



## New insights and applications of survival cumulative Sharma–Taneja–Mittal entropy

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**Abstract.** The literature has introduced several extensions of Shannon function entropy, including the Sharma–Taneja–Mittal entropy and its extensions. This paper presents new findings related to the survival cumulative Sharma–Taneja–Mittal entropy measure, including bounds, convergence, equivalent expressions, normalized survival cumulative Sharma–Taneja–Mittal entropy, its relationship with differential entropy, stochastic comparisons, and the excess wealth transform involving this measure. Additionally, the paper addresses the challenge of estimating the survival cumulative Sharma–Taneja–Mittal entropy using the empirical cumulative distribution function. Moreover, this entropy measure is estimated using two distinct empirical estimators of the cumulative distribution function. In addition, the measure is used to assess test uniformity, yielding an approximation of the distribution of the test statistic as well as the derivation of the limit distribution. The study also contains a simulation study to evaluate the power of the suggested test with other uniformity tests, and it covers the percentage points and power versus seven different distributions for this test statistic.

### 1. Introduction

Through information measures, one can evaluate the uncertainty associated with a random variable. With Shannon's introduction of the idea of entropy, or differential entropy, by [17], the foundational work in information theory was started. The survival cumulative entropy, as reported with Rao et al. [14], is supplied as a function of the cumulative distribution function (PDF)  $F$  and a non-negative random variable  $Y$  in the form

$$SCn(Y) = - \int_0^{\infty} \bar{F}(y) \log \bar{F}(y) dy, \quad (1)$$

taking into consideration that  $\bar{F}(y) = 1 - F(y)$ . Numerous extensions of Shannon function of entropy have been proposed in the literature, incorporating additional parameters to make these measures responsive to

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various configurations of probability distributions and characteristics.

A significant extension of Shannon measure of entropy, introduced by Tsallis [21], is known as the  $\delta$ -order generalized Tsallis entropy measure. This generalized Tsallis entropy of order  $\delta$  for a continuous random variable  $Y$  with probability density function (PDF)  $f$  is presented as

$$Tsn_{\delta}(Y) = \frac{1}{\delta - 1} \left( 1 - \int_0^{\infty} f^{\delta}(y) dy \right), \delta \neq 1, \quad (2)$$

when  $\delta$  tends to 1, then the Tsallis entropy of order  $\delta$  turns to Shannon entropy.

Numerous modifications and extensions have been introduced for  $Tsn_{\delta}$ . Order  $\delta$  for the survival cumulative Tsallis entropy was proposed by Sati and Gupta [15], while Rajesh and Sunoj [13] adapted this concept to realize the  $\delta$ -order survival cumulative Tsallis entropy as follows

$$SCTn_{\delta}(Y) = \frac{1}{\delta - 1} \left( \int_0^{\infty} (\bar{F}(y) - \bar{F}^{\delta}(y)) dy \right), \delta > 0, \delta \neq 1. \quad (3)$$

Furthermore, Asadi et al. [2] specifically studied the proportional hazards (PHz) representative to determine the survival cumulative Tsallis entropy, connecting the Bayes risks of the equilibrium distribution's Fisher information and the mean excess of a random variable with the PHz distribution. Although various studies have explored this measure, there is still room for further enhancement in the results related to it.

In the subsequent process, Sharma, Taneja [18], and Mittal [11] separately introduced the Sharma-Taneja-Mittal entropy.

$$SMn_{\delta_1, \delta_2}(Y) = \frac{1}{\delta_1 - \delta_2} \int_0^{\infty} (f^{\delta_1}(y) - f^{\delta_2}(y)) dy, \quad (4)$$

where  $\delta_1 \neq \delta_2 > 0$ . Furthermore, if  $\delta_1 = 1$ , then  $SMn_{1, \delta_2}(Y) = -Tsn_{\delta_2}(Y)$ , which is evident that the negative Tsallis entropy can be seen as extending to the Sharma-Taneja-Mittal entropy model. The survival cumulative and cumulative entropies for Sharma-Taneja-Mittal were recently introduced by Kattumannil et al. [9] as

$$SCSMn_{\delta_1, \delta_2}(Y) = \frac{1}{\delta_1 - \delta_2} \int_0^{\infty} (\bar{F}^{\delta_1}(y) - \bar{F}^{\delta_2}(y)) dy, \quad (5)$$

$$CSMn_{\delta_1, \delta_2}(Y) = \frac{1}{\delta_1 - \delta_2} \int_0^{\infty} (F^{\delta_1}(y) - F^{\delta_2}(y)) dy, \quad (6)$$

where  $\delta_1 \neq \delta_2 > 0$ .

The purpose of this paper is to explore additional findings related to the survival cumulative Sharma-Taneja-Mittal entropy measure. To this end, we begin by examining the connection between risk Bayesian formula of the random variable mean excess and the PH representative. Following this, we go through additional ramifications for the survival cumulative Sharma-Taneja-Mittal entropy measure, including aspects such as stochastic ordering, expressions, bounds, convergence, and further relevant results. Moreover, the suggested measure is also presented in a normalized form. Additionally, a precise formula linking the survival cumulative Sharma-Taneja-Mittal entropy measure with Shannon differential entropy is presented. The relationship between the excess wealth order and the measure is also investigated. Lastly, we address the estimation of the measure using the empirical CDF, proposing two different empirical estimators for this purpose and applying the estimator in a test of uniformity.

The remaining portions of this work are organized as follows: The bounds and convergence of the surviving Sharma-Taneja-Mittal entropy are discussed in Section 2. It also presents some initial findings related to linking risk Bayesian formula of the random variable mean excess and the PH representative. Additionally, a normalized version and excess wealth transform of the survival Sharma-Taneja-Mittal entropy are introduced, accompanied by illustrative examples. In Section 3, the survival Sharma-Taneja-Mittal entropy is estimated using two distinct empirical estimators of the CDF. A central limit theorem for the empirical model based on random samples from an exponential distribution is derived for the first estimator. A central limit theorem for the empirical model based on random samples from an unknown distribution is given for the second estimator. Finally, the obtained estimator is applied in the uniformity test in Section 4.

**2. Survival cumulative Sharma–Taneja–Mittal entropy properties**

Several aspects of survival Sharma-Taneja-Mittal entropy will be covered in this section. Including the sign, bounds, convergence, and relating to other uncertainty measures of the measure.

**Proposition 2.1.** *The survival Sharma–Taneja–Mittal entropy is non-positive.*

*Proof.* From (5), since  $\bar{F}(y) \in [0, 1]$ , we have the following cases:

- 1. If  $\delta_1 > \delta_2$ , then  $\bar{F}^{\delta_1}(y) < \bar{F}^{\delta_2}(y)$ , and we have

$$\frac{1}{\delta_1 - \delta_2} \int_0^\infty (\bar{F}^{\delta_1}(y) - \bar{F}^{\delta_2}(y))dy < 0.$$

- 2. If  $\delta_1 < \delta_2$ , then  $\bar{F}^{\delta_1}(y) > \bar{F}^{\delta_2}(y)$ , and we have

$$\frac{1}{\delta_1 - \delta_2} \int_0^\infty (\bar{F}^{\delta_1}(y) - \bar{F}^{\delta_2}(y))dy < 0.$$

For all  $\delta_1 \& \delta_2 > 0$ . □

The following remark discusses the bounds of the survival Sharma-Taneja-Mittal entropy.

**Remark 2.2.** *We can figure out that the function*

$$S(Y; \delta_1, \delta_2) = \frac{1}{\delta_1 - \delta_2} (y^{\delta_1} - y^{\delta_2}), \tag{7}$$

*has its minimum value*

$$P(\delta_1, \delta_2) = \frac{1}{\delta_1 - \delta_2} \left( \left( \frac{\delta_2}{\delta_1} \right)^{\frac{\delta_1}{\delta_1 - \delta_2}} - \left( \frac{\delta_2}{\delta_1} \right)^{\frac{\delta_2}{\delta_1 - \delta_2}} \right), \tag{8}$$

*at  $\left( \frac{\delta_2}{\delta_1} \right)^{\frac{1}{\delta_1 - \delta_2}}$ . Consequently, we obtain*

$$P(\delta_1, \delta_2) \leq S(Y; \delta_1, \delta_2) \leq 0, \tag{9}$$

*for all  $0 \leq y \leq 1$ , and  $\delta_1 \neq \delta_2 \geq 0$ .*

**Example 2.3.** *Let Y follows a survival function characterized by an exponential distribution by  $\bar{F}(y) = e^{-\rho y}$ ,  $y \geq 0$ . Then, from (5), we get*

$$SCSMn_{\delta_1, \delta_2}(Y) = \frac{-1}{\rho \delta_1 \delta_2}.$$

*Moreover, the survival cumulative Sharma-Taneja-Mittal entropy for various values of  $\delta_1$  and  $\delta_2$  is displayed in Figure 1.*

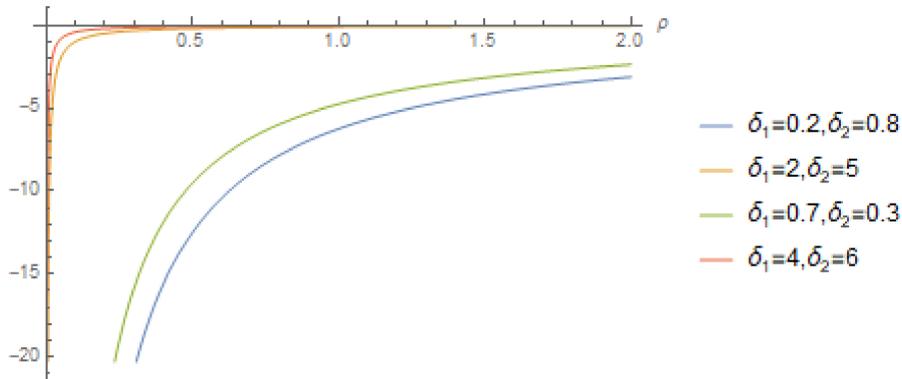


Figure 1: Plot of  $SCSMn_{\delta_1, \delta_2}(Y)$  of exponential distribution for different values of  $\delta_1$  and  $\delta_2$ .

