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Equivariant extensions

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Abstract. This paper introduces and studies equivariant versions of several fundamental topological extensions, namely, the Katětov extension (κX), the Hewitt realcompactification (υX), the almost realcompactification (aX), and the Dieudonné completion (μX) for topological transformation groups. The main results establish natural identifications between the H-orbit space of the equivariant extension of X and the corresponding extension of the orbit space X/H if G is a compact group acting on a Hausdorff (or Tychonoff) space X, and H is a closed normal subgroup of G.

By setting $H = G$, we obtain $(\upsilon_G X)/G \cong \upsilon(X/G)$, $(\mu_G X)/G = (\mu X)/G \cong \mu(X/G)$, $(a_G X)/G \cong a(X/G)$, and $(\kappa_G X)/G \cong \kappa(X/G)$. These results extend the classical theorem $(\beta_G X)/G \cong \beta(X/G)$ for the Stone-Čech compactification.

For finite and discrete groups, we obtain some well-known results: $\nu_G X = \upsilon X$ and $(\upsilon X)/G = \nu(X/G)$.

1. Introduction

The theory of extensions, in particular that of compactifications, constitutes a significant and expansive area of study within the field of topology, with applications across a range of other disciplines. In this article we will deal with the equivariant versions of some well-known extension concepts such as the Hewitt realcompactification, the almost realcompactification, the Dieudonné completion, and the Katětov extension. The existence of G-extensions is still one of the topics that is being actively studied.

If a topological group G acts continuously on a Hausdorff space X, it is an important problem under which conditions this action can be continuously extended to some of extensions of X. Another issue concerns the compactification of orbit space of a topological transformation group. The following question, attributed to Zambakhidze, has been partially answered in Antonyan ([7] and later fully addressed by Antonian and Smirnov in [8] (see [3] for the proof).

Question: Let G be a compact Hausdorff group acting on a Tychonoff G-space X. Does there exist a G-compactification $B_G(X)$ of X is such that $B_G(X)/G = B(X/G)$, where $B(X/G)$ be an arbitrary compactification of the orbit space X/G?

Theorem 1.1. ([3]) *Let G be a compact group acting on a Tychonoff space X. Then $\beta_G(X)/G = \beta(X/G)$.*

This article is motivated by the question of whether the above question can be replaced by well-known extensions of compactification. In this paper, the following theorems will be proven.

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Theorem 1.2. *Suppose that G is a compact group, X is a Tychonoff G-space, and H is a closed normal subgroup of G. Then*

1. $(\upsilon_G X)/H = \upsilon_{G/H}(X/H)$.
2. $(\mu X)/H = (\mu_G X)/H = \mu_{G/H}(X/H)$.
3. $(a_G X)/H = a_{G/H}(X/H)$.

Theorem 1.3. *Suppose that G is a compact group, X is a Hausdorff G-space, and H is a closed normal subgroup of G. Then* $(\kappa_G X)/H = \kappa_{G/H}(X/H)$.

Here $\upsilon_G X$, $\mu_G X$, and $a_G X$ denote the G-Hewitt realcompactification and the maximal Dieudonné G-completion, and the maximal G-almost realcompactification of a Tychonoff G-space X. Also $\kappa_G X$ denote the equivariant Katětov extension of a Hausdorff G-space X.

2. Preliminaries

It is assumed that all the topological spaces (including groups) under consideration are Hausdorff, and all maps are continuous. Necessarily basic notations, terminology and information related to G-spaces can be found in the artifacts of Bredon [11], Palais [35] and de Vries [43].

Nevertheless, for the sake of readability, it would be appropriate to specify some definitions and notations that will be used throughout the paper.

A non-empty collection $\mathcal{U}$ of open sets on a topological space is called *open filter* if the following conditions are satisfied

1. $\emptyset \notin \mathcal{U}$
2. If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.
3. If $A, B \in \mathcal{U}$ and $C = Int(C) \supset A \cap B$, then $C \in \mathcal{U}$.

An open filter which is a maximal element in the family of all open filters (with respect to set inclusion) on a space X is called an *open ultrafilter*.

A space Y is called *extension* of a space X if there exists a homeomorphism f from X to Y such that $f(X)$ is dense in Y. If Y possesses some topological property $\mathcal{P}$, then Y is called *$\mathcal{P}$-extension* of X.

Two extensions Y and Z of X are called *equivalent* if there exists a homeomorphism of Y onto Z which keeps X pointwise fixed, i.e., $f(x) = x$ for all $x \in X$. This gives an "equivalence relation" on the class $E(X)$ of extensions of X. We will identify two equivalent extensions of X. With this identification, the family $E(X)$ of extensions of X is a set [39, 4.1]. We will now define a partial ordering on $E(X)$.

For Y and Z to be extensions of X, Y is said to be *projectively larger* than Z (written $Y \geq Z$ or $Z \leq Y$) if there exists a map $f : Y \longrightarrow Z$ such that $f|X = 1_X$.

It is clear that two extensions Y and Z of X are equivalent if and only if $Y \geq Z$ and $Z \geq Y$.

It is well-known (see, for example, [37, 38]) that $E(X)$ is partially ordered by $\geq$. For each Hausdorff space X, $(E(X), \geq)$ is a complete upper semilattice [39, 4.1].

