Filomat 40:6 (2026), 2155–2170
https://doi.org/10.2298/FIL2606155Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Super-biderivations of the Hamiltonian Lie superalgebra $H(m, n; \underline{t})$

Jingyi Zhang[a], Jixia Yuan[b], Liangyun Chen[a,*]

[a]School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
[b]School of Mathematical Sciences, Heilongjiang University, Harbin 150080, China

Abstract. Let H denote the Hamiltonian Lie superalgebra $H(m, n; \underline{t})$ over a field of characteristic $p \geq 3$, which has a finite $\mathbb{Z}$-grading structure. In this paper, we take a canonical torus T_H of H, which is an abelian subalgebra of H. By the decomposition of the weight space of H with respect to T_H, we show the action of the unique linear map related to the symmetric super-biderivation on the elements of the generators of H. Moreover, we prove that each symmetric super-biderivation of H is zero. Further, we get that each super-biderivation of H is inner. As applications, the super-commutative post-Lie superalgebra structures and the linear super-commuting maps on H are described.

1. Introduction

Lie superalgebra is a generalization of Lie algebra, which comes from the super-symmetry in mathematical physics. In recent years, many scholars have been studying derivations and the generalization of derivations of Lie superalgebras[1, 2, 9, 11, 14–16, 23]. The super-biderivations are a class of generalized derivations. In [3], the concept of biderivations was introduced and studied. In [22], Yuan, Chen and Cao characterized the super-biderivations of Cartan type Lie superalgebras over the complex field $\mathbb{C}$ and proved that all super-biderivations of Cartan type simple Lie superalgebras are inner. It is well known that any biderivation can be decomposed into a skew-symmetric biderivation and a symmetric one. The skew-symmetric biderivations of Lie algebras and Lie superalgebras have been sufficiently studied. It is worth noting that in [4], skew-symmetric biderivations over a field of characteristic p different from 2 are described in a unified way. In [20], Xu and Wang generalized the notion of biderivations of Lie algebras to the super case. In [9, 16], the authors proved all skew-symmetric super-biderivations on the centerless super-Virasoro algebra and the well-known super-Virasoro algebra are inner. Moreover, Li and Sun found that there exist non-inner skew-symmetric super-biderivations of the super Heisenberg-Virasoro algebra. In [15], Tang et al. proved that all skew-symmetric biderivations on any perfect and centerless Lie superalgebras are inner. The study of symmetric super-biderivations of Lie superalgebras is less than that of

2020 *Mathematics Subject Classification*. Primary 17B05; Secondary 17B40, 17B50.
Keywords. Torus, weight space decomposition, super-biderivation, the Hamiltonian Lie superalgebras, super-commutative post-Lie superalgebra structures.
Received: 01 April 2025; Revised: 18 December 2025; Accepted: 10 January 2026
Communicated by Dijana Mosić
This work is supported by NNSF of China (No. 12271085).
* Corresponding author: Liangyun Chen
Email addresses: jyzhang831@nenu.edu.cn (Jingyi Zhang), yuanjixia138@163.com (Jixia Yuan), chenly640@nenu.edu.cn (Liangyun Chen)
ORCID iDs: https://orcid.org/0009-0008-9845-6984 (Jingyi Zhang), https://orcid.org/0000-0001-8450-4991 (Jixia Yuan), https://orcid.org/0000-0002-2941-1087 (Liangyun Chen)

skew-symmetric super-biderivations. The study of symmetric super-biderivations can be reduced to the study of the related linear maps. In [8], all symmetric super-biderivations of some Lie superalgebras related to the Virasoro algebra are determined by a general method. As an application, commutative post-Lie superalgebra structures on these Lie superalgebras are also obtained. However, there is not a unified method to characterize the symmetric super-biderivations of Lie superalgebras, so there are still many problems to be studied about the super-biderivations on Lie superalgebras.

The simple modular Lie superalgebra of Cartan type is a class of modular Lie superalgebras, which is over prime characteristic fields. In [24], Zhang and Liu constructed four classes of Lie superalgebras of Cartan type, which are W, S, H and K. Four other families of $\mathbb{Z}$-graded Lie superalgebras of Cartan type were constructed over a field of characteristic $p > 3$, which are HO, KO, SHO and SKO [10, 12–14]. In [5], the derivations of modular Lie algebras of Cartan-type were studied. In [2], super-derivations for the eight families of finite or infinite dimensional graded Lie superalgebras of Cartan-type over a field of characteristic $p > 3$ are completely determined by a uniform approach. All skew-symmetric super-biderivations of gengralized Witt Lie superalgebra $W(m, n; \underline{t})$ and contact Lie superalgebra $K(m, n; \underline{t})$ were proved to be inner in [6, 25]. In [17, 18, 21], there were similar results for $H(m, n; \underline{t})$, $S(m, n; \underline{t})$ and $KO(n, n+1; \underline{t})$. In [1], Bai and Liu proved that all skew-symmetric super-biderivations are inner and all symmetric super-biderivations are zero for simple modular Lie superalgebras of Witt type and special type. As applications, the linear super-commuting maps and the super-commutative post-Lie superalgebra structures on simple modular Lie superalgebras of Witt type and special type are described. It is natural to wonder whether several other classes of Cartan-type simple modular Lie superalgebras have similar results.

In this paper, we study the super-biderivations of the Hamiltonian Lie superalgebra $H(m, n; \underline{t})$ (denoted as H for short). We use the weight space decomposition of H with respect to a canonical torus T_H and prove each symmetric super-biderivation of H is zero. Furthermore, we get that each super-biderivation of H is inner. The paper is organized as follows. In Section 2, we review the basic notations. In Section 3, we take a canonical torus T_H of H, which is an abelian subalgebra of H. By the decomposition of the weight space of H with respect to T_H, we show the action of the unique linear map related to the symmetric super-biderivation on the elements of the generators of H. Moreover, we prove that each symmetric super-biderivation of H is zero. Furthermore, we get that each super-biderivation of H is inner. In Section 4, as applications, the linear super-commuting maps and the super-commutative post-Lie superalgebra structures on Hamiltonian superalgebra are described.

2. Preliminaries

In this paper, we always assume that the base field $\mathbb{F}$ is algebraically closed and of characteristic $p \geq 3$. All vector spaces V are over $\mathbb{F}$ and $\mathbb{Z}_2$-graded, i.e., $V = V_{\bar{0}} \oplus V_{\bar{1}}$. For each $\mathbb{Z}_2$-homogeneous element $v \in V$, we denote by $\mathrm{d}(v)$ the parity of v. If $V = \oplus_{i\in\mathbb{Z}} V_i$ is a $\mathbb{Z}$-graded vector space and $x \in V$ is a $\mathbb{Z}$-homogeneous, write $\mathrm{zd}(x)$ for the $\mathbb{Z}$-degree of x. Once the symbol $\mathrm{d}(x)$ or $\mathrm{zd}(x)$ appears in this paper, it implies that x is a $\mathbb{Z}_2$-homogeneous element or that x is a $\mathbb{Z}$-homogeneous element.

2.1. *Definitions and notions*

First, we recall the definitions of derivation and biderivation. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$, a linear map $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a derivation if

$$D([x, y]) = [D(x), y] + [x, D(y)],$$

for all $x, y \in \mathfrak{g}$. A bilinear map $\psi : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is a biderivation if it is a derivation with respect to both components, meaning that

$$\psi(x, [y, z]) = [\psi(x, y), z] + [y, \psi(x, z)],$$
$$\psi([x, y], z) = [\psi(x, z), y] + [x, \psi(y, z)],$$

for all $x, y, z \in \mathfrak{g}$.

Let us recall some facts related to the super-derivation and super-biderivation of Lie superalgebras. A Lie superalgebra is a vector superspace $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with an even bilinear mapping $[\cdot,\cdot] : L \times L \longrightarrow L$ satisfying the following axioms:

$$[x, y] = (-1)^{\mathrm{d}(x)\mathrm{d}(y)}[y, x],$$
$$[x, [y, z]] = [[x, y], z] + (-1)^{\mathrm{d}(x)\mathrm{d}(y)}[y, [x, z]],$$

for all $x, y, z \in L$. A homogeneous super-derivation D of L is a homogeneous linear transformation of L such that

$$D([x, y]) = [D(x), y] + (-1)^{\mathrm{d}(D)\mathrm{d}(x)}[x, D(y)],$$

for all $x, y \in L$. Write $\mathrm{Der}_{\bar{0}}(L)$ (resp. $\mathrm{Der}_{\bar{1}}(L)$) for the space of all super-derivations of $\mathbb{Z}_2$-degree $\bar{0}$ (resp. $\bar{1}$) of L. Clearly,

$$\mathrm{Der}(L) = \mathrm{Der}_{\bar{0}}(L) \oplus \mathrm{Der}_{\bar{1}}(L).$$

Definition 2.1. *[9] Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra. We call a bilinear mapping $\phi : L \times L \longrightarrow L$ a super-biderivation of L if it satisfies the following equations:*

$$\phi([x, y], z) = (-1)^{\mathrm{d}(\phi)\mathrm{d}(x)}[x, \phi(y, z)] + (-1)^{\mathrm{d}(y)\mathrm{d}(z)}[\phi(x, z), y],$$
$$\phi(x, [y, z]) = [\phi(x, y), z] + (-1)^{(\mathrm{d}(\phi)+\mathrm{d}(x))\mathrm{d}(y)}[y, \phi(x, z)]$$

for all $\mathbb{Z}_2$-homogeneous elements $x, y, z \in L$.

