



## Irreducible completion of topological spaces

Taotao Cao<sup>a</sup>

<sup>a</sup>College of Science, Northwest A&F University, Yangling, 712100, China

**Abstract.** It is well known that in a  $T_0$  space, directed sets with respect to the specialization order are irreducible sets. Motivated by the directed completion of topological spaces, we use irreducible sets to define strong  $SI$ -spaces and the irreducible completion of topological spaces, called the  $SI$ -completion. Then we present an  $SI$ -completion of topological spaces and show that this completion is idempotent. Furthermore, we prove that strong  $SI$ -spaces are exactly sober spaces, and we show that the category  $\mathbf{SOB}^+$ , consisting of all sober spaces with  $SI^+$ -continuous maps, is a reflective subcategory of  $\mathbf{TOP}_0^+$ , consisting of all  $T_0$  spaces with  $SI^+$ -continuous maps. Inspired by the fact that irreducible sets of a poset equipped with the Alexandroff topology are exactly directed sets, and that such spaces are monotone determined, we study the connections between directed completion and irreducible completion. In particular, we find that  $\Gamma(P(SI(L_\gamma))) \leftrightarrow \Gamma(\Sigma(cl_d(\psi(L))))$  for a poset  $L$ .

### 1. Introduction

It is well known that  $T_0$  spaces are extremely important topological spaces satisfying the minimal separation axiom in general topology. Domain theory, which arose from logic and computer science, was studied based on posets. A  $T_0$  space can induce a poset via the specialization order. Conversely, there are many topologies defined on posets, such as Alexandroff topology, Scott topology, upper topology, Lawson topology. In domain theory, topology and order theory have mutually influenced and enriched each other. Hence, there are numerous links between  $T_0$  spaces and domain theory [5, 6].

Scott topology and Alexandroff topology are two classical topologies on posets. The Scott topology on a poset  $L$  is defined to consist of all upper sets which are inaccessible by directed suprema. Motivated by how the Scott topology on a poset is constructed from its Alexandroff topology, a new topology defined on a  $T_0$  space is obtained by using irreducible sets, which is called the irreducibly-derived topology (or for short,  $SI$ -topology) [10]. In [9], Z. Zhang, F. Shi, and Q. Li introduced the directed completion of  $T_0$  spaces, called the Scott completion, and gave a Scott completion for monotone determined spaces. Just as directed sets play a central role in domain theory, irreducible sets are of fundamental importance in the study of  $T_0$  spaces. The motivating ingredient for our present paper is the fact that irreducible sets are generalization of directed sets with respect to the specialization order in  $T_0$  spaces. Based on this idea, we define strong  $SI$ -spaces and put forward the irreducible completion of  $T_0$  spaces, called the  $SI$ -completion. We require that the space be a strong  $SI$ -space and the maps between topological spaces be  $SI^+$ -continuous (that is, the

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*Email address:* [cttmath@163.com](mailto:cttmath@163.com) (Taotao Cao)

ORCID iD: <https://orcid.org/0000-0001-5881-7417> (Taotao Cao)

map is continuous and  $SI$ -continuous). In Section 3, we obtain positive results concerning the irreducible completion of  $T_0$  spaces. Since the directed sets and the irreducible sets are equivalent in partial order sets with Alexandroff topology, this naturally raises the question whether there exist connections between these two types of completion in such special topological spaces. We provide several results addressing this question.

The paper is organized as follows. In Section 2, we briefly recall some fundamental notions and related results from general topology and domain theory that will be used throughout the paper. In Section 3, we first introduce definitions of strongly complete  $T_0$  spaces and strong  $SI$ -spaces, and show that strong  $SI$ -spaces are exactly sober spaces. We then define the irreducible completion of  $T_0$  spaces in the same way as defining the directed completion of topological spaces and present an explicit construction that is idempotent. Moreover, we show that the category  $\mathbf{SOB}^+$ , consisting of all sober spaces with  $SI^+$ -continuous maps, is a reflective subcategory of  $\mathbf{TOP}_0^+$ , consisting of all  $T_0$  spaces with  $SI^+$ -continuous maps. In Section 4, we recall the notion of the directed completion and study the connections between irreducible completion and directed completion. Finally, we prove that there exists an order embedding from the lattice of closed sets of the irreducible completion to the lattice of closed sets of the directed completion.

## 2. Preliminaries

In this section, we briefly review some definitions and known results that are related to the paper. For more details, we refer the reader to [2, 3, 5, 6].

Let  $L$  be a poset. A subset  $D$  of  $L$  is called *directed* if it is non-empty and every finite subset of  $D$  has an upper bound in  $D$ . The collection of all directed sets of  $L$  is denoted by  $\mathcal{D}(L)$ . A poset  $L$  is called a *directed complete poset*, or *dcpo* for short, if every directed subset of  $L$  has a supremum in  $L$ . For any  $X \subseteq L$ , we define  $\downarrow X = \{y \in L : y \leq x \text{ for some } x \in X\}$ , and dually  $\uparrow X = \{y \in L : x \leq y \text{ for some } x \in X\}$ . In particular, we write  $\downarrow x = \downarrow \{x\}$ ,  $\uparrow x = \uparrow \{x\}$ . A subset  $X \subseteq L$  is called a *lower set* if  $X = \downarrow X$ , and an *upper set* if  $X = \uparrow X$ .