In the family $E(X)$ of extensions of X, Z is called *projective maximum* if $Z \in E(X)$ and $Z \geq Y$ for all $Y \in E(X)$. Due to the above-mentioned identifying of equivalent extensions, a projective maximum for $E(X)$ is unique if it exists.

Definition 2.1. Let $\mathcal{P}$ be a property of topological spaces. If X is a space, then a pair $(\gamma X, \gamma)$ is called a maximal (or maximum) $\mathcal{P}$-extension of X if and only if

1. γX is a space with the property $\mathcal{P}$.
2. γ is a homeomorphism from X onto a dense subspace of γX.
3. For every map f from X into any space Y satisfying $\mathcal{P}$, there exists a map $\overline{f} : \gamma X \longrightarrow Y$ with the property $\overline{f} \circ \gamma = f$.

If $\mathcal{P}$ is a topological property then a space X is called $\mathcal{P}$-regular if X is homeomorphic to a subspace of a product of spaces each of which satisfying $\mathcal{P}$.

If a maximal $\mathcal{P}$-extension $(\gamma X, \gamma)$ of X exists, it is uniquely determined (up to homeomorphism) by X, and γ is a homeomorphism onto γX if and only if X has property $\mathcal{P}$.

It is clear that a maximal $\mathcal{P}$-extension of a space X is a projective maximum of the family of $\mathcal{P}$-extensions of X. However, there exist spaces X and topological properties for which the family of $\mathcal{P}$-extensions of X has a projective maximum, but X has no maximal $\mathcal{P}$-extension. For instance, the family of H-closed extensions of $\mathbb{N}$ has a projective maximum, i.e., $\kappa\mathbb{N}$ [39, 4.8(n)], but $\mathbb{N}$ has no maximal H-closed extension [39, 5.8(c)].

A Hausdorff space that is closed in every Hausdorff space that contains it as a subspace is called *H-closed* (*Hausdorff-closed*) or *absolutely closed space*.

Compact Hausdorff spaces are H-closed. The H-closed spaces are closely related to compact spaces: regular H-closed spaces are compact. The notion of H-closed space was first introduced by Alexandroff and Urysohn [1].

For an open ultrafilter $\mathcal{U}$ on any topological space X, if $\bigcap_{U\in\mathcal{U}} \mathrm{Cl}_X U$ is empty, then it is called free (nonconvergent), otherwise $\mathcal{U}$ is called fixed (convergent). If X is a Hausdorff space, $\bigcap_{U\in\mathcal{U}} \mathrm{Cl}\, U$ is either empty or singleton.

Now let

$$X^{\vee} = \{\mathcal{U} : \mathcal{U} \text{ is a free open ultrafilter on } X\}.$$

Let $\kappa X = X \cup X^{\vee}$ be the disjoint union. It can easily be verified that

$$\{U : U \text{ is open in } X\} \cup \{U \cup \{\mathcal{U}\} : U \in \mathcal{U}, \mathcal{U} \in \kappa X \setminus X\}$$

is an open base for some topology on κX. We will consider this topology on κX. Note that X is open in κX. κX is called the Katětov extension of X [26]. See also [29, 36] for the construction of κX.

It was shown by Liu [29] that κX is a projective maximum in the class $\mathcal{H}(X)$ of H-closed extensions of X (see also [23, 36]).

Every Tychonoff space X has a maximal compactification, which is the Stone-Čech compactification βX of X. Every Tychonoff space X has a maximal realcompactification, which is the Hewitt realcompactification υX of X [22].

An open filter $\mathcal{U}$ on a space X is said to be have the *countable closure intersection property* (abbreviated CCIP) in case $\bigcap \{\mathrm{Cl}_X A : A \in \mathcal{A}\} \neq \emptyset$ for each countable subcollection $\mathcal{A} \subset \mathcal{U}$.

A Hausdorff space X is called as *almost realcompact* if every open ultrafilter on X with CCIP converges. This concept was defined and studied by Frolík [17]. It should be noted that, in contrast to realcompact spaces, an almost realcompact space need not satisfy the Tychonoff separation property.

Woods in [46] showed that each Tychonoff space X can correspond to an almost realcompact Tychonoff space aX with the following properties:

1. $X \subset aX \subset \beta X$, and
2. every map $f : X \longrightarrow Y$ into an almost realcompact Tychonoff space Y, has a continuous extension $f^a : aX \longrightarrow Y$, where f^a is the restriction to aX of the Stone extension $f^\beta : \beta X \longrightarrow \beta Y$ of f.

Therefore, every Tychonoff space has a maximal almost realcompactification.

A topological space X is called the *Dieudonné complete* (or the *topologically complete*) if there exists a complete uniformity on the space X, generating the topology of X. It is easy to see that a space X is Dieudonné complete if and only if X is a Tychonoff space and universal (finest) uniformity on the space X is complete.

The *topological completion* of a Tychonoff space X is the completion of X with respect to its finest uniformity, as defined by Morita [33], and will be denoted by μX, and which is also called the *Dieudonné completion* of X. Every map $f : X \longrightarrow Y$ induces a map $\mu f : \mu X \longrightarrow \mu Y$, which is the restriction of the Stone extension βf [33]. For more information on the map μf, see [32]. Note that $X \subset \mu X \subset \upsilon X \subset \beta X$. In fact, μX is the smallest Dieudonné complete subspace of υX containing X. A paracompact space is Dieudonné complete [13], and so it is a realcompact space.

