note that for any dendriform algebra D, the product defined by $x * y := x \prec y + x \succ y$ is associative.

Definition 2.2. *Let* D, D' *be two dendriform algebras, a **dendriform morphism** is a linear map* $f : D \to D'$ *such that* $f(x \prec y) = f(x) \prec' f(y)$ *and* $f(x \succ y) = f(x) \succ' f(y)$ *for any* $x, y \in D$.

Definition 2.3. *A linear map* $\varphi : D \to D$ *for a denderiform algebra D is a **derivation** if* $\varphi(x \circ y) = \varphi(x) \circ y + x \circ \varphi(y)$ *for any* $x, y \in D$ *and* $\circ \in \{\prec, \succ\}$.

A derivation for a dendriform algebra $(D, \prec, \succ)$ is a derivation for the associated associative algebra $(D, *)$ with $* = \prec + \succ$.

Definition 2.4. *A dendriform algebra D together with a derivation* φ *is called a **DendDer pair** or simply a **DD pair**.*

Definition 2.5. *Let* $(D, \prec, \succ, \varphi)$ *and* $(D', \prec', \succ', \varphi')$ *be two DD pairs, a **morphism of DD pairs** is a dendriform morphism* $f : D \to D'$ *that commute with derivations, that is,* $f \circ \varphi = \varphi' \circ f$.

Definition 2.6. *Let* $(D, \prec_D, \succ_D, \varphi_D)$ *be a dendriform algebra with derivation, a D-**representation** of D is a vector space M with actions*

$$\prec: D \otimes M \to M, \quad \prec: M \otimes D \to M, \quad \succ: D \otimes M \to M, \quad \succ: M \otimes D \to M$$

and a linear map $\varphi_M : M \to M$ *such that*

$$(m \prec y) \prec z = m \prec (y \prec z + y \succ z)$$
$$(m \succ y) \prec z = m \succ (y \prec z)$$
$$(m \prec y + m \succ y) \succ z = m \succ (y \succ z)$$

and

$$(x \prec m) \prec z = x \prec (m \prec z + m \succ z)$$
$$(x \succ m) \prec z = x \succ (m \prec z)$$
$$(x \prec m + x \succ m) \succ z = x \succ (m \succ z)$$

and

$$(x \prec y) \prec m = x \prec (y \prec m + y \succ m)$$
$$(x \succ y) \prec m = x \succ (y \prec m)$$
$$(x \prec y + x \succ y) \succ m = x \succ (y \succ m)$$

and

$$\varphi_M(x \prec m) = \varphi_D(x) \prec m + x \prec \varphi_M(m)$$
$$\varphi_M(x \succ m) = \varphi_D(x) \succ m + x \succ \varphi_M(m)$$
$$\varphi_M(m \prec x) = \varphi_M(m) \prec x + m \prec \varphi_D(x)$$
$$\varphi_M(m \succ x) = \varphi_M(m) \succ x + m \succ \varphi_D(x)$$

for any $x, y, z \in D$ *and* $m \in M$.

Note that any DD pair is a representation over itself.

Proposition 2.7. *Let $(D, \prec_D, \succ_D, \varphi_D)$ be a DD pair and $(M, \prec, \succ, \varphi_M)$ a D-representation, define $(x, m) \prec_\oplus (y, n) := (x \prec_D y, x \prec n + m \prec y)$ and $(x, m) \succ_\oplus (y, n) := (x \succ_D y, x \succ n + m \succ y)$. Then $(D \oplus M, \prec_\oplus, \succ_\oplus, \varphi_D \oplus \varphi_M)$ is a DD pair.*

Proof. The identity $((x, m) \prec_\oplus (y, n)) \prec_\oplus (z, l) = (x, m) \prec_\oplus ((y, n) \prec_\oplus (z, l) + (y, n) \succ_\oplus (z, l))$ is in fact

$$((x \prec_D y) \prec_D z, (x \prec_D y) \prec l + (x \prec n) \prec z + (m \prec y) \prec z) = (x \prec_D (y \prec_D z + y \succ_D z),$$

$$+x \prec (y \prec l + y \succ l) + x \prec (n \prec z + n \succ z) + m \prec (y \prec_D z + y \succ_D z))$$

so it certainly holds. Other identities for being a DD pair can also be checked directly. □

3. Cohomology

3.1. *Dendriform algebra cohomology*

Dendriform algebra cohomology was introduced in [10] and further discussed with more details in [3]. Recall we denote $C_n = \{[1], [2], \cdots, [n]\}$ to be the set of n symbols and the module of n-cochain, for any dendriform algebra D, is defined to be $C_D(n) := Hom_k(k[C_n] \otimes D^{\otimes n}, D)$ for $n \geqslant 1$. Recall there is a **pre-operadic system** [16] on $\{C_n : n \geqslant 1\}$, that is, a collection of functions

$$R_0(k; n_1, \cdots, n_k) : C_{n_1+\cdots+n_k} \to C_k$$

$$R_i(k; n_1, \cdots, n_k) : C_{n_1+\cdots+n_k} \to k[C_{n_i}]$$

for $k, n_i \geqslant 1, 1 \leqslant i \leqslant k$ satisfying certain conditions, which in turn gives an operad structure for $\{C_D(n)\}_{n \geqslant 1}$ (Theorem 1.5 [16]). In particular, we have

$$\begin{array}{cccc} R_0(2;1,1) : C_2 \to C_2\ , & R_1(2;1,1), R_2(2;1,1) : C_2 \to C_1\ , & R_0(1;2) : C_2 \to C_1\ , & R_1(1;2) : C_2 \to C_1 \\ & & & \\ [r] \mapsto [r] & [r] \mapsto [1] & [r] \mapsto [1] & [r] \mapsto [r] \end{array}$$

There is a multiplication $\pi \in C_D(2)$ of the operad $\{C_D(n)\}_{n \geqslant 1}$ defined by

$$\pi([r]; a, b) := \begin{cases} a \prec b, \text{ if } [r] = [1] \\ a \succ b, \text{ if } [r] = [2] \end{cases}$$

and the differential of dendriform algebra cochain complex is $d_\pi(-) := [\pi, -] : C_D(n) \to C_D(n+1)$ and given explicitly by

$$(d_\pi f)([r]; x_1, \cdots, x_{n+1}) = \pi(R_0(2; 1, n)[r]; x_1, f(R_2(2; 1, n)[r]; x_2, \cdots, x_{n+1}))$$

$$+ \sum_{i=1}^{n} (-1)^i f(R_0(n; 1, \cdots, \underbrace{2}_{ith}, \cdots, 1)[r]; x_1, \cdots, x_{i-1}, \pi(R_i(n; 1, \cdots, 2, \cdots, 1)[r]; x_i, x_{i+1}), x_{i+2}, \cdots, x_{n+1})$$

$$+(-1)^{n+1} \pi(R_0(2; n, 1)[r]; f(R_1(2; n, 1)[r]; x_1, \cdots, x_n), x_{n+1})$$

for $[r] \in C_{n+1}$ and $x_1, \cdots, x_{n+1} \in D$.

