



On S -Artinian and S -Noetherian dimensions

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Abstract. Let S be a multiplicative subset of a ring R , and let M be an R -module. In this article, we introduce and study the concepts of S -Artinian dimension and S -Noetherian dimension of an R -module M . These dimensions are ordinal numbers, and in essence, they measure the deviation of an R -module M from being S -Artinian and S -Noetherian. We observe some basic facts for modules with these dimensions, similar to the basic properties of modules with the Krull and Noetherian dimensions.

1. Introduction

Due to the importance of Noetherian and Artinian rings and modules, there are several attempts to generalize these concepts. Anderson and Dumitrescu introduced the concept of S -Noetherian modules over commutative rings, [3]. Ahmed and Sanna in [1], tried to characterize the concept of S -Noetherian modules via a suitable chain condition and a special kind of maximality. An ascending chain $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$ of submodules of M is called S -stationary if there exists a positive integer k and $s \in S$ such that for each $n \geq k$ we have $sN_n \subseteq N_k$. Let F be a set of submodules of M . A submodule N of F is called S -maximal if there exists $s \in S$ such that for every $L \in F$ with $N \subseteq L$ we have $sL \subseteq N$. They showed that if every nonempty set of ideals of R has an S -maximal element then R is S -Noetherian and the latter implies that every ascending chain of submodules of M is S -stationary. In [7], Bilgin, Reyes, and Tekir characterized S -Noetherian modules over noncommutative rings. A commutative ring R is called an S -Noetherian ring if for any ideal I of R , there exist $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$, see [3]. In [31], Sevim, Tekir, and Koc studied the duality of the S -Artinian concept. They introduced the concept of S -Artinian rings and finitely S -cogenerated rings. Let R be a commutative ring and $S \subseteq R$ be a multiplicative closed subset. R is called an S -Artinian ring if for each descending chain of ideals $\{I_n\}_{n \in \mathbb{N}}$ of R , there exist $s \in S$ and $k \in \mathbb{N}$ such that $sI_k \subseteq I_n$ for all $n \geq k$. Also in [29], M. Özen, O.A. Nazi, Ü. Tekir, and K.P. Shum studied S -Artinian modules.

The Krull dimension (resp., Noetherian dimension) measures how close a module is to being Artinian (resp., Noetherian). For the reader's convenience, we now recall these definitions.

The *Krull dimension* of a module was defined by Gordon and Robson [15] in the following way.

Definition 1.1. Let M be a left R -module. The *Krull dimension* of M , which will be denoted by $k\text{-dim}(M)$, is defined by transfinite recursion as follows:

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1. if $M = 0$, then $k\text{-dim}(M) = -1$;
2. if α is an ordinal number and $k\text{-dim}(M) \not\leq \alpha$, then $k\text{-dim}(M) = \alpha$, provided there is no infinite descending chain $M = M_0 \supseteq M_1 \supseteq \dots$ of submodules M_i such that $k\text{-dim}(M_{i-1}/M_i) \not\leq \alpha$ for all $i \in \mathbb{N}$;
3. it is possible that there is no ordinal α such that $k\text{-dim}(M) = \alpha$. In this case, M has no Krull dimension.

Following [18], the Noetherian dimension (the terms dual Krull dimension [2], and N -dimension [8] are also used) of a module is defined inductively in the following way.

Definition 1.2. Let M be a left R -module. The Noetherian dimension of M , which will be denoted by $n\text{-dim}(M)$, is defined as follows:

1. if $M = 0$, then $n\text{-dim}(M) = -1$;
2. if α is an ordinal number and $n\text{-dim}(M) \not\leq \alpha$, then $n\text{-dim}(M) = \alpha$, provided there is no infinite ascending chain $M_0 \subseteq M_1 \subseteq \dots$ of submodules M_i such that $n\text{-dim}(M_i/M_{i-1}) \not\leq \alpha$ for all $i \in \mathbb{N}$;
3. it is possible that there is no ordinal α such that $n\text{-dim}(M) = \alpha$. In this case, M has no Noetherian dimension.

Motivated by these generalizations, in this paper, we introduce and study the concepts of S -Artinian dimension and S -Noetherian dimension of an R -module M , denoted by $Sa\text{-dim}(M)$ and $Sn\text{-dim}(M)$, respectively. These dimensions are ordinal numbers and measure how close a module is to being S -Artinian (resp., S -Noetherian). In fact, these dimensions are the S -version of Krull dimension and Noetherian dimension. In Section 2 of this paper, we investigate some basic properties of S -Artinian dimension. We extend most of the basic results of S -Artinian modules to S -Artinian dimension. If S is a multiplicative closed subset of a ring R , we observe that every R -module with Krull dimension has S -Artinian dimension. However, the converse is not true in general, see Example 2.3. We also observe that the behavior of S -Artinian dimension is similar to the behavior of the Krull dimension. We show that $k\text{-dim} M = \alpha$ if and only if $Sa\text{-dim}(M) = \alpha$, where $S = R - \mathcal{M}$ for each $\mathcal{M} \in \text{Max}(R)$, see Theorem 2.14. If a nonzero R -module M has Krull dimension and α is an ordinal number, then M is called α -critical if $k\text{-dim}(M) = \alpha$ and $k\text{-dim} \frac{M}{N} < \alpha$ for all nonzero N of M . An R -module M is called critical if M is α -critical for some ordinal α . Section 3 is devoted to a brief study of S -critical modules, which are the S -version of critical modules. In Section 4, we study some basic properties of modules with S -Noetherian dimension. The S -Noetherian dimension in fact is the dual of S -Artinian dimension. In essence, it measures the deviation of an R -module from being S -Noetherian. We also observe that the behavior of this dimension is similar to the behavior of the Noetherian dimension. We show that $n\text{-dim}(M) = \alpha$ if and only if $Sn\text{-dim}(M) = \alpha$, where $S = R - \mathcal{M}$ for each $\mathcal{M} \in \text{Max}(R)$, see Theorem 4.13. If an R -module M has Noetherian dimension and α is an ordinal number, then M is called α -atomic if $n\text{-dim}(M) = \alpha$ and $n\text{-dim}(N) < \alpha$ for all proper submodule N of M . An R -module M is called atomic if it is α -atomic for some ordinal α . Finally, in the last section, we study the concept of S -atomic modules. In fact, these modules are the S -version of atomic modules.

2. On S -Artinian dimension of modules

In this section, we define a dimension for modules inspired by a definition found in [29]. Before addressing it, we recall the definition of an S -Artinian module in the following.

Definition 2.1. Let S be a multiplicative closed subset of a ring R . An R -module M is said to be S -Artinian if for every descending chain of submodules

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$$

of M , there exist $s \in S$ and $k \in \mathbb{N}$ such that $sM_k \subseteq M_n$ for every $n \geq k$.

Given the above definition, we can now define the S -Artinian dimension as follows.

Definition 2.2. Let S be a multiplicative closed subset of R and M be an R -module. The S -Artinian dimension of M , which will be denoted by $Sa\text{-dim}(M)$, is defined by transfinite recursion as follows:

1. $Sa\text{-dim}(M) = 0$ if and only if M is S -Artinian.
2. If α is an ordinal number and $Sa\text{-dim}(M) \not\leq \alpha$, then $Sa\text{-dim}(M) = \alpha$ provided that there is no infinite descending chain of submodules $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ such that $Sa\text{-dim}(\frac{M_i}{M_{i+1}}) \not\leq \alpha$ for each $i \in \mathbb{N}$.
3. It is possible that there is no ordinal α such that $Sa\text{-dim}(M) = \alpha$. In this case, M has no S -Artinian dimension.

We define the (right) S -Artinian dimension of a ring to be its dimension as a (right) module over itself.

We recall that, the definition of the S -Artinian dimension of an R -module M is modeled after the definition of Krull dimension, see [15]. Specifically, we define $Sa\text{-dim}(M) = 0$ if and only if M is an S -Artinian module.

We define $Sa\text{-dim}M = 1$ if M is not S -Artinian, but for every descending chain of submodules

$$M_1 \supseteq M_2 \supseteq \dots,$$

almost all of the quotients $\frac{M_i}{M_{i+1}}$ are S -Artinian.

For example, consider the \mathbb{Z} -module \mathbb{Z} with $S = \mathbb{Z} \setminus \{0\}$. This module is not S -Artinian, since the chain

$$2\mathbb{Z} \supseteq 2^2\mathbb{Z} \supseteq 2^3\mathbb{Z} \supseteq 2^5\mathbb{Z} \supseteq 2^7\mathbb{Z} \supseteq 2^{11}\mathbb{Z} \supseteq \dots$$

is not S -stationary. However, for each submodule $n\mathbb{Z}$ of \mathbb{Z} , the quotient $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is Artinian, and hence S -Artinian, see [6, Example 1]. This implies that $Sa\text{-dim} \mathbb{Z} = 1$.

Example 2.3. 1. Let S be a multiplicative closed subset of a ring R . Every R -module with Krull dimension has S -Artinian dimension. However, the converse is not true in general. To see this, let $R = (\bigoplus_{i \in I} \mathbb{Z}) \oplus \mathbb{Q}$ and $S = \{((m_i)_{i \in I}, \frac{a}{b}) \mid 0 \neq \frac{a}{b} \in \mathbb{Q}, m_i \in \mathbb{Z} \text{ for all } i \in I\}$. Then, $M = R$ as a left R -module is an S -Artinian module and $Sa\text{-dim}(M) = 0$. But, it does not have Krull dimension.