The space μX is characterized as a space Y with the following properties:

1. Y is a topological complete space containing X as a dense subspace, and
2. any map from X into a topologically complete space Z can be extended to a map from Y into Z.

Therefore, every Tychonoff space has a maximal Dieudonné completion.

The following result proved by Ishii [24] (cf. [32]) will be used to prove our theorems.

A map f of a space X onto a space Y is called *quasi-perfect* (*perfect*) if f is continuous, closed, and the inverse images of points are countably compact (respectively compact).

Theorem 2.2. *If $f : X \longrightarrow Y$ is an open and quasi-perfect map, then $\mu f : \mu X \longrightarrow \mu Y$ and $\upsilon f : \upsilon X \longrightarrow \upsilon Y$ are open perfect maps.*

Corollary 2.3. *If $f : X \longrightarrow Y$ is an open perfect map, then the following hold.*

1. *X is a Dieudonné complete space if and only if Y is a Dieudonné complete space.*
2. *X is a realcompact space if and only if Y is a realcompact space [18].*

A property $\mathcal{P}$ of topological spaces is defined as *productive* if every product of an arbitrary family of spaces having $\mathcal{P}$ also has property $\mathcal{P}$. It is called *closed-hereditary* if every closed subspace of a space satisfying $\mathcal{P}$ also has $\mathcal{P}$. Compactness, realcompactness, almost realcompactness, and Dieudonné completeness are examples of properties which are closed-hereditary and productive (see [14, 17, 22]).

The following theorem appears in [21] (cf. [47]).

Theorem 2.4. *Let $\mathcal{P}$ be a topological property of Tychonoff spaces. Then the following are equivalent.*

1. *Each $\mathcal{P}$-regular space X has a maximal $\mathcal{P}$-extension.*
2. *$\mathcal{P}$ is closed-hereditary and productive.*

This leads to the following result.

Corollary 2.5. ([21, 41]) *If $\mathcal{P}$ is a property of Tychonoff spaces, then the following statements are equivalent.*

1. *Every Tychonoff space has a maximal $\mathcal{P}$-extension.*
2. *$\mathcal{P}$ is closed-hereditary, productive, and possessed by all compact spaces.*

Now let us remind some concepts related to topological transformation groups. A topological transformation group is a triple (G, X, θ) consisting of a topological group G, a topological space X, and a (continuous) map $\theta : G \times X \longrightarrow X$ which satisfies

a. $\theta(e, x) = x$ for all $x \in X$, where e is the identity element of G.

b. $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $x \in X, g, h \in G$.

As usual, we shall supress the map θ, and we shall write gx instead of $\theta(g, x)$. A (continuous) map $\varphi : X \longrightarrow Y$ between G-spaces is referred to as an *equivariant map* (or *G-map*) if $\varphi(gx) = g\varphi(x)$ for all $g \in G, x \in X$. An equivariant map that is a homeomorphism is called G-homeomorphic.

Let us now examine the extension problem of a topological action. Let G be a topological group acting on a topological space X. For each $g \in G$, $\theta_g : X \to X, x \mapsto gx$ is a homeomorphism. Consequently, it can be uniquely extended to the homeomorphism $\beta\theta_g : \beta X \longrightarrow \beta X$. It is evident that the map which takes $g \in G$ to $\beta\theta_g$ is a homomorphism of G into the group of all autohomeomorphisms of βX. It can be seen that the natural map $\psi : G \times \beta X \longrightarrow \beta X$, $(g, a) \mapsto \beta\theta_g(a)$ satisfies the two algebraic conditions a. and b. of an action. It can be shown that the algebraic action $G \times \beta X \longrightarrow \beta X$ is an extension of the given action $G \times X \to X$. Note that this algebraic action is not continuous in general. An counterexample, given by M. Jerison, is the multiplication action of the circle group on complex plane (see [35, p. 23] or [4, p. 220]). It can be easily shown that a necessary and sufficient condition for the extended algebraic action to be continuous is that $\beta_G X = \beta X$. In certain significant instances, the extended algebraic action $G \times \beta X \longrightarrow \beta X$ is continuous. For example, when G is a discrete group, $\beta_G X = \beta X$ [43, 7.3.10(ii)].

Since $\psi^{-1}(U) = \bigcup_{g\in G}\left(\{g\} \times \left(\beta\theta_g\right)^{-1}(U)\right)$ for $U \subset \beta X$, it is clear that ψ is continuous if G is a discrete.

It was proved by Palais [35, §1.5] that every Tychonoff G-space admits a maximal G-compactification if G is a compact Lie group (see [6] for a compact group G). J. de Vries subsequently extended this result to the case of arbitrary locally compact Hausdorff groups (see [42, 44]). It is crucial here that G is locally compact (see [30, 31]).

The study of the equality $\beta_G X = \beta X$ was first addressed by Antonyan [5]. Subsequently, de Vries posed in [45] the problem of determining necessary and sufficient conditions on a G-space X for the equality $\beta_G X = \beta X$. In the same paper, de Vries in [45] provided a partial answer to the question by proving that $\beta_G X = \beta X$ under the condition that the group G is a k-space and X is a pseudocompact G-space. Furthermore, N. Antonyan in [2] proved that $\beta_G X = \beta X$ when both the group G and the space X are pseudocompact. Subsequently, González and Sanchis in [20] unified these two results by J. de Vries and N. Antonyan. We refer the interested reader to the comprehensive survey conducted by N. Antonyan, S. Antonyan and M. Sanchis [4].