**Theorem 2.4.** Suppose that the random vector  $Y = (Y_1, Y_2, \dots, Y_n)$  in  $\mathbb{R}^n$ . Then, from (5), and under the conditions  $1 \leq \delta_1 < \delta_2$ , we get: for few  $w > \delta_1, \delta_2, Y_l \in \mathbb{L}^w, E[|Y_l|^w] < \infty, 1 \leq l \leq n$ . Therefore,  $SCSMn_{\delta_1, \delta_2}(Y) > -\infty$ .

*Proof.* given  $0 \leq y \leq 1, 1 \leq \delta_1 < \delta_2$ , and by using the function (7) and the relation (9). We want to prove the inequality

$$0 \geq S(Y; \delta_1, \delta_2) \geq P(\delta_1, \delta_2) \frac{y^\eta}{(1-\eta)}, \tag{10}$$

where  $\eta = \frac{\delta_1}{\delta_2}$  (note here that we choose  $\eta$  such that its range will cover from 0 to 1). In the sequel, we have two cases in the inequality (10):

1. If  $\frac{y^\eta}{(1-\eta)} \leq 1$ , the inequality holds.
2. If  $\frac{y^\eta}{(1-\eta)} > 1$ , then, we obtain  $y > (1-\eta)^{\frac{1}{\eta}}$ . Moreover, using the relation (9), we can see that the ratio

$$\begin{aligned} \frac{S(Y; \delta_1, \delta_2)}{P(\delta_1, \delta_2)} \frac{(1-\eta)}{y^\delta} &= \frac{(1-\eta)}{P(\delta_1, \delta_2)(\delta_1 - \delta_2)} (y^{\delta_1 - \eta} - y^{\delta_2 - \eta}) \\ &\geq \frac{1}{P(\delta_1, \delta_2)(\delta_1 - \delta_2)} \left( (1-\eta)^{\frac{\delta_1}{\eta}} - (1-\eta)^{\frac{\delta_2}{\eta}} \right) \\ &\geq \frac{1}{P(\delta_1, \delta_2)} P(\delta_1, \delta_2) = 1. \end{aligned} \tag{11}$$

Noting that, the final expression in (11) can be derived from equation (9) as

$$\frac{1}{(\delta_1 - \delta_2)} \left( (1-\eta)^{\frac{\delta_1}{\eta}} - (1-\eta)^{\frac{\delta_2}{\eta}} \right) \geq P(\delta_1, \delta_2).$$

Moreover, the behavior of the functions is depicted in Figure 2. Subsequently, for every  $1 \leq \delta_1 < \delta_2$ , utilizing the inequality (10), we obtain

$$\begin{aligned} \frac{1}{\delta_1 - \delta_2} \left[ P[|Y_l| > y_l, 1 \leq l \leq n]^{\delta_1} - P[|Y_l| > y_l, 1 \leq l \leq n]^{\delta_2} \right] &\geq P(\delta_1, \delta_2) \frac{P[|Y_l| > y_l, 1 \leq l \leq n]^\eta}{(1-\eta)} \\ &\geq \frac{P(\delta_1, \delta_2)}{(1-\eta)} \prod_{l=1}^n \bar{F}_{|Y_l|}^\eta(y_l). \end{aligned} \tag{12}$$

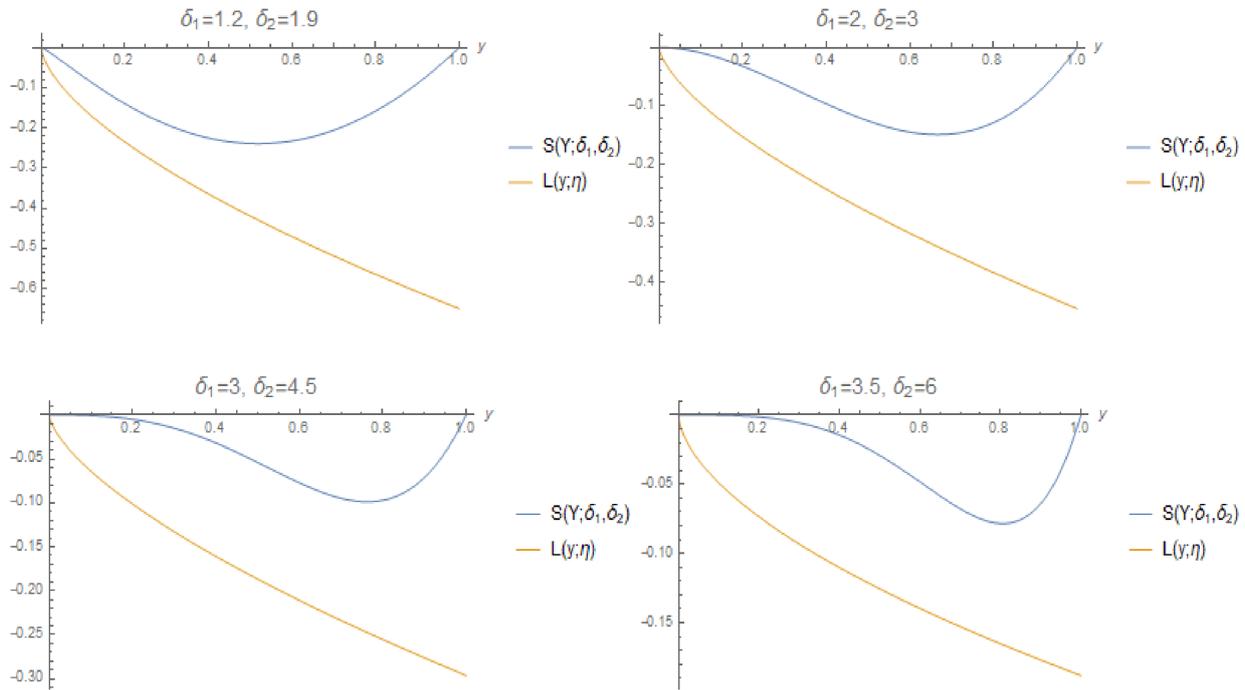


Figure 2: The connection between the functions  $S(Y; \delta_1, \delta_2)$  and  $L(y; \eta) := P(\delta_1, \delta_2) \frac{y^\eta}{1-\eta}$  depends on the specific values of  $1 \leq \delta_1 < \delta_2$  and  $0 < \eta < 1$ .

Integrating (12) across  $\mathbb{R}_+^n = \{y_l \in \mathbb{R}^n; y_l \geq 0\}$ . Besides applying the Markov inequality, the result obtained is:

$$\begin{aligned} \text{SCSM}n_{\delta_1, \delta_2}(Y) &\geq \frac{P(\delta_1, \delta_2)}{(1-\eta)} \int_{\mathbb{R}_+^n} \prod_{l=1}^n \bar{F}_{|Y_l|}^\eta(y_l) dy_l = \frac{P(\delta_1, \delta_2)}{(1-\eta)} \prod_{l=1}^n \left\{ \int_0^\infty \bar{F}_{|Y_l|}^\eta(y_l) dy_l \right\} \\ &= \frac{P(\delta_1, \delta_2)}{(1-\eta)} \prod_{l=1}^n \left\{ \int_0^1 \bar{F}_{|Y_l|}^\eta(y_l) dy_l + \int_1^\infty \bar{F}_{|Y_l|}^\eta(y_l) dy_l \right\} \\ &\geq \frac{P(\delta_1, \delta_2)}{(1-\eta)} \prod_{l=1}^n \left\{ 1 + \int_1^\infty \left[ \frac{1}{y_l^w} E[|Y_l|^w] \right]^\frac{\eta}{n} dy_l \right\} \\ &= \frac{P(\delta_1, \delta_2)}{(1-\eta)} \prod_{l=1}^n \left\{ 1 + [E[|Y_l|^w]]^\frac{\eta}{n} \int_1^\infty \left[ \frac{1}{y_l^w} \right]^\frac{\eta}{n} dy_l \right\}, \end{aligned}$$

When  $\frac{w\eta}{n} > 1$ , the value is finite. Therefore, for any  $w > n$ , it is possible to pick  $\eta$  to be less than 1 (but still close to it) so that  $\frac{w\eta}{n}$  exceeds 1, ensuring that the condition is met.  $\square$

**Remark 2.5.** Given any random variable  $Y$ , it is commonly understood that if  $\text{Var}(Y)$  exists, then  $Y$  is in  $L^2$ . Consequently, the existence of  $\text{Var}(Y)$  is sufficient for ensuring that  $\text{SCSM}n_{\delta_1, \delta_2}(Y) > -\infty$  where  $1 \leq \delta_1 < \delta_2$ .