The *Alexandroff topology* on a poset  $L$  is defined as the collection of all upper sets of  $L$  and is denoted by  $\gamma(L)$ . We write the topological space  $(L, \gamma(L))$  by  $L_\gamma$  for short. A subset  $U$  of a poset  $L$  is called *Scott open* if (i)  $U = \uparrow U$ , (ii)  $\forall D \in \mathcal{D}(L)$  implies  $D \cap U \neq \emptyset$  whenever  $\vee D$  exists in  $L$ . The collection of all Scott open sets of  $L$  forms a topology on  $L$ , and the topology will be called the *Scott topology* of  $L$ , denoted by  $\sigma(L)$ . The space  $\Sigma(L) = (L, \sigma(L))$  is called the *Scott space* of poset  $L$ . The complement of a Scott open set is called *Scott closed set*. It is well known that a subset  $A \subseteq L$  is Scott closed if and only if it is a lower set that is closed under existing directed suprema.

Throughout this paper, topological spaces are always assumed to be  $T_0$ , unless otherwise stated. Let  $X$  be a topological space, we denote the collection of all open sets of  $X$  by  $\mathcal{O}(X)$  and the set of all closed sets of  $X$  by  $\Gamma(X)$ . The partial order  $\leq$  on a  $T_0$  space  $X$ , defined by  $x \leq y$  if and only if  $x \in cl(\{y\})$ , is called the *specialization order* of  $X$ . We denote by  $\Omega(X) = (X, \leq)$  the poset induced by the specialization order of  $X$ . In particular, the specialization order of  $\Sigma(L)$  and  $L_\gamma$  coincide with the original order on the poset  $L$ . A non-empty subset  $A$  of a topological space  $X$  is called *irreducible* if for any closed sets  $B$  and  $C$ ,  $A \subseteq B \cup C$  implies  $A \subseteq B$  or  $A \subseteq C$ . We denote the set of all irreducible (irreducible closed) subsets of  $X$  by  $Irr(X)$  ( $Irr_c(X)$ ). For any  $x \in X$ , the closure of a singleton  $cl(\{x\})$  is always an irreducible closed set. A topological space  $X$  is called *sober* if for every irreducible closed set  $C$  of  $X$ , there exists a unique point  $x \in X$  such that  $C = cl(\{x\})$ . Noted that every sober space is necessarily  $T_0$ , since  $cl(\{x\}) = cl(\{y\})$  always implies  $x = y$ .

**Definition 2.1.** ([10]) Let  $(X, \tau)$  be a  $T_0$  space. A subset  $U$  of  $X$  is called *SI-open* if the following conditions are satisfied:

- (i)  $U \in \tau$ .
- (ii) For any  $F \in Irr(X)$ ,  $\forall F \in U$  implies  $F \cap U \neq \emptyset$  whenever  $\vee F$  exists.

The set of all  $SI$ -open sets of  $(X, \tau)$  forms a topology on  $X$ , which is denoted by  $\tau_{SI}$  and called the *SI-topology* of  $X$ . The topological space  $(X, \tau_{SI})$  will be denoted by  $SI(X)$  for short. The complements of  $SI$ -open sets are called *SI-closed* sets. A closed subset  $C$  is *SI-closed* if and only if, for any  $F \in Irr(X)$ ,  $F \subseteq C$

implies that  $\forall F \in C$  whenever  $\forall F$  exists. For any subset  $C \subseteq X$ , we denote by  $cl_{SI}(C)$  the closure of  $C$  with respect to  $SI$ -topology.

**Proposition 2.2.** Let  $X, Y$  be  $T_0$  spaces.

- (1)  $F \in Irr(X)$  iff  $cl_X(F) \in Irr(X)$ .
- (2) For all  $x \in X$ ,  $cl(\{x\}) = cl_{SI}(\{x\}) = \downarrow x$ .
- (3) If  $F \in Irr(X)$ , then  $F \in Irr(SI(X))$ .
- (4) A subset directed with respect to the specialization order of  $X$  is irreducible.
- (5) Open sets are upper sets.
- (6) Closed sets are lower sets.
- (7) Continuous images of irreducible sets are irreducible.
- (8) If  $f: X \rightarrow Y$  is continuous, then for any  $A \subseteq X$ ,  $f(cl(A)) \subseteq cl(f(A))$ .

For the following results, please refer to [4].

**Proposition 2.3.** A  $T_0$  space is *monotone determined* if and only if any subset  $U$  meeting all directed sets with respect to the specialization order whose closure meets  $U$  is open.

