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Characterization and stability of multi-Euler-Lagrange-Jensen-cubic functional equations

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Abstract. The paper aims to investigate an alternative form of multi-Euler-Lagrange-cubic mappings. We provide a characterization of multi-Euler-Lagrange cubic mappings and multi-Euler-Lagrange-Jensen-cubic mappings by unifying the corresponding systems of equations into a single defining equation. We investigate the Hyers-Ulam stability for multi-Euler-Lagrange-Jensen- cubic mappings by applying a fixed-point method in Banach spaces. Also, we deduce several other results corresponding to well-known stability results, and provide a suitable counterexample to demonstrate a failure case of stability.

1. Introduction

In 1940, Ulam [26] posed the fundamental question concerning the stability of functional equations. In response, Hyers [12] provided the first affirmative solution to the Ulam problem in 1941 for Banach spaces. This result was later generalized by Rassias [21], and a further extension was proposed by Găvruţa [10], leading to what is now known as the generalized Hyers-Ulam-Rassias stability. Since then, the stability of various functional equations has been the subject of extensive research, resulting in significant developments in the field.

We recall that a functional equation $\mathfrak{F}$ is said to be stable if any mapping ϕ fulfilling $\mathfrak{F}$ approximately is near to an exact solution of $\mathfrak{F}$. Moreover, $\mathfrak{F}$ is called hyperstable if any function ϕ satisfying $\mathfrak{F}$ approximately is an exact solution of $\mathfrak{F}$. Significant examples of some functional equations include [1, 9, 15, 22]:

(i) The Cauchy equation:

$$A(x + y) = A(x) + A(y); \tag{1.1}$$

(ii) The quadratic equation:

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y); \tag{1.2}$$

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(iii) The cubic equation:

$$C(x+2y)+C(x-2y)=4C(x+y)+4C(x-y)-6C(x); \tag{1.3}$$

(iv) The Jensen equation

$$J\left(\frac{x+y}{2}\right)=\frac{J(x)+J(y)}{2}. \tag{1.4}$$

Jun and Kim introduced a cubic equation different from (1.3) as follows:

$$C(2x+y)+C(2x-y)=2C(x+y)+2C(x-y)+12C(x); \tag{1.5}$$

in [13].

The generalized case of the Jensen-type functional equation is given by

$$r\left[J\left(\frac{x+y}{r}\right)+J\left(\frac{x-y}{r}\right)\right]=2J(x), \tag{1.6}$$

where $r\in(0,\infty)$. The equation

$$\mathcal{J}\left(\frac{x+y}{2}\right)+\mathcal{J}\left(\frac{x-y}{2}\right)=\mathcal{J}(x), \tag{1.7}$$

is a special case of (1.6) when $r=2$, and we focus on it in Sections 2 and 3.

Throughout this paper, $\mathbb{N}$ and $\mathbb{Q}$ are the sets of all positive integers and rationals, respectively, $\mathbb{N}_0 := \mathbb{N}\cup 0$, $\mathbb{R}_+ := [0,\infty)$. Moreover, for the set X, we denote $\overbrace{X\times X\times\cdots\times X}^{n\text{-times}}$ by X^n.

Let V be a commutative group, W be a linear space over $\mathbb{Q}$ and $n\in\mathbb{N}$ with $n\geq 2$. A mapping $f: V^n\to W$ is called

(i) multi-additive, if it satisfies (1.1) in each variable [6];
(ii) multi-quadratic, if it satisfies (1.2) in each variable [8];
(iii) multi-cubic, if it satisfies (1.5) in each variable [4, 11].
(iv) multi-Jensen, if it satisfies (1.4) in each variable [18].

The multicubic mappings were first introduced by Ghaemi et al. in [11]. In [4], the authors investigated the structure of multicubic mappings and proved every multicubic functional equation can be stable and hyperstable.

Prager and Schwaiger [18] introduced the notion of multi-Jensen mappings with the connection with generalized polynomials and obtain their general form. The aim was to study the stability of the multi-Jensen equation. The multi-m-Jensen mappings for $m\geq 2$, along with their generalizations, have been investigated in [16] and [25]. For further results concerning the characterization and stability of multi-Jensen, multi-Cauchy-Jensen, multi-quadratic-Jensen, multi-Jensen-quartic, multi-mixed additive-quadratic Jensen type mappings, we refer to [2, 7, 17, 23, 24].

Rassias [19, 20] solved the stability problem of Ulam for the Euler–Lagrange type quadratic functional equation

$$f(rx+sy)+f(sx-ry)=(r^2+s^2)[f(x)+f(y)]$$

for fixed reals r,s with $r\neq 0,\ s\neq 0$.

In [14], Jun and Kim studied the generalized cubic functional equation

$$f(ax+by)+f(bx+ay)=(a+b)(a-b)^2[f(x)+f(y)]+ab(a+b)f(x+y) \tag{1.8}$$

for fixed integers a,b with $a\neq 0,\ b\neq 0$, and $a\pm b\neq 0$. Eq. (1.5) is called a Euler–Lagrange type cubic functional equation and its solution a Euler-Lagrange type cubic mapping.

Bodaghi and Sahami [3], worked on multi-Jensen and multi-Euler-Lagrange additive mappings. They unified the system of n equations defining each of the mentioned mappings as a single equation and investigated the Hyers-Ulam stability for the multi-Euler-Lagrange-Jensen mappings in the setting of Banach spaces.

In this work, we investigate an alternative form Euler-Lagrange type cubic functional equation inspired by equation (1.8):

$$f\left(\frac{ax+by}{2}\right)+f\left(\frac{bx+ay}{2}\right)=\frac{(a+b)(a-b)^2}{8}[f(x)+f(y)]+\frac{ab(a+b)}{8}f(x+y) \tag{1.9}$$

for fixed integers a, b with $a \neq 0,\ b \neq 0$, and $a \pm b \neq 0$.

Remark 1.1. *It is worth noting that, by taking a and b as even integers in equation (1.9), one can directly obtain equation (1.8).*

The rest of the article is organized as follows: In Section 2, we first recall the ideas of multi-Jensen and the multi-Euler-Lagrange cubic mappings. We describe the structure of such mappings and indeed we prove that every multi-Jensen and multi-Euler-Lagrange cubic mapping can be shown a single equation. Section 3 is devoted to the study of structure of multi-Euler-Lagrange-cubic-Jensen mappings. In other words, we reduce the system of n equations defining multi-Euler-Lagrange-Jensen- cubic mappings to obtain a single equation. In section 4, we investigate the Hyers-Ulam stability for multi-Euler-Lagrange-Jensen-cubic mappings and establish several related corollaries. In Section 5, we construct a counterexample to one of the corollaries established in Section 4 to demonstrate a failure case of stability.

2. Characterization of multi-Jensen and multi-Euler-Lagrange cubic mappings

Let V and W be real vector spaces. We observe that the functional equation (1.5) is equivalent to the functional equation (1.8) [14].