The convergence of the surviving Sharma-Taneja-Mittal entropy (Weak convergence) is discussed in the following theorem. As  $n \rightarrow \infty$ , the symbol  $(\xrightarrow{\frac{d}{n}})$  in the forthcoming discussion represents convergence in distribution.

**Theorem 2.6.** Let us assume that  $Y_n$  represents the series of random vectors with dimensions of  $n$  that converges to

a random vector  $Y$  in distribution, and that for each  $n$ ,  $Y_n$  belongs to  $L^w$ , where  $w > n$ , it follows that:

$$\lim_{n \rightarrow +\infty} \text{SCSM}n_{\delta_1, \delta_2}(Y_n) = \text{SCSM}n_{\delta_1, \delta_2}(Y).$$

*Proof.* Given that  $Y_n \xrightarrow[n]{d} Y$ , we get

$$\lim_{n \rightarrow +\infty} (\bar{F}_{|Y_n|}^{\delta_1}(y) - \bar{F}_{|Y_n|}^{\delta_2}(y)) = \bar{F}_{|Y|}^{\delta_1}(y) - \bar{F}_{|Y|}^{\delta_2}(y), y \in \mathbb{R}_+^N.$$

In the meantime, by referring to (12), we get

$$\begin{aligned} \frac{1}{\delta_1 - \delta_2} (\bar{F}_{|Y_n|}^{\delta_1}(y) - \bar{F}_{|Y_n|}^{\delta_2}(y)) &\geq \frac{P(\delta_1, \delta_2)}{(1 - \eta)} \prod_{l=1}^n \bar{F}_{|Y_{n_l}|}^{\frac{\eta}{n}}(y_l) \\ &\leq \frac{P(\delta_1, \delta_2)}{(1 - \eta)} \prod_{l=1}^n [U_{[0,1]}(y_l) + y_l^{-w} U_{[1,\infty)}(y_l) E(|Y_{n_l}|^w)]^{\frac{\eta}{n}}, \end{aligned}$$

such that  $U_\phi(y)$  is the indicator function and  $Y_{n_l}$  is the  $l$ th component of the random vector  $Y_n$ ; that is,  $U_\phi(y) = 1, y \in \phi$ , and  $U_\phi(y) = 0, y \notin \phi$ . Therefore,  $\frac{1}{\delta_1 - \delta_2} (\bar{F}_{|Y_n|}^{\delta_1}(y) - \bar{F}_{|Y_n|}^{\delta_2}(y))$ , if  $\frac{w\eta}{n} > 1$ , is an integrable function limits. Furthermore, we can choose  $\eta < 1$  to be sufficiently close to one for each  $w > n$  in order to guarantee  $\frac{w\eta}{n} > 1$ . The proof is completed by applying the dominated convergence theorem.  $\square$

In the following, we illustrate that the model  $\text{SCSM}n_{\delta_1, \delta_2}(Y)$  exceeds the differential Shannon entropy, which could be present when  $Y$  obtain a density.

**Theorem 2.7.** *Examine a random variable that is not negative. Using  $Y$  with PDF  $f$  and CDF  $F$ , we can obtain*

$$\text{SCSM}n_{\delta_1, \delta_2}(Y) \geq \xi(\delta_1, \delta_2) e^{DSn(Y)}, \delta_1 \neq \delta_2 > 0,$$

where  $0 \leq \xi(\delta_1, \delta_2) = e^{\int_0^1 \log\left(\frac{1}{\delta_1 - \delta_2} (y^{\delta_1} - y^{\delta_2})\right) dy} > -\infty$  and

$$DSn(Y) = - \int_0^\infty f(y) \log f(y) dy, \tag{13}$$

is the differential Shannon entropy.

*Proof.* Using the log-sum inequality, we are able to derive

$$\int_0^\infty f(y) \log \left( \frac{f(y)}{\frac{1}{\delta_1 - \delta_2} (\bar{F}^{\delta_1}(y) - \bar{F}^{\delta_2}(y))} \right) dy \geq \log \frac{1}{\int_0^\infty \frac{1}{\delta_1 - \delta_2} (\bar{F}^{\delta_1}(y) - \bar{F}^{\delta_2}(y)) dy} = - \log \text{SCSM}n_{\delta_1, \delta_2}(Y). \tag{14}$$

Additionally, the expression on the left sidelong of the equation (14) can be acquired in the manner described below:

$$\int_0^\infty f(y) \log \left( \frac{f(y)}{\frac{1}{\delta_1 - \delta_2} (\bar{F}^{\delta_1}(y) - \bar{F}^{\delta_2}(y))} \right) dy = -DSn(Y) - \int_0^1 \log \left( \frac{1}{\delta_1 - \delta_2} (y^{\delta_1} - y^{\delta_2}) \right) dy.$$

Therefore,

$$\log \text{SCSM}n_{\delta_1, \delta_2}(Y) \geq DSn(Y) + \int_0^1 \log \left( \frac{1}{\delta_1 - \delta_2} (y^{\delta_1} - y^{\delta_2}) \right) dy. \tag{15}$$

After exponentiating both sides of Equation (15) and applying Equation (9), the result is obtained. Here, the expression  $0 \leq \xi(\delta_1, \delta_2) = e^{\int_0^1 \ln\left(\frac{1}{\delta_1 - \delta_2} (y^{\delta_1} - y^{\delta_2})\right) dy} \geq P(\delta_1, \delta_2)$  which is finite. This finalizes the proof.  $\square$

2.1. Features on the proportional hazards distribution

The PHz model is crucial in the reliability framework. In certain cases, PH models naturally emerge, such as when illustrating the proportional effect of environmental factors on the hazard rate. These models are commonly utilized because they facilitate easier analysis and inference. Let  $Y$  be a nonnegative random variable with a finite mean  $\mu = E(Y)$  and a survival function  $F(y)$ . The PHz model is described as follows if  $Y_\delta$  denotes a nonnegative random variable with a survival function  $\bar{F}_\delta(y) = P(Y_\delta > y)$  for  $y \geq 0$ , then

$$\bar{F}_\delta(y) = [\bar{F}(y)]^\delta, y \geq 0, \delta > 0, \tag{16}$$

where  $\delta$  is the PHz parameter. The residual or excess lifespan of  $Y$ , if it is above a threshold  $u$ , is written as  $Y_u = [Y - u | Y > u]$  where the random variable  $Y$  represents the lifetime of a system or component. As usual, a random variable that, when conditioned on  $\phi$ , has the same distribution as  $Y$  is denoted by  $[Y|\phi]$ . For any  $y > u$ ,  $f(y|u) = f(y)/F(u)$  is the PDF of  $Y_u$ . In the event that  $Y$  has a finite mean, the mean residual life function can be found as follows:

$$\begin{aligned} \mu^*(u) &= \mu^*(Y; u) = E_{Y>u}[Y - u | Y > u] \\ &= \frac{\int_u^\infty \bar{F}(y)dy}{\bar{F}(u)}, \end{aligned} \tag{17}$$

where  $u \geq 0$ . By replacing  $Y$  in equation (17) with  $Y_\delta$ , one may derive the mean residual life function of  $Y_\delta$ , which is represented as  $\mu_\delta^*(u)$ . It is significant to remember that under the quadratic loss function  $Q(h, Y_\delta|u) = (Y_\delta - u - h)^2$ , the mean residual life function acts as the best criterion for making decisions, where  $Y_\delta > u$ , for predicting the excess as

$$h^*(u) = \arg \min_h E_{Y>u}[Q(h, Y_\delta|u)] = \mu_\delta^*(u), \delta > 0.$$

As mentioned by Asadi et al. [2], based on the threshold  $u, \mu_\delta^*(u)$  serves as a risk local measure. The overall risk, known as the risk Bayesian, is expressed as  $SCn(\mu_\delta^*) = E_\tau[\mu_\delta^*(u)]$ , where, given the threshold  $u, \tau(u)$  represents a prior distribution. The surviving Sharma-Taneja-Mittal entropy can be expressed as the Bayes risk, as demonstrated by the following theorem.

**Theorem 2.8.** Given the baseline prior  $\tau(u) = f(u)$ , the risk Bayesian of  $\mu_\delta^*(u)$  is expressed through the survival Sharma-Taneja-Mittal entropy, from (5), as follows

$$\begin{aligned} SCSMn_{\delta_1, \delta_2}(Y) &= \frac{1}{\delta_1 - \delta_2} (\mu_{\delta_1} - \mu_{\delta_2}) \\ &= \frac{\delta_2 - 1}{\delta_1 - \delta_2} E(\mu_{\delta_2}^*(Y)) - \frac{\delta_1 - 1}{\delta_1 - \delta_2} E(\mu_{\delta_1}^*(Y)), \end{aligned} \tag{18}$$

where  $\mu_{\delta_i} = E(Y_{\delta_i}) = \int_0^\infty \bar{F}^{\delta_i}(y)dy, i = 1, 2.$

*Proof.* For  $\tau(u) = f(u)$ , and utilizing Fubini's theorem, we have

$$\begin{aligned} E(\mu_\delta^*(U)) &= \int_0^\infty \left( \frac{\int_u^\infty \bar{F}^\delta(y)dy}{\bar{F}^\delta(u)} \right) f(u)du \\ &= \int_0^\infty \bar{F}^\delta(y) \left( \int_0^y \frac{f(u)}{\bar{F}^\delta(u)} du \right) dy \\ &= \int_0^\infty \bar{F}^\delta(y) \left( \frac{1}{\delta - 1} (\bar{F}^{-\delta+1}(y) - 1) \right) dy \\ &= \frac{1}{\delta - 1} \int_0^\infty (\bar{F}(y) - \bar{F}^\delta(y)) dy \\ &= \frac{1}{\delta - 1} (\mu - \mu_\delta). \end{aligned}$$

From (5), and after simple calculations, we can see that

$$SCSMn_{\delta_1, \delta_2}(Y) = \frac{\delta_2 - 1}{\delta_1 - \delta_2} E(\mu_{\delta_2}^*(Y)) - \frac{\delta_1 - 1}{\delta_1 - \delta_2} E(\mu_{\delta_1}^*(Y)).$$

□

The relationship between the survival cumulative entropy and the survival cumulative Sharma–Taneja–Mittal entropy is demonstrated by the following theorem.