Definition 2.2. *A bilinear map $\phi : L \times L \longrightarrow L$ is called a symmetric biderivation of L, if*

$$\phi(x, y) = (-1)^{\mathrm{d}(x)\mathrm{d}(y)}\phi(y, x),$$

for all $\mathbb{Z}_2$-homogeneous elements $x, y \in L$.

Definition 2.3. *A bilinear map $\phi : L \times L \longrightarrow L$ is called a skew-symmetric biderivation of L, if*

$$\phi(x, y) = -(-1)^{\mathrm{d}(x)\mathrm{d}(y)}\phi(y, x),$$

for all $\mathbb{Z}_2$-homogeneous elements $x, y \in L$.

A super-biderivation ϕ of $\mathbb{Z}_2$-degree γ of L is a super-biderivation such that $\phi(L_\alpha, L_\beta) \subset L_{\alpha+\beta+\gamma}$ for any $\alpha, \beta \in \mathbb{Z}_2$. Denote by $\mathrm{BDer}_\gamma(L)$ the space of all symmetric super-biderivations of $\mathbb{Z}_2$-degree γ. Obviously,

$$\mathrm{BDer}(L) = \mathrm{BDer}_{\bar{0}}(L) \oplus \mathrm{BDer}_{\bar{1}}(L).$$

Notice that every super-biderivation ϕ of a Lie superalgebra L can be written as the sum of a symmetric one and a skew-symmetric one, namely,

$$\phi(x, y) = \frac{\phi(x, y) + (-1)^{\mathrm{d}(x)\mathrm{d}(y)}\phi(y, x)}{2} + \frac{\phi(x, y) - (-1)^{\mathrm{d}(x)\mathrm{d}(y)}\phi(y, x)}{2},$$

for all $x, y \in L$.

Definition 2.4. *[9] Let $\lambda \in \mathbb{F}$, if the bilinear map $\phi_\lambda : L \times L \longrightarrow L$ is defined by $\phi_\lambda(x, y) = \lambda[x, y]$, for all homogenous $x, y \in L$, then ϕ_λ is a super-biderivation of L. This class of super-biderivations is called inner. Denote by* $\mathrm{IBDer}(L)$ *the space of all inner super-biderivations.*

Definition 2.5. *[1] A super-biderivation ϑ of L is called weakly inner if there exists a Lie superalgebra $\widehat{L}$ containing L and a map ϕ_ϑ, said to be related to ϑ, from L to $\widehat{L}$ such that*

$$\vartheta(x,y) = [\phi_\vartheta(x), y], \ \forall\ x, y \in L.$$

It is said to be inner if, in addition, ϕ_ϑ is a scalar transformation, i.e., $\vartheta(x,y) = \lambda[x,y]$, for all $x, y \in L$ and $\lambda \in \mathbb{F}$.

Definition 2.6. *[1] A Lie superalgebra L is called weakly complete if there is a Lie superalgebra $\widetilde{L}$ such that L embeds in $\widetilde{L}$ as an ideal and satisfies*

$$\operatorname{Der} L = \operatorname{ad}_L \widetilde{L}, \quad Ann_{\widetilde{L}} L = \{x \in \widetilde{L} \mid [x, L] = 0\} = \{0\}.$$

Definition 2.7. *[16] For a Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$, a linear map $\varphi : L \longrightarrow L$ is said to be super-commuting if*

$$[\varphi(x), x] = 0, \ \forall\ x \in L.$$

If ϕ is a linear super-commuting map of the Lie superalgebra L, then we have

$$[\varphi(x+y), x+y] = 0, \ \forall\ x, y \in L.$$

This implies that

$$[\varphi(x), y] = (-1)^{\mathrm{d}(\varphi)\mathrm{d}(x)}[x, \varphi(y)], \ \forall\ x, y \in L.$$

Definition 2.8. *[19] Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra. A super-commutative post-Lie superalgebra structure on L is a bilinear product $x \cdot y$ on L satisfying the following identities:*

$$x \cdot y = (-1)^{\mathrm{d}(x)\mathrm{d}(y)} y \cdot x,$$
$$[x,y] \cdot z = x \cdot (y \cdot z) - (-1)^{\mathrm{d}(x)\mathrm{d}(y)} y \cdot (x \cdot z),$$
$$x \cdot [y,z] = [x \cdot y, z] + (-1)^{\mathrm{d}(x)\mathrm{d}(y)}[y, x \cdot z],$$

for all $x, y, z \in L$.

The super-commutative post-Lie superalgebra structure on L defined by $x \cdot y = 0$, for all $x, y \in L$ is said to be trivial.

2.2. Hamiltonian Lie superalgebra $H(m, n; \underline{t})$

Let us recall the generalized Witt Lie superalgebra and the Hamiltonian Lie superalgebra in simple modular Lie superalgebras. Fix two positive integers $m > 1$ and $n > 1$. Put $I_0 = \{1, \dots, m\}, I_1 = \{m+1, \dots, m+n\}, I = I_0 \cup I_1$. Set two m-tuples $\underline{t} = (t_1, \dots, t_m)$ and $\pi = (\pi_1, \dots, \pi_m)$, where $\pi_i = p^{t_i} - 1$. Put $|\alpha| = \sum_{i=1}^{m} \alpha_i$ and write

$$A(m; \underline{t}) = \{(\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m \mid 0 \le \alpha_i \le \pi_i, i = 1, \dots, m\},$$

where $\mathbb{N}^m$ denote the space of natural numbers. Let $O(m; \underline{t})$ be the divided power algebra with $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in A(m; \underline{t})\}$ under the formul:

$$x^{(\alpha)} x^{(\beta)} = \binom{\alpha+\beta}{\alpha} x^{(\alpha+\beta)}, \ \forall\ \alpha, \beta \in \mathbb{N}^m. \tag{1}$$

Let $\Lambda(n)$ be the exterior superalgebra of n variables $x_{m+1}, \dots, x_s$, where $s = m + n$. Set $\Lambda(m, n; \underline{t}) = O(m; \underline{t}) \otimes_{\mathbb{F}} \Lambda(n)$, which is an associative superalgebra with respect to the usual $\mathbb{Z}_2$-grading. Set

$$\mathbb{B}_k = \{< i_1, i_2, \dots, i_k > \mid k \in \{0, \dots, n\}, m+1 \le i_1 < i_2 < \dots < i_k \le s\}$$

be the set of k-tuples of strictly increasing integers in I_1 and $\mathbb{B} = \cup_{k=0}^{n}\mathbb{B}_k$, where $\mathbb{B}_0 = \emptyset$. Write $x^u = x_{i_1}x_{i_2}\cdots x_{i_k}$ ($x^\emptyset = 1$), where $|u| := k$ and $u =< i_1, i_2, \ldots, i_k >\in \mathbb{B}_k$. Note that $\mathrm{zd}(x^u) = |u|$. Denote the number of times j appears in u be $\tau(u, j)$ for $j \in u$. Specially, we define $|\emptyset| = 0, x^\emptyset = 0, |\omega| = 0$ and $x^\omega = x_{m+1}\cdots x_s$. Obviously, $\Lambda(m, n; \underline{t})$ is super-commutative. For $g \in O(m; \underline{t}), f \in \Lambda(n)$, it is customary to write gf instead of $g \otimes f$. Clearly, $\Lambda(m, n; \underline{t})$ has a $\mathbb{Z}_2$-homogeneous $\mathbb{F}$-basis $\{x^{(\alpha)}x^u \mid \alpha \in A(m; \underline{t}), u \in \mathbb{B}_k\}$. $\Lambda(m, n; \underline{t})$ possesses a $\mathbb{Z}$-graded structure:

$$\Lambda(m, n; \underline{t}) = \bigoplus_{i=0}^{\xi} \Lambda(m, n; \underline{t})_{[i]},$$

where $\xi := |\pi| + n$ and

$$\Lambda(m, n; \underline{t})_{[i]} = \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \mid |\alpha| + |u| = i\}.$$

Including the formula (1), the following formulas also hold in $\Lambda(m, n; \underline{t})$:

$$x_k x_l = -x_l x_k,\ \forall\ k, l \in \{m + 1, \ldots, s\};$$

$$x^{(\alpha)}x_k = x_k x^{(\alpha)},\ \forall\ \alpha \in \mathbb{N}^m, k \in \{m + 1, \ldots, s\}.$$

For $i \in I_0$ and $\varepsilon_i = (\delta_{i,1}, \delta_{i,2} \ldots, \delta_{i,m})$, where $\delta_{i,j}$ is the Kronecker symbol, we write $x^{(\varepsilon_i)}$ as x_i. Let $D_1, D_2 \ldots, D_s$ be the linear transformations of $\Lambda(m, n; \underline{t})$ such that

$$D_i(x^{(\alpha)}x^u) = \begin{cases} x^{(\alpha-\varepsilon_i)}x^u, & i \in I_0, \\ (-1)^{i-1}x^{(\alpha)} \cdot \partial x^u \setminus \partial x_i, & i \in I_1. \end{cases}$$