**Proposition 2.4.** Posets endowed with the Alexandroff topology are monotone determined.

### 3. The irreducible completion

In [9], Z. Zhang, F. Shi, and Q. Li introduced a type of directed completion for  $\Delta$ -space, called the Scott completion, where a  $\Delta$ -space is a monotone determined, weak monotone convergence space. They constructed a directed completion for monotone determined spaces. In this section, we introduce the definition of irreducible completion of  $T_0$  spaces, called the  $SI$ -completion.

Let  $X$  be a  $T_0$  space. If  $D \subseteq X$  is a directed subset with respect to its specialization order, then  $D$  is irreducible. Thus, irreducible sets can be regarded as a generalization of directed sets. Motivated by this observation, we introduce the notion of strong  $SI$ -spaces by replacing directed sets of Scott spaces with irreducible sets. Unless otherwise stated, throughout this paper, whenever order-theoretic concepts are mentioned in the context of a  $T_0$  space  $X$ , it is to be interpreted with respect to the specialization order on  $X$ .

Inspired by the definition of strongly complete dcpo in [7], H. Andradi and W. Ho proposed a corresponding definition in the setting of topological spaces in [11].

**Definition 3.1.** A space  $X$  is called *strongly complete* if every irreducible subset has a supremum.

The following definition is motivated by the Scott space introduced in [9] and by the close relation between directed sets and irreducible sets in a topological space.

**Definition 3.2.** A space  $X$  is called a *strong  $SI$ -space* if  $X$  is strongly complete and the topology  $\mathcal{O}(X)$  is exactly the  $SI$ -topology on  $X$ .

We list two results that will be used in subsequent sections about strong  $SI$ -spaces.

**Remark 3.3.** (1) Let  $L$  be a poset. Then  $SI((L, \gamma(L))) = \Sigma(L)$ .

(2) Let  $X$  be a strong  $SI$ -space. Then  $F \in Irr(X)$  if and only if  $F \in Irr(SI(X))$ .

Sober spaces play a fundamental role in general topology. The following proposition establishes a precise relation between sober spaces and strong  $SI$ -spaces.

**Proposition 3.4.** *Strong SI-spaces are exactly sober spaces.*

*Proof.* Suppose that  $X$  is a strong SI-space and let  $F \in Irr_c(X)$ . Then  $F$  is SI-closed. Thus,  $\vee F \in F$ , which implies that  $F = cl(\{\vee F\})$ . Conversely, let  $X$  is a sober space and  $F \in Irr(X)$ . Then there exists a unique point  $x \in X$  such that  $cl(F) = cl(\{x\})$ . It is straightforward to verify that  $\vee F = x$  indeed. Consequently, if  $A$  is closed, then for any  $F \subseteq A$ ,  $cl(F) = cl(\{\vee F\}) \subseteq A$ , which shows that  $A$  is SI-closed. Therefore,  $X$  is a strong SI-space.  $\square$

**Definition 3.5.** ([10]) *Let  $X, Y$  be two  $T_0$  spaces. A map from  $X$  to  $Y$  is called  $SI^+$ -continuous if it is continuous with respect to both the original topology and the corresponding SI-topology.*

**Definition 3.6.** An SI-completion  $(Y, f)$  of a space  $X$  is a pair  $(Y, f)$ , where  $Y$  is a strong SI-space and  $f: X \rightarrow Y$  is an  $SI^+$ -continuous such that for any strong SI-space  $Z$  and any  $SI^+$ -continuous map  $g: X \rightarrow Z$ , there exists a unique  $SI^+$ -continuous map  $\tilde{g}: Y \rightarrow Z$  satisfying  $g = \tilde{g} \circ f$ .

We emphasize that an SI-completion of a space is not its sobrification (This distinction will be further clarified after presenting a concrete construction of the SI-completion), and the sobrification of a space is also not its SI-completion. We now provide an example illustrating that the sobrification of a space differs from its SI-completion. Let  $(X, \pi)$  be a space. Then  $((Irr_c(X), \tau), f)$  is the sobrification of  $X$ , where  $\tau = \{U^S \mid U \in \pi\}$ ,  $U^S = \{A \in Irr_c(X) \mid A \cap U \neq \emptyset\}$ , and  $f: X \rightarrow Irr_c(X)$  defined by  $f(x) = cl(\{x\})$  for any  $x \in X$ . It is easy to show that for any  $U \in \pi$ ,  $f^{-1}(U^S) = U$ . Therefore, to distinguish SI-completion from sobrification, it suffices to find a  $T_0$  space  $X$  which is not sober and whose SI-topology is different from the original topology on  $X$ .