2. Let S be a multiplicative closed subset of a ring R . Every Noetherian module has S -Artinian dimension. We first claim that every module with Krull dimension has S -Artinian dimension, and in fact $Sa\text{-dim}M \leq k\text{-dim}M$. Let M be a nonzero R -module with $k\text{-dim}M = \alpha$. We proceed by transfinite induction on α .

- If $\alpha = 0$, then M is Artinian, and hence S -Artinian, see [6, Example 2]. Therefore, $Sa\text{-dim}M = 0$.
- Now assume $\alpha > 0$ and the claim holds for all ordinals less than α . Let

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots$$

be a descending chain of submodules of M . Since M has Krull dimension, there exists an integer n such that for all $k \geq n$, we have

$$k\text{-dim}(M_k/M_{k+1}) < \alpha.$$

By the induction hypothesis, it follows that for each $k \geq n$,

$$Sa\text{-dim}(M_k/M_{k+1}) < \alpha.$$

Hence the chain satisfies the condition for $Sa\text{-dim}M \leq \alpha$.

Since every Noetherian module has Krull dimension, see [15, Proposition 1.3], we conclude that every Noetherian module has S -Artinian dimension.

3. Let $S = \mathbb{Z} \setminus \{0\}$. Then, the \mathbb{Z} -module \mathbb{Z} is not an S -Artinian module but $Sa\text{-dim}(\mathbb{Z}) = 1$.
4. By [15, Theorem 9.8], for every ordinal number α , there exists a Noetherian domain R such that $k\text{-dim}(R) = \alpha$. Taking $S = \{1\}$, we obtain an R -module M with $Sa\text{-dim}(M) = \alpha$.

Example 2.4. Let $R = \mathbb{Q}[x_1, x_2, \dots, x_n]$ be a polynomial ring over the field of rational numbers. Then, $M = R$ as an R -module is Noetherian. Take $S = \{x_1^n : n \geq 0\}$.

By [15, Corollary 9.4], R has Krull dimension and $k\text{-dim}(R) = 1$. Consider the descending chain of submodules:

$$(x_1) \supseteq (x_1^2) \supseteq (x_1^3) \supseteq (x_1^5) \supseteq (x_1^7) \supseteq (x_1^{11}) \supseteq \dots,$$

which is not S -stationary. Hence R is not an S -Artinian module. However, by the argument in Example 2.3(2), we conclude that $Sa\text{-dim}(R) = 1$.

Let S be a multiplicative closed subset of R . In [6, Theorem 3.14], it is proved that an R -module M is an S -Artinian module if and only if N and $\frac{M}{N}$ are S -Artinian modules for a submodule N of M . Before we prove a closely related result, we need the following lemma.

Lemma 2.5. *Let S be a multiplicative closed subset of a ring R . Let N be a nonzero proper submodule of an R -module M and α be an ordinal number. If $Sa\text{-dim}(M) \leq \alpha$, then N and $\frac{M}{N}$ have S -Artinian dimension, $Sa\text{-dim}(N) \leq \alpha$, and $Sa\text{-dim}(\frac{M}{N}) \leq \alpha$.*

Proof. Assume $Sa\text{-dim}(M) \leq \alpha$. Suppose that $N_0 \supseteq N_1 \supseteq N_2 \supseteq \dots \supseteq N_n \supseteq \dots$ is a descending chain of submodules of N . From $M \supset N$ and $Sa\text{-dim}(M) \leq \alpha$, there exists $k \in \mathbb{N}$ such that $Sa\text{-dim}(\frac{N_i}{N_{i+1}}) < \alpha$ for each $i \geq k$. Hence, N has S -Artinian dimension and $Sa\text{-dim}(N) \leq \alpha$. Now, suppose that $\frac{M_0}{N} \supseteq \frac{M_1}{N} \supseteq \frac{M_2}{N} \supseteq \dots \supseteq \frac{M_n}{N} \supseteq \dots$ is a descending chain of submodules of $\frac{M}{N}$. Thus, $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ is a descending chain of submodules of M . Since $Sa\text{-dim}(M) \leq \alpha$, there exists $k \in \mathbb{N}$ such that $Sa\text{-dim}(\frac{M_i}{M_{i+1}}) < \alpha$ for each $i \geq k$. We note that

$$Sa\text{-dim}(\frac{\frac{M_i}{N}}{\frac{M_{i+1}}{N}}) = Sa\text{-dim}(\frac{M_i}{M_{i+1}}) < \alpha.$$

Therefore, $\frac{M}{N}$ has S -Artinian dimension and $Sa\text{-dim}(\frac{M}{N}) \leq \alpha$. \square

Now, we are in a position to prove the following result.

Theorem 2.6. *Let S be a multiplicative closed subset of a ring R . Let N be a nonzero proper submodule of an R -module M . The following statements are equivalent.*

1. M has S -Artinian dimension.
2. N and $\frac{M}{N}$ have S -Artinian dimension and $Sa\text{-dim}(M) = \max\{Sa\text{-dim}(N), Sa\text{-dim}(\frac{M}{N})\}$.

Proof. (1) \Rightarrow (2) It follows from Lemma 2.5.

(2) \Rightarrow (1) Let $\max\{Sa\text{-dim}(N), Sa\text{-dim}(\frac{M}{N})\} = \alpha$. We proceed by induction on α . If $\alpha = 0$, then N and $\frac{M}{N}$ are S -Artinian modules. With the help of [6, Theorem 3.14], there is nothing to prove. Now, assume that for all ordinals less than α , the statement holds. If $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ is an arbitrary descending chain of submodules of M , then $M_0 \cap N \supseteq M_1 \cap N \supseteq M_2 \cap N \supseteq \dots \supseteq M_n \cap N \supseteq \dots$ and $\frac{M_0+N}{N} \supseteq \frac{M_1+N}{N} \supseteq \frac{M_2+N}{N} \supseteq \dots \supseteq \frac{M_n+N}{N} \supseteq \dots$ are descending chains of submodules of N and $\frac{M}{N}$, respectively. Since N and $\frac{M}{N}$ have S -Artinian dimension, so there exists $k \in \mathbb{N}$ such that $Sa\text{-dim}(\frac{M_{n+N}}{M_{n+1+N}}) < \alpha$ and $Sa\text{-dim}(\frac{M_n \cap N}{M_{n+1} \cap N}) < \alpha$ for all $n \geq k$. By the modular law, we have

$$\frac{M_n \cap N}{M_{n+1} \cap N} = \frac{M_n \cap N}{M_{n+1} \cap (M_n \cap N)} \simeq \frac{M_{n+1} + (M_n \cap N)}{M_{n+1}}$$

and

$$\frac{M_n + N}{M_{n+1} + N} = \frac{M_n + (M_{n+1} + N)}{(M_{n+1} + N)} \simeq \frac{M_n}{M_n \cap (M_{n+1} + N)} = \frac{M_n}{M_{n+1} + (M_n \cap N)}.$$

So,

$$Sa\text{-dim}(\frac{M_{n+1} + (M_n \cap N)}{M_{n+1}}) < \alpha \quad \text{and} \quad Sa\text{-dim}(\frac{M_n}{M_{n+1} + (M_n \cap N)}) < \alpha.$$

Thus,

$$\max\{Sa\text{-dim}(\frac{M_{n+1} + (M_n \cap N)}{M_{n+1}}), Sa\text{-dim}(\frac{M_n}{M_{n+1} + (M_n \cap N)})\} < \alpha.$$

Now, by the induction hypothesis

$$Sa\text{-dim}(\frac{M_n}{M_{n+1}}) \leq \max\{Sa\text{-dim}(\frac{M_{n+1} + (M_n \cap N)}{M_{n+1}}), Sa\text{-dim}(\frac{M_n}{M_{n+1} + (M_n \cap N)})\} < \alpha.$$

Hence, we deduce $Sa\text{-dim}(\frac{M_n}{M_{n+1}}) < \alpha$ for each $n \geq k$. This yields that $Sa\text{-dim}(M) \leq \alpha$, and we are done. \square

Theorem 2.6 leads the following corollary.

Corollary 2.7. *Let $M = \bigoplus_{i=1}^n M_i$ be a nonzero R -module. If each M_i has S -Artinian dimension, then M has S -Artinian dimension and $Sa\text{-dim}(M) = \max\{Sa\text{-dim}(M_i)\}_{i=1}^n$.*

Corollary 2.8. *Let S be a multiplicative closed subset of a ring R . The following statements hold for an R -module M .*

1. *If every nonzero proper submodule of M has S -Artinian dimension, then M has S -Artinian dimension and $Sa\text{-dim}(M) = \sup\{Sa\text{-dim}(N) : N \subsetneq M\}$.*
2. *If every proper quotient of M has S -Artinian dimension, then M has S -Artinian dimension and $Sa\text{-dim}(M) \leq \sup\{Sa\text{-dim}(\frac{M}{N}) : 0 \neq N \subsetneq M\} + 1$.*
3. *If for every nonzero proper submodule N of M , either N or $\frac{M}{N}$ has S -Artinian dimension, then M has S -Artinian dimension.*

Proof. (1) We put $\alpha = \sup\{Sa\text{-dim}(N) : 0 \neq N \subsetneq M\}$. Assume $M \supset M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ is an arbitrary descending chain of submodules of M . So, $M_2 \supseteq M_3 \supseteq \dots \supseteq M_n \supseteq \dots$ is a descending chain of submodules of M_1 . Since M_1 is a proper submodule of M , then M_1 has S -Artinian dimension. Thus, there exists $k \in \mathbb{N}$ such that

$$Sa\text{-dim}(\frac{M_i}{M_{i+1}}) < Sa\text{-dim}(M_1) \leq \alpha$$

for each $i \geq k$. This implies that $Sa\text{-dim}(M) \leq \alpha$. By hypothesis, for every nonzero proper submodule N of M , we have $Sa\text{-dim}(N) \leq Sa\text{-dim}(M)$. Thus, we infer that $\alpha \leq Sa\text{-dim}(M)$ and so $Sa\text{-dim}(M) = \alpha$, as desired.

