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Spectral Turán results on triangle-free graphs and hypergraphs

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Abstract. One of the most classical spectral Turán problems is determining the maximum spectral radius of triangle-free graphs and hypergraphs. As the hypergraph analogue of triangles, Fan^k is a linear k-uniform hypergraph with k hyperedges $f_1, \ldots, f_k$ which pairwise intersect in a common vertex v, and an additional hyperedge g which intersects all f_i in a vertex different from v. Let $K_{s,t}^-$ be the graph obtained from a complete bipartite graph $K_{s,t}$ by deleting an edge. Motivated by the classic theorems on triangle-free graphs due to Erdős and Nosal, and by the Turán number of Fan^k on linear k-uniform hypergraphs determined by Füredi and Gyárfás respectively, we prove that $\rho(G) \le \rho(K^-_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$ where $\rho(G)$ is the spectral radius of a connected triangle-free graph G with order n and diameter 3, and $spex_k^{lin}(m, Fan^k) = \sqrt{m}$ where $spex_k^{lin}(m, Fan^k)$ denotes the maximum spectral radius of Fan^k-free linear k-uniform hypergraphs with size m.

1. Introduction

A hypergraph $H = (V(H), E(H))$ consists of a vertex set $V(H)$ and a hyperedge (edge) set $E(H)$, where each hyperedge is a nonempty subset of $V(H)$. A hypergraph is called k-uniform if each hyperedge is a k-element subset of $V(H)$. Obviously, a graph is a 2-uniform hypergraph. So, we usually refer to a hyperedge simply as an edge in 2-uniform hypergraph. Two vertices x and y are said to be adjacent if there is a hyperedge that contains both of these vertices. A hypergraph H is called linear if every two hyperedges have at most one vertex in common.

Turán type extremal problems in graphs and hypergraphs are the central topic of extremal combinatorics and have a vast literature, which asks to maximize the number of edges in a graph that does not contain fixed forbidden subgraphs. Given a family of graphs $\mathcal{F}$, the Turán number of $\mathcal{F}$, denoted $ex(n, \mathcal{F})$, is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices. As an extension of graphs, for a family $\mathcal{F}$ of linear k-uniform hypergraphs, the linear Turán number of $\mathcal{F}$, denoted $ex_k^{lin}(n, \mathcal{F})$, is the maximum number of hyperedges in an $\mathcal{F}$-free linear k-uniform hypergraph on n vertices. Over a century old, a classic example

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of Turán type results is Mantel's theorem, which states that $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the unique graph maximizing the number of edges over all triangle-free graphs.

As a generalization of triangles in hypergraphs, Fan^k is a linear k-uniform hypergraph with k hyperedges $f_1, \ldots, f_k$ which pairwise intersect in a common vertex v, and an additional hyperedge g which intersects all f_i in a vertex different from v. Obviously, for $k = 2$, Fan^2 is a triangle. We assume that n is a multiple of k. A transversal design $T(n, k)$ is a linear k-partite k-uniform hypergraph on n vertices where the vertices are divided into k groups, each containing $\frac{n}{k}$ vertices, and where each pair of vertices from different groups belongs to exactly one hyperedge. In particular, Füredi and Gyárfás studied the linear Turán number of Fan^k in [9].

Theorem 1.1 ([9]). *One has $ex_k^{lin}(n, Fan^k) \leq \frac{n^2}{k^2}$ for all $k \geq 2$. The only extremal hypergraphs are the transversal designs on n vertices with k groups.*

Analogous to the Turán-type problem, the spectral Turán-type problems for adjacency spectral radius are a natural extension. Given a family of graphs $\mathcal{F}$, let $spex(n, \mathcal{F})$ and $spex(m, \mathcal{F})$ denote the maximum adjacency spectral radius in a family of $\mathcal{F}$-free graphs with given order n and size m, respectively. In 1970, Nosal [15] proved that a graph G has a triangle if $\rho(G) \geq \sqrt{m}$, which can be considered the spectral version of Mantel's theorem. Erdős (see [1], Ex.12.2.7) improved Mantel's theorem on non-bipartite triangle-free graphs. Motivated by Erdős's theorem, Lin, Ning, and Wu [14] confirmed the spectral version of Erdős's theorem using tools from doubly stochastic matrix theory and characterized all families of extremal graphs. They showed that $\rho(G) \geq \rho(S(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}))$ where G is a non-bipartite graph of order n and $S(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$ denotes a subdivision of $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ at one edge. Motivated by the spectral extremal results on given diameter, such as [4, 10] and the survey [18], we will continue studying the spectral extremal problem for triangle-free non-bipartite graphs. Notice that the extremal graph $S(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$ in [14] is still a triangle-free graph with diameter 2, this paper will study the spectral extremal problem for triangle-free non-bipartite graphs with diameter 3.