**Theorem 2.9.** *Let  $Y$  be a nonnegative continuous random variable. Then, from (1) and (5), we obtain*

$$SCSMn_{\delta_1, \delta_2}(Y) \geq -SCn(Y),$$

under the restriction  $\delta_2 > \delta_1 \geq 1$ .

*Proof.* Since  $\delta_2 > \delta_1 \geq 1$ , then  $E(\mu_{\delta_2}^*(Y)) \leq E(\mu_{\delta_1}^*(Y))$ . Furthermore, it's evident that  $\mu_{\delta}^*(y) < \mu^*(y)$ . Therefore, From (18), we have

$$\begin{aligned} SCSMn_{\delta_1, \delta_2}(Y) &= \frac{\delta_2 - 1}{\delta_1 - \delta_2} E(\mu_{\delta_2}^*(Y)) - \frac{\delta_1 - 1}{\delta_1 - \delta_2} E(\mu_{\delta_1}^*(Y)) \\ &\geq \frac{\delta_2 - 1}{\delta_1 - \delta_2} E(\mu^*(Y)) - \frac{\delta_1 - 1}{\delta_1 - \delta_2} E(\mu^*(Y)) = -E(\mu^*(Y)) = -SCn(Y). \end{aligned}$$

□

**Remark 2.10.** *If the PDF of  $Y_{\delta}$  is  $f_{\delta}(y)$ , its hazard function is given by  $\kappa_{\delta}(y) = \frac{f_{\delta}(y)}{\bar{F}_{\delta}(y)}$ , where  $\bar{F}_{\delta}(y) > 0$ . Thus, the PHz model can be expressed as  $\kappa_{\delta}(y) = \delta\kappa(y)$ , with  $\delta > 0$ , where  $\kappa(y)$  represents the baseline hazard function. Moreover, we can see that*

$$E(\mu_{\delta}^*(Y)) = \frac{1}{\delta} E(\mu^*(Y_{\delta})). \tag{19}$$

Another valuable way to express the survival cumulative Sharma–Taneja–Mittal entropy is through the mean residual life function of  $Y$  by

$$SCSMn_{\delta_1, \delta_2}(Y) = \frac{\delta_2 - 1}{\delta_2(\delta_1 - \delta_2)} E(\mu^*(Y_{\delta_2})) - \frac{\delta_1 - 1}{\delta_1(\delta_1 - \delta_2)} E(\mu^*(Y_{\delta_1})). \tag{20}$$

We can establish the following result that incorporates  $\mu_{\delta}^*(u)$  and its derivative  $\mu_{\delta}'^*(u) = \kappa_{\delta}(u)\mu_{\delta}^*(u) - 1$ . Moreover, the mean residual lifetime functions are  $\mu_{Y_1}^*(u)$  and  $\mu_{Y_2}^*(u)$ , respectively, for the absolutely continuous nonnegative random variables  $Y_1$  and  $Y_2$ , which have CDFs  $F_1$  and  $F_2$ , PDFs  $f_1$  and  $f_2$ . If, for every  $u > 0$ ,  $\bar{F}_1(u) \leq \bar{F}_2(u)$ , we say that  $Y_1$  is smaller than  $Y_2$  in the standard or usual stochastic order, expressed as  $Y_1 \leq_{st} Y_2$ , see Shaked and Shanthikumar [16].

**Theorem 2.11.** *Think of  $Y_1$  and  $Y_2$  as finite, unequal mean nonnegative random variables, where  $Y_1 \leq_{st} Y_2$ . If  $E(\mu_{\delta}^*(Y_2))$  is finite for all  $\delta > 0$ , then it follows that*

$$\begin{aligned} SCSMn_{\delta_1, \delta_2}(Y_1) &= \frac{\delta_2 - 1}{\delta_1 - \delta_2} [E(\mu_{\delta_2}^*(Y_2)) + E(\mu_{\delta_2}'^*(Z))][E(Y_1) - E(Y_2)] \\ &\quad - \frac{\delta_1 - 1}{\delta_1 - \delta_2} [E(\mu_{\delta_1}^*(Y_2)) + E(\mu_{\delta_1}'^*(Z))][E(Y_1) - E(Y_2)], \end{aligned}$$

with noting that  $Z$  is a completely continuous, nonnegative random variable with a pdf

$$f_Z(y) = \frac{\bar{F}_2(y) - \bar{F}_1(y)}{E(Y_2) - E(Y_1)}, y > 0.$$

*Proof.* Using the identity (18), the proof is based on the probabilistic mean value theory given in Di Crescenzo [5], theory 4.1.  $\square$

An analogous statement for the cumulative survival In the next theorem, Sharma–Taneja–Mittal entropy is derived using equations (19) and (20), with respect to the expectation of the PHz random variable.

**Theorem 2.12.** *Think of  $Y$  as an absolutely continuous, nonnegative random variable, and its PDF is  $f(y)$ . In that case, the following holds:*

$$SCSMn_{\delta_1, \delta_2}(Y_1) = \frac{\delta_2 - 1}{\delta_1 - \delta_2} [E(Y_{\delta_2}) + E(g_{\delta_2}(Y))] - \frac{\delta_1 - 1}{\delta_1 - \delta_2} [E(Y_{\delta_1}) + E(g_{\delta_1}(Y))],$$

where

$$g_{\delta}(y) = \int_0^y \mu_{\delta}^*(u) \bar{F}^{\delta-1}(u) du, y > 0,$$

$$E(Y_{\delta}) = \int_0^{\infty} [\bar{F}(y)]^{\delta} dy. \tag{21}$$

*Proof.* Using (19) and the relation  $\kappa_{\delta}(u)\mu_{\delta}^*(u) = 1 + \mu_{\delta}^*(u)$ , we have that:

$$E(\mu_{\delta}^*(Y)) = \int_0^{\infty} \mu_{\delta}^*(y)\kappa(y)[\bar{F}(y)]^{\delta} dy = E(Y_{\delta}) + \int_0^{\infty} \mu_{\delta}^*(y)[\bar{F}(y)]^{\delta} dy,$$

By applying Fubini’s theorem, we achieve

$$\begin{aligned} \int_0^{\infty} \mu_{\delta}^*(y)[\bar{F}(y)]^{\delta} dy &= \int_0^{\infty} \mu_{\delta}^*(y) \int_y^{\infty} f(u)[\bar{F}(y)]^{\delta-1} du dy \\ &= \int_0^{\infty} f(u) \int_0^u \mu_{\delta}^*(y)[\bar{F}(y)]^{\delta-1} dy du, \end{aligned}$$

this achieves the intended outcome.  $\square$

The following theorem presents a characterization result concerning the survival cumulative Sharma–Taneja–Mittal entropy under the dispersive order. Furthermore, the dispersive order, represented by  $Y_1 \leq_d Y_2$ , indicates that the random variable  $Y_1$  is less than  $Y_2$  if  $F_1^{-1}(u) - F_2^{-1}(u)$  increases with  $u$  over the interval  $(0, 1)$ . In this case, the right-continuous inverses of the functions  $F_1$  and  $F_2$ , respectively, are represented by  $F_1^{-1}$  and  $F_2^{-1}$ , [16].

**Theorem 2.13.** *Consider two continuous random variables with distribution functions  $F_1$  and  $F_2$  are represented by  $Y_1$  and  $Y_2$ , respectively, which are strictly increasing. If  $Y_1 \leq_d Y_2$ , and*

$$SCSMn_{\delta_1, \delta_2}(Y_1) = SCSMn_{\delta_1, \delta_2}(Y_2), \tag{22}$$

with  $\delta_1 \neq \delta_2 > 0$  is constant, then  $Y_1$  and  $Y_2$  have identical distributions apart from a location parameter.