(2) We put $\alpha = \sup\{Sa\text{-dim}(\frac{M}{N}) : 0 \neq N \subsetneq M\}$. Assume $M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ is an infinite descending chain of submodules of M . Then, we have $\frac{M_i}{M_{i+1}} \subseteq \frac{M}{M_{i+1}}$ for each i . Since $\frac{M}{M_{i+1}}$ has S -Artinian dimension, then $Sa\text{-dim}(\frac{M_i}{M_{i+1}}) \leq Sa\text{-dim}(\frac{M}{M_{i+1}}) \leq \alpha < \alpha + 1$ for all i . Hence, we conclude that $Sa\text{-dim}(M) \leq \alpha + 1$.

(3) Let $M \supseteq N_1 \supseteq N_2 \supseteq \dots \supseteq N_n \supseteq \dots$ be an infinite descending chain of submodules of M . If N_k has S -Artinian dimension for some k , then $\frac{N_i}{N_{i+1}}$ has S -Artinian dimension for each $i \geq k$. Otherwise, $\frac{M}{N_{i+1}}$ and $\frac{N_i}{N_{i+1}}$ have S -Artinian dimension for each i . Thus, there exists $k \in \mathbb{N}$ such that $\frac{N_i}{N_{i+1}}$ has S -Artinian dimension for all $i \geq k$. Hence, M has S -Artinian dimension and we are done. \square

Corollary 2.9. *Let S be a multiplicative closed subset of R . Assume M is an R -module such that $Sa\text{-dim}(M) \geq 1$. If f is an injective endomorphism of M then $Sa\text{-dim}(M) \geq Sa\text{-dim}(\frac{M}{f(M)}) + 1$.*

Proof. Put $\alpha = Sa\text{-dim}(\frac{M}{f(M)})$. Consider the descending chain $M \supseteq f(M) \supseteq f^2(M) \supseteq \dots \supseteq f^n(M) \supseteq \dots$ of submodules of M . From $\frac{f^i(M)}{f^{i+1}(M)} \simeq \frac{M}{f(M)}$, we have

$$\alpha = Sa\text{-dim}(\frac{M}{f(M)}) = Sa\text{-dim}(\frac{f^i(M)}{f^{i+1}(M)})$$

for each i . Since $Sa\text{-dim}(\frac{f^i(M)}{f^{i+1}(M)}) \geq \alpha$ for each i , we deduce that $Sa\text{-dim}(M) > \alpha$. Hence, $Sa\text{-dim}(M) \geq \alpha + 1$ and we conclude that

$$Sa\text{-dim}(M) \geq Sa\text{-dim}(\frac{M}{f(M)}) + 1.$$

\square

Let S_1 and S_2 be two multiplicative closed subsets of a ring R . Below, we discuss the relationship between two different S_1 -Artinian dimension and S_2 -Artinian dimension for an R -module. Before proceeding, we recall a definition from [13]. Let S be a multiplicative closed subset of a ring R . The set $S^* = \{a \in R : a \text{ is a unit of } S^{-1}R\}$ is called *the saturation of S* . Obviously, a multiplicative closed subset $S \subseteq R$ is a saturated set if $S = S^*$.

Theorem 2.10. Let S_1 and S_2 be two multiplicative closed subsets of a ring R such that $S_1 \subseteq S_2$. The following statements hold for a nonzero R -module M .

1. If $S_1a\text{-dim}(M) = \alpha$, then M has S_2 -Artinian dimension and $S_2a\text{-dim}(M) \leq \alpha$.
2. M has S^* -Artinian dimension and $S^*a\text{-dim}(M) \leq \alpha$ if and only if $Sa\text{-dim}(M) \leq \alpha$.

Proof. (1) Assume $S_1a\text{-dim}(M) = \alpha$. We show that $S_2a\text{-dim}(M) \leq \alpha$. We proceed by induction on α . If $\alpha = 0$, then M is an S_1 -Artinian R -module. By [29, Lemma 1], M is an S_2 -Artinian R -module and so $S_2a\text{-dim}(M) = 0$. Now, assume that for all ordinals less than α , the statement holds. Let $M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ be an arbitrary descending chain of submodules of M . Since M has S_1 -Artinian dimension, there exists $k \in \mathbb{N}$ such that $S_1a\text{-dim}(\frac{M_i}{M_{i+1}}) < \alpha$, for each $i \geq k$. Thus, by the induction hypothesis, $\frac{M_i}{M_{i+1}}$ has S_2 -Artinian dimension and $S_2a\text{-dim}(\frac{M_i}{M_{i+1}}) < \alpha$. Therefore, M has S_2 -Artinian dimension and $S_2a\text{-dim}(M) \leq \alpha$.

(2) From $S \subseteq S^*$, we have $S^*a\text{-dim}(M) \leq Sa\text{-dim}(M)$ by part (1). Now assume $S^*a\text{-dim}(M) = \alpha$. We proceed by induction on α . If $\alpha = 0$, then M is an S^* -Artinian R -module. By [29, Lemma 1, ii], M is an S -Artinian R -module and so $Sa\text{-dim}(M) = 0$. Now, suppose that for all ordinals less than α , the statement holds. Let $M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ is an arbitrary descending chain of submodules of M . Since M is an S^* -Artinian R -module, there exists $k \in \mathbb{N}$ such that $S^*a\text{-dim}(\frac{M_i}{M_{i+1}}) < \alpha$, for each $i \geq k$. Thus, by the induction hypothesis, $\frac{M_i}{M_{i+1}}$ has S -Artinian dimension and $Sa\text{-dim}(\frac{M_i}{M_{i+1}}) < \alpha$. Therefore, M has S -Artinian dimension and $Sa\text{-dim}(M) \leq \alpha$. \square

We recall the following result from [29, Lemma 1].

Lemma 2.11. Let M be an S -Artinian module with $\text{ann}_R(M) \cap S = \emptyset$. Then $S^{-1}M$ is an Artinian $S^{-1}R$ -module.

In view of the previous lemma we have the following result.

Proposition 2.12. Let S be a multiplicative closed subset of a ring R . Assume M is an R -module such that $\text{ann}_R(M) \cap S = \emptyset$. If M has S -Artinian dimension and $Sa\text{-dim}(M) = \alpha$, then the $S^{-1}R$ -module $S^{-1}M$ has Krull dimension and $k\text{-dim}(S^{-1}M) \leq \alpha$.

Proof. Assume $Sa\text{-dim}(M) = \alpha$. We proceed by induction on α . If $\alpha = 0$, then M is an S -Artinian R -module. By Lemma 2.11, $S^{-1}M$ is an Artinian $S^{-1}R$ -module and so $k\text{-dim}(S^{-1}M) = 0$. Now, assume that the statement holds for all ordinals less than α . Let $M_1 \supseteq M_2 \supseteq \dots$ be a descending chain of submodules of the $S^{-1}R$ -module $S^{-1}M$. For each i , there exists an R -submodule N_i of M such that $M_i = S^{-1}N_i$. Hence, we have the descending chain $N_1 \supseteq N_1 \cap N_2 \supseteq \dots$ of submodules of the R -module M . Since $Sa\text{-dim}(M) = \alpha$, there exists an integer k such that $Sa\text{-dim}(\frac{N_1 \cap \dots \cap N_i}{N_1 \cap N_2 \cap \dots \cap N_i \cap N_{i+1}}) < \alpha$, for each $i \geq k$.

By the induction hypothesis, we have $k\text{-dim}(S^{-1}(\frac{N_1 \cap \dots \cap N_i}{N_1 \cap N_2 \cap \dots \cap N_i \cap N_{i+1}})) < \alpha$, for each $i \geq k$. But whenever A and B are submodules of the R -module M and $A \subseteq B$, we have $S^{-1}(\frac{A}{B}) \simeq \frac{S^{-1}A}{S^{-1}B}$, this implies that $k\text{-dim}(\frac{S^{-1}(N_1 \cap \dots \cap N_i)}{S^{-1}(N_1 \cap \dots \cap N_i \cap N_{i+1})}) < \alpha$, for each $i \geq k$. Thus, $k\text{-dim}(\frac{M_1 \cap \dots \cap M_i}{M_1 \cap M_2 \cap \dots \cap M_{i+1}}) < \alpha$, for each $i \geq k$, (note, $S^{-1}(N_1 \cap \dots \cap N_i) = S^{-1}N_1 \cap \dots \cap S^{-1}N_i$). This implies that $k\text{-dim}(\frac{M_i}{M_{i+1}}) < \alpha$ for each $i \geq k$. Therefore $k\text{-dim}(S^{-1}M) \leq \alpha$, as desired. \square

As noted earlier, if M has Krull dimension, then M has S -Artinian dimension. In the following, we give a sufficient condition under which the converse is true. To state it, we need recall some definitions. An R -module M is called *multiplication* if for every submodule N of M there exists an ideal I of R such that $N = IM$. For an R -module M , we let $U_M(R) := \{r \in R : rM = M\}$. Clearly, $U(R) \subseteq U_M(R)$ and $U_M(R)$ is a saturated multiplicative closed subset.