Proposition 2.1. *If a mapping $f : V \to W$ satisfies the functional equation (1.5), then f satisfies the functional equation (1.9).*

Let S be a subset of $\mathbb{R}$. For any $l \in \mathbb{N}_0,\ m \in \mathbb{N},\ t = (t_1, \ldots, t_m) \in S^m$ and $x = (x_1, \ldots, x_m) \in V^m$, we write $lx := (lx_1, \ldots, lx_m)$ and $tx := (t_1x_1, \ldots, t_mx_m)$, where lx stands, as usual, for the scalar product of l on x in the linear space V.Throughout the work, it is assumed that V and W are vector spaces over $\mathbb{R}$, $n \in \mathbb{N}$ with $n \geq 2$ and $x_i^n = (x_i1, \ldots, x_in) \in V^n$, where $i \in \{1, 2\}$. We denote x_i^n by x_i if there is risk of ambiguity.

2.1. *Multi-Jensen mappings*

Motivated by equation (1.7), we bring a new definition of multi-Jensen map pings as follows.

Definition 2.2. *A mapping $f : V^n \to W$ is called multi-Jensen if it satisfies Jensen equation (1.7) in each of its n arguments, that is*

$$f(v_1, \ldots, v_{i-1}, \frac{v_i+v_i'}{2}, v_{i+1}, \ldots, v_n) + f(v_1, \ldots, v_{i-1}, \frac{v_i-v_i'}{2}, v_{i+1}, \ldots, v_n) = f(v_1, \ldots, v_n).$$

Definition 2.3. *We say mapping $f : V^n \to W$ satisfies linear condition in the jth variable if*

$$f(z_1, \ldots, z_{i-1}, 2z_i, z_{i+1}, \ldots, z_n) = 2f(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n), \qquad (z_1, \ldots, z_n) \in V^n.$$

Theorem 2.4. *A mapping $f : V^n \to W$ is multi-Jensen if it satisfies the equation*

$$\sum_{s\in\{-1,1\}^n} f\left(\frac{x_1^n + sx_2^n}{2}\right) = f(x_1^n), \tag{2.1}$$

for all $x_1^n,\ x_2^n \in V^n$. Converse is true provided that f has linear condition in each variable.

Proof. Suppose that f is multi-Jensen mapping. We proceed this implication by induction on n. For $n = 1$, the result is trivial. Let us assume that (2.1) is true for some positive integer $n > 1$, that is,

$$\sum_{s\in\{-1,1\}^n} f\left(\frac{x_1^n + sx_2^n}{2}, z\right) = f(x_1^n, z), \tag{2.2}$$

for all $x_1^n,\ x_2^n \in V^n$ and $z \in V$. Then

$$\begin{aligned}\sum_{s\in\{-1,1\}^{n+1}} f\left(\frac{x_1^{n+1} + sx_2^{n+1}}{2}\right) &= \sum_{s\in\{-1,1\}^n}\sum_{t\in\{-1,1\}} f\left(\frac{x_1^n + sx_2^n}{2}, \frac{x_{1,n+1} + tx_{2,n+1}}{2}\right)\\ &= \sum_{s\in\{-1,1\}^n} f\left(\frac{x_1^n + sx_2^n}{2}, x_{1,n+1}\right)\\ &= f(x_1^{n+1})\end{aligned}$$

Hence, (2.1) holds for $n + 1$.

Conversely, let f satisfies (2.1). Fix $j = \{1, ..., n\}$, and $x_{2k} = 0$ for all $k \in \{1, ..., n\}\setminus\{j\}$. We get

$$\begin{aligned}&2^{n-1}\Big[f\left(\frac{x_{11}}{2}, ..., \frac{x_{1,j-1}}{2}, \frac{x_{1j} + x_{2j}}{2}, \frac{x_{1,j+1}}{2}, ..., \frac{x_{1n}}{2}\right)\\ &\qquad + f\left(\frac{x_{11}}{2}, ..., \frac{x_{1,j-1}}{2}, \frac{x_{1j} - x_{2j}}{2}, \frac{x_{1,j+1}}{2}, ..., \frac{x_{1n}}{2}\right)\Big]\\ &= f(x_{11}, ..., x_{1,j-1}, x_{1j}, x_{1,j+1} . . . , x_{1n})\end{aligned}$$

Or,

$$\begin{aligned}&f\left(x_{11}, ..., x_{1,j-1}, \frac{x_{1j} + x_{2j}}{2}, x_{1,j+1}, ..., x_{1n}\right) + f\left(x_{11}, ..., x_{1,j-1}, \frac{x_{1j} - x_{2j}}{2}, x_{1,j+1}, ..., x_{1n}\right)\\ &= f(x_{11}, ..., x_{1,j-1}, x_{1j}, x_{1,j+1}, ..., x_{1n})\end{aligned} \tag{2.3}$$

Relation (2.3) implies that f is Jensen in jth variable. As j is arbitrary, we obtain the required result. □

2.2. *Multi-Euler-Lagrange cubic mappings*

Definition 2.5. *A mapping $f : V^n \to W$ is called multi-Euler-Lagrange cubic if it satisfies Euler-Lagrange cubic equation (1.9) in each of their n arguments, namely*

$$\begin{aligned}&f\left(v_1, ..., v_{i-1}, \frac{a_iv_i + b_iv_i'}{2}, v_{i+1}, ..., v_n\right) + f\left(v_1, ..., v_{i-1}, \frac{b_iv_i + a_iv_i'}{2}, v_{i+1}, ..., v_n\right)\\ &= \frac{(a_i + b_i)(a_i - b_i)^2}{8}\left[f(v_1, ..., v_{i-1}, v_i, v_{i+1}, ..., v_n) + f(v_1, ..., v_{i-1}, v_i', v_{i+1}, ..., v_n)\right]\\ &+ \frac{a_ib_i(a_i + b_i)}{8} f\left(v_1, ..., v_{i-1}, v_i + v_i', v_{i+1}, ..., v_n\right)\end{aligned}$$

where $a_j,\ b_j \in \mathbb{R}\setminus\{0\}$ are fixed with $a_j \pm b_j \neq 0$.

Consider $a_i^n = (a_{i1}, ..., a_{in}) \in \mathbb{R}^n\setminus\{0, ..., 0\}$ such that $a_{1j} \pm a_{2j} \neq 0$ where $i \in \{1, 2\}$ and $j \in \{1, . . . , n\}$. For simplicity, we write a_i^n as a_i. For $x_1,\ x_2 \in V^n$ and $a_1,\ a_2$ as in the above, we consider the following notation

$$A_j = \sum_{i=1}^{2} \frac{a_{ij}}{2} x_{ij} \quad \text{and} \quad A_j' = \sum_{i=1}^{2} \frac{a_{3-i,j}}{2} x_{ij} \tag{2.4}$$

where $j \in \{1, ..., n\}$.We put,

$$\mathcal{M}^n = \left\{\mathfrak{M}_n = (M_1, \dots, M_n) | M_j \in \{x_{1j}, x_{2j}, x_{1j} + x_{2j}\}\right\},$$

where $j \in \{1, ..., n\}$. Consider the subset $\mathcal{M}^n_{(\sum_i r_i, r_1, ..., r_n)}$ as

$$\mathcal{M}^n_{(\sum_i r_i, r_1, ..., r_n)} := \left\{\mathfrak{M}_n \in \mathcal{M}^n \,\middle|\, \operatorname{card}\{x_{1j} + x2j\} = \sum_j r_j;\ r_j = \begin{cases} 1, & \text{if } x_{1j} + x_{2j} \text{ apears in the } j\text{th position} \\ 0, & \text{otherwise} \end{cases}\right\} \tag{2.5}$$

We claim that the equation

$$\sum_{\substack{\mathcal{A}_j \in \{A_j, A'_j\} \\ j \in \{1, ..., n\}}} f(\mathcal{A}_1, ..., \mathcal{A}_n) = \prod_{p,q} \frac{a_{1p} a_{2p} (a_{1p} + a_{2p})(a_{1q} + a_{2q})(a_{1q} - a_{2q})^2}{8^n} \sum_{\substack{r_i \in \{0,1\} \\ i \in \{1, ..., n\}}} f\left(\mathcal{M}^n_{(\sum_i r_i, r_1, ..., r_n)}\right) \tag{2.6}$$

where, $p = \{i \mid r_i = 1\}$ and $q = \{i \mid r_i = 0\}$.