Since the independent introduction of tensor eigenvalues and eigenvectors by Lim [13] and [20], and Cooper and Dutle's [6] definition of the adjacency tensor for uniform hypergraphs, research on spectral Turán problems for linear hypergraphs has also seen advanced rapidly in recent years. In particular, several results on spectral Turán-type problems of linear uniform hypergraphs with fixed order have been obtained, such as [3, 11, 12, 17, 19] and references therein. Similarly, we denote by $spex_k^{lin}(n, \mathcal{F})$ and $spex_k^{lin}(m, \mathcal{F})$ the maximum spectral radius in a family of $\mathcal{F}$-free linear k-uniform hypergraphs with given order n and size m respectively. There are many excellent works on triangle-free hypergraphs, such as [8] and references therein. However, the corresponding problems for hypergraphs with fixed size remain challenging and underdeveloped. In this paper, we will focus on determining $spex_k^{lin}(m, Fan^k)$.

The rest of this paper is organized as follows. In Section 2, the necessary notation and basic facts have been presented, including the definitions and properties of eigenvalues of tensors and hypergraphs. In Section 3, we consider spectral Turán-type problems for triangle-free graphs with fixed order and diameter. In Section 4, the maximum spectral radius and extremal hypergraphs of Fan^k-free linear k-uniform hypergraphs with given size are determined. In Section 5, we conclude this article with some open problems for further study.

2. Spectra of tensors

In 2005, Qi [20] and Lim [13] independently introduced the concept of tensor eigenvalues and the spectra of tensors. We set $[n] = \{1, 2, \ldots, n\}$. An kth-order n-dimensional real tensor $\mathcal{T} = (\mathcal{T}_{i_1 \ldots i_k})$ consists of n^k real entries $\mathcal{T}_{i_1 \ldots i_k}$ for $i_1, i_2, \ldots, i_k \in [n]$. Obviously, a vector of dimension n is a tensor of order 1 and a matrix is a tensor of order 2. $\mathcal{T}$ is called symmetric if the value of $\mathcal{T}_{i_1 \ldots i_k}$ is invariant under any permutation of its indices $i_1, i_2, \ldots, i_k$. Given a vector $x \in \mathbb{R}^n$, $\mathcal{T}x^k$ is a real number and $\mathcal{T}x^{k-1}$ is an n-dimensional vector. $\mathcal{T}x^k$

and the ith component of $\mathcal{T}x^{k-1}$ are defined as follows:

$$\mathcal{T}x^k = \sum_{i_1,i_2,\ldots,i_k\in[n]} \mathcal{T}_{i_1i_2\ldots i_k}x_{i_1}x_{i_2}\cdots x_{i_k}.$$

$$(\mathcal{T}x^{k-1})_i = \sum_{i_2,\ldots,i_k\in[n]} \mathcal{T}_{ii_2\ldots i_k}x_{i_2}\cdots x_{i_k}.$$

Let $\mathcal{T}$ be a k th-order n-dimensional real tensor. For some $\lambda \in \mathbb{C}$, if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying the following eigenequation

$$\mathcal{T}x^{k-1} = \lambda x^{[k-1]},$$

then λ is an eigenvalue of $\mathcal{T}$ and x is its corresponding eigenvector, where $x^{[k-1]} = (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1})^T \in \mathbb{C}^n \setminus \{0\}$.