Corollary 2.13. Let S be a multiplicative closed subset of a ring R and M be a multiplication R -module. The following statements hold.

1. If $S \subseteq U_M(R)$, then M has Krull dimension if and only if M has S -Artinian dimension.
2. If $S \subseteq U_M(R)$ and M has S -Artinian dimension, then M has finite Goldie dimension.

Proof. (1) The proof follows from [29, Lemma 1, iii].
 (2) It follows from part (1). \square

Let M be a nonzero R -module. Clearly, $S_P = R \setminus P$ is a multiplicative closed subset of R for a prime ideal of R . Following [29], we say M has P -Artinian dimension, denoted by $Pa\text{-dim}(M)$, whenever M has S_P -Artinian dimension.

Theorem 2.14. *Let M be a nonzero R -module. The following statements are equivalent.*

1. M has Krull dimension and $k\text{-dim}(M) = \alpha$.
2. M has P -Artinian dimension for each $P \in \text{Spec}(R)$ and $Pa\text{-dim}(M) = \alpha$.
3. M has \mathcal{M} -Artinian dimension for each $\mathcal{M} \in \text{Max}(R)$ and $\mathcal{M}a\text{-dim}(M) = \alpha$.

Proof. (1) \Rightarrow (2) \Rightarrow (3) Clear.

(3) \Rightarrow (1) We proceed by induction on α , where α is an ordinal number. If $\mathcal{M}a\text{-dim}(M) = 0$, then M is an \mathcal{M} -Artinian module. By [29, Theorem 2], M is an Artinian module. Thus, it has Krull dimension and $k\text{-dim}(M) = 0$. Now, assume that the statement holds for all ordinals less than α . Suppose that $\mathcal{M}a\text{-dim}(M) = \alpha$ and $M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ is a descending chain of submodules of M . Since M has \mathcal{M} -Artinian dimension, there exists $k \in \mathbb{N}$ such that $\mathcal{M}a\text{-dim}(\frac{M_i}{M_{i+1}}) < \alpha$ for each $i \geq k$. By the induction hypothesis, $\frac{M_i}{M_{i+1}}$ has Krull dimension and $k\text{-dim}(\frac{M_i}{M_{i+1}}) < \alpha$. Hence, we have $k\text{-dim}(M) \leq \alpha$. Since $\mathcal{M}a\text{-dim}(M) \leq k\text{-dim}(M)$, we conclude that $\mathcal{M}a\text{-dim}(M) = k\text{-dim}(M) = \alpha$. \square

We close this section by two results about rings with S -Artinian dimension.

Corollary 2.15. *Let S be a multiplicative closed subset of a ring R . Let M be a finitely generated R -module. If R has S -Artinian dimension, then $Sa\text{-dim}(M) \leq Sa\text{-dim}(R)$.*

Proof. Since M is a finitely generated R -module, we infer that $M \simeq \frac{F}{N}$, where $F \simeq R^k$ and $N \subseteq F$. So, we have $Sa\text{-dim}(M) = Sa\text{-dim}(\frac{F}{N}) \leq Sa\text{-dim}(F) = Sa\text{-dim}(R^k)$. By Corollary 2.7, $Sa\text{-dim}(F) = \max\{Sa\text{-dim}(R), \dots, Sa\text{-dim}(R)\}$. Hence $Sa\text{-dim}(M) \leq Sa\text{-dim}(R)$. \square

Corollary 2.16. *Let S be a multiplicative closed subset of a ring R . If R has S -Artinian dimension and $c \in R$ is a non-zero-divisor of R , then $Sa\text{-dim}(\frac{R}{Rc}) < Sa\text{-dim}(R)$.*

Proof. Consider the descending chain of left ideals $Rc \supseteq Rc^2 \supseteq \dots \supseteq Rc^n \supseteq \dots$. Since R has S -Artinian dimension, there exists $k \in \mathbb{N}$ such that $Sa\text{-dim}(\frac{Rc^n}{Rc^{n+1}}) < Sa\text{-dim}(R)$ for each $n \geq k$. But, $\frac{Rc^n}{Rc^{n+1}} \simeq \frac{R}{Rc}$ for each n , and so $Sa\text{-dim}(\frac{R}{Rc}) < Sa\text{-dim}(R)$. \square

3. S,α -critical modules

We recall from [15], that an R -module M is called α -critical if $k\text{-dim}(M) = \alpha$, which $\alpha \geq 0$ is an ordinal number, and $k\text{-dim}(\frac{M}{N}) < \alpha$ for every submodule N of M . This section introduces the concept of an S,α -critical submodule and explores its relation to the S -Artinian dimension.

Definition 3.1. *Let S be a multiplicative closed subset of a ring R and let $\alpha \geq 1$ be an ordinal number. We say an R -module M is S,α -critical if $Sa\text{-dim}(M) = \alpha$ and $Sa\text{-dim}(\frac{M}{N}) < \alpha$ for every proper nonzero submodule N of M .*

Lemma 3.2. *Let S be a multiplicative closed subset of a ring R and M be an S,α -critical R -module. Every nonzero submodule of M is an S,α -critical R -module.*

Proof. Let N be a submodule of M . Since M is an S,α -critical R -module, we have $Sa\text{-dim}(\frac{M}{N}) < \alpha$. In view of Lemma 2.6, we infer that

$$\alpha = Sa\text{-dim}(M) = \max\{Sa\text{-dim}(N), Sa\text{-dim}(\frac{M}{N})\}.$$

This yields that $Sa\text{-dim}(N) = \alpha$. If B is a nonzero submodule of N , then $Sa\text{-dim}(\frac{N}{B}) \leq Sa\text{-dim}(\frac{M}{B}) < \alpha$. Thus, $Sa\text{-dim}(\frac{N}{B}) < \alpha$ for each $0 \neq B \subset N$. This yields that N is an $S.\alpha$ -critical R -module, as desired. \square

We recall that a uniform module is a nonzero module M , such that the intersection of any two nonzero submodules of M is nonzero. Now we have the following result.

Lemma 3.3. *Let S be a multiplicative closed subset of a ring R . If M is an $S.\alpha$ -critical R -module, then M is uniform.*

Proof. Assume, for a contradiction, M is not uniform. Hence, there are two nonzero submodules A_1 and A_2 of M such that $A_1 \cap A_2 = 0$. Since $0 \neq A_1 \subseteq M$ and $0 \neq A_2 \subseteq M$, we have $A_1 \oplus A_2 \subseteq M$. From Lemma 3.2, we deduce that A_1 and A_2 are $S.\alpha$ -critical submodules and $Sa\text{-dim}(A_1) = Sa\text{-dim}(A_2) = Sa\text{-dim}(A_1 \oplus A_2) = \alpha$. By Lemma 3.2, we have $A_1 \oplus A_2$ is an $S.\alpha$ -critical R -module. Since $\frac{(A_1 \oplus A_2)}{A_1} \simeq A_2$, we conclude that

$$\alpha = Sa\text{-dim}(A_2) = Sa\text{-dim}(\frac{(A_1 \oplus A_2)}{A_1}) < Sa\text{-dim}(A_1 \oplus A_2) = \alpha.$$

That is a contradiction. \square

Lemma 3.4. *Let S be a multiplicative closed subset of a ring R . If M is an $S.\alpha$ -critical R -module and $f : M \rightarrow M$ is a nonzero endomorphism, then f is one to one.*

Proof. Assume, for a contradiction, $\text{Ker } f \neq 0$. Since M is an $S.\alpha$ -critical R -module, $Sa\text{-dim}(M) = \alpha$ and $Sa\text{-dim}(\frac{M}{N}) < \alpha$ for each $0 \neq N \subset M$. From $0 \neq \text{Ker } f \subset M$, we have $Sa\text{-dim}(\frac{M}{\text{Ker } f}) < Sa\text{-dim}(M)$. But, $(\frac{M}{\text{Ker } f}) \simeq f(M) \subseteq M$, and so $Sa\text{-dim}(\frac{M}{\text{Ker } f}) = Sa\text{-dim}(f(M))$. Clearly, every submodule of an $S.\alpha$ -critical module is an $S.\alpha$ -critical module. Thus, we infer that $Sa\text{-dim}(f(M)) = Sa\text{-dim}(M) = \alpha$, a contradiction. \square

Lemma 3.4 has the following corollary.

Corollary 3.5. *Let S be a multiplicative closed subset of a ring R . If R has S -Artinian dimension and I is a left $S.\alpha$ -critical ideal of R , then one of the following occurs:*

1. $I^2 = 0$.
2. There exists $0 \neq c \in I$ such that $l.\text{ann}(c) \cap I = 0$.