More generally, let D be a dendriform algebra and M a D-representation, the n-cochain of D with coefficients in M is defined to be $C^n_D(D, M) := Hom_k(k[C_n] \otimes D^{\otimes n}, M)$ for $n \geqslant 1$ and the differential is: for any $[r] \in C_{n+1}$ and $x_1, \cdots, x_{n+1} \in D$

$$(df)([r]; x_1, \cdots, x_{n+1}) = \alpha(R_0(2; 1, n)[r]; x_1, f(R_2(2; 1, n)[r]; x_2, \cdots, x_{n+1}))$$

$$+\sum_{i=1}^{n}(-1)^i f(R_0(n;1,\cdots,\underbrace{2}_{ith},\cdots,1)[r];x_1,\cdots,x_{i-1},\pi(R_i(n;1,\cdots,2,\cdots,1)[r];x_i,x_{i+1}),x_{i+2},\cdots,x_{n+1})$$

$$+(-1)^{n+1}\beta(R_0(2;n,1)[r];f(R_1(2;n,1)[r];x_1,\cdots,x_n),x_{n+1})$$

where $\alpha : k[C_2]\otimes(D\otimes M)\to M$ and $\beta : k[C_2]\otimes(M\otimes D)\to M$ are maps defined by

$$\alpha([r];x,m) := \begin{cases} x \prec m, & \text{if } [r]=[1] \\ x \succ m, & \text{if } [r]=[2] \end{cases} \qquad \beta([r];m,x) := \begin{cases} m \prec x, & \text{if } [r]=[1] \\ m \succ x, & \text{if } [r]=[2] \end{cases}$$

In particular, a linear map $f \in C^1_D(D,M) = Hom_k(k[C_1]\otimes D, D) \cong Hom(D,D)$ is a 1-cocycle (i.e., $df = 0$) if f is a derivation for D, i.e., $f(x \prec y) = f(x) \prec y + x \prec f(y)$ and $f(x \succ y) = f(x) \succ y + x \succ f(y)$ for any $x, y \in D$.

Similarly, $f \in C^2_D(D,M)$ is a 2-cocycle if

$$x \prec f([1]+[2];y,z) - f([1];x \prec y,z) + f([1];x,y \prec z + y \succ z) - f([1];x,y) \prec z = 0 \quad (1)$$

$$x \succ f([1];y,z) - f([1];x \succ y,z) + f([2];x,y \prec z) - f([2];x,y) \prec z = 0 \quad (2)$$

$$x \succ f([2];y,z) - f([2];x \prec y + x \succ y,z) + f([2];x,y \succ z) - f([1]+[2];x,y) \succ z = 0 \quad (3).$$

Lemma 3.1. *Let $(D,\prec,\succ)$ be a dendriform algebra and π the multiplication of the dendriform algebra cochain operad, then $\varphi \in C_D(1) = Hom_k(k[C_1]\otimes D, D)$ is a derivation if and only if $[\pi,\varphi] = 0$.*

Proof. Indeed, we have

$$\begin{aligned}
[\pi,\varphi]([r];a,b) &= (\pi\circ_1\varphi + \pi\circ_2\varphi - \varphi\circ_1\pi)([r];a,b) = (\pi\circ_1\varphi)([r];a,b) + (\pi\circ_2\varphi)([r];a,b) - (\varphi\circ_1\pi)([r];a,b) \\
&= \pi(R_0(2;1,1)[r];\varphi(R_1(2;1,1)[r];a),b) + \pi(R_0(2;1,1)[r];a,\varphi(R_2(2;1,1)[r];b)) \\
&\quad -\varphi(R_0(1;2)[r];\pi(R_1(1;2)[r];a,b)) \\
&= \pi([r];\varphi(1;a),b) + \pi([r];a,\varphi(1;b)) - \varphi(1;\pi([r];a,b)) \\
&= \pi([r];\varphi(a),b) + \pi([r];a,\varphi(b)) - \varphi(\pi([r];a,b)) \\
&= \begin{cases} \varphi(a) \prec b + a \prec \varphi(b) - \varphi(a \prec b), & \text{if } [r]=[1] \\ \varphi(a) \succ b + a \succ \varphi(b) - \varphi(a \succ b), & \text{if } [r]=[2] \end{cases}
\end{aligned}$$

Then $[\pi,\varphi] = 0$ precisely when φ is a derivation for D. □

3.2. DD pair cohomology

Fix a DD pair (D,φ) and a D-representation M, define a cochain complex as follows:

$$\begin{array}{lccccccc} & 0 & & 1 & & 2 & & \cdots \\ C^\bullet_{DD}(D,M): & 0 & \longrightarrow & C^1_D(D,M) & \longrightarrow & C^2_D(D,M)\oplus C^1_{DD}(D,M) & \longrightarrow & \cdots \end{array}$$

The differential $\partial : C^1_{DD}(D,M) \to C^2_{DD}(D,M)$ is defined to be $\partial(f_1) := (df_1, -\delta f_1)$ and for $i \geqslant 2$,

$$\partial : C^i_{DD}(D,M) \to C^{i+1}_{DD}(D,M)$$

$$(f_i, g_{i-1}) \mapsto (df_i, dg_{i-1} + (-1)^i\delta f_i)$$

for any $f_1 \in C^1_D(D,M), f_i \in C^i_D(D,M)$ and $g_{i-1} \in C^{i-1}_D(D,M)$, where d is the differential of the dendriform algebra cochain complex ($d = d_\pi$ if the coefficient M is taken to be the dendriform algebra D) and $\delta : C^n_{DD}(D,M) \to C^n_{DD}(D,M)$ is the map that sends an element f to $\sum_{i=1}^{n} f\circ(id\otimes\cdots\otimes\varphi_D\otimes\cdots\otimes id) - \varphi_M\circ f$. Note that we have $\delta(f) := [\varphi_D, f]$ when the coefficient M is taken to be D.

Lemma 3.2. *$d(\delta f) = \delta(df)$ for any $f \in C^i_{DD}(D, M)$.*

Proof. The lemma holds if $(M, \varphi_M) = (D, \varphi_D)$ as $d = d_\pi = [\pi, -]$ and $\delta f = [\varphi_D, f]$ in this case, so the $LHS = [\pi, [\varphi_D, f]] = [[\pi, \varphi_D], f] + [\varphi_D, [\pi, f]] = [0, f] + [\varphi_D, [\pi, f]] = RHS$ by Lemma 3.1.