Put

$$W(m, n; \underline{t}) = \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u D_s \mid 0 \le \alpha \le \pi, u \in \mathbb{B}_k, s \in I\},$$

which is a finite-dimensional simple superalgebra, called the generalized Witt Lie superalgebra. Note that $|\pi| = \sum_{i=1}^{m} \pi_i$. Then $W(m, n; \underline{t})$ possesses a $\mathbb{Z}$-graded structure:

$$W(m, n; \underline{t}) = \bigoplus_{i=-1}^{\xi-1} W(m, n; \underline{t})_{[i]},$$

where $\xi := |\pi| + n$ and

$$W(m, n; \underline{t})_{[i]} = \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u D_s \mid |\alpha| + |u| = i + 1, s \in I\}.$$

Hereafter, suppose $m = 2r$ is even, where $r \in \mathbb{N}$. For $i \in I$, put

$$i' = \begin{cases} i + r, & 1 \le i \le r, \\ i - r, & r < i \le m, \\ i + \tau, & m < i \le m + \tau, \\ i - \tau, & m + \tau < i \le m + 2\tau, \\ i, & \text{others}, \end{cases} \qquad \sigma(i) = \begin{cases} 1, & 1 \le i \le r, \\ -1, & r < i \le m, \\ 1, & m < i \le s, \end{cases}$$

where $\tau = \lfloor \frac{n}{2} \rfloor$. The restricted linear mapping of D_H on $\Lambda(m, n; \underline{t})$ is denoted by D_H, that is,

$$D_H(f) = \sum_{i=1}^{s} \sigma(i)(-1)^{\mathrm{d}(D_i)\mathrm{d}(f)} D_i(f) D_{i'}.$$

It is obvious that D_H is an even linear map. For any $f, g \in \mathrm{hg}(\Lambda(m,n;\underline{t}))$, we have

$$[D_H(f), D_H(g)] = D_H(\sum_{i=1}^{s} \sigma(i)(-1)^{\mathrm{d}(D_i)\mathrm{d}(f)} D_i(f) D_{i'}(g)) \\ = D_H(D_H(f)(g)). \quad (2)$$

Set

$$H(m,n;\underline{t}) = \{D_H(f) \mid f \in \Lambda(m,n;\underline{t}) \setminus \{x^{(\pi)}x^{\omega}\}\},$$

where $\pi = (\pi_1, \ldots, \pi_m) \in A(m;\underline{t}), \omega =< m+1, m+2, \ldots, s >\in \mathbb{B}_k$. It follows from the Equation (2) that $H(m,n;\underline{t})$ is a simple Lie subalgebra of $W(m,n;\underline{t})$, which called Hamiltonian Lie superalgebra. $H(m,n;\underline{t})$ possesses a $\mathbb{Z}$-graded structure:

$$H(m,n;\underline{t}) = \bigoplus_{i=-1}^{\xi-2} H(m,n;\underline{t})_{[i]},$$

where $\xi := |\pi| + n$ and

$$H(m,n;\underline{t})_{[i]} = \mathrm{span}_{\mathbb{F}}\{D_H(x^{(\alpha)}x^u) \mid |\alpha| + |u| = i+2\}.$$

For convenience, we use H and H_i to denote $H(m,n;\underline{t})$ and its $\mathbb{Z}$-graded subspace $H(m,n;\underline{t})_{[i]}$, respectively.

Put $h_i = x_i D_i$ and set $\overline{T} = \oplus_{i\in I}\mathbb{F}h_i$. Choose the torus T_H of H, where $T_H = \overline{T} \cap H = \mathrm{span}_{\mathbb{F}}\{D_H(x_i x_{i'}) \mid i \in I\}$. Obviously, $T_H \subseteq H(m,n;\underline{t})_{[0]} \cap H(m,n;\underline{t})_{\bar{0}}$. T_H is an abelian subalgebra of H. For any $D_H(x^{(\alpha)}x^u) \in H$, we have

$$[D_H(x_i x_{i'}), D_H(x^{(\alpha)}x^u)] = \sigma(i)(\alpha_{i'} - \alpha_i + \delta_{i'\in u} - \delta_{i\in u}) D_H(x^{(\alpha)}x^u), \quad (3)$$

where $\delta_{\mathrm{P}} = 1$ if the proposition P is true, $= 0$ if the proposition P is false. Fixed an m-tuple α, where $\alpha \in \mathbb{N}^m, 0 \leq \alpha \leq \pi$ and $u \in \mathbb{B}$, we define a linear function $(\alpha + \langle u \rangle) : T_H \to \mathbb{F}$ such that

$$(\alpha + \langle u \rangle) D_H(x_i x_{i'}) = \sigma(i)(\alpha_{i'} - \alpha_i + \delta_{i'\in u} - \delta_{i\in u}).$$

Further, H has a weight space decomposition with respect to T_H:

$$H = \bigoplus_{\alpha\in\mathbb{N}^m, u\in\mathbb{B}_k} H_{(\alpha+\langle u\rangle)} = H_{(\theta)} \oplus \bigoplus_{\gamma\in\Omega_H} H_{(\gamma)},$$

where Ω_H is the root system of H. The elements in $H_{(\gamma)}$ are called T_H^*-homogenous with weight γ. Notice that $\mathrm{BDer}(H)$ inherits the T_H^*-grading from H as above, that is, $\mathrm{BDer}(H) = \oplus_{a\in T_H^*} \mathrm{BDer}(H)_{(a)}$, where

$$\mathrm{BDer}(H)_{(a)} = \{\vartheta \in \mathrm{BDer}(H) \mid \vartheta(H_{(b)}, H_{(c)}) \subseteq H_{(a+b+c)},\ \forall\, b, c \in T_H^*\};$$
$$\mathrm{BDer}(H)_{[i]} = \{\vartheta \in \mathrm{BDer}(H) \mid \vartheta(H_{[j]}, H_{[k]}) \subseteq H_{[i+j+k]},\ \forall\, j, k \in \mathbb{Z}\}.$$

3. Super-biderivations of $H(m,n;\underline{t})$

Lemma 3.1. [7] *Let $M = \{D_H(x^{(k\varepsilon_i)}) \mid 1 \leq k \leq \pi_i,\ i \in I_0\}$ and $N = \{D_H(x^{(2\varepsilon_1)}x_{m+1})\}$. Then H is generated by $M \cup H_0 \cup N$, where $H_0 = \{D_H(x_i x_j) \mid i, j \in I\}$.*

Lemma 3.2. *Set*

$$\alpha = \{(\alpha_1 \ldots, \alpha_k, \alpha_{1'}, \ldots, \alpha_{k'}) \in \mathbb{N}^{2k} \mid \alpha_{i'} - \alpha_i \equiv -q_i \,(\mathrm{mod}\, p),\ 0 \leq k \leq r,\ 1 \leq i \leq k,\ 0 \leq q_i \leq \pi_i\},$$

$$\beta = \{(\beta_1 \ldots, \beta_t, \beta_{1'}, \ldots, \beta_{t'}) \in \mathbb{N}^{2t} \mid \beta_{i'} - \beta_i \equiv 0 \,(\mathrm{mod}\, p),\ 0 \leq t \leq r,\ 1 \leq i \leq t\},$$

and $u \in \mathbb{B}$. Then the following statements hold:

(i) If n is even, we have a weight space decomposition with respect to T_H:

$$H_{(\alpha+\langle u\rangle)} = \sum_{\substack{0\leq\beta\leq\pi \\ \bar{u}\in\mathbb{B}}} \mathbb{F}D_H(x^{(\alpha+\beta)}x^u x^{\bar{u}}),$$

where l and l' are both in $\bar{u}$ for $l \in I_1$.

(ii) If n is odd, we have a weight space decomposition with respect to T_H:

$$H_{(\alpha+\langle u\rangle)} = \delta_{s\in u} \sum_{\substack{0\leq\beta\leq\pi \\ \bar{u}\in\mathbb{B}}} \mathbb{F}D_H(x^{(\alpha+\beta)}x^u x^{\bar{u}}) + \delta_{s\notin u} \sum_{\substack{0\leq\beta\leq\pi \\ \bar{u}\in\mathbb{B} \\ s\in\bar{u}}} \mathbb{F}D_H(x^{(\alpha+\beta)}x^u x^{\bar{u}}),$$

where l and l' are both in $\bar{u}$ for $l \in J_1$.

Proof. (i) Set $D_H(x^{(\alpha)}x^u) = D_H(x^{(q_1\varepsilon_1+\cdots+q_{k'}\varepsilon_{k'})}x_{j_1}\cdots x_{j_v}) \in H$, where $0 \leq q_i \leq p^{t_i}-1$ for $i \in I_0$, $u = \{j_1,\dots,j_v\} \in \mathbb{B}_v$, the inclusion "$\supset$"is straightforward. To show the converse, from Equation (2), we have

$$[D_H(x_l x_{l'}), D_H(x^{(\alpha)}x^u)] = -\sigma(l)(q_1\delta_{l,1}+\cdots+q_k\delta_{l,k}+\delta_{l,j_1}+\cdots+\delta_{l,j_v})D_H(x^{(\alpha)}x^u). \tag{4}$$

It follows from Equation (3) that

$$\alpha_{l'} - \alpha_l + \delta_{l'\in u} - \delta_{l\in u} = -(q_1\delta_{l,1}+\cdots+q_k\delta_{l,k}+\delta_{l,j_1}+\cdots+\delta_{l,j_v}).$$

Then we try to discuss the choice of $l \in I$. If $l = i$ for $i \in \{1,\dots,k\}$, it is obvious that $\alpha_{i'} - \alpha_i \equiv -q_i \pmod p$. If $l = i$ for $i \in I_0 \setminus \{1,\dots,k\}$, it is obvious that $\alpha_{i'} - \alpha_i \equiv 0 \pmod p$. If $l \in \{j_1,\dots,j_v\}$, it is clear that $l' \notin u$. If $l \in I_1 \setminus \{j_1,\dots,j_v\}$, we have that l and l' are both in $\bar{u}$. Then the assertion follows.