Definition 2.6. *We say a mapping $f : V^n \to W$*

(i) has 3-power condition in the jth variable if

$$f(z_1, \dots, z_{i-1}, a z_i, z_{i+1}, \dots, z_n) = a^3 f(z_1, \dots, z_{i-1}, z_i, z_{i+1}, \dots, z_n),$$

for all $(z_1, \dots, z_n) \in V^n$, where $a \in \{a_{1j}, a_{2j}, a_{1j} + a_{2j}\}$.

(ii) has zero condition if $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.

Remark 2.7. *It is obvious that if a mapping $f : V^n \to W$ satisfies the 3-power condition in the jth variable then it has zero condition in the same variable. Thus, if f has 3-power condition in each variable, then it has zero condition.*

Theorem 2.8. *If a mapping $f : V^n \to W$ is multi-Euler-Lagrange cubic, then f satisfies equation (2.6). The converse is true provided that f has 3-power condition in each variable.*

Proof. Let us consider f is multi-Euler-Lagrange cubic. We proceed the proof by induction on n so that f satisfies equation (2.6). For $n = 1$, it is obvious that f satisfies (1.9). Let us assume that (2.6) is valid for some positive integer $n > 1$. Then

$$\sum_{\substack{\mathcal{A}_j \in \{A_j, A'_j\} \\ j \in \{1, ..., n+1\}}} f(\mathcal{A}_1, ..., \mathcal{A}_{n+1}) = \sum_{\substack{\mathcal{A}_j \in \{A_j, A'_j\} \\ j \in \{1, ..., n\}}} f(\mathcal{A}_1, ..., \mathcal{A}_n, A_{n+1}) + \sum_{\substack{\mathcal{A}_j \in \{A_j, A'_j\} \\ j \in \{1, ..., n\}}} f(\mathcal{A}_1, ..., \mathcal{A}_n, A'_{n+1})$$

$$= \frac{(a_{1,n+1} + a_{2,n+1})(a_{1,n+1} - a_{2,n+1})^2}{8} \sum_{\substack{\mathcal{A}_j \in \{A_j, A'_j\} \\ j \in \{1, ..., n\}}} [f(\mathcal{A}_1, ..., \mathcal{A}_n, x_{1,n+1}) + f(\mathcal{A}_1, ..., \mathcal{A}_n, x_{2,n+1})]$$

$$+ \frac{a_{1,n+1} a_{2,n+1} (a_{1,n+1} + a_{2,n+1})}{8} \sum_{\substack{\mathcal{A}_j \in \{A_j, A'_j\} \\ j \in \{1, ..., n\}}} f(\mathcal{A}_1, ..., \mathcal{A}_n, x_{1,n+1} + x_{2,n+1})$$

$$= \frac{(a_{1,n+1}+a_{2,n+1})(a_{1,n+1}-a_{2,n+1})^2}{8} \prod_{p,q} \frac{a_{1p}a_{2p}(a_{1p}+a_{2p})(a_{1q}+a_{2q})(a_{1q}-a_{2q})^2}{8^n}$$
$$\times \sum_{\substack{r_i\in\{0,1\}\\ i\in\{1,\dots,n\}}} \left[f\left(\mathcal{M}^n_{(\sum_i r_i, r_1,\dots,r_n)}, x_{1,n+1}\right) + f\left(\mathcal{M}^n_{(\sum_i r_i, r_1,\dots,r_n)}, x_{2,n+1}\right)\right]$$
$$+ \frac{a_{1,n+1}a_{2,n+1}(a_{1,n+1}+a_{2,n+1})}{8} \prod_{p,q} \frac{a_{1p}a_{2p}(a_{1p}+a_{2p})(a_{1q}+a_{2q})(a_{1q}-a_{2q})^2}{8^n}$$
$$\times \sum_{\substack{r_i\in\{0,1\}\\ i\in\{1,\dots,n\}}} f\left(\mathcal{M}^n_{(\sum_i r_i, r_1,\dots,r_n)}, x_{1,n+1}+x_{2,n+1}\right)$$
$$= \prod_{p,q} \frac{a_{1p}a_{2p}(a_{1p}+a_{2p})(a_{1q}+a_{2q})(a_{1q}-a_{2q})^2}{8^{n+1}} \sum_{\substack{r_i\in\{0,1\}\\ i\in\{1,\dots,n+1\}}} f\left(\mathcal{M}^{n+1}_{(\sum_i r_i, r_1,\dots,r_{n+1})}\right)$$

Conversely, let f satisfies (2.6). Fix $j \in \{1,\dots,n\}$. Putting $x_{2k} = 0$, for all $k \in \{1,\dots,n\}\setminus\{j\}$ in (2.6) and using remark 2.7, we have the left side of (2.6) as follows

$$\sum_{u_k\in\left\{\frac{a_{1k}}{2},\frac{a_{2k}}{2}\right\}} \left[f(u_1x_{11},\dots,A_j,\dots,u_nx_{1n}) + f(u_1x_{11},\dots,A'_j,\dots,u_nx_{1n})\right]$$
$$= \prod_{\substack{k=1\\ k\neq j}} \frac{a_{1k}^3+a_{2k}^3}{8^{n-1}} \left[f(x_{11},\dots,A_j,\dots,x_{1n}) + f(x_{11},\dots,A'_j,\dots,x_{1n})\right] \tag{2.7}$$

On the other hand, we have the right side of (2.6) as

$$= \prod_{\substack{k=1\\ k\neq j}} \frac{a_{1k}^3+a_{2k}^3}{8^{n-1}} \left[\frac{(a_{1j}+a_{2j})(a_{1j}-a_{2j})^2}{8}\left(f(x_{11},\dots,x_{1,j-1},x_{1j},x_{1,j+1},\dots,x_{1n})\right.\right.$$
$$\left.\left. +f(x_{11},\dots,x_{1,j-1},x_{2j},x_{1,j+1},\dots,x_{1n})\right) + \frac{a_{1j}a_{2j}(a_{1j}+a_{2j})}{8} f(x_{11},\dots,x_{1n})\right] \tag{2.8}$$