If x is a real eigenvector of $\mathcal{T}$, then clearly the corresponding eigenvalue λ is real. In this case, λ is called an H-eigenvalue and x is called an H-eigenvector associated with λ. Furthermore, if x is nonnegative and real, we say λ is an H^+-eigenvalue of $\mathcal{T}$. If x is positive and real, λ is said to be an H^{++}-eigenvalue of $\mathcal{T}$. The maximal absolute value of the eigenvalues of $\mathcal{T}$ is called the spectral radius of $\mathcal{T}$, denoted by $\rho(\mathcal{T})$.

In 2012, Cooper and Dutle [6] defined the adjacency tensor of a k-uniform hypergraph H. The adjacency tensor $\mathcal{A} = \mathcal{A}(H)$ is a k-th order n-dimensional symmetric tensor, where:

$$\mathcal{A}_{i_1\cdots i_k} = \begin{cases} \frac{1}{(k-1)!} & if\ \{i_1, \ldots i_k\} \in E, \\ 0 & otherwise. \end{cases}$$

For a vector x of dimension n and a subset $U \subseteq V$, we write

$$x^U = \prod_{v_i\in U} x_i.$$

The product $\mathcal{A}x^k$ has an interpretation as follows:

$$\mathcal{A}x^k = \sum_{e\in E} kx^e.$$

For nonnegative tensors, we have a generalization of the Perron-Frobenius theorem, see [5–7, 21]. Let $\mathcal{T} = (\mathcal{T}_{i_1\ldots i_k})$ be an order k dimension n nonnegative tensor. If for any nonempty proper index subset $\alpha \subset \{1, 2, \ldots, n\}$, there is at least an entry $\mathcal{T}_{i_1\ldots i_k} > 0$, where $i_1 \in \alpha$ and at least an $i_j \notin \alpha$ for $j = 2, \ldots, k$, then $\mathcal{T}$ is called nonnegative weakly irreducible tensor. It was proved that a k-uniform hypergraph H is connected if and only if its adjacency tensor $\mathcal{A}(H)$ is weakly irreducible. For a k-uniform hypergraph H, the spectral radius of H, denoted by $\rho(H)$, is defined as the maximum absolute value of the eigenvalues of the adjacency tensor $\mathcal{A}(H)$. By the Perron-Frobenius theorem, if H is connected, the eigenvector $x = (x_1, x_2, \ldots, x_n)^T$ with $||x||_k = 1$ corresponding to $\rho(H)$, known as the Perron vector, can be chosen to be unique positive eigenvector. Throughout the paper, we only consider connected and simple hypergraphs.

3. Spectral extremal results for triangle-free graphs with fixed diameter

Let $G = (V(G), E(G))$ be a simple graph of order n and $\rho(G)$ be the spectral radius of G. A path $P = v_0e_1v_1e_2\cdots v_{l-1}e_lv_l$ of length l is an alternating sequence of distinct vertices and edges such that $v_iv_{i+1} \in E$ for $i = 0, \ldots, l-1$. G is connected if for every pair of vertices $u, v \in V(G)$, there exists a path between u and v. Let $d(u, v)$ be the length of shortest path between two vertices u and v. The diameter $D(G) = \max\{d(u, v)|u, v \in V(G)\}$.

For a vertex v and a subgraph S of G, let $N_S(v)$ be the neighborhood of v in S. Let $F \subseteq V(G)$, denote by $G[F]$ the subgraph of G induced by F. We write $G - F$ for the subgraph of G induced by $V(G) \setminus F$. For an edge e, we denote by $G-e$ the graph obtained from G by deleting the edge e. Let $K_{s,t}$ be a complete bipartite graph whose vertex set is partitioned into two disjoint and independent sets S and T with $|S| = s$ and $|T| = t$. For simplicity, we write $K^-_{s,t}$ for $K_{s,t} - e$.

Extremal spectral problems on diameters played an important inspirational role in the development of extremal graph theory. Let $\mathscr{B}(n,d)$ be the set of bipartite graphs with order n and diameter d. Zhai, Liu and Shu (see [23], Lemma 2.8) characterized the extremal graph with the maximal spectral radius in $\mathscr{B}(n,d)$. In particular, for $d = 3$, the extremal graph with the maximal spectral radius is $K^-_{\lfloor \frac{n}{2}\rfloor,\lceil \frac{n}{2}\rceil}$. For a more comprehensive conclusion, we rewrite this result as follows.