3. Main results

Let us start by defining the equivariant versions of the extensions mentioned in the introduction. By an equivariant extension (or G-extension) of a Hausdorff G-space X, we mean an arbitrary Hausdorff G-space EX containing X as a dense invariant subspace. More preciesly, an equivariant extension of a Hausdorff space X is a pair (EX, φ) of a Hausdorff G-space EX and a G-homeomorphic embedding $\varphi : X \longrightarrow EX$ such that $\overline{\varphi(X)} = EX$. In the family of all G-extensions of a given space X, a partial order is defined by setting $(E_1X, \varphi_1) \geq (E_2X, \varphi_2)$ when there is an equivariant map $\Phi : E_1X \longrightarrow E_2X$ such that $\Phi(\varphi_1(x)) = \varphi_2(x)$ for all $x \in X$. We say that E_1X is greater than E_2X if $(E_1X, \varphi_1) \geq (E_2X, \varphi_2)$. Two G-extensions (E_1X, φ_1) and (E_2X, φ_2) of the same topological space X are called *equivalent* to each other if there exists a G-homeomorphism $\Phi : E_1X \longrightarrow E_2X$ such that $\Phi(\varphi_1(x)) = \varphi_2(x)$ for all $x \in X$. For brevity, we shall denote extensions of a space X by E_1X, E_2X, etc. It is easy to see that E_1X and E_2X are equivalent if and only if $E_1X \geq E_2X$ and $E_2X \geq E_1X$.

It is evident that the equivalence of G-extensions constitutes an equivalence relation within the class of all G-extensions of X. For the sake of simplicity, it will be assumed in the sequel that equivalent G-extensions of a space X are equal.

If we take the space X to be a Hausdorff G-space and EX to be a H-closed G-space in the above definition, then EX will be called the *H-closed G-extension* of X.

It can be shown in the same way as in [28] that the set $\mathcal{H}_G(X)$ of all H-closed G-extensions of a Hausdorff G-space X is a sublattice of the set $\mathcal{H}(X)$ of all H-closed extensions of X. Therefore, every non-empty family of H-closed G-extensions has a least upper bound with respect to the order $\geq$. In particular, if a Hausdorff G-space X has a H-closed G-extension, then there exists a largest H-closed G-extension with respect to the order $\geq$. We will call this the *equivariant Katětov extension* of X, and it will be denoted by $\kappa_G X$.

Note that this lattice relation is also true for the G-extensions that we will define below.

If we take the space X to be a Tychonoff G-space and EX to be a compact Hausdorff G-space in the above definition, then the maximal element of the family of G-compactifications of X is called the *maximal G-compactification*, and it is denoted by $\beta_G X$ (see [3, 35]). Note here that the reason why this definition is called maximal instead of projective maximum is its below characterization Theorem 3.1, i.e. it is also a maximal extension in the sense of Definition 2.1.

The following characterization of the maximal G-compactification $\beta_G X$ was established in [42] (see also [3]).

Theorem 3.1. *Let G be a locally compact group which acts on a Tychonoff space X. Then*

a. *Each G-map $f : X \longrightarrow Y$ to a compact G-space can be uniquely extended to a G-map $F : \beta_G X \longrightarrow Y$.*

b. *Let bX be a G-compactification of X. If every G-map $f : X \longrightarrow Y$ to a compact G-space can be equivariantly extended over bX, then bX is equivalent to $\beta_G X$.*

If we take X to be a Tychonoff G-space and EX to be a realcompact G-space in the above definition, then the maximal element of the family of G-realcompactications of X will be called the *G-Hewitt realcompactification of X*, and it will be denoted by $\upsilon_G X$. We will call the space X a *G-Hewitt space* if $\upsilon_G X = X$.

Remark 3.2. Note that our proposed definition of G-Hewitt realcompactification is slightly different from the definition given by Antonyan [5] , but is more suitable for us since it avoids very pathological cases. For example, our definition of G-Hewitt space requires that it be Hewitt; however, Antonyan's definition allows for the possibility that it may not be.

If we take X to be a Tychonoff G-space and EX to be an almost realcompact G-space in the above definition, then the maximal element of the family of G-almost realcompactifications of X will be called the *maximal G-almost realcompactification* of X, and it will be denoted by $a_G X$. We will call the space X a *G-almost realcompact space* if $a_G X = X$.

Note again that the reason we use the term maximal instead of equivariant projective maximum is that it is in fact maximal in the sense of Definition 2.1 (see Proposition 3.20 below).

If we take X to be a Tychonoff G-space and EX to be a Dieudonné complete G-space in the above definition, then the maximal element of the family of *Dieudonné G-completions of X* will be called the *maximal Dieudonné G-completion* of X, and it will be denoted by $\mu_G X$.

3.1. Equivariant Hewitt realcompactification

The following characterization of the G-Hewitt realcompactification of a Tychonoff G-space X is used in proving the main theorem. It states that G-Hewitt realcompactification $\upsilon_G X$ is (uniquely determined) maximal G-realcompactification of X. It is presented without proof in [5]; however, we will provide a proof here for the reader's convenience. Recall that realcompactness is closed-hereditary and productive.