Proof. Assume that $I^2 \neq 0$. Thus, there exists $0 \neq c \in I$ such that $cI \neq 0$. Define $f : I \rightarrow I$ such that $f(x) = xc$. It is clear that f is a homomorphism. In view of Lemma 3.4, f is a monomorphism, so $l.\text{ann}(c) \cap I = \text{Ker } f = 0$. \square

Proposition 3.6. *Let S be a multiplicative closed subset of a ring R . If an R -module M has S -Artinian dimension then it is an $S.\alpha$ -critical submodule for an ordinal number α .*

Proof. Let N be a nonzero submodule of M with the least S -Artinian dimension α . Assume, for a contradiction, N is not an $S.\alpha$ -critical submodule. Hence, there exists $0 \neq N_1 \subset N$ such that $Sa\text{-dim}(\frac{N}{N_1}) = \alpha$. By our assumption, we have $Sa\text{-dim}(N_1) = \alpha$. If N_1 is not an $S.\alpha$ -critical submodule, then there exists $0 \neq N_2 \subset N_1$ such that $Sa\text{-dim}(\frac{N_1}{N_2}) = \alpha$ and $Sa\text{-dim}(N_2) = \alpha$. By continuing this method, M does not have any $S.\alpha$ -critical submodule and there exists $N_1 \supset N_2 \supset \dots$, which is an infinite descending chain of submodules of M and $Sa\text{-dim}(\frac{N_i}{N_{i+1}}) = \alpha$ for each i . This is a contradiction. \square

Lemma 3.7. *Let S be a multiplicative closed subset of a ring R . If M is a nonzero Noetherian R -module and $Sa\text{-dim}(M) = \alpha$, then M has a proper nonzero submodule N such that $\frac{M}{N}$ is an $S.\alpha$ -critical module.*

Proof. Consider the set $\Sigma = \{N \subset M : Sa\text{-dim}(\frac{M}{N}) = \alpha\}$. From $0 \in \Sigma$, we have $\Sigma \neq \emptyset$. Since M is a Noetherian R -module, we infer that Σ has a maximal element, say N . Thus, $Sa\text{-dim}(\frac{M}{N}) = \alpha$ and $N \subset M$. It suffices to show that $Sa\text{-dim}(\frac{M}{K}) < \alpha$ for each $0 \neq K \subset N$. In view of Lemma 2.6, we infer that $Sa\text{-dim}(\frac{M}{N}) = \max\{Sa\text{-dim}(\frac{K}{N}), Sa\text{-dim}(\frac{M}{K})\}$. However, $\frac{M}{K} \simeq \frac{\frac{M}{N}}{\frac{K}{N}}$ and so $Sa\text{-dim}(\frac{M}{K}) = Sa\text{-dim}(\frac{\frac{M}{N}}{\frac{K}{N}})$. Thus,

$$Sa\text{-dim}\left(\frac{M}{N}\right) = \max\{Sa\text{-dim}\left(\frac{K}{N}\right), Sa\text{-dim}\left(\frac{M}{K}\right)\}.$$

From $Sa\text{-dim}\left(\frac{M}{N}\right) = \alpha$, we deduce that

$$Sa\text{-dim}\left(\frac{K}{N}\right) \leq \alpha, Sa\text{-dim}\left(\frac{M}{K}\right) \leq \alpha.$$

By our assumption, $K \subset M$ and so $K \in \Sigma$ if $Sa\text{-dim}\left(\frac{M}{K}\right) = \alpha$. Notice that $N \subset K$ contradicts the maximality of N . Thus, $Sa\text{-dim}\left(\frac{M}{K}\right) < \alpha$ and $Sa\text{-dim}\left(\frac{M}{N}\right) < \alpha$. Hence, $\frac{M}{N}$ is an S - α -critical module, as desired. \square

4. On S -Noetherian dimension of modules

In this section, we define a dimension for modules inspired by a definition found in [7]. Before proceeding, we recall the definition an S -Noetherian module in the following.

Definition 4.1. Let R be a ring and $S \subseteq R$ a multiplicative subset. A left R -module M is said to be S -finite if $sM \subseteq F$ for some $s \in S$ and some finitely generated submodule F of M , and M is said to be S -Noetherian if every submodule of M is an S -finite module. A left ideal I of R is S -finite if it is S -finite as a left R -module, and R is called left S -Noetherian if the left R -module $M = R$ is S -Noetherian.

We recall from [7], a chain of submodules $\{N_i\}_i \in I$ of a left R -module M is said to be S -stationary if there exists $j \in I$ and $s \in S$ such that $sN_i \subseteq N_j$ for all $i \in I$.

Theorem 4.2. Let S be a multiplicative subset of R and M a left R -module. The following are equivalent:

1. M is S -Noetherian.
2. Every nonempty chain of submodules of M is S -stationary.
3. Every nonempty set \mathcal{F} of submodules of M has an S -maximal element.

Proof. See [7, Theorem 2.3]. \square

With the help of the above definition, we can now define the S -Noetherian dimension as follows.

Definition 4.3. Let S be a multiplicative closed subset of R and M be an R -module. The S -Noetherian dimension of M , which will be denoted by $Sn\text{-dim}(M)$, is defined by transfinite recursion as follows:

1. $Sn\text{-dim}(M) = 0$ if and only if M is an S -Noetherian module.
2. If α is an ordinal number and $Sn\text{-dim}(M) \not\leq \alpha$, then $Sn\text{-dim}(M) = \alpha$ provided that there is no infinite ascending chain of submodules $M_0 \subseteq M_1 \subseteq \dots$ of submodules M_i such that $Sn\text{-dim}\left(\frac{M_i}{M_{i-1}}\right) \not\leq \alpha$ for each $i \in \mathbb{N}$.
3. It is possible that there is no ordinal α such that $Sn\text{-dim}(M) = \alpha$. In this case, M has no S -Noetherian dimension.

We define the (left) S -Noetherian dimension of a ring to be its dimension as a (left) module over itself.

Remark 4.4. Obviously, every module with Noetherian dimension has S -Noetherian dimension. However, the converse is not true in general. For example, consider $R = \prod_{n=0}^{\infty} \mathbb{F}_{p^n}$ and $S = \{1\} \cup aA^*$, where $A^* = \prod_{n=0}^{\infty} \mathbb{F}_{p^n}^*$. By [6, Example 5], R is an S -Noetherian ring and $Sn\text{-dim}(R) = 0$, but it does not have Noetherian dimension.

Let S be a multiplicative closed subset of a ring R . In [6, Theorem 3.1], it is proved that an R -module M is an S -Noetherian module if and only if N and $\frac{M}{N}$ are S -Noetherian modules for a submodule N of M .

Lemma 4.5. Let S be a multiplicative closed subset of a ring R . Let N be a nonzero proper submodule of an R -module M and α be an ordinal number. Then, $Sn\text{-dim}(M) = \sup\{Sn\text{-dim}(N), Sn\text{-dim}\left(\frac{M}{N}\right)\}$.

Proof. The proof is similar to the proof of Theorem 2.6. \square

Proposition 4.6. Let S be a multiplicative closed subset of a ring R and M be an R -module. The following statements are equivalent.

1. $Sn\text{-dim}(M) = \alpha$.
2. (a) $Sn\text{-dim}(M) \neq \alpha$.
 (b) For each family \mathcal{F} of submodules of M , there exists $N \in \mathcal{F}$ such that $Sn\text{-dim}(\frac{P}{N}) < \alpha$ for each $N \subset P \in \mathcal{F}$.

Proof. (2) \Rightarrow (1) Let $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ be an infinite ascending chain of submodules of M . Consider the family $\mathcal{F} = \{M_1, M_2, \dots\}$. Thus, there exists $M_i \in \mathcal{F}$ such that $Sn\text{-dim}(\frac{M_j}{M_i}) < \alpha$ for each $j \geq i$. Thus, $Sn\text{-dim}(\frac{M_{j+1}}{M_j}) < \alpha$ for each $j \geq i$, and hence $Sn\text{-dim}(M) = \alpha$, as desired.

(1) \Rightarrow (2) Let $Sn\text{-dim}(M) = \alpha$ and \mathcal{F} be a family of submodules of M . Take $N \in \mathcal{F}$. Thus, either $Sn\text{-dim}(\frac{P}{N}) < \alpha$ for each $N \subset P \in \mathcal{F}$ or there exists $N \subset P_1 \in \mathcal{F}$ such that $Sn\text{-dim}(\frac{P_1}{N}) \geq \alpha$. The proof is complete if P_1 satisfies the condition. Otherwise, there exists $P_1 \subset P_2 \in \mathcal{F}$ such that $Sn\text{-dim}(\frac{P_2}{P_1}) \geq \alpha$. By continuing this method, if the assertion is not true, then we have the chain $P_1 \subseteq P_2 \subseteq P_3 \subseteq \dots$ where $Sn\text{-dim}(\frac{P_{i+1}}{P_i}) \geq \alpha$ for each i , a contradiction. \square

The proof of the following result is similar to the proof of Corollary 2.15, and thus it is omitted.

Corollary 4.7. *Let S be a multiplicative closed subset of a ring R . If R has S -Noetherian dimension, then $Sn\text{-dim}(R) = \sup\{Sn\text{-dim}(M) \mid M \text{ is a finitely generated } R\text{-module}\}$.*

Corollary 4.8. *Let S be a multiplicative closed subset of a ring R . The following statements hold for an R -module M .*

1. *If every nonzero proper submodule of M has S -Noetherian dimension, then M has S -Noetherian dimension and $Sn\text{-dim}(M) \leq \sup\{Sn\text{-dim}(N) : N \subset M\} + 1$.*
2. *If every proper quotient of M has S -Noetherian dimension, then M has S -Noetherian dimension and $Sn\text{-dim}(M) = \sup\{Sn\text{-dim}(\frac{M}{N}) : 0 \neq N \subsetneq M\}$.*
3. *If for every nonzero proper submodule N of M either N or $\frac{M}{N}$ has S -Noetherian dimension, then M has S -Noetherian dimension.*

Proof. (1) Assume $\sup\{Sn\text{-dim}(N) : N \subset M\} = \alpha$. Let $M_1 \subseteq M_2 \subseteq \dots \subseteq M_n \subseteq \dots$ be an arbitrary ascending chain of submodules of M . Thus, $Sn\text{-dim}(\frac{M_{i+1}}{M_i}) \leq Sn\text{-dim}(M_{i+1}) \leq \alpha < \alpha + 1$ for each i . Hence, we have $Sn\text{-dim}(M) \leq \alpha + 1$.