Lemma 3.1 ([23]). *Let $K^-_{s,t}$ be a bipartite graph obtained from $K_{s,t}$ by deleting an edge. For $n \geqslant 4$, we have*

$$\rho(K^-_{s,t}) \leq \rho(K^-_{\lfloor \frac{n}{2}\rfloor,\lceil \frac{n}{2}\rceil}).$$

The equality holds if and only if $K^-_{s,t} \cong K^-_{\lfloor \frac{n}{2}\rfloor,\lceil \frac{n}{2}\rceil}$.

For our main proof, we first adopt the following Zykov symmetrization [22]. For two non-adjacent vertices $v_1, v_2 \in V(G)$, let $Z_{v_1,v_2}(G)$ be the graph obtained from G by deleting all edges incident to vertex v_1 and adding new edges from v_1 to $N_G(v_2)$. Notice that if G is triangle-free, then $Z_{v_1,v_2}(G)$ is also triangle-free. The following Lemma is crucial in our proof.

Lemma 3.2. *Let G be a simple connected graph and x be the Perron vector corresponding to $\rho(G)$. For a pair of non-adjacent vertices v_1 and v_2 with $\sum_{w\in N_G(v_2)} x_w \geq \sum_{w\in N_G(v_1)} x_w$, then $\rho(Z_{v_1,v_2}(G)) \geq \rho(G)$.*

Proof. Since $\sum_{w\in N_G(v_2)} x_w \geq \sum_{w\in N_G(v_1)} x_w$ and $\rho x_{v_i} = \sum_{w\in N_G(v_i)} x_w$ for $i = 1, 2$, it is easy to verify that

$$\begin{aligned}\rho(Z_{v_1,v_2}(G)) - \rho(G) &\geq x^T(A(Z_{v_1,v_2}(G)) - A(G))x \\ &\geq 2x_{v_1}(\sum_{w\in N_G(v_2)} x_w - \sum_{w\in N_G(v_1)} x_w) \\ &\geq 0.\end{aligned}$$

This completes the proof. □

Let $\mathscr{G}^3_n$ denote the set of triangle-free simple connected graphs with order n and diameter 3. Now we determine the extremal graph that attains the maximum spectral radius in $\mathscr{G}^3_n$.

Theorem 3.3. *For $n \geq 4$ and $G \in \mathscr{G}^3_n$,*

$$\rho(G) \leq \rho(K^-_{\lfloor \frac{n}{2}\rfloor,\lceil \frac{n}{2}\rceil}).$$

The equality holds if and only if $G \cong K^-_{\lfloor \frac{n}{2}\rfloor,\lceil \frac{n}{2}\rceil}$.

Proof. Let G be the extremal graph in $\mathscr{G}^3_n$ with maximal adjacency spectral radius $\rho(G)$. Assume that $x = (x_1, \ldots, x_n)^T$ is the Perron vector of G and u is a vertex with maximum eigenvector entry x_u. Since $G \in \mathscr{G}^3_n$, there exist two vertices $w_1, w_4 \in V(G)$ such that $d(w_1, w_4) = 3$. Furthermore, we can obtain the following result.

Claim 3.4. *Except for the vertex pair $\{w_1, w_4\}$, for any other vertex pair $\{u, v\}$, we have $d(u, v) \leq 2$ in G.*

Proof. We prove this by contradiction. Suppose that there exist a vertex pair $\{u, v\}$ such that $d(u, v) \geq 3$ and $\{u, v\} \neq \{w_1, w_4\}$. Since $G \in \mathscr{G}^3_n$, it is easy to see that $d(u, v) = 3$. Let G' be a graph obtained by adding an edge uv. Clearly, $G' \in \mathscr{G}^3_n$ and G is a subgraph of G'. Then $\rho(G') > \rho(G)$, a contradiction. □

Let $P = w_1w_2w_3w_4$ be the shortest path from w_1 to w_4. In the sequence, we will show that G is a bipartite graph obtained by deleting one edge from a complete bipartite graph.

If $V(G) = V(P)$, then G is an induced path of length 3 with diameter 3. Then $G \cong K^-_{2,2}$.