Now it follows from (2.7) and (2.8) that f is Euler-Lagrange cubic in the jth variable. Since j is arbitrary, we obtain the desired result. □

3. Characterization of multi-Euler-Lagrange-Jensen-Cubic mappings

Definition 3.1. *Let V and W be linear spaces, $n \in \mathbb{N}$ and $k \in \{0,\dots,n\}$. A mapping $f : V^n \to W$ is called k-Euler-Lagrange cubic and $n-k$-Jensen or, briefly, multi-Euler-Lagrange-Jensen-cubic if f is Euler-Lagrange cubic in each of some k variables and is Jensen in each of the other variables. In the above definition, for simplicity, we assume that f is Euler-Lagrange cubic in the first k variables. Let us note that for $k = n (k = 0)$ the above definition leads to the so-called multi-Euler-Lagrange cubic(multi-Jensen) mappings which are defined in the previous section.*

From now on, let V and W be vector spaces over $\mathbb{Q}$. Moreover, we identify $x := (x_1,\dots,x_n) \in V^n$ with $(x^k, x^{n-k}) \in V^k \times V^{n-k}$ where $x^k := (x_1,\dots,x_k)$ and $x^{n-k} := (x_{k+1},\dots,x_n)$. We use the convention that $(x^n, x^0) := x^n := (x^0, x^n)$. We put, $x_i^k = (x_{i1},\dots,x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1},\dots,x_{in}) \in V^{n-k}$, where $i \in \{1,2\}$.

In the following result, we reduce the system of n equations defining k-Euler-Lagrange cubic and $n-k$-Jensen mappings to obtain a single functional equation.

Theorem 3.2. *Let $n \in \mathbb{N}$ and $k \in \{0, ..., n\}$. If a mapping $f : V^n \to W$ is k-Euler-Lagrange cubic and $n-k$-Jensen mapping, then f satisfies the equation*

$$\sum_{\substack{\mathcal{A}_j \in \{A_j, A'_j\} \\ j \in \{1,...,k\}}} \sum_{s \in \{-1,1\}^{n-k}} f\left(\mathcal{A}_1, ..., \mathcal{A}_k, \frac{x_1^{n-k} + s x_2^{n-k}}{2}\right) = \prod_{p,q} \frac{a_{1p} a_{2p}(a_{1p} + a_{2p})(a_{1q} + a_{2q})(a_{1q} - a_{2q})^2}{8^k} \sum_{\substack{r_i \in \{0,1\} \\ i \in \{1,...,k\}}} f\left(\mathcal{M}^k_{(\sum_i r_i, r_1, ..., r_k)}, x_1^{n-k}\right) \tag{3.1}$$

for all x_1^{n-k}, $x_2^{n-k} \in V^{n-k}$, where $p = \{i \mid r_i = 1\}$, $q = i \mid r_i = 0$ and A_j, A'_j, $\mathcal{M}^k_{(\sum_i r_i, r_1, ..., r_k)}$ are defined in (2.4) and (2.5). The converse is true provided that f has 3-power condition in the first k variables and linear condition in the other variables.

Proof. For $k \in \{0, n\}$, our assertion follows from Theorem 2.4 and Theorem 2.8. We can assume that $k \in \{1, ..., n-1\}$. For any $x^{n-k} \in V^{n-k}$, define the mapping $g_{x^{n-k}} : V^k \to W$ by $g_{x^{n-k}}(x^k) := f(x^k, x^{n-k})$ for $x^k \in V^k$.

By assumption, $g_{x^{n-k}}$ is k-Euler-Lagrange cubic and hence by Theorem 2.8, we have

$$\sum_{\substack{\mathcal{A}_j \in \{A_j, A'_j\} \\ j \in \{1,...,k\}}} g_{x^{n-k}}(\mathcal{A}_1, ..., \mathcal{A}_k) = \prod_{p,q} \frac{a_{1p} a_{2p}(a_{1p} + a_{2p})(a_{1q} + a_{2q})(a_{1q} - a_{2q})^2}{8^k} \times \sum_{\substack{r_i \in \{0,1\} \\ i \in \{1,...,k\}}} g_{x^{n-k}}\left(\mathcal{M}^k_{(\sum_i r_i, r_1, ..., r_k)}\right)$$

It follows from the above equality that

$$\sum_{\substack{\mathcal{A}_j \in \{A_j, A'_j\} \\ j \in \{1,...,k\}}} f\left(\mathcal{A}_1, ..., \mathcal{A}_k, x^{n-k}\right) = \prod\nolimits_{p,q} \frac{a_{1p} a_{2p}(a_{1p} + a_{2p})(a_{1q} + a_{2q})(a_{1q} - a_{2q})^2}{8^k} \times \sum_{\substack{r_i \in \{0,1\} \\ i \in \{1,...,k\}}} f\left(\mathcal{M}^k_{(\sum_i r_i, r_1, ..., r_k)}, x^{n-k}\right) \tag{3.2}$$

for all $x^{n-k} \in V^{n-k}$. Similar to the above, for any $x^k \in V^k$. Consider the mapping $h_{x^k} : V^{n-k} \to W$ defined by $h_{x^k} := f(x^k, x^{n-k})$ for $x^{n-k} \in V^{n-k}$. By our assumption, h_{x^k} is $n-k$-Jensen. Hence Theorem 2.4 implies that

$$\sum_{s \in \{-1,1\}^{n-k}} h_{x^k}\left(\frac{x_1^{n-k} + s x_2^{n-k}}{2}\right) = h_{x^k}\left(x^{n-k}\right)$$

for all x_1^{n-k}, $x_2^{n-k} \in V^{n-k}$. By the definition of h_{x^k}, we have

$$\sum_{s \in \{-1,1\}^{n-k}} f\left(x^k, \frac{x_1^{n-k} + s x_2^{n-k}}{2}\right) = f\left(x^k, x^{n-k}\right) \tag{3.3}$$

For all x_1^{n-k}, $x_2^{n-k} \in V^{n-k}$ and $x^k \in V^k$. Substituting (3.2) into (3.3), we obtain

$$\sum_{\substack{\mathcal{A}_j \in \{A_j, A'_j\} \\ j \in \{1,\dots,k\}}} \sum_{s \in \{-1,1\}^{n-k}} f\left(\mathcal{A}_1, \dots, \mathcal{A}_k, \frac{x_1^{n-k} + s x_2^{n-k}}{2}\right)$$
$$= \prod_{p,q} \frac{a_{1p}a_{2p}(a_{1p}+a_{2p})(a_{1q}+a_{2q})(a_{1q}-a_{2q})^2}{8^k}$$
$$\times \sum_{s \in \{-1,1\}^{n-k}} \sum_{\substack{r_i \in \{0,1\} \\ i \in \{1,\dots,k\}}} f\left(\mathcal{M}^k_{(\sum_i r_i, r_1, \dots, r_k)}, \frac{x_1^{n-k} + s x_2^{n-k}}{2}\right)$$
$$= \prod_{p,q} \frac{a_{1p}a_{2p}(a_{1p}+a_{2p})(a_{1q}+a_{2q})(a_{1q}-a_{2q})^2}{8^k} \sum_{\substack{r_i \in \{0,1\} \\ i \in \{1,\dots,k\}}} f\left(\mathcal{M}^k_{(\sum_i r_i, r_1, \dots, r_k)}, x_1^{n-k}\right)$$

which proves that f satisfies the equation (3.1).