(ii) By similar calculation and discussion in (i), (ii) can also be proved. □

Lemma 3.3. [1] *Suppose that L is a centerless Lie superalgebra. Then L is weakly complete and any super-biderivation of L is weakly inner. Furthermore, if $\vartheta \in \mathrm{BDer}(L_i)_{(\varepsilon)}$, where $\varepsilon \in T^*, i \in \mathbb{Z}$ and ϕ_ϑ is the linear map related to ϑ, then ϕ_ϑ is T_L^*-homogenous with weight ε, $\mathrm{zd}(\phi_\vartheta) = i$ and $\mathrm{d}(\phi_\vartheta) = \mathrm{d}(\vartheta)$.*

Put $\mathfrak{B} = \mathrm{span}_{\mathbb{F}}\{D_i^{p^{k_i}} \mid i \in I_0,\ 1 \leq k_i \leq t_i - 1\}$ and $h = \sum_{i\in I} h_i$. Then the following results hold.

Lemma 3.4. [7] *Let $H = H(m,n;\underline{t})$ be the Hamiltonian Lie superalgebras over a field of characteristic $p \geq 3$. Then* $\mathrm{Der}\,H = \mathrm{ad}_H(\mathfrak{B} \ltimes \widehat{H})$, *where $\widehat{H} = H \oplus \mathbb{F}D_H(x^{(\pi)}x^\omega) \oplus \sum_{i=1}^m \mathbb{F}D_H(x^{(p^{k_i}\varepsilon_i)}) \oplus \overline{T}$, $i \in I_0$, $1 \leq k_i \leq t_i - 1$.*

Lemma 3.5. *For the weight space decompositions of H and $\widehat{H}$ with respect to T_H, we have $\widehat{H}_{(a)} \neq 0$ if and only if $a \in \Omega_H \cup \theta$.*

Proof. By a direct computation, we have

$$\overline{T} \cup \{D_H(x^{(\pi)}x^\omega)\} \cup \{D_H(x^{(p^{k_i}\varepsilon_i)}) \mid i \in I_0, 1 \leq k_i \leq t_i - 1\} \subset \widehat{H}_{(\theta)},$$

as desired. □

From Lemma 3.3 we know that any super-biderivation of H is weakly inner, then we have the following conclusion.

Lemma 3.6. *Let $H = H(m,n;\underline{t})$ be the Hamiltonian Lie superalgebra, then there exists a unique linear map ϕ_ϑ : $H \longrightarrow \mathfrak{B} \ltimes \widehat{H}$ related to ϑ. If ϑ is symmetric, we have*

$$\vartheta(x,y) = [\phi_\vartheta(x), y] = (-1)^{\mathrm{d}(x)\mathrm{d}(y)}[\phi_\vartheta(y), x],\ \forall\ x, y \in H. \tag{5}$$

Lemma 3.7. *Let $\vartheta \in \mathrm{BDer}(H)$ be a symmetric super-biderivation. Then $\vartheta = 0$ if $\phi_\vartheta(M \cup H_0 \cup N) = 0$.*

Proof. It follows from Lemma 3.3 that

$$[\phi_\vartheta([x,y]),z] = (-1)^{\mathrm{d}(x)\mathrm{d}(y)}[[\phi_\vartheta(x),z],y] + (-1)^{\mathrm{d}(\vartheta)\mathrm{d}(x)}[x,[\phi_\vartheta(y),z]],$$

for all $x,y,z \in H$, which imples that $\phi_\vartheta(\langle X\rangle) = 0$ if $\phi_\vartheta(X) = 0$ for any subset X of H. It follows from Lemma 3.1 that $\phi_\vartheta(H) = 0$. Accordingly, we have $\vartheta = 0$. □

Proposition 3.8. *Suppose that $\vartheta \in \mathrm{BDer}\,(H)_{(\varepsilon)}$ is symmetric, where $\varepsilon \in T_H^*$ and $\varepsilon \notin \Omega_H \cup \{\theta\}$. Then $\vartheta = 0$.*

Proof. By Lemmas 3.3 and 3.5, we know that $\phi_\vartheta(T_H) \subset \widehat{H}_{(\varepsilon)} = \{0\}$. Therefore, for all $t \in T_H$ and $x \in H_{(\eta)}$, where $\eta \in \Omega_H \cup \{\theta\}$, we have $0 = [\phi_\vartheta(t),x] = \pm[t,\phi_\vartheta(x)]$. Further we get $\phi_\vartheta(x) \in (\mathfrak{B} \ltimes \widehat{H})_{(\theta)} \cap (\mathfrak{B} \ltimes \widehat{H})_{(\varepsilon+\eta)}$. Recognizes that

$$M \subset \sum_{i\in I_0} H_{(k\varepsilon_i)},\ \ H_0 \subset \sum_{i,j\in I_0} H_{(\varepsilon_i+\varepsilon_j)} \oplus \sum_{i\in I_0, j\in I_1} H_{(\varepsilon_i+\langle j\rangle)} \oplus \sum_{i,j\in I_1} H_{(\langle i\rangle+\langle j\rangle)},\ \ N \subset H_{(2\varepsilon_1+\langle m+1\rangle)},$$

and $(p-k)\varepsilon_i, (p-1)\varepsilon_i + (p-1)\varepsilon_j, (p-1)\varepsilon_i - \langle j\rangle, -\langle i\rangle - \langle j\rangle, (p-2)\varepsilon_1 - \langle m+1\rangle \in \Omega_H$, which implies that $(M \cup H_0 \cup N) \cap H_{(-\varepsilon)} = \{0\}$. So we can get $\phi_\vartheta(M \cup H_0 \cup N) = 0$. Therefore, it can be known from Lemma 3.7 that $\vartheta = 0$. □

Theorem 3.9. *Every symmetric super-biderivation ϑ of H is zero.*

Proof. By Lemma 3.8, we only need to consider $\vartheta \in \mathrm{BDer}H_{(\varepsilon)}$ and $\varepsilon \in \Omega_H \cup \{\theta\}$. Notice that $\phi_\vartheta(h) \in \mathfrak{B} \ltimes \widehat{H}$ for any $h \in H$. Recall the weight space decompositions of H with respect to T_H mentioned in Lemma 3.2 and $M = \{D_H(x^{(k\varepsilon_i)}) \mid 1 \le k \le p^{t_i} - 1, i \in I_0\}$, $N = \{D_H(x^{(2\varepsilon_1)}x_{m+1})\}$, $H_0 = \{D_H(x_ix_j) \mid i,j \in I\}$. We claim that $\phi_\vartheta(M \cup H_0 \cup N) = 0$.

In fact, we consider the following two types.

Type I : n is even.

Case 1 **:** $\varepsilon = \alpha + \langle u\rangle$, where $u \in \mathbb{B}$ is not empty and $|u| \le \frac{n}{2} - 1$. Suppose that, for $i,h \in I_0$ and $i \ne h$, $j,l \in I_1 \setminus \{u\}$ and $j \ne l$, $e' \in u$, $1 \le k \le \pi_i$,

$$\phi_\vartheta(D_H(x^{(k\varepsilon_i)})) = \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B}}} b^{\alpha,u,k\cdot i}_{\beta,\bar{u}} D_H(x^{(\alpha+k\varepsilon_i+\beta)}x^u x^{\bar{u}});$$

$$\phi_\vartheta(D_H(x_e)) = \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B}}} b^{\alpha,u,e}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)}x^u x_e x^{\bar{u}});$$

$$\phi_\vartheta(D_H(x_ix_h)) = \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B}}} b^{\alpha,u,i,h}_{\beta,\bar{u}} D_H(x^{(\alpha+\varepsilon_i+\varepsilon_h+\beta)}x^u x^{\bar{u}});$$

$$\phi_\vartheta(D_H(x_ix_j)) = \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B} \\ j\notin\bar{u}}} b^{\alpha,u,i,j}_{\beta,\bar{u}} D_H(x^{(\alpha+\varepsilon_i+\beta)}x^u x_j x^{\bar{u}});$$

$$\phi_\vartheta(D_H(x_jx_l)) = \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B} \\ j,l\notin\bar{u}}} b^{\alpha,u,j,l}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)}x^u x_j x_l x^{\bar{u}});$$

$$\phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})) = \delta_{m+1\notin u} \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B} \\ m+1\notin\bar{u}}} b^{\alpha,u,2\cdot 1,m+1}_{\beta,\bar{u}} D_H(x^{(\alpha+2\varepsilon_1+\beta)}x^u x_{m+1} x^{\bar{u}}),$$

where l and l' are both in $\bar{u}$ for $l \in I_1$.