*Proof.* Assuming that (22) is true, we can observe through a change of variable that

$$\begin{aligned} SCSMn_{\delta_1, \delta_2}(Y_1) - SCSMn_{\delta_1, \delta_2}(Y_2) &= \frac{1}{\delta_1 - \delta_2} \int_0^1 [(1-u)^{\delta_1} - (1-u)^{\delta_2}] d[F_1^{-1}(u) - F_2^{-1}(u)] \\ &= 0, \end{aligned}$$

with a fixed  $\delta_1 \neq \delta_2 > 0$ . It follows that  $F_1^{-1}(u) - F_2^{-1}(u)$  is an increasing function of  $u$ , given that  $Y_1 \leq_d Y_2$ . For any  $0 \leq u \leq 1$ , we claim that  $F_1^{-1}(u) - F_2^{-1}(u)$  stays constant. Furthermore, assume there is an interval

$(\alpha, \beta) \subset [0, 1]$  where  $F_1^{-1}(u) - F_2^{-1}(u)$  is not constant throughout  $(\alpha, \beta)$ . This will serve as an example of contradiction. Therefore,

$$\begin{aligned} 0 &= \frac{1}{\delta_1 - \delta_2} \int_0^1 [(1-u)^{\delta_1} - (1-u)^{\delta_2}] d[F_1^{-1}(u) - F_2^{-1}(u)] \\ &\leq \frac{1}{\delta_1 - \delta_2} \int_\alpha^\beta [(1-u)^{\delta_1} - (1-u)^{\delta_2}] d[F_1^{-1}(u) - F_2^{-1}(u)] < 0, \end{aligned}$$

which creates a contradiction. Thus,  $F_1^{-1}(u) - F_2^{-1}(u) = \text{constant}$  for every  $0 \leq u \leq 1$ , indicating that  $Y_1$  and  $Y_2$  possess the same distribution with the exception of a location parameter.  $\square$

Generally speaking, the maximum entropy of any single variable is less than the standard Shannon differential entropy of the sum of independent random variables. Similarly, there is an analogous result for the survival cumulative Sharma–Taneja–Mittal entropy, taking into account negative values, as stated in the following theorem. This result is not mentioned here because it is similar to Klein [10]’s Theorem 4.

**Theorem 2.14.** *If  $Y_1$  and  $Y_2$  are two random variables that are independent of each other, then  $SCSMn_{\delta_1, \delta_2}(Y_1 + Y_2) \leq \min(SCSMn_{\delta_1, \delta_2}(Y_1), SCSMn_{\delta_1, \delta_2}(Y_2))$ .*

Next, we will present a normalized processing of Equation (5), which may be expressed in terms of (3), based on the form provided in (18). To be more precise, we define the normalized survival cumulative Sharma–Taneja–Mittal entropy for a finite, non-zero mean of a nonnegative random variable  $Y$ , for all  $\delta > 0$ .

$$\begin{aligned} \text{NSCSMn}_{\delta_1, \delta_2}(Y) &= \frac{\delta_2 - 1}{\delta_1 - \delta_2} \frac{SCTn_{\delta_2}(Y)}{E(Y_{\delta_2})} - \frac{\delta_1 - 1}{\delta_1 - \delta_2} \frac{SCTn_{\delta_1}(Y)}{E(Y_{\delta_1})} \\ &= \frac{\delta_2 - 1}{\delta_1 - \delta_2} \frac{E(\mu_{\delta_2}^*(Y))}{E(Y_{\delta_2})} - \frac{\delta_1 - 1}{\delta_1 - \delta_2} \frac{E(\mu_{\delta_1}^*(Y))}{E(Y_{\delta_1})}, \end{aligned} \tag{23}$$

where  $E(Y_\delta)$  is defined in (21). Using the features of the survival cumulative Sharma–Taneja–Mittal entropy, we explore certain characteristics of the normalized presented in equation (23). It is established that for any  $\delta > 0$ , the survival cumulative Sharma–Taneja–Mittal entropy satisfies the property  $SCSMn_{\delta_1, \delta_2}(aY + b) = a SCSMn_{\delta_1, \delta_2}(Y)$ , where  $a > 0$  and  $b \geq 0$ . Thus, for every  $\delta > 0$ , the following holds:

$$\text{NSCSMn}_{\delta_1, \delta_2}(aY + b) = \frac{\delta_2 - 1}{\delta_1 - \delta_2} \frac{a SCTn_{\delta_2}(Y)}{a E(Y_{\delta_2}) + b} - \frac{\delta_1 - 1}{\delta_1 - \delta_2} \frac{a SCTn_{\delta_1}(Y)}{a E(Y_{\delta_1}) + b}.$$

**Example 2.15.** *We suppose that  $Y$  has a mean of  $1/\rho$  and is distributed exponentially, where  $\rho > 0$ . Then, from (23), we have  $\text{NSCSMn}_{\delta_1, \delta_2}(Y) = -1$ .*

### 2.2. Features on the wealth order excess

In this subsection, we explore the relationship between the wealth order excess (also known as the spread right order) and the survival cumulative Sharma–Taneja–Mittal entropy. Typically, the easiest way to compare the of two distribution functions variability is by examining their standard deviations. However, because numerical measures alone may not always provide enough insight, stochastic orders and various transforms have been proposed and extensively examined by Shaked and Shanthikumar [16] for contrasting the variability. Among these, the excess wealth order serves as one method for assessing spread. For a nonnegative random variable  $Y$  with a CDF  $F$ , the wealth excess convert is expressed as follows (cf. Fernandez-Ponce et al. [6])

$$\Psi_Y(u) = \int_{F^{-1}(u)}^\infty \bar{F}(y) dy = \int_u^1 (F^{-1}(l) - F^{-1}(u)) dl,$$

where

$$F^{-1}(u) = \inf\{y : F(y) \geq u\}, \quad u \in (0, 1).$$

represents  $F$ 's quantile function. The following relation also connects this function the average residual life function in this way

$$\mu_Y^*(F^{-1}(u)) = \frac{\Psi_Y(u)}{1-u}, \quad 0 < u < 1.$$

Consequently, we may use (20) to express the survival cumulative Sharma–Taneja–Mittal entropy in terms of the wealth excess convert. This can be understood from the following theorem, which lacks evidence but is nonetheless straightforward.

**Theorem 2.16.** *If  $Y$  is an absolutely continuous, non-negative random variable with a CDF  $F$ , then*

$$SCSMn_{\delta_1, \delta_2}(Y) = \frac{\delta_2 - 1}{\delta_1 - \delta_2} \left[ \int_0^1 \mu_Y^*(F^{-1}(u))(1-u)^{\delta_2-1} du \right] - \frac{\delta_1 - 1}{\delta_1 - \delta_2} \left[ \int_0^1 \mu_Y^*(F^{-1}(u))(1-u)^{\delta_1-1} du \right], \quad (24)$$

for all  $\delta_1 \neq \delta_2 > 0$ .

**Example 2.17.** 1. *Let's suppose  $Y$  follows a uniform distribution over the interval  $[0, \beta]$ . It is straightforward to observe that*

$$\mu_Y^*(F^{-1}(u)) = \frac{\beta(1-u)}{2}.$$

Therefore, by applying (24), we derive

$$\begin{aligned} SCSMn_{\delta_1, \delta_2}(Y) &= \frac{\delta_2 - 1}{\delta_1 - \delta_2} \left[ \frac{\beta}{2} \int_0^1 (1-u)^{\delta_2} du \right] - \frac{\delta_1 - 1}{\delta_1 - \delta_2} \left[ \frac{\beta}{2} \int_0^1 (1-u)^{\delta_1} du \right] \\ &= \frac{\delta_2 - 1}{\delta_1 - \delta_2} \left[ \frac{\beta}{2(\delta_2 + 1)} \right] - \frac{\delta_1 - 1}{\delta_1 - \delta_2} \left[ \frac{\beta}{2(\delta_1 + 1)} \right]. \end{aligned}$$

2. *Let's examine the Pareto distribution, characterized by a scale parameter  $c > 0$  and a shape parameter  $\theta > 0$ , where the function of survival is provided by  $\bar{F}(y) = \frac{c^\theta}{(y+c)^\theta}$  for  $y \geq 0$ . It is straightforward to observe that*

$$\mu_Y^*(F^{-1}(u)) = \frac{c(1-u)^{\frac{1}{\theta}}}{(\theta - 1)}.$$

Therefore, by applying (24), we derive

$$\begin{aligned} SCSMn_{\delta_1, \delta_2}(Y) &= \frac{\delta_2 - 1}{\delta_1 - \delta_2} \left[ \int_0^1 (1-u)^{\delta_2 - \frac{1}{\theta} - 1} du \right] - \frac{\delta_1 - 1}{\delta_1 - \delta_2} \left[ \int_0^1 (1-u)^{\delta_1 - \frac{1}{\theta} - 1} du \right] \\ &= \frac{\delta_2 - 1}{\delta_1 - \delta_2} \left[ \frac{c \theta}{(\theta \delta_2 - 1)(\theta - 1)} \right] - \frac{\delta_1 - 1}{\delta_1 - \delta_2} \left[ \frac{c \theta}{(\theta \delta_1 - 1)(\theta - 1)} \right]. \end{aligned}$$

### 3. Empirical survival cumulative Sharma–Taneja–Mittal entropy

The survival cumulative Sharma–Taneja–Mittal entropy is estimated in this section using the empirical survival cumulative entropy. Let  $Y_1, Y_2, \dots, Y_n$  be perfectly continuous, independently distributed, nonnegative random variables. Given the indicator function,  $\mathbb{I}_\chi(y) = 1, y \in \chi$ , the empirical distribution function for the sample corresponding to  $F$  is represented by

$$\widehat{F}_n(y) = \sum_{i=1}^{n-1} \frac{i}{n} \mathbb{I}_{[y_i, y_{i+1})}(y), \quad y \geq 0.$$

For any  $\delta_1 \neq \delta_2 > 0$ , the empirical measure of  $SCSMn_{\delta_1, \delta_2}(F)$  is derived as

$$\begin{aligned} SCSMn_{\delta_1, \delta_2}(\widehat{F}_n) &= \frac{1}{\delta_1 - \delta_2} \int_0^\infty [\widehat{F}_n^{\delta_1}(y) - \widehat{F}_n^{\delta_2}(y)] dy \\ &= \frac{1}{\delta_1 - \delta_2} \sum_{i=1}^{n-1} \int_{Y_{(i)}}^{Y_{(i+1)}} \left[ \left(1 - \frac{i}{n}\right)^{\delta_1} - \left(1 - \frac{i}{n}\right)^{\delta_2} \right] dy \\ &= \frac{1}{\delta_1 - \delta_2} \sum_{i=1}^{n-1} \Delta_{i+1} \left[ \left(1 - \frac{i}{n}\right)^{\delta_1} - \left(1 - \frac{i}{n}\right)^{\delta_2} \right], \end{aligned} \tag{25}$$

with considering that the sample spacings are  $\Delta_{i+1} = Y_{(i+1)} - Y_{(i)}$ , and the associated order statistics of the random sample are represented by  $Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(n)}$ . At this point, we can present a central limit theorem for the empirical survival cumulative Sharma–Taneja–Mittal entropy, which is obtained using an exponential distribution random sample.