For any coefficient $(M, \prec, \succ, \varphi_M)$, consider the DD pair $(D\oplus M, \prec_\oplus, \succ_\oplus, \varphi_D \oplus \varphi_M)$ in Proposition 2.7. Extend the map f to a map $\bar{f} \in C^i_{DD}(D \oplus M)$ by $\bar{f}([r]; (x_1, m_1), \cdots, (x_i, m_i)) := (0, f([r]; x_1, \cdots, x_i))$ (note that f is the restriction of $\bar{f}$ to $k[C_i] \otimes D^{\otimes i}$ and $\bar{f} = 0$ implies that $f = 0$). Observe that $\overline{df} = d_\pi \bar{f}$ and $\overline{\delta f} = \delta \bar{f}$ so $\overline{d(\delta f)} = d_\pi \overline{\delta f} = d_\pi(\delta \bar{f}) = \delta(d_\pi \bar{f}) = \delta(\overline{df}) = \overline{\delta(df)}$ and it follows that $d(\delta f) = \delta(df)$ for any f. □

Proposition 3.3. *The map ∂ satisfies $\partial^2 = 0$.*

Proof. Indeed, $\partial(\partial(f_1)) = \partial((df_1, -\delta f_1)) = (d(df_1), d(-\delta f_1) + \delta(df_1)) = 0$ for $f_1 \in C^1_D(D, M)$ and

$$\partial(\partial(f_i, g_{i-1})) = \partial(df_i, dg_{i-1} + (-1)^i \delta f_i) = (d(df_i), d(dg_{i-1}) + (-1)^i d(\delta f_i) + (-1)^{i+1}\delta(df_i)) = 0$$

for any $f_i \in C^i_{DD}(D, M)$ and $g_{i-1} \in C^{i-1}_{DD}(D, M)$. □

Therefore $C^\bullet_{DD}(D, M)$ is indeed a complex and we define a cohomology theory for $(D, \prec_D, \succ_D, \varphi_D)$ with coefficients in $(M, \prec, \succ, \varphi_M)$ as the homology of this complex and denote the **DD pair cohomology** by $H^\bullet_{DD}(D, M)$. It is denoted by $C^\bullet_{DD}(D)$ and $H^\bullet_{DD}(D)$ if the representation is taken to be the DD pair itself. Similar cohomology theories were defined for Lie algebra with a derivation in [13] and Leibniz algebra with a derivation in [2].

4. Deformation theory

We only talk about cohomology whose coefficient is the DD pair itself when discussing deformation theory, so we fix a DD pair $(D, \prec, \succ, \varphi)$ throughout Sections four and five. Denote by $D[[t]]$ the space of formal power series with coefficients in the dendriform algebra D. An element $f_0 + f_1 t + f_2 t^2 + \cdots$ of $D[[t]]$ is sometimes denoted by $(f_0, f_1, f_2, \cdots)$ for convenience.

Definition 4.1. *A **formal one-parameter deformation** of a DD pair $(D, \prec, \succ, \varphi)$ consists of three formal power series*

$$\prec_t = \prec + \sum_{i=1}^{\infty} \prec_i t^i \text{ with } \prec_i \in C^2_{DD}(D)$$

$$\succ_t = \succ + \sum_{i=1}^{\infty} \succ_i t^i \text{ with } \succ_i \in C^2_{DD}(D)$$

$$\varphi_t = \varphi + \sum_{i=1}^{\infty} \varphi_i t^i \text{ with } \varphi_i \in C^1_{DD}(D)$$

such that $(D[[t]], \prec_t, \succ_t, \varphi_t)$ is a DD pair, that is, for any $x, y, z \in D$ we have

$$(x \prec_t y) \prec_t z = x \prec_t (y \prec_t z + y \succ_t z)$$

$$(x \succ_t y) \prec_t z = x \succ_t (y \prec_t z)$$

$$(x \prec_t y + x \succ_t y) \succ_t z = x \succ_t (y \succ_t z)$$

and

$$\varphi_t(x \prec_t y) = \varphi_t(x) \prec_t y + x \prec_t \varphi_t(y)$$

$$\varphi_t(x \succ_t y) = \varphi_t(x) \succ_t y + x \succ_t \varphi_t(y).$$

Therefore, by comparing coefficients for the t^n terms on both sides gives

$$\sum_{i+j=n}((x \prec_i y) \prec_j z - x \prec_i (y \prec_j z + y \succ_j z)) = 0$$

$$\sum_{i+j=n}((x \succ_i y) \prec_j z - x \succ_i (y \prec_j z)) = 0$$

$$\sum_{i+j=n}((x \prec_i y + x \succ_i y) \succ_j z - x \succ_i (y \succ_j z)) = 0$$

and

$$\sum_{i+j=n}\varphi_i(x \prec_j y) = \sum_{i+j=n}\left(\varphi_j(x) \prec_i y + x \prec_i \varphi_j(y)\right)$$

$$\sum_{i+j=n}\varphi_i(x \succ_j y) = \sum_{i+j=n}\left(\varphi_j(x) \succ_i y + x \succ_i \varphi_j(y)\right).$$

For $i \geqslant 0$, define $\pi_i : k[C_2] \otimes D^{\otimes 2} \to D$ by

$$\pi_i([r]; x, y) := \begin{cases} x \prec_i y, \text{ if } [r] = [1] \\ x \succ_i y, \text{ if } [r] = [2] \end{cases}$$

(note that $\pi_0 = \pi$). The above identities can then be expressed as

$$\sum_{i+j=n}\pi_i \circ \pi_j = 0 \quad \text{and} \quad \sum_{i+j=n}(\varphi_i \circ \pi_j - \pi_i \circ (\varphi_j \otimes id) - \pi_i \circ (id \otimes \varphi_j)) = 0 \quad (4)$$

for all $n \geqslant 0$.

The case when $n = 1$ means exactly that

$$d_\pi(\pi_1) = \pi \circ \pi_1 + \pi_1 \circ \pi = 0$$

$$d_\pi(\varphi_1) + \delta(\pi_1) = \varphi \circ \pi_1 - \pi \circ \varphi_1 + \varphi_1 \circ \pi - \pi_1 \circ \varphi = 0$$

that is, (π_1, φ_1) is a 2-cocycle of the DD pair cohomology of $(D, \prec, \succ, \varphi)$.

The element (π_1, φ_1) is called the **infinitesimal of the deformation**. Apparently, (π_n, φ_n) is a 2-cocycle if $(\pi_1, \varphi_1) = \cdots = (\pi_{n-1}, \varphi_{n-1}) = 0$ and $(\pi_n, \varphi_n) \neq 0$.

Definition 4.2. *Two deformations* $(\prec_t, \succ_t, \varphi_t)$ *and* $(\prec'_t, \succ'_t, \varphi'_t)$ *of* $(D, \prec, \succ, \varphi)$ *are* ***equivalent*** *if there is a formal isomorphism* $f_t = id + \sum_{i=1}^{\infty} f_i t^i$ *with* $f_i \in C^1_{DD}(D)$ *such that*

$$f_t(x \prec_t y) = f_t(x) \prec_t f_t(y), \quad f_t(x \succ_t y) = f_t(x) \succ_t f_t(y) \quad \text{and} \quad \varphi'_t(f_t(x)) = f_t(\varphi_t(x))$$

for any $x, y \in D$.