Proposition 3.3. *Let G be a locally compact group acting on a Tychonoff space X. Then the following hold.*

a. *Every G-map $f : X \longrightarrow Y$ into a realcompact G-space can be uniquely extended to a G-map $F : \upsilon_G X \longrightarrow Y$.*

b. *Let rX be a G-realcompactification of X. If every G-map $f : X \longrightarrow Y$ into a realcompact G-space can be equivariantly extended over rX, then rX is equivalent to $\upsilon_G X$.*

Proof. **a.** Let $\upsilon_G : X \hookrightarrow \upsilon_G X$ be the natural inclusion. Since υ_G is a G-embedding, then the diagonal product $e = \upsilon_G \Delta f : X \longrightarrow \upsilon_G X \times Y$ is also a G-embedding. It follows that $rX = \overline{r(X)} \subset \upsilon_G X \times Y$ is a G-realcompactification of X. By the maximality of $\upsilon_G X$, there exists a G-map $\varphi : \upsilon_G X \longrightarrow rX$ such that $\varphi \upsilon_G = e$. Let $p : rX \longrightarrow Y$ be the restriction to rX of the second projection $\upsilon_G X \times Y \longrightarrow Y$ and let $F = p\varphi : \upsilon_G X \longrightarrow Y$. Since $F\upsilon_G = p\varphi\upsilon_G = pe = f$, the map F is an extension of f. Its uniqueness is obvious.

b. The hypothesis implies the existence of a G-extension $\varphi : rX \longrightarrow \upsilon_G X$ of the G-embedding $\upsilon_G : X \hookrightarrow \upsilon_G X$. Consequently, $rX \geq \upsilon_G X$. Conversely, $\upsilon_G X \geq rX$, which implies that $\upsilon_G X$ is equivalent to rX. □

We now show that taking quotients by normal subgroups commutes with taking equivariant extensions.

Theorem 3.4. *Let G be a compact group acting on any Tychonoff space X, and H be a closed normal subgroup of G. Then $(\upsilon_G X)/H = \upsilon_{G/H}(X/H)$, i.e., $(\upsilon_G X)/H$ and $\upsilon_{G/H}(X/H)$ are equivalent G/H-Hewitt realcompactifications of X/H.*

Proof. The compactness of H requires that the orbit space X/H is a Tychonoff G/H-space. Let $i : X \hookrightarrow \upsilon_G X$ be the standard G-embedding and let $p : X \to X/H$ and $q : \upsilon_G X \longrightarrow (\upsilon_G X)/H$ be the H-orbit maps. The map i gives rise to a G/H-map $j : X/H \longrightarrow (\upsilon_G X)/H$ such that $qi = jp$. Furthermore, j is a G/H-embedding since i is a G-embedding. The continuity of q requires that

$$(\upsilon_G X)/H = q(\upsilon_G X) = q\left(\overline{i(X)}\right) \subset \overline{q(i(X))} = \overline{j(p(X))} = \overline{j(X/H)}.$$

Therefore, $(\upsilon_G X)/H = \overline{j(X/H)}$, and since $(\upsilon_G X)/H$ is realcompact by Corollary 2.3, we infer that $(\upsilon_G X)/H$ is a G/H-realcompactification of X/H. In order to prove the required equality, by Proposition 3.1, it is sufficient to show that each G/H-map $f : X/H \longrightarrow Y$ into a realcompact G/H-space Y has a G/H-equivariant extension $F : (\upsilon_G X)/H \longrightarrow B$. Note that the map $\phi = fp : X \longrightarrow B$ is H-invariant, i.e. $\phi(hx) = \phi(x)$ for all $h \in H, x \in X$.

We may regard Y as a G-space via the quotient homomorphism $G \longrightarrow G/H$, namely $gy := (gH)y$ for all $g \in G$ and $y \in Y$. Then, it is clear that the map ϕ is G-equivariant. In accordance with Proposition 3.1, there exists a G-equivariant extension $\Phi : \upsilon_G X \longrightarrow Y$. Since Φ is G-equivariant and H acts trivially on Y, it is constant on the H-orbits of $\upsilon_G X$. Hence it induces a G/H-map $F : (\upsilon_G X)/H \longrightarrow Y$ such that $Fq = \Phi$. Since Φ is an extension of ϕ, it can be showed that F is an extension of f, as required. This completes the proof. □

As a consequence of the theorem, the following immediately follows.

Corollary 3.5. *Let G be a compact group which acts on a Tychonoff space X. Then $(\upsilon_G X)/G = \upsilon(X/G)$.*

The following result was proved in [10], this result was independently obtained in [15] (see also [34]).

Corollary 3.6. *If G is a finite group which acts on a Tychonoff space X, then we have that $\upsilon_G X = \upsilon X$ and $(\upsilon X)/G = \upsilon(X/G)$.*

See [27] for sufficient conditions for an action of G on X to extend on υX.
It is not difficult to verify the equivariant version of Corollary 2.4.