**Theorem 3.1.** Consider a random sample  $Y_1, Y_2, \dots, Y_n$  taken from a common exponential distribution. Then,

$$\frac{SCSMn_{\delta_1, \delta_2}(\widehat{F}_n) - E[SCSMn_{\delta_1, \delta_2}(\widehat{F}_n)]}{\sqrt{Var[SCSMn_{\delta_1, \delta_2}(\widehat{F}_n)]}} \rightarrow N(0, 1),$$

under the restriction  $\delta_1 \neq \delta_2 > \frac{2}{3}$ , where the standard or usual normal distribution is denoted by  $N(0, 1)$ .

*Proof.* The total of independent exponential random variables  $Y_j$  can be used to represent the empirical measure  $SCSMn_{\delta_1, \delta_2}(\widehat{F}_n)$ , where the expected value and variance are specified by

$$E[Y_j] = \frac{1}{n\rho(\delta_1 - \delta_2)} \left( \left(1 - \frac{j}{n}\right)^{\delta_1 - 1} - \left(1 - \frac{j}{n}\right)^{\delta_2 - 1} \right),$$

and

$$Var[Y_j] = \frac{1}{(n\rho(\delta_1 - \delta_2))^2} \left( \left(1 - \frac{j}{n}\right)^{\delta_1 - 1} - \left(1 - \frac{j}{n}\right)^{\delta_2 - 1} \right)^2.$$

Suppose that  $\Xi_{j,p} = E[|Y_j - E(Y_j)|^p]$ ,  $p = 2, 3$ . The following approximations can be obtained for a big  $n$  as

$$\begin{aligned} \sum_{j=1}^n \Xi_{j,2} &= \sum_{j=1}^n E[|Y_j - E(Y_j)|^2] = \frac{1}{(n\rho(\delta_1 - \delta_2))^2} \sum_{j=1}^n \left( \left(1 - \frac{j}{n}\right)^{\delta_1 - 1} - \left(1 - \frac{j}{n}\right)^{\delta_2 - 1} \right)^2 \\ &\approx \frac{1}{n(\rho(\delta_1 - \delta_2))^2} b_2. \end{aligned}$$

Furthermore, for any random variable  $Y_j$  that follows an exponential distribution, by noting that  $E[|Y_j - E(Y_j)|^3] = \frac{2(6-e)E(Y_j)^3}{e}$ , the following outcome can be obtained

$$\begin{aligned} \sum_{j=1}^n \Xi_{j,3} &= \frac{2(6-e)}{e} \sum_{j=1}^n E[|Y_j - E(Y_j)|^3] = \frac{2(6-e)}{e(n\rho(\delta_1 - \delta_2))^3} \sum_{j=1}^n \left( \left(1 - \frac{j}{n}\right)^{\delta_1 - 1} - \left(1 - \frac{j}{n}\right)^{\delta_2 - 1} \right)^3 \\ &\approx \frac{(6-e)}{en^2(\rho(\delta_1 - \delta_2))^3} b_3. \end{aligned}$$

With noting that

$$b_p = \int_0^1 (y^{\delta_1} - y^{\delta_2})^p dy = \begin{cases} \frac{2(\delta_1 - \delta_2)^2}{(-1+2\delta_1)(-1+\delta_1+\delta_2)(-1+2\delta_2)}, p = 2, \\ \frac{6(\delta_1 - \delta_2)^3}{(-2+3\delta_1)(-2+2\delta_1+\delta_2)(-2+\delta_1+2\delta_2)(-2+3\delta_2)}, p = 3, \end{cases}$$

where  $\delta_1 \neq \delta_2 > \frac{2}{3}$ . Thus, for an appropriate function  $P$ , it is true that for large  $n$

$$\frac{(\sum_{j=1}^n \Xi_{j,3})^{\frac{1}{3}}}{(\sum_{j=1}^n \Xi_{j,2})^{\frac{1}{2}}} \approx \frac{P}{n^{\frac{1}{6}}} \xrightarrow{n \rightarrow \infty} 0.$$

This concludes the proof and satisfies the Lyapunov requirement for the central limit theorem.  $\square$

If the random variables follow an exponential distribution, the findings from Theorem 3.1 is able to calculate confidence intervals for  $SCSMn_{\delta_1, \delta_2}(F)$ . Specifically, for a given  $\delta_1 \neq \delta_2 > \frac{2}{3}$ , it holds that

$$SCSMn_{\delta_1, \delta_2}(\widehat{F}_n) \pm z_{\frac{\gamma}{2}} \sqrt{Var[SCSMn_{\delta_1, \delta_2}(\widehat{F}_n)]}, \tag{26}$$

where  $z_{\frac{\gamma}{2}}$  denotes the critical value of the  $N(0, 1)$  distribution at  $\frac{\gamma}{2}$ .

### 3.1. Second estimator

In this subsection, a different nonparametric estimator can be developed as follows. Zardasht [22] presented the survival cumulative Tsallis entropy as an L-functional. In the same manner, we can represent the survival cumulative Sharma–Taneja–Mittal entropy as

$$\begin{aligned} SCSMn_{\delta_1, \delta_2}^*(F) &= \frac{1}{\delta_1 - \delta_2} \int_0^\infty y \left( \delta_1 \bar{F}^{\delta_1 - 1}(y) - \delta_2 \bar{F}^{\delta_2 - 1}(y) \right) dF(y) \\ &= \int_0^\infty y L_{\delta_1, \delta_2}(F(y)) dF(y), \end{aligned} \tag{27}$$

where  $L_{\delta_1, \delta_2}(u) = \frac{1}{\delta_1 - \delta_2} (\delta_1(1 - u)^{\delta_1 - 1} - \delta_2(1 - u)^{\delta_2 - 1})$ ,  $0 \leq u \leq 1$ . In the sequel, substituting  $F$  in (27) with  $\widehat{F}_n$  allows us to provide an estimator for  $SCSMn_{\delta_1, \delta_2}^*(F)$  through the following L-statistic:

$$SCSMn_{\delta_1, \delta_2}^*(\widehat{F}_n) = \int_0^\infty y L_{\delta_1, \delta_2}(\widehat{F}_n(y)) d\widehat{F}_n(y) = \frac{1}{n} \sum_{j=1}^n L_{\delta_1, \delta_2} \left( \frac{j}{n} \right) Y_{(j)}, \tag{28}$$

The estimator’s asymptotic normality in (28) is confirmed by the next theorem. The proof is not presented because it follows simply from Theorems 2 and 3 in [20].

**Theorem 3.2.** Consider a set of absolutely continuous, nonnegative, identically distributed, and independent random variables denoted by  $Y_1, Y_2, \dots, Y_n$ , with a finite second moment,  $E(Y_i^2) < \infty$ . The asymptotic distribution of  $\sqrt{n}(SCSMn_{\delta_1, \delta_2}^*(\widehat{F}_n) - SCSMn_{\delta_1, \delta_2}^*(F))$  is normally distributed, with considering the mean to be zero and a finite variance  $Var(F, L_{\delta_1, \delta_2}) > 0$ , such that

$$Var(F, L_{\delta_1, \delta_2}) = \int_0^\infty \int_0^\infty (F(\min(y_1, y_2)) - F(y_1)F(y_2)) L_{\delta_1, \delta_2}(y_1) L_{\delta_1, \delta_2}(y_2) dy_1 dy_2. \tag{29}$$

Furthermore, it is clear that

$$\widehat{Var}(F, L_{\delta_1, \delta_2}) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left( \min \left( \frac{i}{n}, \frac{j}{n} \right) - \frac{i}{n} \frac{j}{n} \right) L_{\delta_1, \delta_2} \left( \frac{i}{n} \right) L_{\delta_1, \delta_2} \left( \frac{j}{n} \right) (Y_{(i+1)} - Y_{(i)})(Y_{(j+1)} - Y_{(j)}).$$

provides a consistent estimator of (29). Using Theorem 3.2's outcome, confidence intervals approximation for  $SCSMn_{\delta_1, \delta_2}^*(F)$  may be created by

$$SCSMn_{\delta_1, \delta_2}^*(\widehat{F}_n) \pm z_{\frac{\gamma}{2}} \sqrt{\frac{\widehat{Var}(F, L_{\delta_1, \delta_2})}{n}}, \tag{30}$$