(2) Assume $\sup\{Sn\text{-dim}(\frac{M}{N}) : 0 \neq N \subsetneq M\} = \alpha$. Let $M_1 \subseteq M_2 \subseteq \dots \subseteq M_n \subseteq \dots$ be an infinite ascending chain of submodules of M . Thus, we have the chain $\frac{M_{i+1}}{M_i} \subseteq \frac{M_{i+2}}{M_i} \subseteq \dots$ of submodules of $\frac{M}{M_i}$. Since $Sn\text{-dim}(\frac{M}{M_i}) \leq \alpha$, there exists an integer number k such that $Sn\text{-dim}(\frac{M_{j+1}}{M_j}) = Sn\text{-dim}(\frac{M_{i+1}}{M_i}) < Sn\text{-dim}(\frac{M}{M_i}) \leq \alpha$ for each $j \geq k$.

This implies that $Sn\text{-dim}(M) \leq \alpha$, as desired.

(3) Let $N_1 \subseteq N_2 \subseteq \dots \subseteq N_n \subseteq \dots$ be an infinite ascending chain of submodules of M . If $\frac{N}{N_k}$ has S -Noetherian dimension for some k , then $\frac{N_{i+1}}{N_i}$ has S -Noetherian dimension for each $i \geq k$. Otherwise, assume that N_i and $\frac{N_{i+1}}{N_i}$ have S -Noetherian dimension for each i . Thus, there exists $k \in \mathbb{N}$ such that $\frac{N_{i+1}}{N_i}$ has S -Noetherian dimension for all $i \geq k$. This yields that M has S -Noetherian dimension and we are done. \square

In view of [5, Theorem 3.1], now we have the following result.

Corollary 4.9. *Let R be a commutative ring and S be a multiplicative closed subset of R which contains no zero-divisors of R . We recall that $Sa\text{-dim}(R) = 0$ if and only if R is S -Artinian and $Sn\text{-dim}(R) = 0$ if and only if R is S -Noetherian. In view of [5, Theorem 3.1], if R is an S -Artinian ring, then R is an S -Noetherian ring. So by above result $Sa\text{-dim}(R) = 0$ implies that $Sn\text{-dim}(R) = 0$.*

Lemma 4.10. *Let M be an R -module. The following statements hold.*

1. *If $S_1 \subseteq S_2$ are two multiplicative closed subsets of a ring R and M is an S_1 -Noetherian module, then M is an S_2 -Noetherian module.*

2. Let S^* be the saturation of a multiplicative closed subset S of a ring R . An R -module M is S -Noetherian if and only if it is S^* -Noetherian.
3. Let M be a multiplication R -module and $S \subseteq U_M(R)$ be a multiplicative closed subset of R . An R -module M is Noetherian if and only if M is an S -Noetherian module.
4. Let M be an S -Noetherian module with $\text{ann}_R(M) \cap S = \emptyset$. Then $S^{-1}M$ is a Noetherian $S^{-1}R$ -module.

Proof. (1) It is clear.

(2) Let M be an S -Noetherian module. Since $S \subseteq S^*$ are two multiplicative closed subsets of R , thus M is an S^* -Noetherian module by part (1). For the converse, assume that M is an S^* -Noetherian module so every submodule of M is an S^* -finite module. Let N be a submodule of M , then N is an S^* -finite. So $rN \subseteq F$ for some $r \in S^*$ and some finitely generated submodule F of M . Since $r \in S^*$, we infer that $\frac{r}{1}$ is a unit of $S^{-1}R$ and so $\frac{a}{s} \frac{r}{1} = 1$ for some $s \in S$ and $a \in R$. This implies that $uar = us$ for some $u \in S$. Put $s' = us$. Then, $s' \in S$ and $uarN = s'N \subseteq uaF \subseteq F$.

(3) It is straightforward.

(4) A submodule of $S^{-1}M$ is of the form $S^{-1}N$ where N is a submodule of M . Since N is an S -finite submodule of M , there exist $s \in S$ and a finitely generated submodule F of M such that $sN \subseteq F \subseteq N$. Then $S^{-1}N = S^{-1}(sN) \subseteq S^{-1}F \subseteq S^{-1}N$, so $S^{-1}N = S^{-1}F$ is a finitely generated submodule. \square

Theorem 4.11. 1. Let $S_1 \subseteq S_2$ be two multiplicative closed subsets of a ring R and let M be an R -module. If $S_1n\text{-dim}(M) = \alpha$, then M has S_2 -Noetherian dimension and $S_2n\text{-dim}(M) \leq \alpha$.

2. Let S be a multiplicative closed subset of a ring R . An R -module M has S^* -Noetherian dimension and $S^*n\text{-dim}(M) \leq \alpha$ if and only if $Sn\text{-dim}(M) \leq \alpha$.
3. Let S be a multiplicative closed subset of a ring R . Assume M is a multiplication R -module such that $S \subseteq U_M(R)$. Then, M has Noetherian dimension if and only if M has S -Noetherian dimension.
4. Let S be a multiplicative closed subset of a ring R . Assume M is an R -module such that $\text{ann}_R(M) \cap S = \emptyset$. If M has S -Noetherian dimension and $Sn\text{-dim}(M) = \alpha$, then the $S^{-1}R$ -module $S^{-1}M$ has Noetherian dimension and $n\text{-dim}(S^{-1}M) \leq \alpha$.

Proof. (1) Assume $S_1n\text{-dim}(M) = \alpha$. We proceed by induction on α . If $\alpha = 0$, then M is an S_1 -Noetherian R -module. By Lemma 4.10, M is an S_2 -Noetherian R -module and so $S_2n\text{-dim}(M) = 0$. Now, assume that for all ordinals less than α , the statement holds. Let $M_1 \subseteq M_2 \subseteq \dots \subseteq M_n \subseteq \dots$ be an arbitrary ascending chain of submodules of M . Since M has S_1 -Noetherian dimension, there exists $k \in \mathbb{N}$ such that $S_1n\text{-dim}(\frac{M_{i+1}}{M_i}) < \alpha$, for each $i \geq k$. Thus, by the induction hypothesis, $\frac{M_{i+1}}{M_i}$ has S_2 -Noetherian dimension and $S_2n\text{-dim}(\frac{M_{i+1}}{M_i}) < \alpha$. Therefore, M has S_2 -Noetherian dimension and $S_2n\text{-dim}(M) \leq \alpha$.

(2) From $S \subseteq S^*$, we have $S^*n\text{-dim}(M) \leq Sn\text{-dim}(M)$ by part (1). Now, suppose that $S^*n\text{-dim}(M) = \alpha$. We proceed by induction on α . If $\alpha = 0$, then M is an S^* -Noetherian R -module. By Lemma 4.10, M is an S -Noetherian R -module and so $Sn\text{-dim}(M) = 0$. Now, assume that for all ordinals less than α , the statement holds. Let $M_1 \subseteq M_2 \subseteq \dots \subseteq M_n \subseteq \dots$ is an arbitrary ascending chain of submodules of M . Since $S^*n\text{-dim}(M) = \alpha$, there exists $k \in \mathbb{N}$ such that $S^*n\text{-dim}(\frac{M_{i+1}}{M_i}) < \alpha$, for each $i \geq k$. Thus, by the induction hypothesis, $\frac{M_{i+1}}{M_i}$ has S -Noetherian dimension and $Sn\text{-dim}(\frac{M_{i+1}}{M_i}) < \alpha$. Hence, M has S -Noetherian dimension and $Sn\text{-dim}(M) \leq \alpha$.

(3) The result follows from Lemma 4.10.

(4) Assume $Sn\text{-dim}(M) = \alpha$. We proceed by induction on α . If $\alpha = 0$, then M is an S -Noetherian R -module. By Lemma 4.10(4), we infer that $S^{-1}M$ is a Noetherian $S^{-1}R$ -module, and so $n\text{-dim}(S^{-1}M) = 0$. Now, assume that for all ordinals less than α , the statement holds. Let $M_1 \subseteq M_2 \subseteq \dots$ be an ascending chain of submodules of the $S^{-1}R$ -module $S^{-1}M$. For each i , there exists an R -submodule N_i of M such that $M_i = S^{-1}N_i$. Hence we have the ascending chain $N_1 \subseteq N_1 + N_2 \subseteq \dots$ of submodules of the R -module M . Since $Sn\text{-dim}(M) = \alpha$, there exists an integer k such that $Sn\text{-dim}(\frac{N_1 + \dots + N_i + N_{i+1}}{N_1 + \dots + N_i}) < \alpha$, for each $i \geq k$.