Proposition 3.7. *If $\mathcal{P}$ is a property of Tychonoff G-spaces, then the following statements are equivalent.*

1. *Every Tychonoff G-space has a maximal $\mathcal{P}$-extension.*
2. *$\mathcal{P}$ is invariant closed-hereditary, productive, and possessed all compact G-spaces.*

If the property $\mathcal{P}$ is taken as G-realcompactness, where G is a locally compact group, every Tychonoff G-space has a maximal G-realcompactification (Proposition 3.3), so the following are obtained.

Corollary 3.8. *Each closed invariant subspace of a G-Hewitt space is also G-Hewitt space.*

Corollary 3.9. *A product $\prod_{i \in I} X_i$ of non-empty G-spaces X_i is a G-Hewitt space if all the X_i are G-Hewitt spaces.*

As a consequence of the property of being closed-hereditary and productive, the following are obtained.

Corollary 3.10. *If X is a G-space and $\{A_i : i \in I\}$ is a family of invariant G-Hewitt subspaces of X, then the intersection $\bigcap_{i \in I} A_i$ is also a G-Hewitt space.*

It is not difficult to obtain the counterpart of Proposition 1 of Herrlich and van der Slot [21] (cf. [41, Theorem 1.5]) in the realm of G-spaces.

Proposition 3.11. *Let $\mathcal{P}$ be a property of Tychonoff G-spaces which is inherited invariant closed subsets and the forming of finite topological products. If f is a (continuous) G-map from a G-space X with the property $\mathcal{P}$ into a G-space Y, then the inverse image under f of each invariant subset of Y with the property $\mathcal{P}$ also satifies $\mathcal{P}$.*

The following is a simple consequence of this.

Proposition 3.12. *Let $\mathcal{P}$ be a property of Tychonoff G-spaces which is invariant closed-hereditary. Assume that for any G-space Y with the property $\mathcal{P}$, each product of Y with any compact G-space Z has the property $\mathcal{P}$. If whenever f is perfect G-map of a space X onto a space Y, then X has the property $\mathcal{P}$ if Y has the property $\mathcal{P}$.*

If the property $\mathcal{P}$ is taken as G-Hewitt space, the following is obtained.

Corollary 3.13. *If there exists a perfect G-map $f : X \longrightarrow Y$ of a Tychonoff G-space X onto a G-Hewitt space Y, then X is a G-Hewitt space.*

Now, using Corollary 3.10 and Proposition 3.11, as in the last paragraph of Herrlich and van der Slot [21], it can be shown that $\upsilon_G X$ can be identified with the intersection of all G-Hewitt subspaces of $\beta_G X$ that contain X.

Corollary 3.14. *For any Tychonoff G-space, the space $\upsilon_G X$ is the smallest G-Hewitt space between X and $\beta_G X$, and $\beta_G (\upsilon_G X) = \beta_G X$.*

3.2. Equivariant Dieudonné completion of an orbit space

Since the Dieudonné completeness is both closed-hereditary [14] and productive, similar to Proposition 3.3, the following is easily obtained.

Proposition 3.15. *Let G be a locally compact group acting on a Tychonoff space X . Then the following hold.*

a. *Each G-map $f : X \longrightarrow Y$ into a Dieudonné complete G-space can be uniquely extended to a G-map $F : \mu_G X \longrightarrow Y$.*

b. *Let dX be a Dieudonné G-completion of X. If every G -map $f : X \longrightarrow Y$ into a Dieudonné complete G-space can be equivariantly extended over dX, then dX is equivalent to $\mu_G X$.*

To prove our result we need the following theorem.

Theorem 3.16. *[33] Let G be a compact group and X be any space. Then*

$$\mu (G \times X) = G \times \mu X.$$

An elegant proof of this result can be found in [9, Theorem 6.7.8]. For more general results, see [40]. For results of this type see also [25]. As in [45, Lemma 5.5], we obtain the following.

Theorem 3.17. *If G is a compact group and X be a Tychonoff G-space, then $\mu_G X = \mu X$.*

Proof. Let θ denote the action of G on X. For every $g \in G$, the map $\theta_g : X \longrightarrow X$ extends to a map $\overline{\theta}_g : \mu X \longrightarrow \mu X$. Then we obtain a map $\overline{\theta} : G \times \mu X \longrightarrow \mu X$ which can be easily seen to have conditions a. and b. for an action, but may not satisfy the continuity condition. We will show that $\overline{\theta}$ is continuous. Since $\theta : G \times X \longrightarrow X$ is continuous, it has a continuous extension $\mu\theta : \mu (G \times X) = G \times \mu X \longrightarrow \mu X$. Since for $g \in G$, the continuous maps $(\mu\theta)_g$ and $\overline{\theta}_g$ are equal on X, they are equal on μX, that is $\overline{\theta} = \mu\theta$. This shows that μX is a G-space. Now, it is easily seen that this is the maximal Dieudonné G-completion of X. □

In fact, Kozlov [27, Corollary 3.5] proved that every action of a locally compact and paracompact group G on a Tychonoff space X has an extension over μX.

Similar to Theorem 3.4, the following can be easily proved.