Conversely, by putting $x_2^{n-k} = (0, \dots, 0)$ in the left side of (3.1), we obtain

$$2^{n-k} \sum_{\substack{\mathcal{A}_j \in \{A_j, A'_j\} \\ j \in \{1,\dots,k\}}} f\left(\mathcal{A}_1, \dots, \mathcal{A}_k, \frac{x_1^{n-k}}{2}\right) = \sum_{\substack{\mathcal{A}_j \in \{A_j, A'_j\} \\ j \in \{1,\dots,k\}}} f\left(\mathcal{A}_1, \dots, \mathcal{A}_k, x_1^{n-k}\right)$$
$$= \prod_{p,q} \frac{a_{1p}a_{2p}(a_{1p}+a_{2p})(a_{1q}+a_{2q})(a_{1q}-a_{2q})^2}{8^k} \sum_{\substack{r_i \in \{0,1\} \\ i \in \{1,\dots,k\}}} f\left(\mathcal{M}^k_{(\sum_i r_i, r_1, \dots, r_k)}, x_1^{n-k}\right)$$

for all $x_1^{n-k} \in V^{n-k}$. Thus, in view of Theorem 2.8, f is Euler-Lagrange cubic in each of the first k variables. Moreover, by putting $x_1^k = x_2^k$ in (3.1), we get

$$2^k \prod_{j=1}^{k} \left(\frac{a_{ij} + a_{2j}}{2}\right)^3 \sum_{s \in \{-1,1\}^{n-k}} f\left(x_1^k, \frac{x_1^{n-k} + s x_2^{n-k}}{2}\right) = \frac{1}{2^{2k}} \prod_{j=1}^{k} (a_{1j} + a_{2j})^3 f(x_1^k, x_1^{n-k}),$$

for all $x_1^k \in V^k$ and x_1^{n-k}, $x_2^{n-k} \in V^{n-k}$. Hence the proof is complete by Theorem 2.4. □

4. Stability Analysis

For fixed $a,\ b \in \mathbb{R}_+$ where $a,\ b \neq 0$ and $a \pm b \neq 0$, substituting $a_{1j} = a$ and $a_{2j} = b$ into (3.1) for every $j \in \{1, \dots, k\}$, the equation transforms into

$$\sum_{t_1,\dots,t_k \in \{(a,b),(b,a)\}} \sum_{s \in \{-1,1\}^{n-k}} f\left(A_1^{t_1}, \dots, A_k^{t_k}, \frac{x_1^{n-k} + s x_2^{n-k}}{2}\right)$$
$$= \sum_{\substack{r_i \in \{0,1\} \\ i \in \{1,\dots,k\}}} \frac{[ab(a+b)]^{\sum r_i}[(a+b)(a-b)^2]^{k-\sum r_i}}{8^k} f\left(\mathcal{M}^k_{(\sum_i r_i, r_1, \dots, r_k)}, x_1^{n-k}\right) \tag{4.1}$$

where, $A_j^{(a,b)} = \frac{ax_{1j} + bx_{2j}}{2}$, $A_j^{(b,a)} = \frac{bx_{1j} + ax_{2j}}{2}$ and $x_i^{n-k} = (x_{i,k+1}, \dots, x_{in}) \in V^{n-k}$, where $i \in \{1, 2\}$ and $j \in \{1, \dots, k\}$.

In this section, we establish the generalized Hyers-Ulam stability of equation (4.1) using a known fixed point result in Banach space. Throughout, for two sets X and Y, the set of all mappings from X to Y is denoted Y^X. In the following, we present a fixed point theory result that plays a key role in achieving our objective in this paper [5].

Theorem 4.1. *Suppose that the following hypotheses hold.*

(H1) Y *is a Banach space,* X *is non-empty set,* $j \in \mathbb{N}$, $g_1, ..., g_j : X \to X$ *and* $L_1, ..., L_2 : X \to \mathbb{R}_+$*;*

(H2) $\mathcal{T} : Y^X \to Y^X$ *is an operator satisfying the inequality*

$$\|T\lambda(x) - T\mu(x)\| \leq \sum_{i=1}^{j} L_i(x)\|\lambda(g_i(x)) - \mu(g_i(x))\|, \quad \lambda,\ \mu \in Y^X.$$

(H3) $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ *is an operator defined through*

$$\Lambda\delta(x) := \sum_{i=1}^{j} L_i(x)\delta(g_i(x)), \qquad \delta \in \mathbb{R}_+^X, x \in X.$$

Moreover, the function $\theta : X \to \mathbb{R}_+$ *and the mapping* $\varphi : X \to Y$ *fulfill the following two conditions:*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \theta(x), \qquad \theta^*(x) = \sum_{l=0}^{\infty} \Lambda^l \theta(x), \quad (x \in X),$$

Then, there exists a unique fixed point Ψ *of* $\mathcal{T}$ *such that*

$$\|\varphi(x) - \psi(x)\| \leq \theta^*(x) \quad (x \in X),$$

and, $\Psi(x) = \lim_{l\to\infty} \mathcal{T}^l \varphi(x)$ *for all* $x \in X$.

From now on, for the mapping $f : V^n \to W$, we consider the difference operator $\mathcal{D}f : V^n \times V^n \to W$ by

$$\mathcal{D}f(x_1^n, x_2^n) := \sum_{t_1,...,t_k \in \{(a,b),(b,a)\}} \sum_{s\in\{-1,1\}^{n-k}} f\left(A_1^{t_1}, ..., A_k^{t_k}, \frac{x_1^{n-k} + s x_2^{n-k}}{2}\right)$$
$$- \sum_{\substack{r_i\in\{0,1\} \\ i\in\{1,...,k\}}} \frac{[ab(a+b)]^{\sum r_i}[(a+b)(a-b)^2]^{k-\sum r_i}}{8^k} f\left(\mathcal{M}^k_{(\sum_i r_i, r_1,...,r_k)}, x_1^{n-k}\right)$$