**Example 3.7.** Let  $L = \mathbb{N} \times \mathbb{N}$ , where  $\mathbb{N}$  is the set of natural numbers. Define an order  $\leq$  on  $L$  by for any  $(m, n), (p, q) \in L$ ,  $(m, n) \leq (p, q)$  if and only if  $m < p$  or  $m = p$  and  $n \leq q$ . Then  $(L, \leq)$  is a chain and  $(L, \gamma(L))$  is a  $T_0$  space. For any  $n \in \mathbb{N}$ , the set  $\downarrow(\{n\} \times \mathbb{N})$  is an irreducible closed subset. However, there is no element  $(p, q) \in L$  such that  $\downarrow(\{n\} \times \mathbb{N}) = \downarrow(p, q)$ . Hence,  $(L, \gamma(L))$  is not sober. Moreover, we know that  $\bigvee(\downarrow(\{n\} \times \mathbb{N})) = (n + 1, 1)$ . However,  $(n + 1, 1) \notin \downarrow(\{n\} \times \mathbb{N})$ . Therefore, the closed set  $\downarrow(\{n\} \times \mathbb{N})$  is not an SI-closed set of  $(L, \gamma(L))$ .

Finding a dcpo whose Scott topology is not sober has long been an important problem in domain theory. In Proposition 3.4, we give a simple characterization of sober spaces in terms of SI-topology. As a consequence, we obtain an equivalent condition for a dcpo whose Scott topology is not sober.

**Proposition 3.8.** *Let  $L$  be a dcpo. Then  $\Sigma(L)$  is not sober if and only if there exists an irreducible closed set  $F$  such that  $F$  is not a directed set of  $L$ .*

*Proof.* Suppose that  $L$  is a dcpo and that  $\Sigma(L)$  is not sober. Then there exists an irreducible closed set  $F$  such that  $F \neq cl(\{x\})$  for any  $x \in L$ . If  $F$  is directed, then  $\vee F$  exists and  $\vee F \in F$ , since  $F$  is Scott closed. It follows that  $F = cl(\{\vee F\})$ , a contradiction. Conversely, suppose that there exists an irreducible closed subset  $F_0 \subseteq L$ . If there exists  $x \in L$  such that  $F_0 = cl(\{x\})$ , then  $\vee F_0 = x = max(F_0)$ . Thus,  $F_0$  is directed. It is in contradiction with the condition.  $\square$

Next, we will present an SI-completion of  $T_0$  space. Let  $(X, \tau)$  be a  $T_0$  space. Denote by  $X^S$  the set of all irreducible closed sets of  $SI(X)$ , that is,

$$X^S = \{A \subseteq X \mid A \in \Gamma(SI(X)) \text{ and } A \text{ is irreducible in } SI(X)\}.$$

For an SI-open set  $U$  of  $X$ , define

$$U^S = \{A \in X^S \mid A \cap U \neq \emptyset\}.$$

It is straightforward to verify that  $\pi = \{U^S \mid U \in \tau_{SI}\}$  forms a topology on  $X^S$ . We denote the resulting topological space  $(X^S, \pi)$  by  $P(SI(X))$ . The closed sets of  $P(SI(X))$  are those in the form of  $B^S = \{A \in X^S \mid A \subseteq B\}$ , where  $B$  is an SI-closed set of  $X$ . In fact,  $P(SI(X))$  coincides with the standard sobrification of  $SI(X)$ .

**Proposition 3.9.** *The specialization order on  $P(SI(X))$  is the inclusion order.*

*Proof.* This follows from the definition of the closed subsets of  $P(SI(X))$ .  $\square$

**Proposition 3.10.** *The space  $P(SI(X))$  is strongly complete.*

*Proof.* Let  $\mathcal{F}$  be an irreducible subset of  $P(SI(X))$ . We claim that  $\bigvee \mathcal{F} = cl_{SI}(\bigcup \mathcal{F})$ . It suffices to prove that  $cl_{SI}(\bigcup \mathcal{F})$  is irreducible in  $SI(X)$ . By Proposition 2.2(1), this reduces to proving that  $\bigcup \mathcal{F}$  is irreducible in  $SI(X)$ . Suppose that  $\bigcup \mathcal{F} \subseteq B \cup C$ , where  $B$  and  $C$  are  $SI$ -closed subsets of  $X$ . For any  $A \in \mathcal{F}$ , since  $A$  is irreducible in  $SI(X)$ , we have  $A \subseteq B$  or  $A \subseteq C$ . Thus,  $\mathcal{F} \subseteq B^S \cup C^S$ . Since  $\mathcal{F}$  is irreducible, it follows that  $\mathcal{F} \subseteq B^S$  or  $\mathcal{F} \subseteq C^S$ . Consequently,  $\bigcup \mathcal{F} \subseteq B$  or  $\bigcup \mathcal{F} \subseteq C$ , which proves that  $\bigcup \mathcal{F}$  is irreducible. Therefore,  $cl_{SI}(\bigcup \mathcal{F})$  is irreducible and is the least upper bound. This completes the proof.  $\square$

**Proposition 3.11.** *The topology  $\mathcal{O}(P(SI(X)))$  is exactly the  $SI$ -topology on the space  $P(SI(X))$ .*

*Proof.* It suffices to prove that  $\Gamma(SI(P(SI(X)))) = \Gamma(P(SI(X)))$ . By the definition of  $SI$ -closed sets, it is obvious that  $\Gamma(SI(P(SI(X)))) \subseteq \Gamma(P(SI(X)))$ . Conversely, let  $B^S$  is a closed set of  $P(SI(X))$  such that  $\mathcal{F} \subseteq B^S$ , where  $\mathcal{F}$  is an irreducible subset of  $P(SI(X))$ . It follows that  $\bigcup \mathcal{F} \subseteq B$ . Thus,  $\bigvee \mathcal{F} = cl_{SI}(\bigcup \mathcal{F}) \subseteq B$ , that is  $\bigvee \mathcal{F} \in B^S$ .  $\square$

By Propositions 3.10, 3.11, we immediately obtain the following result.