Theorem 3.18. *Let G be a compact group, H be a closed normal subgroup of G , and X be a Tychonoff G-space. Then $(\mu_G X)/H = \mu_{G/H} (X/H)$, i.e., $(\mu_G X)/H$ and $\mu_{G/H} (X/H)$ are equivalent maximal Dieudonné G/H-completions of X/H.*

Corollary 3.19. *Let G be a compact group which acts on a Tychonoff space X. Then $(\mu X)/G = (\mu_G X)/G = \mu (X/G)$.*

3.3. Maximal equivariant almost realcompactification

Since the almost realcompactness is both closed-hereditary and productive [17], similar to Proposition 3.3, the following is easily obtained. It states that G-almost realcompactification $a_G X$ of a Tychonoff G-space X is (uniquely determined) maximal G-almost reelcompactification of X.

Proposition 3.20. *Let G be a locally compact group acting on a Tychonoff space X . Then the following hold.*

a. *Each G-map $f : X \longrightarrow Y$ into an almost realcompact G-space can be uniquely extended to a G-map $F : a_G X \longrightarrow Y$.*

b. *Let sX be an almost realcompact G-space of X. If every G -map $f : X \longrightarrow Y$ into an almost realcompact G-space can be equivariantly extended over sX, then sX is equivalent to $a_G X$.*

Similar to Theorem 3.4, the following can be easily proved.

Theorem 3.21. *Let G be a compact group, H be a closed normal subgroup of G, and X be a Tychonoff G-space. Then $(a_G X)/H = a_{G/H}(X/H)$.*

Corollary 3.22. *Let G be a compact group which acts on a Tychonoff space X. Then $(a_G X)/G = a(X/G)$.*

The following was also proved in [34].

Corollary 3.23. *Let G be a finite group which acts on a Tychonoff space X. Then $(aX)/G = a(X/G)$.*

All results following Proposition 3.7 are easily obtained for G-almost realcompactification.

Corollary 3.24. *Each closed invariant subspace of a G-almost realcompact space is also G-almost realcompact space.*

Corollary 3.25. *A product $\prod_{i \in I} X_i$ of non-empty G-spaces X_i is a G-almost realcompact space if all the X_i are G-almost realcompact spaces.*

Corollary 3.26. *If X is a G-space and $\{A_i : i \in I\}$ is a family of invariant G-almost realcompact subspaces of X, then the intersection $\bigcap_{i \in I} A_i$ is also a G-almost realcompact space.*

Corollary 3.27. *If there exists a perfect G-map $f : X \longrightarrow Y$ of a Tychonoff G-space X onto a G-almost realcompact space Y, then X is a G-almost realcompact space.*

Corollary 3.28. *For any Tychonoff G-space, the space $a_G X$ is the smallest G-almost realcompact space between X and $\beta_G X$, and $\beta_G(a_G X) = \beta_G X$.*

3.4. Equivariant Katětov extension

A closed subspace of an H-closed space need not be H-closed, but a regular closed subset of an H-closed space is H-closed [26]. Let $\{X_i : i \in I\}$ be a family of nonempty spaces. Then $\prod_{i \in I} X_i$ is H-closed if and only if X_i is H-closed for each $i \in I$ (see [12]). If $f : X \longrightarrow Y$ is a continuous, open map, then f can be extended to a continuous map $\kappa f : \kappa X \longrightarrow \kappa Y$ [36] (cf. [39, 7.6]). From these facts we obtain the following.

Proposition 3.29. *Let G be a locally compact group acting on a Hausdorff space X . Then the following hold.*

a. *Each open G-map $f : X \longrightarrow Y$ into an H-closed G-space can be uniquely extended to a G-map $F : \kappa_G X \longrightarrow Y$.*

b. *Let kX be an H-closed G-space of X. If every open G-map $f : X \longrightarrow Y$ into an H-closed G-space can be equivariantly extended over kX, then kX is equivalent to $\kappa_G X$.*

Proof. **a.** Let $\kappa_G : X \hookrightarrow \kappa_G X$ be the natural open inclusion. Since κ_G is a G-embedding, then the diagonal product $k = \kappa_G \Delta f : X \longrightarrow \kappa_G X \times Y$ is also a G-embedding. It follows that $kX = \overline{k(X)} \subset \kappa_G X \times Y$ is an H-closed G-extension of X because $k(X)$ is an open in $\kappa_G X \times Y$ and kX is regular closed. By the maximality of $\kappa_G X$, there exists a G-map $\varphi : \kappa_G X \longrightarrow kX$ such that $\varphi\kappa_G = k$. Let $p : \kappa X \longrightarrow Y$ be the restriction to κX of the second projection $\kappa_G X \times Y \longrightarrow Y$ and let $F = p\varphi$. Since $F\kappa_G = p\varphi\kappa_G = pk = f$, the map F is an extension of f. Its uniqueness is obvious.

b. The hypothesis implies the existence of a G-extension $\varphi : kX \longrightarrow \kappa_G X$ of the open G-embedding $\kappa_G : X \hookrightarrow \kappa_G X$. Consequently, $kX \geq \kappa_G X$. Conversely, $\kappa_G X \geq kX$, which implies that $\kappa_G X$ is equivalent to kX. □

Theorem 3.30. *Let G be a compact group acting on a Hausdorff space X, H be a closed normal subgroup of G. Then $(\kappa_G X)/H = \kappa_{G/H}(X/H)$.*