Again, by comparing coefficients of the t^n terms the above equalities gives

$$\sum_{i+j=n} f_i \circ \prec_j = \sum_{i+j+k=n} \prec'_i \circ (f_j \otimes f_k)$$

$$\sum_{i+j=n} f_i \circ \succ_j = \sum_{i+j+k=n} \succ'_i \circ (f_j \otimes f_k)$$

that is, $\sum_{i+j=n} f_i \circ \pi_j = \sum_{i+j+k=n} \pi'_k(f_i \otimes f_j)$ and $\sum_{i+j=n} \varphi'_i \circ f_j = \sum_{i+j=n} f_i \circ \varphi_j$. For $n = 1$, we get

$$f_1 \circ \pi + \pi_1 = \pi'_1 + \pi(f_1 \otimes id) + \pi(id \otimes f_1) \Leftrightarrow (\pi_1 - \pi'_1) = \pi(f_1 \otimes id) + \pi(id \otimes f_1) - f_1 \circ \pi = \pi \circ f_1 - f_1 \circ \pi$$

$$\varphi_1 + f_1 \circ \varphi = \varphi'_1 + \varphi \circ f_1$$

hence $(\pi_1, \varphi_1) - (\pi'_1, \varphi'_1) = \partial(f_1)$. Thus we have shown

Proposition 4.3. *The infinitesimals of two equivalent formal deformations of $(D, \prec, \succ, \varphi)$ correspond to the same cohomology class.*

4.1. Rigidity

Definition 4.4. *A formal deformation $(\prec_t, \succ_t, \varphi_t)$ of a DD pair $(D, \prec, \succ, \varphi)$ is **trivial** if it's equivalent to $(\prec, \succ, \varphi)$. A DD pair $(D, \prec, \succ, \varphi)$ is **rigid** if every formal deformation is trivial.*

Theorem 4.5. *The DD pair $(D, \prec, \succ, \varphi)$ is rigid if $H^2_{DD}(D) = 0$.*

Proof. The infinitesimal (π_1, φ_1) of any formal deformation $(\prec_t, \succ_t, \varphi_t)$ is a 2-cocycle so there is some $f_1 \in C^1_{DD}(D)$ such that

$$(\pi_1, \varphi_1) = -\partial(f_1) = -(\pi \circ (f_1 \otimes id) + \pi \circ (id \otimes f_1) - f_1 \circ \pi, \varphi \circ f_1 - f_1 \circ \varphi) \quad (*)$$

as $H^2_{DD}(D) = 0$. Define $f_t := id + f_1 t$ and $f_t^{-1} = id - f_1 t + f_1^2 t^2 + \cdots$ the inverse, set

$$\begin{aligned} \pi'_t := f_t^{-1} \circ \pi_t \circ (f_t \otimes f_t) &= (id - f_1 t + f_1^2 t^2 + \cdots) \circ (\pi + (\pi \circ (f_1 \otimes id) + \pi \circ (id \otimes f_1) + \pi_1)t + \cdots) \\ &= \pi + (\pi \circ (f_1 \otimes id) + \pi \circ (id \otimes f_1) + \pi_1 - f_1 \circ \pi)t + \cdots \\ &\overset{(*)}{=} \pi + (\cdots)t^2 + \cdots \quad (=: \pi + \pi'_2 t^2 + \cdots) \\ \varphi'_t := f_t^{-1} \circ (\varphi_t \circ f_t) &= (id - f_1 t + f_1^2 t^2 + \cdots) \circ (\varphi + (\varphi \circ f_1 + \varphi_1)t + \cdots) \\ &= \varphi + (\varphi \circ f_1 + \varphi_1 - f_1 \circ \varphi)t + \cdots \\ &\overset{(*)}{=} \varphi + (\cdots)t^2 + \cdots \quad (=: \varphi + \varphi'_2 t^2 + \cdots). \end{aligned}$$

It is obvious that $(\prec'_t, \succ'_t, \varphi'_t)$ with $- \prec'_t - := \pi'_t([1]; -, -)$ and $- \succ'_t - := \pi'_t([2]; -, -)$ defines an equivalent formal deformation of $(\prec_t, \succ_t, \varphi_t)$.

To sum up, we have shown that the degree one terms in π'_t, φ'_t vanish by the fact that (π_1, φ_1) has a preimage f_1. Now, (π'_2, φ'_2) is again a 2-cocycle so by repeating the above argument we conclude that (π_t, φ_t) is equivalent to (π, φ). □

4.2. Finite order deformations and extensions

Definition 4.6. 1. *An **order** n **deformation** of a DD pair $(D, \prec, \succ, \varphi)$ are finite sums $\prec_t = \sum_{i=0}^{n} \prec_i t^i$, $\succ_t = \sum_{i=0}^{n} \succ_i t^i$ and $\varphi_t = \sum_{i=0}^{n} \varphi_i t^i$ that makes $(D[[t]]/(t^{n+1}), \prec_t, \succ_t, \varphi_t)$ into a DD pair.*

2. *Let $(\prec_t, \succ_t, \varphi_t)$ be an order n deformation and $\prec_{n+1}, \succ_{n+1} \in C^2_{DD}(D)$, $\varphi_{n+1} \in C^1_{DD}(D)$ such that $(\prec_t + \prec_{n+1} t^{n+1}, \succ_t + \succ_{n+1} t^{n+1}, \varphi_t + \varphi_{n+1} t^{n+1})$ is an order $n+1$ deformation, we say $(\prec_t, \succ_t, \varphi_t)$ **extends** to an order $n+1$ deformation.*

Let $(\prec_t + \prec_{n+1} t^{n+1}, \succ_t + \succ_{n+1} t^{n+1}, \varphi_t + \varphi_{n+1} t^{n+1}) =: (\prec_{t+1}, \succ_{t+1}, \varphi_{t+1})$ be an extension of $(\prec_t, \succ_t, \varphi_t)$, then equations in (4) on page 7 give, for $i = 1, 2, \cdots, n$, that

$$\pi \circ \pi_i + \pi_1 \circ \pi_{i-1} \cdots + \pi_i \circ \pi = 0 \Leftrightarrow -d_\pi(\pi_i) = \sum_{s+t=i, s,t \geqslant 1} \pi_s \circ \pi_t$$

$$-d_\pi(\pi_{n+1}) = \sum_{p+q=n+1, p,q \geqslant 1} \pi_p \circ \pi_q \qquad (5)$$

and similarly

$$\sum_{s+t=i}(\varphi_s\circ\pi_t-\pi_s\circ(\varphi_t\otimes id)-\pi_s\circ(id\otimes\varphi_t))=0\Leftrightarrow\sum_{\substack{s+t=n\\ s,t\geqslant 1}}[\varphi_s,\pi_t]=[\pi,\varphi_n]-[\varphi,\pi_n]$$