**Example 3.3.** Consider a random sample  $Y_1, Y_2, \dots, Y_n$  drawn from an exponential distribution with a mean of one. For  $j = 1, 2, \dots, n - 1$ , the sample spacings, which are provided by  $\Delta_{j+1} = Y_{(j+1)} - Y_{(j)}$  are independent, and each  $\Delta_{j+1}$  has an distribution of exponential with a parameter of  $(n - j)$ . Consequently, from (25), we have the following

1. The mean of  $SCSMn_{\delta_1, \delta_2}(\widehat{F}_n)$  is

$$E[SCSMn_{\delta_1, \delta_2}(\widehat{F}_n)] = \frac{1}{\delta_1 - \delta_2} \sum_{j=1}^{n-1} \frac{1}{n-j} \left[ \left(1 - \frac{j}{n}\right)^{\delta_1} - \left(1 - \frac{j}{n}\right)^{\delta_2} \right]. \tag{31}$$

2. The variance of  $SCSMn_{\delta_1, \delta_2}(\widehat{F}_n)$  is

$$Var[SCSMn_{\delta_1, \delta_2}(\widehat{F}_n)] = \frac{1}{(\delta_1 - \delta_2)^2} \sum_{j=1}^{n-1} \frac{1}{(n-j)^2} \left[ \left(1 - \frac{j}{n}\right)^{\delta_1} - \left(1 - \frac{j}{n}\right)^{\delta_2} \right]^2. \tag{32}$$

With reference to Eq. (28) and the relations (4.6.6)-(4.6.8) given by Arnold et al. [1], it can be deduced that

1. The mean of  $SCSMn_{\delta_1, \delta_2}^*(\widehat{F}_n)$  is

$$E[SCSMn_{\delta_1, \delta_2}^*(\widehat{F}_n)] = \sum_{j=1}^n \frac{1}{n} L_{\delta_1, \delta_2} \left( \frac{j}{n} \right) \sum_{l=1}^j \frac{1}{n-l+1}. \tag{33}$$

2. The variance of  $SCSMn_{\delta_1, \delta_2}^*(\widehat{F}_n)$  is

$$Var[SCSMn_{\delta_1, \delta_2}^*(\widehat{F}_n)] = \sum_{j=1}^n \left( \frac{1}{n} L_{\delta_1, \delta_2} \left( \frac{j}{n} \right) \right)^2 \sum_{l=1}^j \frac{1}{(n-l+1)^2} + 2 \sum_{j=1}^n \sum_{k=1}^n \frac{1}{n^2} L_{\delta_1, \delta_2} \left( \frac{j}{n} \right) L_{\delta_1, \delta_2} \left( \frac{k}{n} \right) \sum_{l=1}^j \frac{1}{(n-l+1)^2}. \tag{34}$$

Table 1 presents the mean and variance of the empirical survival cumulative Sharma–Taneja–Mittal function of entropy for the mentioned relationships. Varying values of the orders  $\delta_1$  and  $\delta_2$  were selected, with sample sizes of 10, 30, 50, 100, and 200. The exact survival cumulative Sharma–Taneja–Mittal entropy is given by  $\frac{-1}{\rho \delta_1 \delta_2}$ , where  $\rho$  equals one, where the values are  $SCSMn_{1.1,1.8}(Y) = -0.505051$ ,  $SCSMn_{1.7,1.2}(Y) = -0.490196$ ,  $SCSMn_{2,3}(Y) = -0.166667$ ,  $SCSMn_{4,2}(Y) = -0.125$ ,  $SCSMn_{3,5,5}(Y) = -0.0571429$ ,  $SCSMn_{5,5,2}(Y) = -0.0909091$ . It is clear that as sample size rises, the variance of the empirical measure approaches zero and the mean converges to the real value. Additionally, the first estimator provides a more accurate result compared to the second.

**Example 3.4.** Examine the 46-sample dataset that Balakrishnan et al. [3] presented, which shows the active repair timeframes (in hours) for an aerial communication transceiver. Jahanshahi et al. [7] analyzed this data and successfully modeled it using a parameter of  $\rho = 0.2773$  for the distribution of exponential. The theoretical survival cumulative Sharma–Taneja–Mittal entropy, calculated using this distribution of exponential parameter, along with confidence intervals (at  $\gamma = 0.05$ ) derived from equations (26) and (30), are illustrated in Figures 3 and 4 with fixed  $\delta_2 = 1.5$  and  $\delta_1 = 2.5$ , respectively. It is evident that the theoretical value is included in the confidence intervals of the estimators. Therefore, the confidence interval derived from equation (26) is more precise than that obtained from equation (30).

Table 1: Expected value and variance of empirical survival cumulative Sharma–Taneja–Mittal entropy for the distribution of exponential with unit mean for different  $\delta_1 \neq \delta_2 > 0$  values and  $n = 10, 30, 50, 100,$  and  $200$  sample sizes.

Sample Size	$\delta_1, \delta_2$	$E[SCSMn_{\delta_1, \delta_2}(\widehat{F}_n)]$	$Var[SCSMn_{\delta_1, \delta_2}(\widehat{F}_n)]$	$E[SCSMn_{\delta_1, \delta_2}^*(\widehat{F}_n)]$	$Var[SCSMn_{\delta_1, \delta_2}^*(\widehat{F}_n)]$
$n = 10$	1.1, 1.8	-0.459634	0.0109339	-0.267746	0.00586474
	1.7, 1.2	-0.451073	0.0100228	-0.270936	0.00645258
	2, 3	-0.165	0.000285	0.1425	0.00348734
	4, 2	-0.12375	0.000120833	-0.1095	0.002374
	3.5, 5	-0.0563134	0.0000129845	-0.0532954	0.000854995
	5.5, 2	-0.0898392	0.0000514718	-0.0809013	0.00154682
$n = 30$	1.1, 1.8	-0.4912	0.001445	-0.396827	0.0016825
	1.7, 1.2	-0.479198	0.00121174	-0.394646	0.00184012
	2, 3	-0.166481	0.0000117353	-0.158426	0.000543335
	4, 2	-0.124861	$4.78395 \times 10^{-6}$	-0.11958	0.000358106
	3.5, 5	-0.0570503	$4.80999 \times 10^{-7}$	-0.0556084	0.000116718
	5.5, 2	-0.0907901	$2.00722 \times 10^{-6}$	-0.0873333	0.000232
$n = 50$	1.1, 1.8	-0.497108	0.00052313	-0.432736	0.00081753
	1.7, 1.2	-0.484145	0.00041671	-0.427809	0.00088831
	2, 3	-0.1666	$2.5872 \times 10^{-6}$	-0.1617	0.000211768
	4, 2	-0.12495	$1.04667 \times 10^{-6}$	-0.121716	0.00013839
	3.5, 5	-0.0571095	$1.03896 \times 10^{-7}$	-0.0561907	0.0000438149
	5.5, 2	-0.0908662	$4.37914 \times 10^{-7}$	-0.0887333	0.0000894511
$n = 100$	1.1, 1.8	-0.501325	0.00012597	-0.464299	0.00028222
	1.7, 1.2	-0.48752	0.0000928014	-0.456162	0.000302964
	2, 3	-0.16665	$3.2835 \times 10^{-7}$	-0.164175	0.000056588
	4, 2	-0.124988	$1.32083 \times 10^{-7}$	-0.123346	0.0000366958
	3.5, 5	-0.0571345	$1.2987 \times 10^{-8}$	-0.0566549	0.0000113056
	5.5, 2	-0.0908984	$5.51474 \times 10^{-8}$	-0.0898096	0.0000236632
$n = 200$	1.1, 1.8	-0.503307	0.000029305	-0.482612	0.000090962
	1.7, 1.2	-0.489018	0.0000197751	-0.472065	0.0000961653
	2, 3	-0.166663	$4.13547 \times 10^{-8}$	-0.165419	0.0000146934
	4, 2	-0.124997	$1.65885 \times 10^{-8}$	-0.12417	$9.4827 \times 10^{-6}$
	3.5, 5	-0.0571408	$1.62338 \times 10^{-9}$	-0.0568959	$2.87167 \times 10^{-6}$
	5.5, 2	-0.0909064	$6.91893 \times 10^{-9}$	-0.0903565	$6.10528 \times 10^{-6}$

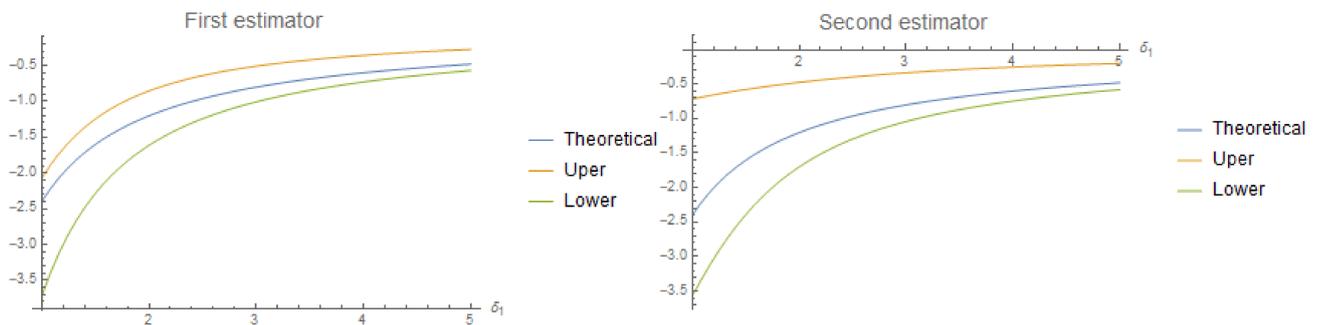


Figure 3: Theoretical value of the cumulative survival Sharma–Taneja–Mittal entropy with 95% confidence intervals using Equations (26) (Left) and (30) (Right) in Example 3.2, with fixed  $\delta_2 = 1.5$ .