Theorem 4.2. *Let* V *be a linear space and* W *be a Banach space. Suppose that* $\varphi : V^n \times V^n \to \mathbb{R}_+$ *is a mapping which satisfies the relations*

$$\lim_{l\to\infty} \left(2^{n-k} m^{-k}\right)^l \sum_{t_1,...,t_l \in \{a_1,a_2\}^k} \varphi\left(\left(t_1...t_l x_1^k, \frac{x_1^{n-k}}{2^l}\right), \left(t_1...t_l x_2^k, \frac{x_2^{n-k}}{2^l}\right)\right) = 0 \tag{4.2}$$

for all $x_1^k,\ x_2^k \in V^k$ *and* $x_1^{n-k},\ x_2^{n-k} \in V^{n-k}$ *and*

$$\Phi(x) = m^{-k} \sum_{l=0}^{\infty} (2^{n-k} m^{-k})^l \sum_{t_1,...,t_l \in \{a_1,a_2\}^k} \varphi\left(\left(t_1...t_l x^k, \frac{x^{n-k}}{2^l}\right), 0\right) < \infty \tag{4.3}$$

for all $x = (x^k, x^{n-k}) \in V^n$*, where* $\frac{a}{2} = a_1$, $\frac{b}{2} = a_2$ *and* $m = \frac{a^3+b^3}{8}$*. Also assume that a mapping* $f : V^n \to W$ *satisfies the inequality*

$$\|\mathcal{D}f(x_1^k, x_1^{n-k}, x_2^k, x_2^{n-k})\| \leq \varphi(x_1^k, x_1^{n-k}, x_2^k, x_2^{n-k}), \tag{4.4}$$

for all $x_1^k,\ x_2^k \in V^k$ *and* $x_1^{n-k},\ x_2^{n-k} \in V^{n-k}$*. Then there exists a solution* $\mathcal{F} : V^n \to W$ *of* (4.1) *such that*

$$\|f(x) - \mathcal{F}(x)\| \leq \Phi(x), \tag{4.5}$$

for all $x \in V^n$*. Furthermore, if* $\mathcal{F}$ *has the 3-power condition in the first k-variables and linear condition in the other variables, then it is a k-Euler-Lagrange cubic and* $n - k$*-Jensen mapping.*

Proof. Putting $x_1^k = x^k$, $x_1^{n-k} = x^{n-k}$ and $(x_2^k, x_2^{n-k}) = (0,0) \in (V^k, V^{n-k})$ in (4.4), we have

$$\left\| 2^{n-k} \sum_{t_1,\ldots,t_k \in \{a,b\}} f\left(\mathcal{A}_1^{t_1}, \ldots, \mathcal{A}_k^{t_k}, \frac{x^{n-k}}{2}\right) - \left(\frac{a^3+b^3}{8}\right)^k f(x) \right\| \le \varphi(x,0) \tag{4.6}$$

where, $\mathcal{A}_j^a = \frac{a}{2}x_j$, $\mathcal{A}_j^b = \frac{b}{2}x_j$. Let $\frac{a}{2} = a_1$, $\frac{b}{2} = a_2$ and $\frac{a^3+b^3}{8} = m$, then (4.6) can be written as

$$\left\| 2^{n-k} \sum_{t_1,\ldots,t_k \in \{a_1,a_2\}} f\left(t_1x_1, \ldots, t_kx_k, \frac{x^{n-k}}{2}\right) - m^k f(x) \right\| \le \varphi(x,0),$$

or,

$$\left\| f(x) - 2^{n-k}m^{-k} \sum_{t \in \{a_1,a_2\}^k} f\left(tx^k, \frac{x^{n-k}}{2}\right) \right\| \le m^{-k}\varphi(x,0). \tag{4.7}$$

Set $\mathcal{T}\theta(x) = 2^{n-k}m^{-k} \sum_{t\in\{a_1,a_2\}^k} f\left(tx^k, \frac{x^{n-k}}{2}\right)$ and $\theta(x) = m^{-k}\varphi(x,0)$, then (4.7) can be rewritten as

$$\|f(x) - \mathcal{T}f(x)\| \le \theta(x) \qquad (x \in V^n).$$

Define, $\Lambda\eta(x) = 2^{n-k}m^{-k} \sum_{t\in\{a_1,a_2\}^k} \eta\left(tx^k, \frac{x^{n-k}}{2}\right)$ for all $\eta \in \mathbb{R}_+^{V^n}$. We observed that Λ has the form presents in (H3) with $X = V^n$, $g_i(x) = g_t(x) = \left(tx^k, \frac{x^{n-k}}{2}\right)$ and $L_i(x) = 2^{n-k}m^{-k}$ for all i and $x \in V^n$. For each λ, $\mu \in W^{V^n}$, we have

$$\begin{aligned}\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| &= \left\| 2^{n-k}m^{-k} \left[\sum_{t\in\{a_1,a_2\}^k} \left(\lambda\left(tx^k, \frac{x^{n-k}}{2}\right) - \mu\left(tx^k, \frac{x^{n-k}}{2}\right)\right)\right] \right\| \\ &\le 2^{n-k}m^{-k} \sum_{t\in\{a_1,a_2\}^k} \left\| \left(\lambda\left(tx^k, \frac{x^{n-k}}{2}\right) - \mu\left(tx^k, \frac{x^{n-k}}{2}\right)\right)\right\|.\end{aligned}$$

Thus, the hypothesis (H2) holds. Induction argument on l shows that

$$\Lambda^l\theta(x) = (2^{n-k}m^{-k})^l \sum_{t_1,\ldots,t_l \in \{a_1,a_2\}^k} \theta\left(t_1\ldots t_l x^k, \frac{x^{n-k}}{2^l}\right) \tag{4.8}$$

For $l = 0$, (4.8) is trivially valid. Further, assume that (4.8) holds for an $l \in \mathbb{N}_0$. Now

$$\begin{aligned}\Lambda^{l+1}\theta(x) &= \Lambda(\Lambda^l\theta(x)) \\ &= \Lambda\left((2^{n-k}m^{-k})^l \sum_{t_1,\ldots,t_l \in \{a_1,a_2\}^k} \theta\left(t_1\ldots t_l x^k, \frac{x^{n-k}}{2^l}\right)\right) \\ &= (2^{n-k}m^{-k})^l 2^{n-k}m^{-k} \sum_{t_{l+1} \in \{a_1,a_2\}^k} \sum_{t_1,\ldots,t_l \in \{a_1,a_2\}^k} \theta\left(t_1\ldots t_l t_{l+1} x^k, \frac{x^{n-k}}{2.2^l}\right) \\ &= (2^{n-k}m^{-k})^{l+1} \sum_{t_1,\ldots,t_{l+1} \in \{a_1,a_2\}^k} \theta\left(t_1\ldots t_{l+1} x^k, \frac{x^{n-k}}{2^{l+1}}\right).\end{aligned}$$

Hence, (4.8) is valid for any $l \in \mathbb{N}_0$. Now, it follows from (4.3) and (4.8) that all assumptions of theorem 4.1 are fulfilled. Hence, there exists a mapping $\mathcal{F} : V^n \to W$ such that

$$\mathcal{F}(x) = \lim_{l\to\infty} \mathcal{T}^l f(x) \qquad (x \in V^n),$$

and moreover (4.5) holds. Now we claim that

$$\|\mathcal{D}(\mathcal{T}^l f)(x_1^k, x_1^{n-k}, x_2^k, x_2^{n-k})\| \leq (2^{n-k}m^{-k})^l \sum_{t_1,\dots,t_l \in \{a_1,a_2\}^k} \varphi\left(\left(t_1 \dots t_l x_1^k, \frac{x_1^{n-k}}{2^l}\right), \left(t_1 \dots t_l x_2^k, \frac{x_2^{n-k}}{2^l}\right)\right) \tag{4.9}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. Clearly, (4.9) is valid for $l = 0$. Let us assume that (4.9) holds for an $l \in \mathbb{N}_0$. Then