Now assume $n \geq 5$ and $V(G) \setminus V(P) \neq \emptyset$. Let $X = N_G(u) \setminus V(P)$ and $Y = V(G) \setminus (N_G(u) \cup V(P))$. Clearly, $X \cap Y = \emptyset$ and X is an independent set. Based on the fact that G is triangle-free, for any vertex $v \in V(G) \setminus V(P)$, it is easy to see that $|N_P(v)| \leq 2$ and $N_P(v)$ is an independent set. In particular, if $Y \neq \emptyset$, we can obtain the following claim.

Claim 3.5. *If $Y \neq \emptyset$, then $N_G(v) = N_G(u)$ for any vertex $v \in Y$.*

Proof. We prove this by contradiction. Suppose there exists a vertex $v' \in Y$ such that $N_G(v') \neq N_G(u)$. Since $x_u \geq x_{v'}$, it is easy to verify that $\sum_{t \in N_G(u)} x_t \geq \sum_{t \in N_G(v')} x_t$. Hence, $N_G(u) \setminus N_G(v') \neq \emptyset$. By Lemma 3.2, we have $\rho(Z_{v',u}(G)) \geq \rho(G)$. Notice that $Z_{v',u}(G) \in \mathscr{G}^3_n$. Let $s \in N_G(u) \setminus N_G(v')$. Then $\rho(G)x_s = \sum_{t \in N_G(s)} x_t < \sum_{t \in N_G(s) \cup \{v'\}} x_t \leq \rho(Z_{v',u}(G))x_s$. Hence $\rho(Z_{v',u}(G)) > \rho(G)$ which contradicts the choice of G. □

By Claim 3.5, we know that all vertices in Y share the same set of neighbors. Notice that the vertex subsets X and Y may be empty, thus we need to consider the following cases.

Case 3.6. *$Y = \emptyset$ or $X = \emptyset$.*

Firstly, we will prove that $G \cong K^-_{2,n-2}$ when $Y = \emptyset$ or $X = \emptyset$.

Suppose $Y = \emptyset$. We get $N_G(u) \cup V(P) = V(G)$, which indicates that $u \in V(P)$. By symmetry, suppose $u = w_i$ where $i = 1, 2$. The discussion for $i = 3$ and $i = 4$ is similar. Clearly, $X \neq \emptyset$ for $n \geq 5$. Let $S = \{w_i, w_{i+2}\}$ and $T = V(G) \setminus S$ for $i = 1, 2$. This means that $T = X \cup (V(P) \setminus S)$. By the choice of G, it is easy to verify that both S and T are independent sets. By Claim 3.4 and $u = w_i$, it is obvious that each vertex in S is adjacent to the vertices in X. In other words, $G[S \cup X] \cong K_{2,n-4}$. Clearly, $P \cong K^-_{2,2}$. Then G is a bipartite graph obtained by deleting an edge w_1w_4 from $K_{2,n-2}$, i.e., $G \cong K^-_{2,n-2}$.

Suppose $X = \emptyset$. Clearly, $Y \neq \emptyset$ for $n \geq 5$. We will show that $|N_P(v)| = 2$ for any vertex $v \in Y$. Otherwise, suppose that there exists a vertex $v' \in Y$ such that $|N_P(v')| \leq 1$. Since $X = \emptyset$ and G is a connected graph, it follows that $|N_P(v')| = 1$. By symmetry, suppose $N_P(v') = \{w_i\}$ where $i = 1, 2$. The discussion for $i = 3$ and $i = 4$ is similar. Now, $d(v', w_{i+2}) = 3$ which contradicts Claim 3.4. So $|N_P(v)| = 2$ for any vertex $v \in Y$. Let $S = \{w_i, w_{i+2}\}$ and $T = V(G) \setminus S$ for $i = 1, 2$. This means that $T = Y \cup (V(P) \setminus S)$. Notice that $G[S \cup Y] \cong K_{2,n-4}$ and $P \cong K^-_{2,2}$. Then G is a bipartite graph obtained by deleting an edge w_1w_4 from $K_{|S|,|T|}$, i.e., $G \cong K^-_{2,n-2}$.