$$\sum_{p+q=n+1}(\varphi_p\circ\pi_q-\pi_p\circ(\varphi_q\otimes id)-\pi_p\circ(id\otimes\varphi_q))=0\Leftrightarrow\sum_{\substack{p+q=n+1\\ p,q\geqslant 1}}[\varphi_p,\pi_q]=[\pi,\varphi_{n+1}]-[\varphi,\pi_{n+1}]\qquad(6)$$

so being extensible it's really asking (5) and (6). Define $ob(\prec_t,\succ_t,\varphi_t):=(\sum\limits_{\substack{s+t=n+1\\ s,t\geqslant 1}}\pi_s\circ\pi_t,\sum\limits_{\substack{p+q=n+1\\ p,q\geqslant 1}}[\varphi_p,\pi_q])\in C^3_{DD}(D)\oplus C^2_{DD}(D)$ to be the **obstruction** to extend the order n deformation $(\prec_t,\succ_t,\varphi_t)$ to an order $n+1$ deformation of the DD pair.

Proposition 4.7. *The obstruction is a 3-cocycle for the cohomology of the DD pair.*

Proof. It's enough to show

$$\partial(\sum_{\substack{s+t=n+1\\ s,t\geqslant 1}}\pi_s\circ\pi_t,\sum_{\substack{p+q=n+1\\ p,q\geqslant 1}}[\varphi_p,\pi_q])=(d_\pi(\sum_{\substack{s+t=n+1\\ s,t\geqslant 1}}\pi_s\circ\pi_t),-\delta(\sum_{\substack{s+t=n+1\\ s,t\geqslant 1}}\pi_s\circ\pi_t)+d_\pi(\sum_{\substack{p+q=n+1\\ p,q\geqslant 1}}[\varphi_p,\pi_q]))=0.$$

Indeed, we have

$$d_\pi(\sum_{\substack{s+t=n+1\\ s,t\geqslant 1}}\pi_s\circ\pi_t)\overset{(5)}{=}-d_\pi(\pi\circ\pi_{n+1}+\pi_{n+1}\circ\pi)=-d_\pi(d_\pi(\pi_{n+1}))=0$$

$$-\delta(\sum_{\substack{s+t=n+1\\ s,t\geqslant 1}}\pi_s\circ\pi_t)+d_\pi(\sum_{\substack{p+q=n+1\\ p,q\geqslant 1}}[\varphi_p,\pi_q])\overset{(5),(6)}{=}\delta(d_\pi(\pi_{n+1}))+d_\pi([\pi,\varphi_{n+1}]-[\varphi,\pi_{n+1}])$$
$$=\delta(d_\pi(\pi_{n+1}))+d_\pi(-\delta(\pi_{n+1})+d_\pi(\varphi_{n+1}))=0$$

by Lemma 3.2 and the fact that d_π is a differential. □

Call the cohomology class $[ob(\prec_t,\succ_t,\varphi_t)]$ defined by the obstruction $ob(\prec_t,\succ_t,\varphi_t)$ the **obstruction class** for $(\prec_t,\succ_t,\varphi_t)$ to be extensible.

Theorem 4.8. *Any order n deformation $(\prec_t,\succ_t,\varphi_t)$ is extensible if and only if the obstruction class is trivial.*

Proof. Suppose $(\prec_t,\succ_t,\varphi_t)$ is extensible so there are $\prec_{n+1},\succ_{n+1}\in C^2_{DD}(D)$ and $\varphi_{n+1}\in C^1_{DD}(D)$ such that equalities (5) and (6) hold true, that is, we have $ob(\prec_t,\succ_t,\varphi_t)=\partial(\pi_{n+1},\varphi_{n+1})$ so the obstruction class $[ob(\prec_t,\succ_t,\varphi_t)]$ is trivial.

Conversely, $[ob(\prec_t,\succ_t,\varphi_t)]=0$ means $ob(\prec_t,\succ_t,\varphi_t)=\partial(\pi_{n+1},\varphi_{n+1})$ for some $(\pi_{n+1},\varphi_{n+1})\in C^2_{DD}(D)\oplus C^1_{DD}(D)$ so equalities (5) and (6) are satisfied, hence $(\widetilde{\prec}_t,\widetilde{\succ}_t,\widetilde{\varphi}_t):=(\prec_t+\prec_{n+1}t^{n+1},\succ_t+\succ_{n+1}t^{n+1},\varphi_t+\varphi_{n+1}t^{n+1})$ is a deformation of order $n+1$. □

Corollary 4.9. *Every finite order deformation is extensible if $H^3_{DD}(D)=0$.*

Corollary 4.10. *Every 2-cocycle of the DD cohomology is the infinitesimal of some formal deformation if $H^3_{DD}(D)=0$.*

Proof. Every 2-cocycle $(\pi_1,\varphi_1)\in C^2_{DD}(D)\oplus C^2_{DD}(D)$ gives rise to an order 1 deformation $(\prec+\prec_1 t,\succ+\succ_1 t,\varphi+\varphi_1 t)$ with $x\prec_1 y:=\pi_1([1];x,y)$, $x\succ_1 y:=\pi_1([2];x,y)$, which in turn extends to a formal deformation as $H^3_{DD}(D)=0$. □

5. An intrinsic characterization

We can define a degree -1 graded Lie bracket on the graded space $\bigoplus_n C^n_{DD}(D)$ by

$$C^m_{DD}(D) \times C^n_{DD}(D) \xrightarrow{[-,-]_{DD}} C^{m+n-1}_{DD}(D)$$

$$((f,g),(f',g')) \longmapsto ([f,f'], (-1)^{m+1}[f,g'] + [g,f'])$$

therefore the shifted graded space $\bigoplus_n C^{n+1}_{DD}(D)$ carries a graded Lie bracket. An element $(f,g) \in C^2_{DD}(D) = C^2_D(D) \oplus C^1_D(D)$ is a **Maurer-Cartan element** of the graded Lie algebra $\bigoplus_n C^{n+1}_{DD}(D)$ if $[(f,g),(f,g)]_{DD} = 0$. As $[(f,g),(f,g)]_{DD} = ([f,f], 2[f,g]) = 0$, this means $f \in C^2_D(D)$ is a multiplication and $g \in C^1_D(D)$ is a derivation with respect to the multiplication f (by defining $- \prec - := f([1];-,-)$ and $- \succ - := f([2];-,-)$, we see g is a derivation). In fact, we have just shown

Proposition 5.1. *Let $(D,\prec,\succ,\varphi)$ be a DD pair, then $(\bigoplus_n C^n_{DD}(D), [-,-]_{DD})$ is a graded Lie algebra and the Maurer-Cartan elements are precisely the DD pair structures on D.*