$$\begin{aligned}
&\|\mathcal{D}(\mathcal{T}^{l+1} f)(x_1^k, x_1^{n-k}, x_2^k, x_2^{n-k})\| \\
&\leq 2^{n-k}m^{-k} \left\| \sum_{t_1 \in \{a_1,a_2\}^k} \mathcal{D}(\mathcal{T}^l f)\left(t_1 x_1^k, \frac{x_1^{n-k}}{2}, t_1 x_2^k, \frac{x_2^{n-k}}{2}\right) \right\| \\
&\leq (2^{n-k}m^{-k})^{l+1} \sum_{t_1 \in \{a_1,a_2\}^k} \sum_{t_2,\dots,t_{l+1} \in \{a_1,a_2\}^k} \varphi\left(\left(t_1 t_2 \dots t_l x_1^k, \frac{x_1^{n-k}}{2.2^l}\right), \left(t_1 t_2 \dots t_l x_2^k, \frac{x_2^{n-k}}{2.2^l}\right)\right) \\
&\leq (2^{n-k}m^{-k})^{l+1} \sum_{t_1,\dots,t_{l+1} \in \{a_1,a_2\}^k} \varphi\left(\left(t_1 \dots t_{l+1} x_1^k, \frac{x_1^{n-k}}{2^{l+1}}\right), \left(t_1 \dots t_{l+1} x_2^k, \frac{x_2^{n-k}}{2^{l+1}}\right)\right),
\end{aligned}$$

for all $x_1^k,\ x_2^k \in V^k$ and $x_1^{n-k},\ x_2^{n-k} \in V^{n-k}$. Taking $l \to \infty$ in (4.9) and using (4.2), we have

$$\mathcal{D}\mathcal{F}(x_1^k, x_1^{n-k}, x_2^k, x_2^{n-k}) = 0$$

for all $x_1^k,\ x_2^k \in V^k$ and $x_1^{n-k},\ x_2^{n-k} \in V^{n-k}$. Hence (4.1) holds for $\mathcal{F}$. If $\mathcal{F}$ has the 3-power condition in the first k variables and linear condition in the other variables, then it is a k-Euler-Lagrange cubic and $n-k$-Jensen mapping by Theorem 3.2 and therefor the proof is complete. □

Corollary 4.3. *Given $\delta > 0$. Let V be a normed space and W be a Banach space. If $f : V^n \to W$ is a mapping satisfying the inequality*

$$\|\mathcal{D}f(x_1^k, x_1^{n-k}, x_2^k, x_2^{n-k})\| \leq \delta,$$

for all $x_1^k,\ x_2^k \in V^k$ and $x_1^{n-k},\ x_2^{n-k} \in V^{n-k}$, then there exists a solution $\mathcal{F} : V^n \to W$ of (4.1) such that

$$\|f(x) - \mathcal{F}(x)\| \leq \frac{1}{m^k - 2^n}\delta,$$

for all $x \in V^n$.

Proof. Let us consider the constant function, $\varphi(x_1^k, x_1^{n-k}, x_2^k, x_2^{n-k}) = \delta$ for all $x_1^k, x_2^k \in V^k$ and $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$ and applying Theorem 4.2, we obtain

$$\begin{aligned}
\Phi(x) &= m^{-k} \sum_{l=0}^{\infty} (2^{n-k}m^{-k})^l \sum_{t_1,\dots,t_l \in \{a_1,a_2\}^k} \varphi\left(\left(t_1 \dots t_l x^k, \frac{x^{n-k}}{2^l}\right), 0\right) \\
&= m^{-k} \sum_{l=0}^{\infty} (2^{n-k}m^{-k})^l 2^{kl} \delta \\
&= m^{-k}\delta \sum_{l=0}^{\infty} (2^n m^{-k})^l \\
&= m^{-k} \frac{m^k}{m^k - 2^n} \\
&= \frac{1}{m^k - 2^n}
\end{aligned}$$

for all $x = (x^k, x^{n-k}) \in V^n$. □

A special case of (4.1) can be obtained by putting $k = n$.

$$\sum_{t_1,\dots,t_n\in\{(a,b),(b,a)\}} f\left(A_1^{t_1},\dots,A_n^{t_n}\right) = \sum_{\substack{r_i\in\{0,1\}\\ i\in\{1,\dots,n\}}} \frac{[ab(a+b)]^{\sum r_i}[(a+b)(a-b)^2]^{n-\sum r_i}}{8^n} f\left(\mathcal{M}^n_{(\sum_i r_i, r_1,\dots,r_n)}\right) \tag{4.10}$$

Let us consider,

$$\begin{aligned}\mathcal{D}_c f\left(x_1^n, x_2^n\right) := &\sum_{t_1,\dots,t_n\in\{(a,b),(b,a)\}} f\left(A_1^{t_1},\dots,A_n^{t_n}\right)\\ &- \sum_{\substack{r_i\in\{0,1\}\\ i\in\{1,\dots,n\}}} \frac{[ab(a+b)]^{\sum r_i}[(a+b)(a-b)^2]^{n-\sum r_i}}{8^n} f\left(\mathcal{M}^n_{(\sum_i r_i, r_1,\dots,r_n)}\right)\end{aligned} \tag{4.11}$$

Putting $k = n$ in Theorem 4.2, we obtain the below result on the Hyers stability of multi-Euler-Lagrange cubic equation (4.10).

Corollary 4.4. *Let V be a linear space and W be a Banach space. Suppose that $\varphi : V^n \times V^n \to \mathbb{R}_+$ is a mapping which satisfies the relations*

$$\lim_{l\to\infty} (m^{-n})^l \sum_{t_1,\dots,t_l\in\{a_1,a_2\}^n} \varphi\left(\left(t_1\dots t_l x_1^n\right), \left(t_1\dots t_l x_2^n\right)\right) = 0$$

for all x_1^n, $x_2^n \in V^n$ and

$$\Phi(x) = m^{-n} \sum_{l=0}^{\infty} (m^{-n})^l \sum_{t_1,\dots,t_l\in\{a_1,a_2\}^n} \varphi\left(\left(t_1\dots t_l x^n\right), 0\right) < \infty$$

for all $x = x^n \in V^n$, where $\frac{a}{2} = a_1$, $\frac{b}{2} = a_2$ and $m = \frac{a^3+b^3}{8}$. Also assume that a mapping $f : V^n \to W$ satisfies the inequality

$$\|\mathcal{D} f(x_1^n, x_2^n)\| \le \varphi(x_1^n, x_2^n),$$

for all x_1^n, $x_2^n \in V^n$. Then there exists a solution $C : V^n \to W$ of (4.10) such that

$$\|f(x) - C(x)\| \le \Phi(x),$$

for all $x \in V^n$. Furthermore, if C has the 3-power condition in each variable, then it is a multi-Euler-Lagrange cubic mapping.