Now we will show that $n = 5$ and $G \cong K^-_{2,3}$. Otherwise, suppose $n \geq 6$. Notice that $K^-_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil} \in \mathscr{G}^3_n$. By Lemma 3.1, we have $\rho(K^-_{2,n-2}) < \rho(K^-_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$, which contradicts the choice of G.

Case 3.7. *$Y \neq \emptyset$ and $X \neq \emptyset$.*

Let z be a vertex in X such that $x_z = \max\{x_v | v \in X\}$. Then we can obtain the following claim.

Claim 3.8. *For any vertex $v \in X$, we have $N_P(v) = N_P(z)$.*

Proof. Since G is triangle-free and X is an independent set, we have $\rho(G)x_v = \sum_{t \in Y} x_t + \sum_{t \in N_P(v)} x_t$ for any vertex $v \in X$. Suppose that there exists a vertex $v' \in X$ such that $N_P(v') \neq N_P(z)$. Since $x_z \geq x_{v'}$, it is easy to verify that $\sum_{t \in N_G(z)} x_t \geq \sum_{t \in N_G(v')} x_t$. Hence, we have that $N_P(z) \setminus N_P(v') \neq \emptyset$. By Lemma 3.2, $\rho(Z_{v',z}(G)) \geq \rho(G)$. Notice that $Z_{v',z}(G) \in \mathscr{G}^3_n$. Let $s \in N_P(z) \setminus N_P(v')$. Then $\rho(G)x_s = \sum_{t \in N_G(s)} x_t < \sum_{t \in N_G(s) \cup \{v'\}} x_t \leq \rho(Z_{v',z}(G))x_s$. Hence $\rho(Z_{v',z}(G)) > \rho(G)$ which contradicts the choice of G. □

By Claims 3.5 and 3.8, we can obtain that each vertex in Y has the same neighbors in P and each vertex in X has the same neighbors in P. Let $P_Y = N_P(v)$ for any vertex $v \in Y$ and $P_X = N_P(v)$ for any vertex $v \in X$. Clearly, $P_Y \cap P_X = \emptyset$, $|P_Y| \leq 2$, and $|P_X| \leq 2$.

Now, we will prove that $|P_X| = |P_Y| = 2$. Otherwise, suppose $|P_X| \leq 1$ or $|P_Y| \leq 1$. Then, by symmetry, there exists a vertex $w_i \in V(P)$ such that $w_i \in V(P) \setminus (P_X \cup P_Y)$, where $i = 1$ or $i = 2$. If $i = 1$, we will show that there is a vertex $v \in V(G) \setminus V(P)$ such that $d(w_1, v) = 3$, which contradicts Claim 3.4. Since G

is triangle-free, $w_2 \notin P_X$ or $w_2 \notin P_Y$. If $w_2 \notin P_X$, then $d(w_1, v) = 3$ for any vertex $v \in X$, a contradiction; else, $d(w_1, v) = 3$ for any vertex $v \in Y$, a contradiction. Hence, suppose $i = 2$. By Claim 3.4, we have that $w_1 \in P_X \cup P_Y$ and $w_3 \in P_X \cup P_Y$. Otherwise, by similar analysis as above, there is a vertex $v \in V(G) \setminus V(P)$ such that $d(w_2, v) = 3$. Without loss of generality, suppose $w_1 \in P_X$ and $w_3 \in P_Y$. Since G is triangle-free and $d(w_1, w_4) = 3$, $w_4 \notin P_X \cup P_Y$. Then $d(w_4, v) = 3$ for any vertex $v \in X$, which contradicts Claim 3.4.

Let $S = X \cup P_Y$ and $T = Y \cup P_X$. It is easy to see that $S \cap T = \emptyset$ and S, T are independent sets. Clearly, $G[X \cup T] \cong K_{|X|,|T|}$ and $G[Y \cup S] \cong K_{|Y|,|S|}$. Thus G is a bipartite graph obtained by deleting an edge w_1w_4 from $K_{|S|,|T|}$, i.e., $G \cong K^-_{|S|,|T|}$. By Lemma 3.1 and the choice of G, $G \cong K^-_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

This completes the proof of Theorem 3.3. □

4. Characterization of linear hypergraphs without Fan^k

In this section, we will use some useful tools and additional notation employed below according to [12].