By the induction hypothesis, we infer that $n\text{-dim}(S^{-1}(\frac{N_1 + \dots + N_i + N_{i+1}}{N_1 + \dots + N_i})) < \alpha$, for each $i \geq k$. But whenever A and B are submodules of the R -module M and $A \subseteq B$, we have $S^{-1}(\frac{A}{B}) \simeq \frac{S^{-1}A}{S^{-1}B}$, this implies that $n\text{-dim}(\frac{S^{-1}(N_1 + \dots + N_i + N_{i+1})}{S^{-1}(N_1 + \dots + N_i)}) < \alpha$, for each $i \geq k$. Thus $n\text{-dim}(\frac{M_1 + \dots + M_i + M_{i+1}}{M_1 + \dots + M_i}) < \alpha$, for each $i \geq k$, (note, we recall

that $S^{-1}(N_1 + N_2 + \cdots + N_k) = S^{-1}N_1 + S^{-1}N_2 + \cdots + S^{-1}N_k$. Hence $n\text{-dim}(\frac{M_{i+1}}{M_i}) < \alpha$ for each $i \geq k$. Therefore, $n\text{-dim}(S^{-1}M) \leq \alpha$, and we are done. \square

Let M be a nonzero R -module and let P be a prime ideal of R . Let S_P denote the set $R \setminus P$. We say M has P -Noetherian dimension, denoted by $Pn\text{-dim}(M)$, whenever M has S_P -Noetherian dimension.

Theorem 4.12. *Let M be a nonzero R -module. The following statements are equivalent.*

1. M is a Noetherian module.
2. M is a P -Noetherian module for each $P \in \text{Spec}(R)$.
3. M is an \mathcal{M} -Noetherian module for each $\mathcal{M} \in \text{Max}(R)$.

Proof. See [23, Proposition 2.2] and [6, Proposition 2.6]. \square

Theorem 4.13. *Let M be a nonzero R -module. The following statements are equivalent.*

1. M has Noetherian dimension and $n\text{-dim}(M) = \alpha$.
2. M has P -Noetherian dimension for each $P \in \text{Spec}(R)$ and $Pn\text{-dim}(M) = \alpha$
3. M has \mathcal{M} -Noetherian dimension for each $\mathcal{M} \in \text{Max}(R)$ and $\mathcal{M}n\text{-dim}(M) = \alpha$

Proof. (1) \Rightarrow (2) \Rightarrow (3) Clear.

(3) \Rightarrow (1) We will proceed by induction on α , where α is an ordinal number. Assume that $\mathcal{M}n\text{-dim}(M) = \alpha$. If $\mathcal{M}n\text{-dim}(M) = 0$, then M is an \mathcal{M} -Noetherian module. By Theorem 4.12, M is a Noetherian module. Thus, it has Noetherian dimension and $n\text{-dim}(M) = 0$. Now, assume that for all ordinals less than α , the statement holds. Suppose that $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$ is an ascending chain of submodules of M . Since M has \mathcal{M} -Noetherian dimension, there exists $k \in \mathbb{N}$ such that $\mathcal{M}n\text{-dim}(\frac{M_{i+1}}{M_i}) < \alpha$ for each $i \geq k$. By the induction hypothesis, $\frac{M_{i+1}}{M_i}$ has Noetherian dimension and $n\text{-dim}(\frac{M_{i+1}}{M_i}) < \alpha$. Hence, we deduce that $n\text{-dim}(M) \leq \alpha$ and $\mathcal{M}n\text{-dim}(M) \leq n\text{-dim}(M)$. This yields that $\mathcal{M}n\text{-dim}(M) = n\text{-dim}(M) = \alpha$. \square

Theorem 4.14. *Let \mathcal{M} be a maximal ideal of a ring R . Then, R has Noetherian dimension if and only if R has \mathcal{M} -Noetherian dimension.*

Proof. The “only if” part is clear. For the “if” part, we will proceed by induction on α , where α is an ordinal number. Assume $\mathcal{M}n\text{-dim}(R) = \alpha$. If $\mathcal{M}n\text{-dim}(R) = 0$, then R is an \mathcal{M} -Noetherian ring. Thus, by [6, Proposition 2.6], R is a Noetherian ring. Hence, it has Noetherian dimension and $n\text{-dim}(R) = 0$. Now, assume that for all ordinals less than α , the statement holds. Consider $\mathcal{M}n\text{-dim}(R) = \alpha$ and $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ is an ascending chain of ideals of R . Since R has \mathcal{M} -Noetherian dimension, there exists $k \in \mathbb{N}$ such that $\mathcal{M}n\text{-dim}(\frac{I_{i+1}}{I_i}) < \alpha$ for each $i \geq k$. Therefore, by the induction hypothesis $\frac{I_{i+1}}{I_i}$ has Noetherian dimension and $n\text{-dim}(\frac{I_{i+1}}{I_i}) < \alpha$. Hence, we conclude that $n\text{-dim}(R) \leq \alpha$ and $\mathcal{M}n\text{-dim}(R) \leq n\text{-dim}(R)$. This implies that $\mathcal{M}n\text{-dim}(R) = n\text{-dim}(R) = \alpha$. \square

By an interesting result due to Lemonnier, is that every module has Noetherian dimension if and only if it has Krull dimension, see [27, Corollary 6]. We end the section with a conjecture. We have been unable to prove it, but it seems likely that it is so.

Conjecture Let S be a multiplicative closed subset of a ring R . An R -module M has S -Artinian dimension if and only if it has S -Noetherian dimension.

5. S - α -atomic modules

We recall from [20], that an R -module M is called α -atomic if $n\text{-dim}(M) = \alpha$, which $\alpha \geq 0$ is an ordinal number and $n\text{-dim}(N) < \alpha$ for every submodule N of M .

Definition 5.1. Let S be a multiplicative closed subset of a ring R and $\alpha \geq 1$ be an ordinal number. We say an R -module M is S - α -atomic if $Sn\text{-dim}(M) = \alpha$ and $Sn\text{-dim}(N) < \alpha$ for every proper nonzero submodule N of M .

Proposition 5.2. Let S be a multiplicative closed subset of a ring R and M be a nonzero Artinian R -module. If $Sn\text{-dim}(M) = \alpha \geq 1$, then M has a S - β -atomic submodule for each $0 \neq \beta \leq \alpha$.

Proof. Consider the set $\mathcal{F} = \{N \subseteq M : Sn\text{-dim}(N) \geq \beta\}$. Clearly, $M \in \mathcal{F}$ and so $\mathcal{F} \neq \emptyset$. Since M is an Artinian module, \mathcal{F} has a minimal element, say N . It suffices to show $Sn\text{-dim}(N) = \beta$. Assume, for a contradiction, that $Sn\text{-dim}(N) > \beta$. Thus, there exists the ascending chain $N_1 \subset N_2 \subset \dots \subset N_n \subset \dots$ of submodules of N such that $Sn\text{-dim}(\frac{N_{i+1}}{N_i}) \geq \beta$ for each i . Since N is a minimal element of \mathcal{F} , we infer that $Sn\text{-dim}(\frac{N_{i+1}}{N_i}) \leq Sn\text{-dim}(N_{i+1}) < \beta$, a contradiction. \square

A submodule K of an R -module M is small (or superfluous) in M , abbreviated $K \subseteq_s M$, in case for any submodule L of M , $K + L = M$, implies that $L = M$. Now, we have the following proposition.

Proposition 5.3. Let S be a multiplicative closed subset of a ring R . If M is an S - α -atomic R -module where $\alpha \neq 0$, then the following statements hold.

1. Every nonzero quotient module of M is an S - α -atomic module.
2. If N is a nonzero proper submodule of M , then $N \subseteq_s M$ and M is indecomposable.
3. For each $0 \neq h \in \text{End}(M)$, we get $h(M) = M$.
4. $\text{End}(M)$ is a domain.
5. If R is a commutative ring, then $rM = M$ or $rM = 0$ for each $r \in R$.
6. If R is a commutative ring, then $P = \text{ann}(M)$ is a prime ideal and M is a divisible $\frac{R}{P}$ -module.
7. For each $0 \neq h \in \text{Hom}(M, \frac{M}{N})$, where $N \neq M$, h is endomorphism.

Proof. (1) Since M is an S - α -atomic module, $Sn\text{-dim}(M) = \alpha$ and for every nonzero proper submodule N of M , $Sn\text{-dim}(N) < \alpha$. From $\alpha = Sn\text{-dim}(M) = \max\{Sn\text{-dim}(N), Sn\text{-dim}(\frac{M}{N})\}$, we have $Sn\text{-dim}(\frac{M}{N}) = \alpha$. We consider the proper submodule $0 \neq \frac{K}{N}$ of $\frac{M}{N}$ such that $N \subset K \subset M$. Thus, $Sn\text{-dim}(K) < \alpha$, $Sn\text{-dim}(\frac{K}{N}) \leq Sn\text{-dim}(K) < \alpha$. (2) Assume, for a contradiction, that $N + N_1 = M$ for some submodule N_1 of M . Since N is a proper nonzero submodule of M , $Sn\text{-dim}(N) < \alpha$. If N_1 is a proper submodule of M , then

$$Sn\text{-dim}(\frac{M}{N}) = Sn\text{-dim}(\frac{N+N_1}{N}) = Sn\text{-dim}(\frac{N_1}{N \cap N_1}) \leq Sn\text{-dim}(N_1) < \alpha.$$

Thus, $Sn\text{-dim}(M) = \max\{Sn\text{-dim}(N), Sn\text{-dim}(\frac{M}{N})\} < \alpha$. This is a contradiction and so $M = N_1$ and $N \subseteq_s M$.

(3) Let $0 \neq h \in \text{End}(M)$ and $\text{Ker } h \neq M$. From (1), we have $Sn\text{-dim}(\frac{M}{\text{Ker } h}) = \alpha$. We note that $\frac{M}{\text{Ker } h} \simeq h(M)$. If $h(M)$ is a proper submodule of M , then $Sn\text{-dim}(h(M)) < \alpha$. But $Sn\text{-dim}(\frac{M}{\text{Ker } h}) = Sn\text{-dim}(h(M)) = \alpha$, a contradiction. Hence, we conclude that $h(M) = M$.