Proof. The closedness of H requires that the orbit space X/H is a Hausdorff G/H-space. Let $i : X \hookrightarrow \kappa_G X$ be the standard G-embedding and let $p : X \to X/H$ and $q : \kappa_G X \longrightarrow (\kappa_G X)/H$ be the H-orbit maps. The map i gives rise to a continuous G/H-map $j : X/H \longrightarrow (\kappa_G X)/H$ such that $qi = jp$. Furthermore, j is a G/H-embedding since i is a G-embedding. The continuity of q requires that

$$(\kappa_G X)/H = q(\kappa_G X) = q\left(\overline{i(X)}\right) \subset \overline{q(i(X))} = \overline{j(p(X))} = \overline{j(X/H)}.$$

Therefore, $(\kappa_G X)/H = \overline{j(X/H)}$, and since $(\kappa_G X)/H$ is H-closed (a continuous image of an H-closed space is H-closed [16]), we infer that $(\kappa_G X)/H$ is an H-closed G/H-extension of X/H. In order to prove the required equality, it suffices to show that each open G/H-map $f : X/H \longrightarrow Y$ into an H-closed G/H-space Y has a G/H-equivariant extension $F : (\kappa_G X)/H \longrightarrow Y$. It is clear that the map $\phi = fp : X \longrightarrow Y$ is H-invariant.

It is possible to regard Y as a G-space via the quotient homomorphism $G \longrightarrow G/H$. Then, it is clear that the map ϕ is G-equivariant. Therefore there exists a G-equivariant extension $\Phi : \kappa_G X \longrightarrow Y$. Since Φ is G-equivariant and H acts trivially on Y, it is constant on the H-orbits of $\kappa_G X$. Hence it induces a G/H-equivariant map $F : (\kappa_G X)/H \longrightarrow Y$ such that $Fq = \Phi$. As Φ is an extension of ϕ, it can be seen that F is an extension of f, as required. This completes the proof. □

Corollary 3.31. *Let G be a compact group which acts on a Hausdorff space X. Then $(\kappa_G X)/G = \kappa(X/G)$.*

We shall now proceed to discuss the extension of the action of a discrete group on the Katětov extension κX.

Suppose that a discrete group G acts on a Hausdorff space X via an action $\theta : G \times X \longrightarrow X$. For some $g \in G$ and a subset A in X, define $gA = \{ga : a \in A\}$, and for a family $\mathcal{A}$ of subsets of X, $g\mathcal{A}$ will denote the family $\{gA : A \in \mathcal{A}\}$ of subsets of X. It is clear the if $\mathcal{U}$ is a free open ultrafiltre, then so is $g\mathcal{U}$.

Proposition 3.32. *[10] Let G be a discrete group acting on a Hausdorff space X. Then the action of G on X induces an action on κX, keeping X invariant.*

Proof. The map $\psi : G \times \kappa X \longrightarrow \kappa X$ defined by $\psi(g, \alpha) = g\alpha$ for $g \in G$ and $\alpha \in \kappa X$ defines an action of G on κX. It is clear that this action leaves X invariant. □

Hence $\kappa_G X = \kappa X$ when G is a discrete group. When G is a finite group, we obtain the following result.

Corollary 3.33. ([10]) *Let X be a Hausdorff G-space, where G is a finite group. Then $(\kappa X)/G = \kappa(X/G)$.*

The following example given in [10] shows that for an infinite group G, $(\kappa X)/G$ may differ from $\kappa(X/G)$.

Example 3.34. Consider the G-space $(\mathbb{Z}, \mathbb{Z}, \theta)$, where $\mathbb{Z}$ is the discrete space of integers, and the group operation is the usual addition of integers and θ is the action of $\mathbb{Z}$ on $\mathbb{Z}$ given by the usual addition of integers. Let $\mathcal{U}$ and $\mathcal{G}$ denote the open ultrafilters containing the open filters generated by the collections

$$\{\{n, n+1, n+2, \cdots\} : n \text{ is a positive integer}\}$$

and

$$\{\{\cdots, -(n+1), -n\} : n \text{ is a positive integer}\}$$

respectively. It is easy to see that $\mathcal{U} \neq n \cdot \mathcal{G}$ for any $n \in \mathbb{Z}$. This shows that $G(\mathcal{U})$ and $G(\mathcal{G})$ are distinct orbits and both belong to $(\kappa\mathbb{Z})/\mathbb{Z}$. Since $\kappa(\mathbb{Z}/\mathbb{Z})$ is singleton, $(\kappa\mathbb{Z})/\mathbb{Z}$ differs from $\kappa(\mathbb{Z}/\mathbb{Z})$.

The answer to the following question may be interesting. In this case, the action of a topological group on a topological space will determine the action of this topological group on the extension of the space.

This result will also provide a generalization of Glicksberg's well-known result ([19]) concerning Stone-Čech compactifications of products to the case of topological transformation groups. The maximal G-compactification was proven by de Vries [45] when G is locally compact and locally connected topological group, and later by N. Antonyan [2] for any Hausdorff topological groups.

Question: Under which conditions $\upsilon_G(X \times Y) = \upsilon_G X \times \upsilon_G Y$ and $a_G(X \times Y) = a_G X \times a_G Y$?

When $X \times Y$ is a pseudocompact G-space, the question is obvious because in this case

$$\upsilon_G(X \times Y) = \beta_G(X \times Y) = \beta_G X \times \beta_G Y = \upsilon_G X \times \upsilon_G Y.$$

See [2, 5, 45]. It is not difficult to see that the same result applies in almost realcompactification.

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