It is obvious that $\phi_\vartheta(D_H(x_i x_{j'})) = \phi_\vartheta(D_H(x_{j'} x_{l'})) = \phi_\vartheta(D_H(x_{j'})) = \phi_\vartheta(D_H(x^{(2\varepsilon_1)} x_{m+1})) = 0$ for $m+1 \in u$. For $e' \in u$ and $e \neq j, l$, from

$$[\phi_\vartheta(D_H(x_j)), D_H(x_e)] = \sum_{\substack{0\leq\beta\leq\pi\\ \bar{u}\in\mathbb{B}\\ j\notin\bar{u}}} (-1)^{\tau(u,e')} b^{\alpha,u,j}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)} x^{u\setminus\{e'\}} x_j x^{\bar{u}}),$$

$$[\phi_\vartheta(D_H(x_e)), D_H(x_j)] = \sum_{\substack{0\leq\beta\leq\pi\\ \bar{u}\in\mathbb{B}\\ j'\notin\bar{u}}} (-1)^{\mathrm{d}(\vartheta)+\tau(u,j')} b^{\alpha,u,e}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)} x^{u\setminus\{j'\}} x_e x^{\bar{u}})$$
$$+ \sum_{\substack{0\leq\beta\leq\pi\\ \bar{u}\in\mathbb{B}\\ j'\in\bar{u}}} (-1)^{\mathrm{d}(\vartheta)+|u|+\tau(\bar{u},j')+\tau(\bar{u},j)} b^{\alpha,u,e}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)} x^{u} x_e x_j x^{\bar{u}\setminus\{j,j'\}}),$$

and

$$[\phi_\vartheta(D_H(x_j)), D_H(x_e)] = -[\phi_\vartheta(D_H(x_e)), D_H(x_j)],$$

we have $\phi_\vartheta(D_H(x_j)) = \phi_\vartheta(D_H(x_e)) = 0$. By

$$0 = [\phi_\vartheta(D_H(x^{(k\varepsilon_i)})), D_H(x_e)] = \sum_{\substack{0\leq\beta\leq\pi\\ \bar{u}\in\mathbb{B}}} (-1)^{\mathrm{d}(\vartheta)+\tau(\bar{u},e')} b^{\alpha,u,k\cdot i}_{\beta,\bar{u}} D_H(x^{(\alpha+k\varepsilon_i+\beta)} x^{u\setminus\{e'\}} x^{\bar{u}}),$$
$$0 = [\phi_\vartheta(D_H(x_i x_h)), D_H(x_e)] = \sum_{\substack{0\leq\beta\leq\pi\\ \bar{u}\in\mathbb{B}}} (-1)^{\mathrm{d}(\vartheta)+\tau(\bar{u},e')} b^{\alpha,u,i,h}_{\beta,\bar{u}} D_H(x^{(\alpha+\varepsilon_i+\varepsilon_h+\beta)} x^{u\setminus\{e'\}} x^{\bar{u}}),$$
$$0 = [\phi_\vartheta(D_H(x_i x_j)), D_H(x_e)] = \sum_{\substack{0\leq\beta\leq\pi\\ \bar{u}\in\mathbb{B}\\ j\notin\bar{u}}} (-1)^{\mathrm{d}(\vartheta)+\tau(\bar{u},e')} b^{\alpha,u,i,j}_{\beta,\bar{u}} D_H(x^{(\alpha+\varepsilon_i+\beta)} x^{u\setminus\{e'\}} x_j x^{\bar{u}}),$$
$$0 = [\phi_\vartheta(D_H(x_j x_l)), D_H(x_e)] = \sum_{\substack{0\leq\beta\leq\pi\\ \bar{u}\in\mathbb{B}\\ j,l\notin\bar{u}}} (-1)^{\mathrm{d}(\vartheta)+\tau(\bar{u},e')} b^{\alpha,u,j,l}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)} x^{u\setminus\{e'\}} x_j x_l x^{\bar{u}}),$$
$$0 = [\phi_\vartheta(D_H(x^{(2\varepsilon_1)} x_{m+1}), D_H(x_e)] = \delta_{m+1\notin u} \sum_{\substack{0\leq\beta\leq\pi\\ \bar{u}\in\mathbb{B}\\ m+1\notin\bar{u}}} b^{\alpha,u,2\cdot 1,m+1}_{\beta,\bar{u}} D_H(x^{(\alpha+2\varepsilon_1+\beta)} x^{u\setminus\{e'\}} x_{m+1} x^{\bar{u}}),$$

we can get

$$\phi_\vartheta(D_H(x^{(k\varepsilon_i)})) = \phi_\vartheta(D_H(x_i x_h)) = \phi_\vartheta(D_H(x_i x_j)) = \phi_\vartheta(D_H(x_j x_l)) = \phi_\vartheta(D_H(x^{(2\varepsilon_1)} x_{m+1}) = 0.$$

Consequently, we get $\phi_\vartheta(M \cup H_0 \cup N) = 0$.

Case 2 : $\varepsilon = \alpha + \langle u\rangle$, where $u \in \mathbb{B}$ is not empty and $|u| = \frac{n}{2}$. Suppose that, for all $i, h \in I_0$ and $i \neq h$, $j, l \in I_1 \setminus \{u\}$ and $j \neq l$, $1 \leq k \leq \pi_i$,

$$\phi_\vartheta(D_H(x^{(k\varepsilon_i)})) = \sum_{0\leq\beta\leq\pi} b^{\alpha,u,k\cdot i}_{\beta} D_H(x^{(\alpha+k\varepsilon_i+\beta)} x^{u});$$
$$\phi_\vartheta(D_H(x_i x_h)) = \sum_{0\leq\beta\leq\pi} b^{\alpha,u,i,h}_{\beta} D_H(x^{(\alpha+\varepsilon_i+\varepsilon_h+\beta)} x^{u});$$

$$\phi_\vartheta(D_H(x_ix_j)) = \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,i,j} D_H(x^{(\alpha+\varepsilon_i+\beta)}x^u x_j);$$
$$\phi_\vartheta(D_H(x_jx_l)) = \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,j,l} D_H(x^{(\alpha+\beta)}x^u x_j x_l);$$
$$\phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})) = \delta_{m+1\notin u} \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,2\cdot 1,m+1} D_H(x^{(\alpha+2\varepsilon_1+\beta)}x^u x_{m+1}),$$

where l and l' are both in $\bar{u}$ for $l \in I_1$.

It is obvious that $\phi_\vartheta(D_H(x_{j'})) = 0$. We have

$$\begin{aligned}
[\phi_\vartheta(D_H(x^{(k\varepsilon_i)})), D_H(x_{j'})] &= [\phi_\vartheta(D_H(x_{j'})), D_H(x^{(k\varepsilon_i)})] = 0,\\
[\phi_\vartheta(D_H(x_ix_h)), D_H(x_{j'})] &= [\phi_\vartheta(D_H(x_{j'})), D_H(x_ix_h)] = 0,\\
[\phi_\vartheta(D_H(x_ix_j)), D_H(x_{j'})] &= [\phi_\vartheta(D_H(x_{j'})), D_H(x_ix_j)] = 0,\\
[\phi_\vartheta(D_H(x_jx_l)), D_H(x_{j'})] &= [\phi_\vartheta(D_H(x_{j'})), D_H(x_jx_l)] = 0,\\
[\phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})), D_H(x_{j'})] &= [\phi_\vartheta(D_H(x_{j'})), D_H(x^{(2\varepsilon_1)}x_{m+1})] = 0,
\end{aligned}$$

which implies that

$$\phi_\vartheta(D_H(x^{(k\varepsilon_i)})) = \phi_\vartheta(D_H(x_ix_h)) = \phi_\vartheta(D_H(x_ix_j)) = \phi_\vartheta(D_H(x_jx_l)) = \phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})) = 0.$$

Case 3 **:** $\varepsilon = \alpha$. Suppose that, for $i, h \in I_0$ and $i \neq h$, $j, l \in I_1$ and $j \neq l$, $1 \le k \le \pi_i$,

$$\begin{aligned}
\phi_\vartheta(D_H(x^{(k\varepsilon_i)})) &= \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}}} b_{\beta,\bar{u}}^{\alpha,k\cdot i} D_H(x^{(\alpha+k\varepsilon_i+\beta)}x^{\bar{u}}) + \delta_{\alpha+k\varepsilon_i=\theta}(c^{k\cdot i}D_H(x^{(\pi)}x^w)\\
&\quad + \sum_{g=1}^m d_g^{k\cdot i} D_H(x^{(p^{k_g}\varepsilon_g)}) + \sum_{a=1}^s e_a^{k\cdot i} x_a D_a + \sum_{g=1}^m f_g^{k\cdot i} D_g^{p^{k_g}});\\
\phi_\vartheta(D_H(x_j)) &= \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}\\ j\notin\bar{u}}} b_{\beta,\bar{u}}^{\alpha,j} D_H(x^{(\alpha+\beta)}x_j x^{\bar{u}});\\
\phi_\vartheta(D_H(x_ix_h)) &= \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}}} b_{\beta,\bar{u}}^{\alpha,i,h} D_H(x^{(\alpha+\varepsilon_i+\varepsilon_h+\beta)}x^{\bar{u}}) + \delta_{\alpha+\varepsilon_i+\varepsilon_h=\theta}(c^{i,h}D_H(x^{(\pi)}x^w)\\
&\quad + \sum_{g=1}^m d_g^{i,h} D_H(x^{(p^{t_g}\varepsilon_g)}) + e_a^{i,h}\sum_{a=1}^s x_a D_a + \sum_{g=1}^m f_g^{i,h} D_g^{p^{k_g}});\\
\phi_\vartheta(D_H(x_ix_j)) &= \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}\\ j\notin\bar{u}}} b_{\beta,\bar{u}}^{\alpha,i,j} D_H(x^{(\alpha+\varepsilon_i+\beta)}x_j x^{\bar{u}});\\
\phi_\vartheta(D_H(x_jx_l)) &= \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}\\ j,l\notin\bar{u}}} b_{\beta,\bar{u}}^{\alpha,j,l} D_H(x^{(\alpha+\beta)}x_j x_l x^{\bar{u}});\\
\phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})) &= \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}\\ m+1\notin\bar{u}}} b_{\beta,\bar{u}}^{\alpha,2\cdot 1,m+1} D_H(x^{(\alpha+2\varepsilon_1+\beta)}x_{m+1} x^{\bar{u}}),
\end{aligned}$$

where l and l' are both in $\bar{u}$ for $l \in I_1$.