**Proposition 3.12.**  *$P(SI(X))$  is a strong  $SI$ -space.*

**Proposition 3.13.** *Let  $X$  be a  $T_0$  space. For any  $F \in Irr(X)$ ,  $\{cl(\{f\}) \mid f \in F\} \in Irr(P(SI(X)))$ .*

*Proof.* Let  $F$  be an irreducible set of  $X$  and  $B_1, B_2$  be two  $SI$ -closed sets of  $X$  such that  $\{cl(\{f\}) \mid f \in F\} \subseteq B_1^S \cup B_2^S$ . It is not difficult to find that  $F \subseteq B_1 \cup B_2$ , which implies that  $F \subseteq B_1$  or  $F \subseteq B_2$ . Hence,  $\{cl(\{f\}) \mid f \in F\} \subseteq B_1^S$  or  $B_2^S$ .  $\square$

**Proposition 3.14.** *Let  $X$  be a  $T_0$  space. For any  $F \in Irr(X)$ , if  $\bigvee F$  exists, then  $\bigvee \{cl(\{f\}) \mid f \in F\} = cl(\{\bigvee F\})$ .*

*Proof.* Clearly,  $cl(\{\bigvee F\})$  is an upper bound of  $\{cl(\{f\}) \mid f \in F\}$  in  $\Omega(X^S)$ . Suppose that  $A \in X^S$  is an upper bound of  $\{cl(\{f\}) \mid f \in F\}$ . Then  $F \subseteq A$ . Since  $A$  is  $SI$ -closed, we have  $\bigvee F \in A$ . Thus,  $cl_X(\{\bigvee F\}) \subseteq A$ , as desired.  $\square$

At present, we have constructed a strong  $SI$ -space. Next, we need to define an  $SI^+$ -continuous map from  $(X, \tau)$  into  $P(SI(X))$ . Define the map  $\eta: X \rightarrow X^S$  by  $\eta(x) = cl(x)$  for all  $x \in X$ . Since the closure of a singleton is an irreducible  $SI$ -closed sets, so the map  $\eta$  is well-defined. Next, we will show that map  $\eta$  is  $SI^+$ -continuous.

**Lemma 3.15.** *The map  $\eta: (X, \tau) \rightarrow P(SI(X))$  is  $SI^+$ -continuous.*

*Proof.* Suppose that  $U^S$  is an open set of  $P(SI(X))$ . Since  $U$  is a upper set, then  $\eta^{-1}(U^S) = U$ . Thus,  $\eta$  is  $SI^+$ -continuous.  $\square$

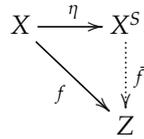
**Lemma 3.16.** *Let  $X, Y$  be  $T_0$  spaces. Then  $f: X \rightarrow Y$  is  $SI^+$ -continuous if and only if for any  $F \in Irr(X)$ ,  $\bigvee f(F)$  exists and  $\bigvee f(F) = f(\bigvee F)$  whenever  $\bigvee F$  exists.*

*Proof.* Because  $f$  is continuous with respect to  $SI$ -topology, then  $f$  is monotone. Thus,  $f(\bigvee F)$  is an upper bound of  $f(F)$ . Assume that  $y \in Y$  is an arbitrary upper bound of  $f(F)$ . Then  $cl(\{y\}) = \downarrow y$  is  $SI$ -closed in  $Y$  and  $F \subseteq f^{-1}(f(F)) \subseteq f^{-1}(\downarrow y)$ . By the  $SI^+$ -continuity of  $f$ , we have  $f^{-1}(\downarrow y)$  is  $SI$ -closed. It follows that  $\bigvee F \in f^{-1}(\downarrow y)$ , that is,  $f(\bigvee F) \leq y$ . Hence,  $f(\bigvee F)$  is the supremum of  $f(F)$ . For the converse, it is obvious.  $\square$

Based on the above results, we return to a main theorem of the paper related to the  $SI$ -completion of  $T_0$  spaces.

**Theorem 3.17.**  *$(P(SI(X)), \eta)$  is an  $SI$ -completion of  $T_0$  space  $X$ .*

*Proof.* Let  $X$  be a  $T_0$  space and  $f$  be an arbitrary  $SI^+$ -continuous map from  $X$  into an arbitrary strong  $SI$ -space  $Z$ . We need to find an  $SI^+$ -continuous map  $\bar{f}$  from  $X^S$  into  $Z$  such that the following diagram is commutative.