Let $(D,\prec,\succ,\varphi)$ be a DD pair, then the graded Jacobi identity of the above graded Lie algebra structure indicates that the map

$$d_{(\pi,\varphi)} : C^n_{DD}(D) \to C^{n+1}_{DD}(D)$$

defined as $d_{(\pi,\varphi)}(f,g) := [(\pi,\varphi),(f,g)]_{DD}$ is a graded derivation of the graded Lie algebra structure such that $d^2_{(\pi,\varphi)} = 0$, that is, we have

Proposition 5.2. *$(\bigoplus_n C^n_{DD}(D), [-,-]_{DD}, d_{(\pi,\varphi)})$ is a dg Lie algebra, for any DD pair $(D,\prec,\succ,\varphi)$.*

Theorem 5.3. *Let $(D,\prec,\succ,\varphi)$ be a DD pair and $(\pi',\varphi') \in C^2_{DD}(D) = C^2_D(D) \oplus C^1_D(D)$. Then $(D, \prec + \prec', \succ + \succ', \varphi + \varphi')$ (with $- \prec' - := \pi'(1;-,-), - \succ' - := \pi'(2;-,-)$) is a DD pair if and only if (π',φ') is a Maurer-Cartan element of the dg Lie algebra $(\bigoplus_n C^n_{DD}(D), [-,-]_{DD}, d_{(\pi,\varphi)})$, i.e., (π',φ') satisfies the Maurer-Cartan equation*

$$d_{(\pi,\varphi)}(\pi',\varphi') + \frac{1}{2}[(\pi',\varphi'),(\pi',\varphi')]_{DD} = 0.$$

Proof.

$$\begin{aligned}(D, \prec + \prec', \succ + \succ', \varphi + \varphi') \text{ is a DD pair } &\Leftrightarrow [(\pi + \pi', \varphi + \varphi'),(\pi + \pi', \varphi + \varphi')]_{DD} = 0\\ &\Leftrightarrow 2[(\pi,\varphi),(\pi',\varphi')]_{Der} + [(\pi',\varphi'),(\pi',\varphi')]_{DD} = 0\\ &\Leftrightarrow d_{(\pi,\varphi)}(\pi',\varphi') + \frac{1}{2}[(\pi',\varphi'),(\pi',\varphi')]_{DD} = 0\end{aligned}$$

as $[(\pi,\varphi),(\pi,\varphi)]_{DD} = 0$. □

6. Abelian extensions

Definition 6.1. *Let $(D,\prec_D,\succ_D,\varphi_D)$ be a DD pair and $(M,\prec_M,\succ_M,\varphi_M)$ a trivial DD pair (i.e., the two dendriform operations $\prec_M = \succ_M = 0$). A **abelian extension** of D by M is an exact sequence of DD pairs*

$$0 \to (M,\prec_M,\succ_M,\varphi_M) \xrightarrow{i} (E,\prec_E,\succ_E,\varphi_E) \xrightarrow{p} (D,\prec_D,\succ_D,\varphi_D) \to 0$$

such that p has a section, that is, a linear map $s : D \to E$ such that $p \circ s = id$.

For any abelian extension, $E \cong D \oplus M$ via $s : D \to E$ and s, i, p are the obvious ones and it induces a D-representation structure on M by

$$x \prec m := s(x) \prec_E i(m) = (x, 0) \prec_E (0, m), \qquad x \succ m := s(x) \succ_E i(m) = (x, 0) \succ_E (0, m)$$
$$m \prec x := i(m) \prec_E s(x) = (0, m) \prec_E (x, 0), \qquad m \succ x := i(m) \succ_E s(x) = (0, m) \succ_E (x, 0)$$

and

$$\varphi_M(m) := \varphi_E(i(m)) = \varphi_E(0, m).$$

Also, notice that we have

$$(0, m) \prec_E (0, n) = i(m) \prec_E i(n) = i(m \prec_M n) = (0, m \prec_M n) = (0, 0)$$
$$(0, m) \succ_E (0, n) = i(m) \succ_E i(n) = i(m \succ_M n) = (0, m \succ_M n) = (0, 0)$$

as i is a DD pair morphism and $\prec_M = \succ_M = 0$.

Definition 6.2. *Two abelian extensions $(E, \prec_E, \succ_E, \varphi_E)$ and $(F, \prec_F, \succ_F, \varphi_F)$ are **equivalent** if there is a DD pair morphism $\eta : (E, \prec_E, \succ_E, \varphi_E) \to (F, \prec_F, \succ_F, \varphi_F)$ such that the following diagram commutes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & E & \xrightarrow{p} & D & \longrightarrow & 0 \\ & & \big\downarrow = & & \big\downarrow \eta & & \big\downarrow = & & \\ 0 & \longrightarrow & M & \xrightarrow[i']{} & F & \xrightarrow[p']{} & D & \longrightarrow & 0 \end{array}$$

For an D-representation M, denote by $Ext(D, M)$ the equivalence classes of abelian extensions of D by M with the above induced D-representation structure on M.

Lemma 6.3. *Suppose E is an abelian extension of D by M. Then*

1. *$(x, 0) \prec_E (y, 0) = (x \prec_D y, f([1]; x, y))$ and $(x, 0) \succ_E (y, 0) = (x \succ_D y, f([2]; x, y))$ for some $f \in C^2_{DD}(D, M)$.*
2. *$\varphi_E(x, 0) = (\varphi_D(x), g(x))$ for some $g \in C^1_{DD}(D, M)$.*
3. *Suppose there is an equivalent abelian extension F as in the Definition 6.2, then $\eta : D \oplus M \to D \oplus M$ must has the form $(x, m) \mapsto (x, m + h(x))$ for some $h \in C^1_{DD}(D, M)$.*

Proof. 1. p is a DD pair morphism gives $p((x, 0) \prec_E (y, 0)) = x \prec_D y$ and $p((x, 0) \succ_E (y, 0)) = x \succ_D y$ hence we know there is such an $f \in C^2_{DD}(D, M)$. Indeed, the second coordinates of $(x, 0) \prec_E (y, 0)$ and $(x, 0) \succ_E (y, 0)$ are elements in M determined by x, y and respectively $\prec_E, \succ_E$ and this means exactly that there is an element f in $C^2_{DD}(D, M)$ gives the second coordinates.

2. Similarly, the existence of $g \in C^1_{DD}(D, M)$ is also a consequence of p is a morphism of DD pairs, this time it comes from $p \circ \varphi_E = \varphi_D \circ p$.

3. From the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & D \oplus M & \xrightarrow{p} & D & \longrightarrow & 0 \\ & & \big\downarrow = & & \big\downarrow \eta & & \big\downarrow = & & \\ 0 & \longrightarrow & M & \xrightarrow[i']{} & D \oplus M & \xrightarrow[p']{} & D & \longrightarrow & 0 \end{array}$$

the commutativity of the right square indicates that $\eta(x, m) = (x, \alpha(x, m))$ for some $\alpha : D \oplus M \to M$. Meanwhile, the commutativity of the left square gives $\eta(0, m) = (0, \alpha(0, m)) = (0, m)$ so $\alpha(0, m) = m$ for any $m \in M$. Now,

$$\eta(x, m) = \eta(x, 0) + \eta(0, m) = (x, \alpha(x, 0)) + (0, \alpha(0, m)) = (x, m + \alpha(x, 0))$$

define $\alpha(x, 0) =: h(x)$ finishes the proof.