From

$$[\phi_\vartheta(D_H(x_ix_j)), D_H(x_{j'})] = \sum_{\substack{0\leq\beta\leq\pi \\ \bar{u}\in\mathbb{B} \\ j\notin\bar{u}}} b^{\alpha,i,j}_{\beta,\bar{u}} D_H(x^{(\alpha+\varepsilon_i+\beta)}x^{\bar{u}}),$$

$$\begin{aligned}[\phi_\vartheta(D_H(x_{j'})), D_H(x_ix_j)] = &\sum_{\substack{0\leq\beta\leq\pi \\ \bar{u}\in\mathbb{B} \\ j'\notin\bar{u}}} \sigma(i')b^{\alpha,j'}_{\beta,\bar{u}} D_H(x^{(\alpha-\varepsilon_{i'}+\beta)}x_{j'}x_jx^{\bar{u}}) \\ &+ \sum_{\substack{0\leq\beta\leq\pi \\ \bar{u}\in\mathbb{B} \\ j'\notin\bar{u}}} (\alpha_i+\beta_i+1)b^{\alpha,j'}_{\beta,\bar{u}} D_H(x^{(\alpha+\varepsilon_i+\beta)}x^{\bar{u}}),\end{aligned}$$

and

$$[\phi_\vartheta(D_H(x_ix_j)), D_H(x_{j'})] = -[\phi_\vartheta(D_H(x_{j'})), D_H(x_ix_j)],$$

we can get $\phi_\vartheta(D_H(x_{j'})) = 0$ for $0 < \alpha_i + \beta_i \leq \pi_i$, $\phi_\vartheta(D_H(x_ix_j)) = 0$ for $0 \leq \alpha_i + \beta_i < \pi_i$. Due to the arbitrariness of $i \in I_0$, we only need to consider the case of $\alpha + \beta = 0$ and $\alpha + \beta = \pi$. Therefore, we assume

$$\phi_\vartheta(D_H(x_{j'})) = \sum_{\substack{\bar{u}\in\mathbb{B} \\ j'\notin\bar{u}}} b^{0,j'}_{0,\bar{u}} D_H(x_{j'}x^{\bar{u}});$$

$$\phi_\vartheta(D_H(x_ix_j)) = \sum_{\substack{\bar{u}\in\mathbb{B} \\ j\notin\bar{u}}} b^{\alpha,i,j}_{\beta,\bar{u}} D_H(x^{(\pi)}x_jx^{\bar{u}}).$$

It follows from

$$[\phi_\vartheta(D_H(x_jx_l)), D_H(x_{j'})] = \sum_{\substack{0\leq\beta\leq\pi \\ \bar{u}\in\mathbb{B} \\ j,l\notin\bar{u}}} b^{\alpha,j,l}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)}x_lx^{\bar{u}}),$$

$$[\phi_\vartheta(D_H(x_{j'})), D_H(x_jx_l)] = \sum_{\substack{\bar{u}\in\mathbb{B} \\ j',l'\notin\bar{u}}} b^{0,j'}_{0,\bar{u}} D_H(x^{\bar{u}}x_l) - \sum_{\substack{\bar{u}\in\mathbb{B} \\ j'\notin\bar{u} \\ l'\in\bar{u}}} (-1)^{\tau(\bar{u},l')} b^{0,j'}_{0,\bar{u}} D_H(x_jx_{j'}x^{\bar{u}\setminus\{l'\}}),$$

and

$$[\phi_\vartheta(D_H(x_jx_l)), D_H(x_{j'})] = [\phi_\vartheta(D_H(x_{j'})), D_H(x_jx_l)],$$

that $\phi_\vartheta(D_H(x_jx_l)) = \phi_\vartheta(D_H(x_{j'})) = 0$. From

$$0 = [\phi_\vartheta(D_H(x_ix_j)), D_H(x_{j'}x_l)] = \sum_{\substack{\bar{u}\in\mathbb{B} \\ j,l\notin\bar{u}}} b^{\alpha,i,j}_{\beta,\bar{u}} D_H(x^{(\pi)}x^{\bar{u}}) - \sum_{\substack{\bar{u}\in\mathbb{B} \\ j\notin\bar{u} \\ l\in\bar{u}}} b^{\alpha,i,j}_{\beta,\bar{u}} D_H(x^{(\pi)}x_{j'}x^{\bar{u}\setminus\{l'\}}),$$

we have $\phi_\vartheta(D_H(x_ix_j)) = 0$. By

$$0 = [\phi_\vartheta(D_H(x^{(k\varepsilon_i)})), D_H(x_j)]$$

$$= \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B}}} b^{\alpha,k\cdot i}_{\beta,\bar{u}} D_H(x^{(\alpha+k\varepsilon_i+\beta)}x^{\bar{u}}) + \delta_{\alpha+k\varepsilon_i=\theta}(c^{k\cdot i}D_H(x^{(\pi)}x^{w\setminus\{j'\}}) + e^{k\cdot i}_{j'}D_{j'}),$$

we can obtain that

$$\phi_\vartheta(D_H(x^{(k\varepsilon_i)})) = \delta_{\alpha+k\varepsilon_i=\theta}(\sum_{g=1}^{m} d^{k\cdot i}_g D_H(x^{(p^{k_g}\varepsilon_g)}) + \sum_{a=1}^{m} e^{k\cdot i}_a x_a D_a + \sum_{g=1}^{m} f^{k\cdot i}_g D^{p^{k_g}}_g).$$

Due to

$$\begin{aligned} 0 =&[\phi_\vartheta(D_H(x^{(k\varepsilon_i)})), D_H(x_i x_j)] \\ =&d^{k\cdot i}_{i'} D_H(x^{((p^{k_{i'}}-1)\varepsilon_{i'})}x_j) - e^{k\cdot i}_i x_i D_{j'} - f^{k\cdot i}_i D^{p^{k_i}-1}_i D_{j'}, \end{aligned}$$

we have $\phi_\vartheta(D_H(x^{(k\varepsilon_i)})) = 0$. Similarly, from

$$\begin{aligned} [\phi_\vartheta(D_H(x_i x_h)), D_H(x_j)] =&[\phi_\vartheta(D_H(x_j)), D_H(x_i x_h)] = 0, \\ [\phi_\vartheta(D_H(x_i x_h)), D_H(x_i x_j)] =&[\phi_\vartheta(D_H(x_i x_j)), D_H(x_i x_h)] = 0, \\ [\phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})), D_H(x_j)] =&[\phi_\vartheta(D_H(x_j)), D_H(D_H(x^{(2\varepsilon_1)}x_{m+1}))] = 0, \\ [\phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})), D_H(x_i x_j)] =&[\phi_\vartheta(D_H(x_i x_j)), D_H(D_H(x^{(2\varepsilon_1)}x_{m+1}))] = 0, \end{aligned}$$

we can get $\phi_\vartheta(D_H(x_i x_h)) = \phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})) = 0$. This implies $\phi_\vartheta(M \cup H_0 \cup N) = 0$.

Type II : n **is odd**.

Case 1 : $\varepsilon = \alpha + \langle u\rangle + \langle s\rangle$, where $u \in \mathbb{B}$ is not empty, $|u| \le \left\lfloor \frac{n}{2} \right\rfloor - 1$ and $s \notin u$. Suppose that, for all $i, h \in I_0$ and $i \ne h$, $j, l \in I_1 \setminus \{u\}$ and $j \ne l$, $e' \in u$, $1 \le k \le \pi_i$,

$$\begin{aligned} \phi_\vartheta(D_H(x^{(k\varepsilon_i)})) =& \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B}}} b^{\alpha,u,k\cdot i}_{\beta,\bar{u}} D_H(x^{(\alpha+k\varepsilon_i+\beta)}x^u x^{\bar{u}} x_s); \\ \phi_\vartheta(D_H(x_e)) =& \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B}}} b^{\alpha,u,e}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)}x^u x_e x^{\bar{u}} x_s); \\ \phi_\vartheta(D_H(x_i x_h)) =& \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B}}} b^{\alpha,u,i,h}_{\beta,\bar{u}} D_H(x^{(\alpha+\varepsilon_i+\varepsilon_h+\beta)}x^u x^{\bar{u}} x_s); \\ \phi_\vartheta(D_H(x_i x_j)) =& \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B} \\ j\notin\bar{u}}} b^{\alpha,u,i,ji}_{\beta,\bar{u}} D_H(x^{(\alpha+\varepsilon_i+\beta)}x^u x_j x^{\bar{u}} x_s); \\ \phi_\vartheta(D_H(x_j x_l)) =& \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B} \\ j,l\notin\bar{u}}} b^{\alpha,u,j,l}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)}x^u x_j x_l x^{\bar{u}} x_s); \\ \phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})) =&\delta_{m+1\notin u} \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B} \\ m+1\notin\bar{u}}} b^{\alpha,u,2\cdot 1,m+1}_{\beta,\bar{u}} D_H(x^{(\alpha+2\varepsilon_1+\beta)}x^u x_{m+1} x^{\bar{u}} x_s), \end{aligned}$$

where l and l' are both in $\bar{u}$ for $l \in I_1$. It is obvious that $\phi_\vartheta(D_H(x_i x_{j'})) = \phi_\vartheta(D_H(x_{j'} x_{l'})) = \phi_\vartheta(D_H(x_{j'})) = \phi_\vartheta(D_H(x_s)) = \phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})) = \phi_\vartheta(D_H(x_i x_s)) = \phi_\vartheta(D_H(x_{j'} x_s)) = 0$ for $m+1 \in u$. Similar to the prove of **Case 1** in **Type I** before, it is clear that $\phi_\vartheta(M \cup H_0 \cup N) = 0$.