(4) Let $h_1, h_2 \in \text{End}(M)$ and $h_1 h_2 = 0$. From (3), $h_2 \neq 0$ and $h_1 h_2(M) = h_1(M) = 0$. Hence, we have $h_1 = 0$.

(5) Assume that $rM \neq 0$. From (3), we deduce that the homomorphism $f : M \rightarrow M$ where $f(m) = rm$ is an epimorphism. Hence, we have $f(M) = rM = M$.

(6) Let $a, b \in R$ and $ab \in \text{ann}(M)$. It suffices to show $a \in \text{ann}(M)$ or $b \in \text{ann}(M)$. Assume $b \notin \text{ann}(M)$. From (5), $bM = M$ and so $abM = aM = 0$. Hence, $a \in \text{ann}(M)$ and $\text{ann}(M)$ is a prime ideal. Now, since $\text{ann}(M) = P \subseteq R$, we deduce that there exists $r \in R \setminus P$ such that $rM = M$. Thus, for each $m \in M$, there exists $m' \in M$ such that $rm' = m$. Hence, $(r + P)m' = m$ and so M is a divisible $\frac{R}{P}$ -module.

(7) Assume $f : M \rightarrow \frac{M}{N}$ such that $0 \neq N \subsetneq M$. Since M and $\frac{M}{N}$ are S -atomic modules, we have $\alpha = Sn\text{-dim}(M) = Sn\text{-dim}(\frac{M}{N})$. Hence, $Sn\text{-dim}(\frac{M}{\text{Ker } f}) = Sn\text{-dim}(\text{Im}(f)) < \alpha$ and so $Sn\text{-dim}(\frac{M}{\text{Ker } f}) < \alpha$. But, $Sn\text{-dim}(M) = \sup\{Sn\text{-dim}(\text{Ker } f), Sn\text{-dim}(\frac{M}{\text{Ker } f})\} < \alpha$, a contradiction. Hence, $\text{Im}(f) = M$ and f is endomorphism. \square

Corollary 5.4. *Let S be a multiplicative closed subset of a ring R and let M be a nonzero R -module. The following statements are equivalent.*

1. M is an S -atomic module.
2. Every proper submodule of M is a small submodule of M and M has a quotient module $\frac{M}{N}$ such that $\text{Sn-dim}(N) < \text{Sn-dim}(\frac{M}{N})$.

Proof. (1) \Rightarrow (2) It follows from parts (1) and (2) of Proposition 5.3.

(2) \Rightarrow (1) First, we note that $\text{Sn-dim}(M) = \max\{\text{Sn-dim}(N), \text{Sn-dim}(\frac{M}{N})\} = \text{Sn-dim}(\frac{M}{N})$. If $0 \neq N \subsetneq M$, then by assumption $N \subseteq_s M$ and we have $\text{Sn-dim}(N) < \text{Sn-dim}(\frac{M}{N}) = \text{Sn-dim}(M)$. Hence, M is an S -atomic module. \square

Theorem 5.5. *Let S be a multiplicative closed subset of a ring R and let M be a nonzero R -module. If $\text{Sn-dim}(M) = \alpha$, then there is a submodule $N \subsetneq M$ such that $\frac{M}{N}$ is an S -atomic module.*

Proof. If M is an S -atomic module, then by Proposition 5.3, $\frac{M}{N}$ is an S -atomic module for every $N \subset M$. Assume, for a contradiction, that $\frac{M}{N}$ is not an S -atomic module for each $N \subset M$. Suppose that $\frac{M}{P}$ is an S -atomic module with the least S -Noetherian dimension. We show $\frac{M}{P}$ has a nonzero S -atomic quotient module that is a nonzero quotient module of M . Without loss of generality, we may assume that M is not an S -atomic module with the least S -Noetherian dimension. By our assumption, there exists $M_1 \subset M$ such that $\text{Sn-dim}(M_1) = \alpha$ and $\frac{M}{M_1}$ is not an S -atomic module. Notice that $\text{Sn-dim}(\frac{M}{M_1}) = \alpha$. By continuing this method, there exists a proper submodule $\frac{M_2}{M_1}$ of $\frac{M}{M_1}$ such that $\text{Sn-dim}(\frac{M_2}{M_1}) = \alpha$ and $\text{Sn-dim}(\frac{M}{M_2}) = \alpha$. With this method, we have the ascending chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$ of submodules of M such that $\text{Sn-dim}(\frac{M_{i+1}}{M_i}) = \alpha$ for every i , that is a contradiction. \square

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References

- [1] H. Ahmed and H. Sana, Modules satisfying the S -Noetherian property and S -ACCR, *Commun. Algebra* 44(5) (2016), 1941-1951.
- [2] T. Albu and S.T. Rizvi, Chain conditions on quotient finite dimensional modules, *Commun. Algebra* 29 (2001), 1909-1928.
- [3] D.D. Anderson and T. Dumitrescu, S -Noetherian rings, *Commun. Algebra* 30(2002), 4407-4416.
- [4] F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules*. Springer-Verlag, New York, 1992.
- [5] A.U. Ansari, S.K. Maurya, and B.K. Sharma, Some Remarks on S -Noetherian Modules, *Ukr. Math. J.*, 77(1) (2025), 77-84.
- [6] A. Benhissi, *Chain Conditions in Commutative Rings*, Springer, 2022.
- [7] Z. Bilgin, M.L. Reyes, and U. Tekir, On right S -Noetherian rings and S -Noetherian modules, *Commun. Algebra* 46(2018), 863-869.
- [8] L. Chambless, *N -Dimension and N -critical modules, Application to Artinian modules*. *Comm. Algebra*, 8 (16) (1980), 1561-1592.
- [9] M. Davoudian, O.A.S. Karamzadeh, and N. Shirali, On α -short modules, *Math. Scand.* 114 (2014), 26-37.
- [10] M. Davoudian, Dimension of non-finitely generated submodules, *Vietnam J. Math.* 44 (2016), 817-827.
- [11] M. Davoudian and O.A.S. Karamzadeh, Artinian serial modules over commutative (or, left Noetherian) rings are at most one step away from being Noetherian, *Commun. Algebra* 44 (2016), 3907-3917.
- [12] M. Davoudian, On countably generated dimension, *Algebra Colloquium* 28 (2021), 361-366.
- [13] R.W. Gilmer, *Multiplicative Ideal Theory*. Queen's Univ., Kingston, Ontario, 1968.
- [14] R. Gordon, Gabriel and Krull dimension, *Ring theory (Proc. Conf., Univ. Oklahoma, Norman, Okla., 1973) Lecture Notes in Pure and Appl. Math.*, Vol. 7, Dekker, New York, 1974, pp. 241-295.
- [15] R. Gordon and J.C. Robson, *Krull Dimension*, *Mem. Amer. Math. Soc.* 1973.
- [16] K.R. Goodearl and R.B. Warfield, *An Introduction to Non-Commutative Noetherian Rings*, Cambridge Univ. Press, 1989.
- [17] J. Hashemi, O.A.S. Karamzadeh, and N. Shirali, Rings over which the Krull dimension and Noetherian dimension of all modules coincide, *Commun. Algebra* 37(2009), 650-662.
- [18] O.A.S. Karamzadeh, *Noetherian-dimension*, Ph.D. Thesis, Exeter, 1974.
- [19] O.A.S. Karamzadeh and Sh. Rahimpour, On λ -finitely embedded modules, *Commun. Algebra* 23 (1995), 3685-3703.
- [20] O.A.S. Karamzadeh and A.R. Sajedinejad, Atomic modules, *Commun. Algebra* 29 (2001), 2757-2773.

- [21] O.A.S. Karamzadeh and A.R. Sajedinejad, On the Loewy length and the Noetherian dimension of Artinian modules, *Commun. Algebra* 30 (2002), 1077-1084.
- [22] O.A.S. Karamzadeh and N. Shirali, On the countability of Noetherian dimension of modules, *Commun. Algebra* 32 (2004), 4073-4083.
- [23] H. Kim, M. O. Kim, J. W. Lim, On S -strong Mori domains, *Journal of Algebra* 416 (2014), 314-332.
- [24] D. Kirby, Dimension and length for Artinian modules, *Quart. J. Math. Oxford.* 41 (1990), 419-429.
- [25] T.Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics, 189. Springer-Verlag, New York, 1999.
- [26] T.Y. Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Mathematics, Vol. 131, 2nd ed. New York: Springer-Verlag, 2001.
- [27] B. Lemonnier, Déviation des ensembles et groupes abéliens totalment ordonnés, *Bull. Sci. Math.* 96 (1972), 289-303.
- [28] B. Lemonnier, Dimension de krull et codeviation. Application au theoreme d'eakin, *Commun. Algebra* 6 (1978), 1647-1665.
- [29] M. Özen, O.A. Nazi, Ü. Tekir, and K.P. Shum, Characterization theorems of S -Artinian modules, *C. R. Acad. Bulgare Sci.* 74 (2021), 496-505
- [30] R.N. Roberts, Krull dimension for Artinian modules over quasi local commutative rings, *Quart. J.Math. Oxford.* 26 (1975), 269-273.
- [31] E.S. Sevim, U. Tekir, and S. Koc, S -Artinian ring and finitely S -cogenerated rings, *J. Algebra Appl.* 19 (2020), 2050051.