Putting $n = 1$, we get the above stability results for the Euler-Lagrange cubic mapping (1.9). In particular, we take the control function $\varphi(x, y)$ as $|x|^\alpha + |y|^\alpha$, where $\varphi : V \times V \to \mathbb{R}_+$ and $\alpha > 0$. Also, define $\mathcal{D}_{(a,b)} f : V \times V \to W$ as

$$\mathcal{D}_{(a,b)} f(x, y) := f\left(\frac{ax+by}{2}\right) + f\left(\frac{bx+ay}{2}\right) - \frac{(a+b)(a-b)^2}{8}[f(x) + f(y)] - \frac{ab(a+b)}{8} f(x+y)$$

Corollary 4.5. *Let $\alpha > 0$ with $\alpha \neq 3$. Also let V be a normed space and W be a Banach space. Suppose that $f : V \to W$ is a mapping satisfying the inequality*

$$\|\mathcal{D}_{(a,b)} f(x, y)\| \le \|x\|^\alpha + \|y\|^\alpha,$$

for all x, $y \in V$, then there exists a solution $C : V \to W$ of (1.9) such that

$$\|f(x) - C(x)\| \le \frac{8}{a^3+b^3} \sum_{l=0}^{\infty} 2^{(3-\alpha)l} \left(\frac{a^\alpha + b^\alpha}{a^3+b^3}\right)^l \|x\|^\alpha.$$

5. Non stability

We will now provide an example to show that the functional equation (1.9) is not stable for $\alpha = 3$ in corollary 4.5.

Example 5.1. *Let $\delta > 0$ and put $\mu = \frac{7}{2^6 k}\delta$, where $k = \left[\frac{16-(a+b)\{2(a-b)^2-ab\}}{8}\right]$ with $|a| + |b| < 1$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\varphi(x) = \begin{cases} \mu x^3, & \text{if } |x| < 1, \\ \mu, & \text{otherwise} \end{cases}$$

Also, define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{l=0}^{\infty} \frac{\varphi(2^l x)}{2^{3l}},$$

for all $x \in \mathbb{R}$. Clearly, φ is bounded by μ. Also, for each $x \in \mathbb{R}$, we have

$$|f(x)| \le \sum_{l=0}^{\infty} \left|\frac{\varphi(2^l x)}{2^{3l}}\right| \le \mu \sum_{l=0}^{\infty} \frac{1}{2^{3l}} = \frac{8}{7}\mu.$$

We claim that

$$|\mathcal{D}_{(a,b)} f(x, y)| \le \delta\left(|x^3| + |y^3|\right) \tag{5.1}$$

for all $x,\ y \in \mathbb{R}$. It is clear that (5.1) holds for $x = y = 0$. Now let us suppose that

$$0 < |x^3| + |y^3| < \frac{1}{2^3}.$$

Then there exists a positive integer N such that

$$\frac{1}{2^{3(N+1)}} \le |x^3| + |y^3| < \frac{1}{2^{3N}} \tag{5.2}$$

Thus, we obtain $2^{3N}|x^3|,\ 2^{3N}|y^3| < 1$, or $2^{3(N-1)}|x^3|,\ 2^{3(N-1)}|y^3| < \frac{1}{2^3}$, so that $2^{N-1}|x|,\ 2^{N-1}|y| < \frac{1}{2}$. Now,

$$2^{N-1}|x + y| \le 2^{N-1}|x| + 2^{N-1}|y| < 1,$$

$$2^{N-1}\left|\frac{ax + by}{2}\right| \le 2^{N-1}|ax| + 2^{N-1}|ay| \le |a| + |b| < 1,$$

Consequently,

$$2^{N-1}\left|\frac{bx + ay}{2}\right| < 1.$$

Thus, for each $l = 0, 1, ..., N - 1$, we have

$$2^l\left(\frac{ax + by}{2}\right),\ 2^l\left(\frac{bx + ay}{2}\right),\ 2^l x,\ 2^l y,\ 2^l(x + y) \in (-1, 1).$$

As for $|x| < 1$, φ is cubic, we have

$$\frac{1}{2^{3l}}\left[\varphi\left(2^l\frac{ax + by}{2}\right) + \left(2^l\frac{bx + ay}{2}\right) - \frac{(a + b)(a - b)^2}{8}\left(\varphi(2^l x) + \varphi(2^l y)\right) - \frac{ab(a + b)}{8}\varphi(2^l(x + y))\right] = 0,$$

for $l = 0, 1, ..., N-1$. From the definition of f and (5.2), we have

$$\frac{|\mathcal{D}_{(a,b)}f(x,y)|}{|x|^3+|y|^3} \leq \sum_{l=N}^{\infty} \frac{\mathcal{D}_{(a,b)}\varphi(2^l x, 2^l y)}{2^{3l}(|x|^3+|y|^3)}$$
$$= \frac{\mu}{|x|^3+|y^3|}\left[\frac{16-(a+b)\left(2(a-b)^2-ab\right)}{8}\right]\sum_{l=0}^{\infty}\frac{1}{2^{3(l+N)}}$$
$$= \frac{\mu k}{2^{3N}}2^{3(N+1)}\frac{8}{7} = \frac{2^6 k}{7}\mu = \delta$$

Thus, f satisfies (5.1) for all $x, y \in \mathbb{R}$ with $0 \leq |x|^3+|y|^3 < \frac{1}{2^3}$. If $|x|^3+|y|^3 \geq \frac{1}{2^3}$, then

$$\frac{|\mathcal{D}_{(a,b)}f(x,y)|}{|x|^3+|y|^3} \leq 2^3\frac{8}{7}\left[\frac{16-(a+b)\left(2(a-b)^2-ab\right)}{8}\right] = \frac{2^6 k}{7}\mu = \delta$$

Thus, f satisfies (5.1) for all $x, y \in \mathbb{R}$ with $|x|^3+|y|^3 \geq \frac{1}{2^3}$. Now, we claim that the cubic functional equation (1.9) is not stable for $\alpha = 3$ in corollary 4.5.

Suppose that there exists a mapping $C : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying

$$|f(x) - C(x)| \leq \beta|x|^3,$$

for all $x \in \mathbb{R}$. As C is cubic, it may be of the form $C(x) = cx^3$, where c is a constant. Thus, we obtain

$$|f(x)| = |f(x) - C(x) + C(x)| \leq \beta|x|^3 + |c||x|^3 = (\beta + |c|)|x|^3. \tag{5.3}$$

But we can choose a positive integer m such that $m\mu > \beta + |c|$. Let $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then we get $2^l x \in (0, 1)$ for all $l = 0, 1, ..., m-1$. Hence, we obtain

$$f(x) = \sum_{l=0}^{\infty}\frac{\varphi(2^l x)}{2^{3l}} \geq \sum_{l=0}^{m-1}\frac{\mu(2^l x)^3}{2^{3l}} = m\mu x^3 > (\beta + |c|)x^3,$$

which is a contradiction to (5.3).

Conclusion

In this paper, we explored an alternative formulation of multi-Euler-Lagrange-cubic mappings by unifying the systems of equations that define multi-Euler-Lagrange cubic and multi-Euler-Lagrange - Jensen-cubic mappings into single, equivalent equations. We provided a detailed structural analysis of these mappings and demonstrated that both the multi-Jensen and multi-Euler-Lagrange cubic mappings can be characterized by a single equation. Furthermore, we investigated the Hyers-Ulam stability of multi-Euler-Lagrange-Jensen-cubic mappings using a fixed-point approach in Banach spaces. Several corollaries related to known stability results were derived, and a counterexample was constructed to illustrate a specific case where stability fails. These findings are expected to contribute to a deeper understanding of the algebraic structure and stability behaviour of advanced functional equations.

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