□

Lemma 6.4. *1. Any 2-cocycle $(f, g) \in C^2_{DD}(D, M)$ gives rise to an abelian extension $0 \to M \to D \oplus M \to D \to 0$.*

2. Any abelian extension of D by M gives rise to a 2-cocycle of the DD pair cochain complex.

Proof. 1. For any such 2-cocycle (f, g), we have $df = 0$, that is, f is a 2-cocycle for the dendriform cohomology of $(D, \prec_D, \succ_D)$ with representation M and $dg + \delta f = 0$. There are operations on $E := D \oplus M$ given by

$$(x, m) \prec_E (y, n) := (x \prec_D y, x \prec n + m \prec y + f([1]; x, y))$$

$$(x, m) \succ_E (y, n) := (x \succ_D y, x \succ n + m \succ y + f([2]; x, y))$$

$$\varphi_E(x, m) := (\varphi_D(x), \varphi_M(m) + g(x)).$$

One checks directly that the first two operations defines a dendriform algebra structure on E as $df = 0$ and $dg + \delta f = 0$. For example,

$$\big((x, m) \prec_E (y, n)\big) \prec_E (z, l) = (x, m) \prec_E \big((y, n) \prec_E (z, l) + (y, n) \succ_E (z, l)\big)$$

is equivalent to

$$((x \prec_D y) \prec_D z, (x \prec_D y) \prec l + (x \prec n) \prec z + (m \prec y) \prec z + f([1]; x, y) \prec z + f([1]; x \prec_D y, z))$$

$$= (x \prec_D (y \prec_D z + y \succ_D z), \quad x \prec (y \prec l + y \succ l) + x \prec (n \prec z + n \succ z) + m \prec (y \prec_D z + y \succ_D z)$$
$$+ x \prec (f([1], y, z) + f([2]; y, z)) + f([1]; x, y \prec_D z) + f([1]; x, y \succ_D z))$$

which means

$$f([1]; x, y) \prec z + f([1]; x \prec_D y, z) = x \prec f([1]; y, z) + x \prec f([2]; y, z) + f([1]; x, y \prec_D z) + f([1]; x, y \succ_D z).$$

By equality (1) on page 5 (since $d_\pi f = 0$), $LHS = x \prec f([1] + [2]; y, z) + f([1]; x, y \prec_D z + y \succ_D z) = RHS$. The remaining two equalities for the dendriform algebra structure can be proved similarly using equalities (2) and (3) on page 5.

To show that φ_E is a derivation on E, we need to prove

$$\varphi_E((x, m) \prec_E (y, n)) = \varphi_E(x, m) \prec_E (y, n) + (x, m) \prec_E \varphi_E(y, n)$$

$$\varphi_E((x, m) \succ_E (y, n)) = \varphi_E(x, m) \succ_E (y, n) + (x, m) \succ_E \varphi_E(y, n)$$

the first equality is in fact

$$(\varphi_D(x \prec_D y), \quad \varphi_M(x \prec n) + \varphi_M(m \prec y) + \varphi_M(f([1]; x, y)) + g(x \prec_D y))$$

$$= \big(\varphi_D(x) \prec_D y + x \prec_D \varphi_D(y), \varphi_D(x) \prec n + x \prec \varphi_M(n)$$
$$+ \varphi_M(m) \prec y + m \prec \varphi_D(y) + g(x) \prec y + x \prec g(y) + f([1]; \varphi_D(x), y) + f([1]; x, \varphi_D(y))\big)$$

that is,

$$g(x \prec_D y) - x \prec g(y) - g(x) \prec y) + \varphi_M(f([1]; x, y)) - f([1]; \varphi_D(x), y) - f([1]; x, \varphi_D(y)) = 0$$

$$\Leftrightarrow dg + \delta f = 0.$$

The second equality can be proved in exactly the same way. Therefore,

$$0 \to (M, \prec, \succ, \varphi_M) \xrightarrow{i} (E, \prec_E, \succ_E, \varphi_E) \xrightarrow{p} (D, \prec_D, \succ_D, \varphi_D) \to 0$$

is an abelian extension with the obvious section s.

2. Recall that $E \cong D \oplus M$ via $s : D \to E$ for any abelian extension $0 \to M \xrightarrow{i} E \xrightarrow{p} D \to 0$ and Lemma 6.3 indicates that there are maps f and g such that

$$(x,0) \prec_E (y,0) = (x \prec_D y, f([1];x,y))$$
$$(x,0) \succ_E (y,0) = (x \succ_D y, f([2];x,y))$$
$$\varphi_E(x,0) = (\varphi_D(x), g(x))$$

we claim that (f,g) is the desired 2-cocycle.

First, the dendriform algebra identities on E implies that $df = 0$. Indeed,

$$((x,0) \prec_E (y,0)) \prec_E (z,0) = (x,0) \prec_E ((y,0) \prec_E (z,0) + (y,0) \succ_E (z,0))$$
$$\Leftrightarrow (x \prec_D y, f([1];x,y)) \prec_E (z,0) = (x,0) \prec_E ((y \prec_D z, f([1];y,z)) + (y \succ_D z, f([2];y,z))).$$

Note that $(x,0) \prec_E (0,m) = (0, x \prec m)$ and $(x,0) \succ_E (0,m) = (0, x \succ m)$ as these are part of the D-representation structure on M in an abelian extension (See the discussion above Definition 6.2). Hence the LHS of the above equality is

$$(x \prec_D y, 0) \prec_E (z,0) + (0, f([1];x,y)) \prec_E (z,0) = ((x \prec_D y) \prec_D z, f([1]; x \prec_D y, z)) + (0, f([1];x,y) \prec z))$$

and the RHS is

$$(x,0) \prec_E (y \prec_D z + y \succ_D z, f([1]+[2];y,z)) = (x,0) \prec_E (y \prec_D z + y \succ_D z, 0) + (x,0) \prec_E (0, f([1]+[2];y,z))$$
$$= (x \prec_D (y \prec_D z + y \succ_D z), f([1];x, y \prec_D z + y \succ_D z)) + (0, x \prec f([1]+[2];y,z)).$$

Therefore the first dendriform identity gives

$$f([1]; x \prec_D y, z) + f([1];x,y) \prec z = f([1]; x, y \prec_D z + y \succ_D z) + x \prec f([1]+[2];y,z)$$

which is the first identity of f being a 2-cocycle (that is, equality (1) on page 5). The remaining identities for f being a 2-cocycle is given by the remaining dendriform identities in a similar way.