Case 2 : $\varepsilon = \alpha + \langle u\rangle$, where $u \in \mathbb{B}$ is not empty, $|u| \le \left\lfloor \frac{n}{2} \right\rfloor - 1$ and $s \notin u$. Suppose that, for $i, h \in I_0$ and $i \ne h$, $j, l \in I_1 \setminus \{u, s\}$ and $j \ne l$, $e' \in u$, $1 \le k \le \pi_i$,

$$\phi_\vartheta(D_H(x^{(k\varepsilon_i)})) = \sum_{\substack{0\le\beta\le\pi \\ \bar{u}\in\mathbb{B}}} b^{\alpha,u,k\cdot i}_{\beta,\bar{u}} D_H(x^{(\alpha+k\varepsilon_i+\beta)}x^u x^{\bar{u}});$$

$$\phi_\vartheta(D_H(x_e)) = \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}}} b^{\alpha,u,e}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)}x^u x_e x^{\bar{u}});$$

$$\phi_\vartheta(D_H(x_s)) = \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}}} b^{\alpha,u,s}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)}x^u x^{\bar{u}} x_s);$$

$$\phi_\vartheta(D_H(x_i x_h)) = \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}}} b^{\alpha,u,i,h}_{\beta,\bar{u}} D_H(x^{(\alpha+\varepsilon_i+\varepsilon_h+\beta)}x^u x^{\bar{u}});$$

$$\phi_\vartheta(D_H(x_i x_j)) = \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}\\ j\notin\bar{u}}} b^{\alpha,u,i,j}_{\beta,\bar{u}} D_H(x^{(\alpha+\varepsilon_i+\beta)}x^u x_j x^{\bar{u}});$$

$$\phi_\vartheta(D_H(x_i x_s)) = \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}\\ j,s\notin\bar{u}}} b^{\alpha,u,i,s}_{\beta,\bar{u}} D_H(x^{(\alpha+\varepsilon_i+\beta)}x^u x_j x^{\bar{u}} x_s);$$

$$\phi_\vartheta(D_H(x_j x_l)) = \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}\\ j,l\notin\bar{u}}} b^{\alpha,u,j,l}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)}x^u x_j x_l x^{\bar{u}});$$

$$\phi_\vartheta(D_H(x_j x_s)) = \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}\\ j,s\notin\bar{u}}} b^{\alpha,u,j,s}_{\beta,\bar{u}} D_H(x^{(\alpha+\beta)}x^u x_j x^{\bar{u}} x_s);$$

$$\phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})) = \delta_{m+1\notin u} \sum_{\substack{0\le\beta\le\pi\\ \bar{u}\in\mathbb{B}\\ m+1\notin\bar{u}}} b^{\alpha,u,2\cdot 1,m+1}_{\beta,\bar{u}} D_H(x^{(\alpha+2\varepsilon_1+\beta)}x^u x_{m+1} x^{\bar{u}}),$$

where l and l' are both in $\bar{u}$ for $l \in I_1$.

It is obvious that $\phi_\vartheta(D_H(x_i x_{j'})) = \phi_\vartheta(D_H(x_{j'} x_{l'})) = \phi_\vartheta(D_H(x_{j'})) = \phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})) = 0$ for $m+1 \in u$. Similar to the prove of **Case 1** in **Type I** before, it is clear that $\phi_\vartheta(D_H(x^{(k\varepsilon_i)})) = \phi_\vartheta(D_H(x_i x_h)) = \phi_\vartheta(D_H(x_e)) = \phi_\vartheta(D_H(x_i x_j)) = \phi_\vartheta(D_H(x_j x_l)) = \phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})) = 0$. It only needs to prove that $\phi_\vartheta(D_H(x_s)) = \phi_\vartheta(D_H(x_i x_s)) = \phi_\vartheta(D_H(x_j x_s)) = 0$. Indeed, by

$$\begin{aligned}
[\phi_\vartheta(D_H(x_s)), D_H(x_e)] &= -[\phi_\vartheta(D_H(x_e)), D_H(x_s)] = 0,\\
[\phi_\vartheta(D_H(x_i x_s)), D_H(x_e)] &= -[\phi_\vartheta(D_H(x_e)), D_H(x_i x_s)] = 0,\\
[\phi_\vartheta(D_H(x_j x_s)), D_H(x_e)] &= [\phi_\vartheta(D_H(x_e)), D_H(x_j x_s)] = 0,
\end{aligned}$$

we can get the result. This imples that $\phi_\vartheta(M \cup H_0 \cup N) = 0$.

Case 3 **:** $\varepsilon = \alpha + \langle u\rangle + \langle s\rangle$, where $u \in \mathbb{B}$ is not empty and $|u| = \left\lfloor \frac{n}{2} \right\rfloor$. Suppose that, for $i, h \in I_0$ and $i \neq h$, $j, l \in I_1 \setminus \{u\}$ and $j \neq l$,

$$\phi_\vartheta(D_H(x^{(k\varepsilon_i)})) = \sum_{0\le\beta\le\pi} b^{\alpha,u,k\cdot i}_{\beta} D_H(x^{(\alpha+k\varepsilon_i+\beta)}x^u x_s);$$

$$\phi_\vartheta(D_H(x_i x_h)) = \sum_{0\le\beta\le\pi} b^{\alpha,u,i,h}_{\beta} D_H(x^{(\alpha+\varepsilon_i+\varepsilon_h+\beta)}x^u x_s);$$

$$\phi_\vartheta(D_H(x_i x_j)) = \sum_{0\le\beta\le\pi} b^{\alpha,u,i,j}_{\beta} D_H(x^{(\alpha+\varepsilon_i+\beta)}x^u x_j x_s);$$

$$\phi_\vartheta(D_H(x_j)) = \sum_{0\le\beta\le\pi} b^{\alpha,u,j}_{\beta} D_H(x^{(\alpha+\beta)}x^u x_j x_s);$$

$$\phi_\vartheta(D_H(x_jx_l)) = \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,j,l} D_H(x^{(\alpha+\beta)}x^u x_j x_l x_s);$$
$$\phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})) = \delta_{m+1\notin u}\sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,2\cdot 1,m+1} D_H(x^{(\alpha+2\varepsilon_1+\beta)}x^u x_{m+1} x_s),$$

where l and l' are both in $\bar{u}$ for $l \in I_1$.

It is obvious that $\phi_\vartheta(D_H(x_{j'})) = \phi_\vartheta(D_H(x_s)) = 0$. Through a similar proof process as in **Case 2** of **Type I**, we can obtain $\phi_\vartheta(M \cup H_0 \cup N) = 0$.

Case 4 **:** $\varepsilon = \alpha + \langle u\rangle$, where $u \in \mathbb{B}$ is not empty and $|u| = \left\lfloor \frac{n}{2} \right\rfloor$. Suppose that, for $i, h \in I_0$ and $i \neq h$, $j, l \notin u$ and $j \neq l$, $1 \le k \le \pi_i$,

$$\phi_\vartheta(D_H(x^{(k\varepsilon_i)})) = \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,k\cdot i} D_H(x^{(\alpha+k\varepsilon_i+\beta)}x^u) + \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,k\cdot i} D_H(x^{(\alpha+k\varepsilon_i+\beta)}x^u x_s);$$
$$\phi_\vartheta(D_H(x_ix_h)) = \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,k\cdot i} D_H(x^{(\alpha+\varepsilon_i+\varepsilon_h+\beta)}x^u) + \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,i,h} D_H(x^{(\alpha+\varepsilon_i+\varepsilon_h+\beta)}x^u x_s);$$
$$\phi_\vartheta(D_H(x_ix_s)) = \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,i,s} D_H(x^{(\alpha+\varepsilon_i+\beta)}x^u x_s);$$
$$\phi_\vartheta(D_H(x_s)) = \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,s} D_H(x^{(\alpha+\beta)}x^u x_s);$$
$$\phi_\vartheta(D_H(x_j)) = \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,j} D_H(x^{(\alpha+\beta)}x^u x_j) + \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,j} D_H(x^{(\alpha+\beta)}x^u x_j x_s);$$
$$\phi_\vartheta(D_H(x_ix_j)) = \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,i,j} D_H(x^{(\alpha+\varepsilon_i+\beta)}x^u x_j) + \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,i,j} D_H(x^{(\alpha+\varepsilon_i+\beta)}x^u x_j x_s);$$
$$\phi_\vartheta(D_H(x_jx_l)) = \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,j,l} D_H(x^{(\alpha+\beta)}x^u x_j x_l) + \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,j,l} D_H(x^{(\alpha+\beta)}x^u x_j x_l x_s);$$
$$\phi_\vartheta(D_H(x_jx_s)) = \sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,j,s} D_H(x^{(\alpha+\beta)}x^u x_j x_s);$$
$$\begin{aligned}\phi_\vartheta(D_H(x^{(2\varepsilon_1)}x_{m+1})) =& \delta_{m+1\notin u}\sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,2\cdot 1,m+1} D_H(x^{(\alpha+2\varepsilon_1+\beta)}x^u x_{m+1})\\ &+ \delta_{m+1\notin u}\sum_{0\le\beta\le\pi} b_\beta^{\alpha,u,2\cdot 1,m+1} D_H(x^{(\alpha+2\varepsilon_1+\beta)}x^u x_{m+1} x_s),\end{aligned}$$

where l and l' are both in $\bar{u}$ for $l \in I_1$.