Define  $\bar{f}$  by  $\bar{f}(A) = \vee f(A)$  for any  $A \in X^S$ . By the fact that continuous images of irreducible sets are irreducible, we immediately have that for any  $A \in X^S$ , the supremum of  $f(A)$  exists, thus  $\bar{f}$  is well-defined. By Lemma 3.15, the map  $\eta$  is  $SI^+$ -continuous. Next, we prove that  $\bar{f}$  is  $SI^+$ -continuous. Let  $V$  be an open set of  $Z$ . Since  $Z$  is a strong  $SI$ -space,  $V$  is also  $SI$ -open. By the  $SI^+$ -continuity of  $f$ ,  $f^{-1}(V)$  is an  $SI$ -open set of  $X$ . We claim that  $\bar{f}^{-1}(V) = (f^{-1}(V))^S$ . For any  $A \in \bar{f}^{-1}(V)$ , we have  $\bar{f}(A) = \vee f(A) \in V$ . Since  $f(A) \in Irr(Z)$  by Remark 3.3(2) and  $V$  is  $SI$ -open, it follows that  $f(A) \cap V \neq \emptyset$  holds. Consequently,  $f^{-1}(V) \cap A \neq \emptyset$ , which implies  $A \in (f^{-1}(V))^S$ . The reverse inclusion follows directly from Proposition 2.2(5). Hence,  $\bar{f}$  is  $SI^+$ -continuous. Since  $(\bar{f} \circ \eta)(x) = \vee f(\downarrow x) = f(\vee \downarrow x) = f(x)$  by Lemma 3.16, we have  $\bar{f} \circ \eta = f$ . It remains to prove the uniqueness of the  $\bar{f}$ . Assume that  $g: X^S \rightarrow Z$  is also an  $SI^+$ -continuous map such that  $g \circ \eta = f$ . Let  $A$  be an arbitrary irreducible  $SI$ -closed set of  $X$ . We claim that  $cl(\{g(A)\}) = cl(\{f(A)\})$ . For any  $x \in A$ ,  $\eta(x) = cl(\{x\}) \in A^S = cl(\{A\})$  in  $P(SI(X))$ . By the continuity of map  $g$ , we have  $f(x) = (g \circ \eta)(x) = g(cl(\{x\})) \in g(cl(\{A\})) \subseteq cl(\{g(A)\})$ , which implies that  $f(A) \subseteq cl(\{g(A)\})$ . Conversely,  $g(A) \in g(cl(\eta(x) \mid x \in A)) \subseteq cl(\{(g \circ \eta)(x) \mid x \in A\}) = cl(\{f(A)\})$ . Thus,  $cl(\{g(A)\}) \subseteq cl(\{f(A)\})$ . Since  $Z$  is a strong  $SI$ -space and  $cl(\{f(A)\})$  is an  $SI$ -closed set of  $Z$ , we have  $cl(\{g(A)\}) = cl(\{f(A)\}) = cl(\{\bar{f}(A)\})$ . By the fact that  $Z$  is  $T_0$ , we obtain  $\bar{f}(A) = g(A)$ . Therefore,  $\bar{f} = g$ .  $\square$

Obviously, sober spaces and  $SI^+$ -continuous maps can form a category denoted by  $\mathbf{SOB}^+$ .  $T_0$  spaces and  $SI^+$ -continuous maps form a category denoted by  $\mathbf{TOP}_0^+$ .

**Corollary 3.18.** *The category  $\mathbf{SOB}^+$  is a reflective subcategory of  $\mathbf{TOP}_0^+$ .*

By the definition of  $SI$ -completion, if  $X$  is a strong  $SI$ -space, then  $X \cong P(SI(X))$ . Hence we have the following corollary.

**Corollary 3.19.** *The  $SI$ -completion is idempotent.*

One remaining issue is that the  $SI$ -completion of a  $T_0$  space  $X$  is not a sobrification of the space. At present, we have given an  $SI$ -completion of  $T_0$  spaces. Actually, we need the condition that  $f$  be an arbitrary  $SI^+$ -continuous map from  $X$  into a strong  $SI$ -space  $Z$ . If the  $SI^+$ -continuity of  $f$  is weakened to continuity, then in the proof of Theorem 3.17, for any  $A \in Irr_c(SI(X))$ ,  $f(A)$  is not irreducible in  $Z$ , since  $F \in Irr(SI(X))$  cannot imply  $F \in Irr(X)$ .

One might also note that a  $T_0$  space  $X$  induces not only the  $SI$ -topology, but also the Scott topology, the Alexandroff topology, and the upper topology on the poset  $X$  equipped with the specialization order. And  $X$  endowed with the Scott topology, Alexandroff topology and upper topology are all  $T_0$  spaces, then we can obtain their standard sobrifications. In fact, standard sobrifications of the three topologies are all not the  $SI$ -completion of the space  $X$  as the method in Theorem 3.17. Indeed, the map  $\eta$  from  $X$  into the corresponding standard sobrification defined above may not be continuous. Open set of  $X$  must be an upper set, however the upper set is not necessarily an open set of  $T_0$  space.