Let $H = (V(H), E(H))$ be a connected simple linear k-uniform hypergraph on n vertices and m hyperedges. For a vertex v, let N_v be the neighborhood of v, i.e., $N_v = \{u \in V(H) \setminus \{v\} | v, u \in l \text{ for some } l \in E(H)\}$. For a set $X \subseteq V$, let $E_t(X) = \{l | l \in E(H) \text{ and } |l \cap X| = t\}$ and $e_t(X)$ be the number of hyperedges in $E_t(X)$, respectively.

Lemma 4.1 ([12]). *Let H be a connected simple linear k-uniform hypergraph and ρ be the spectral radius of the adjacency tensor of H. Let u be the vertex with maximum eigenvector entry. Then*

$$\rho^2 \le \frac{1}{k-1}[e_1(N_u) + 2e_2(N_u) + \cdots + ke_k(N_u)].$$

Theorem 4.2. *Let $spex_k^{lin}(m, Fan^k)$ be the maximum spectral radius in the set of Fan^k-free linear k-uniform hypergraphs with given size m. Then $spex_k^{lin}(m, Fan^k) = \sqrt{m}$.*

Proof. For $n \equiv 0 \mod (k)$, a transversal design $T(n, k)$ is a $\frac{n}{k}$-regular linear hypergraph without Fan^k with size $\frac{n^2}{k^2}$. Then $\rho(T(n, k)) = \frac{n}{k} = \sqrt{m}$. Therefore, $spex_k^{lin}(m, Fan^k) \ge \sqrt{m}$.

Let $H = (V, E)$ be a Fan^k-free linear k-uniform hypergraph on m hyperedges, and let $\rho(H)$ be the spectral radius of H. Set $B_u = V \setminus N_u$, where N_u is the neighborhood of vertex u. It is obvious that $E_k(N_u) = \emptyset$. In other words, $e_k(N_u) = 0$. Otherwise, if there exists a hyperedge $g \in E_k(N_u)$, the vertices of g must belong to k differ hyperedges containing u since H is linear. Denote these k hyperedges as $\{f_1, f_2, \ldots, f_k\}$. Then $\{g, f_1, f_2, \ldots, f_k\}$ is the hyperedge set of a Fan^k, a contradiction. By Lemma 4.1, we have

$$\begin{aligned}
\rho^2(H) &\le \frac{1}{k-1}[e_1(N_u) + 2e_2(N_u) + \cdots + (k-1)e_{k-1}(N_u)] \\
&\le \frac{1}{k-1}[(k-1)e_1(N_u) + (k-1)e_2(N_u) + \cdots + (k-1)e_{k-1}(N_u)] \\
&= e_1(N_u) + e_2(N_u) + \cdots + e_{k-1}(N_u) \\
&\le m,
\end{aligned}$$

which indicates $\rho(H) \le \sqrt{m}$. In other words, $spex_k^{lin}(m, Fan^k) \le \sqrt{m}$.

This completes the proof. □

5. Conclusions and Future Work

In this paper, we determine the maximum spectral radius and characterize the extremal graphs among K_3-free connected graphs with diameter 3. Furthermore, it is a natural problem to determine the maximum spectral radius in K_r-free connected graphs with a given diameter.

One of the most well-known problems in spectral graph theory is the Brualdi-Hoffman-Turán problem, which considers the maximum spectral radius in $\mathcal{F}$-free graphs or hypergraphs of size m. Motivated by the problem, the spectral Turán type problem in terms of the size has been widely studied. Nosal [15] showed that if $\rho(G) > \sqrt{m}$ then G contains a triangle. In 2009, Nikiforov [16] proved that G contains a quadrilateral unless G is a star if $\rho(G) \geq \sqrt{m}$ when G of size $m \geq 10$. In [2], Bai and Lu determined the maximum spectral radius among all k-uniform hypergraph H with size m. However, there are very few known results about Brualdi-Hoffman-Turán problem for hypergraphs. In particular, due to the complexity of hypergraph structures, we only consider the maximum spectral radius in Fan^k-free linear k-uniform hypergraphs with size m. It is also interesting to consider the Brualdi-Hoffman-Turán problem of some classical subhypergraphs, such as Berge cycles or linear cycles.

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