In this case, it only needs to prove that $\phi_\vartheta(D_H(x_s)) = \phi_\vartheta(D_H(x_ix_s)) = \phi_\vartheta(D_H(x_jx_s)) = 0$. Indeed, by

$$\begin{aligned}[\phi_\vartheta(D_H(x_s)), D_H(x_{j'})] &= -[\phi_\vartheta(D_H(x_{j'})), D_H(x_s)] = 0,\\ [\phi_\vartheta(D_H(x_ix_s)), D_H(x_{j'})] &= -[\phi_\vartheta(D_H(x_{j'})), D_H(x_ix_s)] = 0,\\ [\phi_\vartheta(D_H(x_jx_s)), D_H(x_{j'})] &= [\phi_\vartheta(D_H(x_{j'})), D_H(x_jx_s)] = 0,\end{aligned}$$

we can get the result. This imples that $\phi_\vartheta(M \cup H_0 \cup N) = 0$.

Case 5 **:** $\varepsilon = \alpha$. In this case, there is no need to specifically distinguish whether s is in u or $\bar{u}$. Therefore, by similar assumption and prove as in **Case 3** of **Type 1**, we can also obtain $\phi_\vartheta(M \cup H_0 \cup N) = 0$. □

It follows from [17] that every skew-symmetric super-biderivation of H is inner. Combining Theorem 3.9 readily yields the following conclusion for characterizing of every super-biderivation of H.

Theorem 3.10. *Every super-biderivation of H is inner.*

4. Super-commutative post-Lie superalgebra structures and linear super-commuting maps

In this section, we describe the linear super-commuting maps and super-commutative post-Lie superalgebra structures on H, as applications for the results of Section 3.

Since H is simple, from [19] that we have the following result.

Theorem 4.1. *Let $H = H(m, n, \underline{t})$ be the Hamiltonian Lie superalgebra. Then a linear map ϕ on H is super-commuting if and only if ϕ is a scalar transformation.*

Recall the super-commutative post-Lie superalgebra structure "$\cdot$"on a Lie superalgebra mentioned in Definition 2.8, we get the following result.

Theorem 4.2. *Let $H = H(m, n, \underline{t})$ be the Hamiltonian Lie superalgebra. Then any super-commutative post-Lie superalgebra structure on H is trivial.*

Proof. Define a bilinear map $\vartheta : H \times H \longrightarrow H$ by $\vartheta(x, y) = x \cdot y$, for all $x, y \in H$. From

$$\vartheta(x, y) = x \cdot y = (-1)^{\mathrm{d}(x)\mathrm{d}(y)} y \cdot x = \vartheta(y, x) = (-1)^{\mathrm{d}(x)\mathrm{d}(y)} \vartheta(y, x),$$

we get ϑ is symmetric and $\mathrm{d}(\vartheta) = \bar{0}$. We claim that ϑ is a symmetric super-biderivation of H. In fact, for any $x, y, z \in H$, we have

$$\begin{aligned}\vartheta([x, y], z) =& [x, y] \cdot z = (-1)^{\mathrm{d}(z)(\mathrm{d}(x)+\mathrm{d}(y))} z \cdot [x, y]\\ &+(-1)^{\mathrm{d}(z)(\mathrm{d}(x)+\mathrm{d}(y))}[z \cdot x, y] + (-1)^{\mathrm{d}(z)+\mathrm{d}(y)}[x, z \cdot y]\\ =&(-1)^{\mathrm{d}(z)\mathrm{d}(y)}[\vartheta(x, z), y] + [x, \vartheta(x, z)],\end{aligned}$$

$$\begin{aligned}\vartheta(x, [y, z]) =& x \cdot [y, z] = [x \cdot y, z] + (-1)^{\mathrm{d}(x)\mathrm{d}(y)}[y, x \cdot z]\\ =&[\vartheta(x, y), z] + (-1)^{\mathrm{d}(x)\mathrm{d}(y)}[y, \vartheta(x, z)].\end{aligned}$$

Thus, claim holds. It follows from Theorem 3.9 that the conclusion holds. □

References

[1] W. Bai, W. D. Liu, *Superbiderivations of simple modular Lie superalgebras of Witt type and special type*, Algebra Colloq. **30** (2023), 181–192.
[2] W. Bai, W. D. Liu, *Superderivations for modular graded Lie superalgebras of Cartan-type*, Algebr. Represent. Theory **17** (2014), 69–86.
[3] M. Brešar, *On generalized biderivations and related maps*, J. Algebra **172** (1995), 764–786.
[4] M. Brešar, K. M. Zhao, *Biderivations and commuting linear maps on Lie algebras*, J. Lie Theory **28** (2018), 885–900.
[5] M. J. Celousov, *Derivations of Lie algebras of Cartan type (Russian)*, Izv. Vysš. Učebn. Zaved. Matematika **7** (1970), 126–134.
[6] Y. Chang, L. Y. Chen, Y. Cao, *Super-biderivations of the generalized Witt Lie superalgebra $W(m, n; \underline{t})$*, Linear Multilinear Algebra **69** (2021), 233–244.
[7] Y. Z. Chen, Y. Wang, B. L. Ma, *The derivation algebra of the Cartan-type Lie superalgebra H*, Chinese Quart. J. Math. **23** (2008), 409–414.
[8] M. Dilxat, S. L. Gao, D. Liu, *Super-biderivations and post-Lie superalgebras on some Lie superalgebras*, Acta Math. Sin. (Engl. Ser.) **39** (2023), 1736–1754.
[9] G. Z. Fan, X. S. Dai, *Super-biderivations of Lie superalgebras*, Linear Multilinear Algebra **65** (2017), 58–66.
[10] J. Y. Fu, Q. C. Zhang, C. P. Jiang, *The Cartan-type modular Lie superalgebra KO*, Comm. Algebra **34** (2006), 107–128.
[11] W. H. Li, X. M. Tang, J. X. Yuan, *Super-biderivations and linear super-commuting maps on the super W-algebra $\widetilde{W}(2, 2)$*, Colloq. Math. **153** (2018), 273–300.
[12] W. D. Liu, Y. H. He, *Finite-dimensional special odd Hamiltonian superalgebras in prime characteristic*, Commun. Contemp. Math. **11** (2009), 523–546.
[13] W. D. Liu, J. X. Yuan, *Finite dimensional special odd contact superalgebras over a field of prime characteristic*, J. Lie Theory **79** (2012), 113–130.
[14] W. D. Liu, Y. Z. Zhang, *Finite-dimensional odd Hamiltonian superalgebras over a field of prime characteristic*, J. Aust. Math. Soc. **79** (2005), 113–130.
[15] L. M. Tang, L. Y. Meng, L. Y. Chen, *Super-biderivations and linear super-commuting maps on the Lie superalgebras*, Comm. Algebra **48** (2020), 5076–5085.

[16] C. G. Xia, D. Y. Wang, X. Han, *Linear super-commuting maps and super-biderivations on the super-Virasoro algebras*, Comm. Algebra **44** (2016), 5342–5350.

[17] D. Xu, X. N. Xu, *Skew-symmetric super-biderivatives of the Hamiltonian superalgebra* $H(m,n;t)$, (2025), arXiv: 2508.12067.

[18] D. Xu, X. N. Xu, *Skew-symmetric super-biderivations of the special Lie superalgebra* $S(m,n;t)$, (2025), arXiv: 2508.12064.

[19] D. Xu, Q. Y Wang, X. N. Xu, *Super-biderivations and linear super-commuting maps on simple Lie superalgebras*, (2023), arXiv: 2312.13854.

[20] H. Xu, L. Wang, *The properties of biderivations on Heisenberg superalgebras*, Math. Aeterna **5** (2015), 285–291.

[21] X. N. Xu, Q. Y. Wang, *A note on finite dimensional odd contact Lie superalgebra in prime characteristic*, Axioms **12** (2023), 1108.

[22] J. X. Yuan, L. Y. Chen, Y. Cao, *Super-biderivations of Cartan type Lie superalgebras*, Comm. Algebra **49** (2021), 4416–4426.

[23] J. X. Yuan, X. M. Tang, Super-biderivations of classical simple Lie superalgebras, Aequationes Math. **92** (2018), 91–109.

[24] Y. Z. Zhang, W. D. Liu, *Modular Lie superalgebras* (in Chinese), Scientific Press, Beijing, 2004.

[25] X. D. Zhao, Y. Chang, X. Zhou, L. Y. Chen, *Super-biderivations of the contact Lie superalgebra* $K(m,n;\underline{t})$, Comm. Algebra **48** (2020), 3237–3248.