#### 4. The links between directed completion and irreducible completion

Let  $L$  be a poset, then we can define the Alexandroff topology on the poset. By Proposition 2.4, the space  $L_\gamma$  is monotone determined. Thus,  $L_\gamma$  has a Scott completion given in [9]. In Section 3, the space  $L_\gamma$  has a specific  $SI$ -completion. The specialization order of the Alexandroff topology on a poset coincides with the

underlying order. The  $SI$ -topology defined from the space  $L_\gamma$  is exactly the Scott topology on poset  $L$ , that is  $SI(L_\gamma) = \Sigma(L)$ . In this section, we will study the connections between  $SI$ -completion given in Section 3 and Scott completion given in [9] of the space  $L_\gamma$ .

First, we will introduce the definition of Scott completion briefly. A subset  $A$  of a poset  $L$  is called  $D$ -closed if for any  $D \in \mathcal{D}(L)$ , then  $\vee D \in A$  when  $\vee D$  exists. The set of all  $D$ -closed subsets of  $L$  forms the set of closed sets of a topology on  $L$ , which will be called  $D$ -topology. Let  $X$  be a space, then  $\Gamma(X)$  is a complete lattice. Denote  $\psi(X) = \{\downarrow x \mid x \in X\}$ , which can be regarded as a subset of  $\Gamma(X)$ . We denote by  $cl_d(\mathcal{A})$  the closure of  $\mathcal{A} \subseteq \Gamma(X)$  with respect to the  $D$ -topology of  $\Gamma(X)$ , then  $cl_d(\psi(X))$  is a dcpo under the inclusion order [12]. Denote by  $\Sigma(cl_d(\psi(X)))$  the Scott topology on dcpo  $cl_d(\psi(X))$  and define a map  $\delta$  from  $X$  into  $\Sigma(cl_d(\psi(X)))$  by  $\delta(x) = \downarrow x$  for all  $x \in X$ , then  $(\Sigma(cl_d(\psi(X))), \delta)$  is a Scott completion of  $T_0$  space  $X$ . We now turn to the research of the connections between  $SI$ -completion and Scott completion of Alexandroff topology of a poset. If Scott topology of a poset  $L$  is sober, then  $P(SI(L_\gamma)) \cong \Sigma(L)$  holds obviously. However if the Scott topology of  $L$  is not sober,  $L^S \neq L$  fails. So the homeomorphism relation is wrong. Here we list Johnstone’s famous dcpo whose Scott topology is not sober to explain further [8].

**Example 4.1.** Let  $L = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ , where  $(m, n) \leq (p, q)$  if and only if  $m = p, n \leq q$  or  $q = \infty, n \leq p$ . Then  $Irr_c(\Sigma(L)) = \{\downarrow (m, n) \mid (m, n) \in L\} \cup \{L\}$ , which clearly is not lattice isomorphic to  $L$ .

Let  $L$  be a poset. We consider the Scott completion of  $L_\gamma$ . In the following, if it is not enhanced,  $cl_d(\psi(L))$  always refers to the closure of  $\psi(L)$  with respect to  $D$ -topology in  $\Gamma(L_\gamma)$ .

**Lemma 4.2.** Let  $L$  be a poset. Then  $cl_d(\psi(L)) = \{\bigcup D \mid D \subseteq \mathcal{D}(\psi(L))\}$ .

*Proof.* First, it is obvious that  $\psi(L) \subseteq \{\bigcup D \mid D \subseteq \mathcal{D}(\psi(L))\} \subseteq cl_d(\psi(L))$ . It suffices to show that  $\{\bigcup D \mid D \subseteq \mathcal{D}(\psi(L))\}$  is  $d$ -closed. Assume that  $D' \subseteq \{\bigcup D \mid D \subseteq \mathcal{D}(\psi(L))\}$  is directed. Then  $D' = \{\bigcup D_i \mid D_i \in \mathcal{D}(\psi(L))\}_{i \in I}$ . We claim that  $\bigcup D' \in \{\bigcup D \mid D \subseteq \mathcal{D}(\psi(L))\}$ . Clearly,  $\bigcup D' = \bigcup \{\downarrow x \mid \downarrow x \in D_i\}$ . For any  $\downarrow x_1, \downarrow x_2 \in \{\downarrow x \mid \downarrow x \in D_i\}$ , there exists  $i_1, i_2 \in I$  such that  $\downarrow x_1 \in D_{i_1}$  and  $\downarrow x_2 \in D_{i_2}$ . For the directness of  $D'$ , there exists  $i_3 \in I$  satisfying  $\bigcup D_{i_1}, \bigcup D_{i_2} \subseteq \bigcup D_{i_3}$ . It follows that  $\downarrow x_1, \downarrow x_2 \subseteq \bigcup D_{i_3}$ . Since  $D_{i_3}$  is directed, we can find  $\downarrow x_3 \in D_{i_3}$  such that  $\downarrow x_1, \downarrow x_2 \subseteq \downarrow x_3$ . Thus,  $\{\downarrow x \mid \downarrow x \in D_i\}$  is also a directed set of  $\psi(L)$ .  $\square$

**Corollary 4.3.** Let  $L$  be a poset. For any  $\mathcal{A} \subseteq \psi(L)$ ,  $cl_d(\mathcal{A}) = \{\bigcup D \mid D \subseteq \mathcal{D}(\mathcal{A})\}$ .