Now, the fact that φ_E is a derivation on E gives the equality $dg + \delta f = 0$. Indeed,

$$\varphi_E((x,0) \prec_E (y,0)) = \varphi_E(x,0) \prec_E (y,0) + (x,0) \prec_E \varphi_E(y,0)$$

is in fact

$$\varphi_E(x \prec_D y, 0) + \varphi_E(0, f([1];x,y)) = (\varphi_D(x),0) \prec E(y,0) + (0,g(x)) \prec_E (y,0) + (x,0) \prec_E (\varphi_D(y),0) + (x,0) \prec_E (0,g(y))$$

therefore (also from the discussion above Definition 6.2) we get

$$(\varphi_D(x \prec_D y),0) + (0, g(x \prec_D y)) + \varphi_M(f[1];x,y)) = (\varphi_D(x) \prec_D y, 0) + (0, f([1];\varphi_D(x),y))$$
$$+(0, g(x) \prec y) + (x \prec_D \varphi_D(y), 0) + (0, f([1];x,\varphi_D(y))) + (0, x \prec g(y))$$

which means

$$g(x \prec_D y) - g(x) \prec y - x \prec g(y) + \varphi_M(f([1];x,y)) - f([1];\varphi_D(x),y) - f([1];x,\varphi_D(y)) = 0.$$

Similarly, we get

$$g(x \succ_D y) - g(x) \succ y - x \succ g(y) + \varphi_M(f([2];x,y)) - f([2];\varphi_D(x),y) - f([2];x,\varphi_D(y)) = 0$$

from $\varphi_E((x,0) \succ_E (y,0)) = \varphi_E(x,0) \succ_E (y,0) + (x,0) \succ_E \varphi_E(y,0)$ and together these indicate that $dg + \delta f = 0$.

Therefore $(f,g) \in C^2_{DD}(D,M)$ is a 2-cocycle. □

Theorem 6.5. $H^2_{DD}(D,M) \cong Ext(D,M)$.

Proof. From Lemma 6.4 (1) we know there one can construct an abelian extension from any 2-cocycle (f,g) of $C^2_{DD}(D,M)$. Suppose $(f',g') \in C^2_{DD}(D,M)$ is another 2-cocycle whose cohomology class is the same as (f,g) (so there is some $h \in C^1_{DD}(D,M)$ such that $(f,g)-(f',g') = \partial h$) and $(E' = D \oplus M, \prec_{E'}, \succ_{E'}, \varphi_{E'})$ is the DD pair obtained (in the same way as above) from (f',g'). One see immediately that the map $E \to E'$ (where $(x,m) \mapsto (x, m+h(x))$) defines an equivalence between abelian extension E and E', so the map $H^2_{DD}(D,M) \to Ext(D,M)$ is well-defined.

Conversely, there is a 2-cocycle $(f,g) \in C^2_{DD}(D,M)$ from any abelian extension of D by M by Lemma 6.4 (2). Suppose there is an equivalent abelian extension F as in the Definition 6.2, then from Lemma 6.2 (3) we know $\eta : E = D \oplus M \to D \oplus M$ is given by $(x,m) \mapsto (x, m+h(x))$ for some $h \in C^1_{DD}(D,M)$. Now, from

$$\eta((x,0) \prec_E (y,0)) = \eta(x,0) \prec_F \eta(y,0) \quad \eta((x,0) \succ_E (y,0)) = \eta(x,0) \succ_F \eta(y,0)$$

(since η is a DD pair morphism) we get $f'([r];x,y) - f([r];x,y) = (\delta h)([r];x,y)$ with f' the 2-cocycle induced from the extension E', hence the map $Ext(D,M) \to H^2_{DD}(D,M)$ is well-defined.

To sum up, we have established two maps between $H^2_{DD}(D,M)$ and $Ext(D,M)$ and they are obviously inverses to each other. □

Acknowledgements

The author would like to thank the referee for the valuable comments and suggestions on the paper. This work is supported by the Guangdong Basic and Applied Basic Research Foundation (Grant No.2022A1515012176, No.2018A030313581) and Shenzhen University of Information Technology (2023djpj-gyb05).

References

[1] V.Coll, M.Gerstenhaber, A.Giaquinto, An explicit deformation formula with noncommuting derivations. In Ring Theory 1989 (Ramat Gan and Jerusalem, 1988/1989), volume 1 of Israel Math. Conf. Proc., 396-403. Weizmann, Jerusalem, 1989.
[2] A.Das, Leibniz algebras with derivations. J. Homotopy Relat. Struct., 16 (2021), no.2, 245-274.
[3] A.Das, Cohomology and deformations of dendriform algebras, and $Dend_\infty$-algebras. Comm. Algebra 50 (2022), no.4, 1544-1567.
[4] A.Das, A.Mandal, Extensions, deformation and categorification of AssDer pairs. Arxiv preprint (2020), arXiv:2002.11415.
[5] M.Gerstenhaber, A.Voronov, Homotopy G-algebras and moduli space operad. Int. Math. Res. Not. IMRN (1995), 141-153.
[6] S.Guo, Central Extensions and Deformations of Lie Triple Systems with a Derivation. J. Math. Res. Appl., 42 (2022), no.2, 189-198.
[7] S.Guo, R.Saha, On 3-Lie algebras with a derivation. Afr. Mat., 33 (2022), article numer 60.
[8] Y.Li, D.Wang, Hom-Lie algebras with Derivations. Front. Math., 19 (2024), 535-550.
[9] Y.Li, D.Wang, Cohomologies of Hom-Leibniz Algebras with Derivations. Algebra Colloq., 32 (2025), 285-298.
[10] J.L.Loday, Dialgebras. in: Dialgebras and related operads, in: Lecture Notes in Math. 1763 (2001), 7-66.
[11] J.L.Loday, B.Vallette, Algebraic operads. Grundlehren Math. Wiss. 346 (2012), Heidelberg, Springer.
[12] A.Magid, Lectures on differential Galois theory. Univ. Lecture Ser. 7 (1994), American Mathematical Society, Providence, RI.
[13] R.Tang, Y.Frégier, Y.Sheng, Cohomologies of a Lie algebra with a derivation and applications. J. Algebra, 534 (2019), no.2, 65-99.
[14] T.Voronov, Higher derived brackets and homotopy algebras. J. Pure Appl. Algebra 202 (2005), no.1-3, 133-153.
[15] X.Wu, Y.Ma, B.Sun, L.Chen, Cohomology of Leibniz triple systems with derivations. J. Geom. Phys., 179 (2022), 104594.
[16] D.Yau, Gerstenhaber structure and Delignes conjecture for Loday algebras. J. Pure Appl. Algebra 209 (2007), no.3, 739-752.
[17] X.Yu, Operad with a prescribed element. J. Algebra Appl. 17 (2018), no.9, 1850174, 31pp.