In general, the lower set of a  $D$ -closed set is not a Scott closed set of  $L$ . However, in Corollary 4.3, if  $\mathcal{A}$  is a lower set of  $\psi(L)$ , then  $\downarrow cl_d(\mathcal{A})$  is a Scott closed set of  $cl_d(\psi(L))$ .

**Lemma 4.4.** Let  $L$  be a poset and  $\mathcal{A} \in \Gamma(\Sigma(\psi(L)))$ . Then  $\downarrow \{\bigcup D \mid D \text{ is directed and } D \subseteq \mathcal{A}\} = \downarrow cl_d(\mathcal{A}) \in \Gamma(\Sigma(cl_d(\psi(L))))$ .

*Proof.* It is obvious that  $\downarrow cl_d(\mathcal{A})$  is a lower set of  $cl_d(\psi(L))$ . For every directed set  $D \subseteq \downarrow cl_d(\mathcal{A})$ , there exist some directed sets  $\{D_i\}_{i \in I} \subseteq \psi(L)$  such that  $D = \{\bigcup D_i\}_{i \in I}$  and  $\bigcup D_i \subseteq \bigcup D'_i$ , where  $\{D'_i\}_{i \in I}$  is a collection of directed sets of  $\mathcal{A}$ . Thus,  $D_i \subseteq \downarrow D'_i \subseteq \mathcal{A}$ , since  $\mathcal{A}$  is a lower set. As the proof in Lemma 4.2, we have that  $\bigcup_{i \in I} D_i$  is a directed set of  $\mathcal{A}$ . Thus,  $\vee D = \bigcup \bigcup_{i \in I} D_i \in \downarrow cl_d(\mathcal{A})$ .  $\square$

Based on the above results, we can obtain the following proposition.

**Proposition 4.5.** Let  $L$  be a poset. Then  $\Gamma(P(SI(L_\gamma))) \leftrightarrow \Gamma(\Sigma(cl_d(\psi(L))))$ .

*Proof.* Since  $\Gamma(\Sigma(\psi(L))) \cong \Gamma(P(SI(L_\gamma)))$ , it suffices to show that  $\Gamma(\Sigma(\psi(L))) \leftrightarrow \Gamma(\Sigma(cl_d(\psi(L))))$ . For all  $\mathcal{A} \in \Gamma(\Sigma(\psi(L)))$ , denote  $cl_\sigma(\mathcal{A}) = \downarrow cl_d(\mathcal{A}) = \downarrow \{\bigcup \mathcal{D} \mid \mathcal{D} \text{ is directed, } \mathcal{D} \subseteq \mathcal{A}\}$ . Define a map  $f$  from  $\Gamma(\Sigma(\psi(L)))$  into  $\Gamma(\Sigma(cl_d(\psi(L))))$  by  $f(\mathcal{A}) = cl_\sigma(\mathcal{A})$  for all  $\mathcal{A} \in \Gamma(\Sigma(\psi(L)))$ . By Lemma 4.4,  $f$  is well defined. Next, we show that the map  $f$  is an order-embedding, that is, if  $\mathcal{A}_1 \subseteq \mathcal{A}_2$  in  $\Gamma(\Sigma(\psi(L)))$  if and only if  $f(\mathcal{A}_1) \subseteq f(\mathcal{A}_2)$ . If  $\mathcal{A}_1 \subseteq \mathcal{A}_2$ , then  $cl_\sigma(\mathcal{A}_1) \subseteq cl_\sigma(\mathcal{A}_2)$ . Conversely, if  $f(\mathcal{A}_1) \subseteq f(\mathcal{A}_2)$  with  $\mathcal{A}_1, \mathcal{A}_2$  in  $\Gamma(\Sigma(\psi(L)))$ , then for all  $\downarrow x \in \mathcal{A}_1$ ,  $\downarrow x \in f(\mathcal{A}_2)$ . Since  $\mathcal{A}_2$  is Scott-closed, then  $\downarrow x \in \mathcal{A}_2$ . The proof is complete.  $\square$

## 5. Conclusion

In this paper, based on directed completion [9], we introduce irreducible completion of topological spaces and give a specific construction. In Section 4, we compare the two completions on  $L_\gamma$  for a poset  $L$ . However, we want to generalize the result to general monotone determined spaces. Furthermore, the directed completion given by Z. Zhang, F. Shi, and Q. Li applies in monotone determined spaces but not all  $T_0$  spaces at present. In the future, we will do further investigations.

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