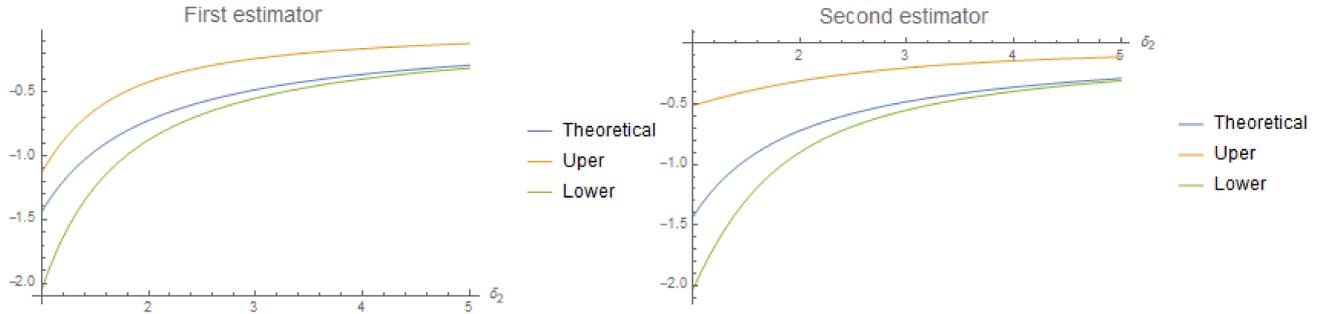


Figure 4: Theoretical value of the cumulative survival Sharma–Taneja–Mittal entropy with 95% confidence intervals using Equations (26) (Left) and (30) (Right) in Example 3.2, with fixed  $\delta_1 = 2.5$ .

#### 4. Uniformity test via test statistics

In this section, we develop a reliable test for the uniformity hypothesis (where the null hypothesis  $H_0$  assumes uniformity), by utilizing a consistent statistical test as introduced in (25).

**Theorem 4.1.** Consider  $Y$  as a nonnegative random variable with a continuous CDF  $F$ , where the support lies within the interval  $[0, 1]$ . Then,  $P(\delta_1, \delta_2) \leq SCSMn_{\delta_1, \delta_2}(F) \leq 0$ , where  $P(\delta_1, \delta_2)$  is defined in (8). Additionally, the expression  $\frac{-1}{(1+\delta_1)(1+\delta_2)}$  is exclusively reached by the standard or usual distribution of uniform  $U(0, 1)$  for any  $\delta_1 \neq \delta_2 \geq 0$ .

*Proof.* Using (5) and the relationship given in (9), it is evident that  $P(\delta_1, \delta_2) \leq SCSMn_{\delta_1, \delta_2}(F) \leq 0$ . Furthermore, by applying the convexity of  $S(Y; \delta_1, \delta_2) = \frac{1}{\delta_1 - \delta_2}(y^{\delta_1} - y^{\delta_2})$ , it follows that  $SCSMn_{\delta_1, \delta_2}(F)$  is a convex function of distributions supported on  $[0, 1]$ . Therefore,  $SCSMn_{\delta_1, \delta_2}(F) = \frac{-1}{(1+\delta_1)(1+\delta_2)}$  is distinctively achieved by the distribution  $U(0, 1)$ . This concludes the proof.  $\square$

**Theorem 4.2.** Let a random sample  $Y_1, Y_2, \dots, Y_n$  be pulled from an unknown continuous cumulative distribution function  $F$  defined on the interval  $[0, 1]$ . Then, according to (25), the following holds:  $P(\delta_1, \delta_2) \leq SCSMn_{\delta_1, \delta_2}(\widehat{F}_n) \leq 0$  for  $\delta_1 \neq \delta_2 \geq 0$ .

*Proof.* Based on (9), it follows that:

$$0 \geq SCSMn_{\delta_1, \delta_2}(\widehat{F}_n) \geq \sum_{i=1}^{n-1} P(\delta_1, \delta_2) \Delta_i = P(\delta_1, \delta_2)(Y_{(n)} - Y_{(1)}) \geq P(\delta_1, \delta_2).$$

This concludes the proof.  $\square$

##### 4.1. Percentage points of the empirical cumulative survival test statistic

This section finds the distribution's asymptotic value for  $SCSMn_{\delta_1, \delta_2}(\widehat{F}_n)$  under the null hypothesis, employing three distinct approaches. For simplicity, from (25), we will write

$$S_n(\delta_1, \delta_2) = SCSMn_{\delta_1, \delta_2}(\widehat{F}_n) = \sum_{k=1}^{n-1} \Delta_{k+1} D_k, \tag{35}$$

where  $D_k = \frac{1}{\delta_1 - \delta_2} \left( \left(1 - \frac{k}{n}\right)^{\delta_1} - \left(1 - \frac{k}{n}\right)^{\delta_2} \right)$ ,  $\delta_1 \neq \delta_2 \geq 0$ , and  $k = 1, 2, \dots, n - 1$ .

**Theorem 4.3.** The variance and expected value of  $S_n(\delta_1, \delta_2)$  under  $H_0$  are provided below, respectively:

$$E[S_n(\delta_1, \delta_2)] = \frac{1}{n+1} \sum_{k=1}^{n-1} D_k, \text{ and } Var[S_n(\delta_1, \delta_2)] = \frac{n}{(n+1)^2(n+2)} \sum_{k=1}^{n-1} D_k^2.$$

*Proof.* Notice that for any  $k = 1, 2, \dots, n - 1$ , the random variable  $\Delta_k$ , which is based on the distribution  $U(0, 1)$ , follows a distribution of beta with parameters  $(1, n)$ , denoted as  $\Delta_k \sim \text{Beta}_d(\cdot; 1, n)$  (see [1]), and the outcome is as follows.  $\square$

**Remark 4.4.** Assuming  $H_0$  is true, as  $n$  approaches infinity, the expected value of  $\mathcal{S}_n(\delta_1, \delta_2)$  converges to  $\frac{-1}{(1+\delta_1)(1+\delta_2)}$ , while the variance of  $\mathcal{S}_n(\delta_1, \delta_2)$  approaches zero.

The critical region that defines the uniformity test is determined by the following conditions:

$$\mathcal{S}_n(\delta_1, \delta_2) \leq \text{SCSM}n_{\delta_1, \delta_2, \frac{\gamma}{2}}^* := \text{lower}, \quad \text{or} \quad \mathcal{S}_n(\delta_1, \delta_2) \geq \text{SCSM}n_{\delta_1, \delta_2, 1-\frac{\gamma}{2}}^* := \text{upper}, \quad (36)$$

where  $\gamma$  represents the chosen significance level, and  $\text{SCSM}n_{\delta_1, \delta_2, \gamma}^*$  is the  $\gamma$ -quantile of the approximated (or asymptotic) CDF of the test statistic  $\mathcal{S}_n(\delta_1, \delta_2)$  under the null hypothesis  $H_0$ .

#### 4.2. The normal approximation

We get the distribution asymptotically of  $\mathcal{S}_n(\delta_1, \delta_2)$  under the null hypothesis  $H_0$ . From equation (35), we express  $\mathcal{S}_n(\delta_1, \delta_2)$  as  $\mathcal{S}_n(\delta_1, \delta_2) = \sum_{k=1}^{n-1} B_k$ , where  $B_k = D_k \Delta_k$  for  $k = 1, 2, \dots, n - 1$ , and  $\Delta_k \sim \text{Beta}_d(\cdot; 1, n)$ . Therefore, the PDF of the random variable  $B_k$  is obtained by

$$f_{B_k}(y) = \frac{n}{D_k} \left(1 - \frac{y}{D_k}\right)^{n-1}, \quad k = 1, 2, \dots, n - 1.$$

The expected value and variance of  $B_k$  are  $\mu_k = E[B_k] = D_k E[\Delta_k] = \frac{D_k}{n+1}$  and  $\sigma_k^2 = \text{Var}[B_k] = D_k^2 \text{Var}[\Delta_k] = \frac{nD_k^2}{(n+1)^2(n+2)}$ , respectively. By applying the central limit theorem of Lyapunov (cf. Billingsley [4]), we derive the following result:

$$\frac{\sum_{k=1}^{n-1} (B_k - \mu_k)}{\sqrt{\sum_{k=1}^{n-1} \sigma_k^2}} = \frac{\mathcal{S}_n(\delta_1, \delta_2) - E[\mathcal{S}_n(\delta_1, \delta_2)]}{\sqrt{\text{Var}[\mathcal{S}_n(\delta_1, \delta_2)]}} \xrightarrow{d} N(0, 1).$$

Thus, under  $H_0$ , the  $\gamma$ -quantile  $\text{SCSM}n_{\delta_1, \delta_2, \gamma}^*$  can be estimated for large  $n$  by utilizing the normalcy asymptotically of  $\mathcal{S}_n(\delta_1, \delta_2)$  by

$$\widehat{\text{SCSM}}n_{\delta_1, \delta_2, \gamma}^* = E[\mathcal{S}_n(\delta_1, \delta_2)] + z_\gamma \sqrt{\text{Var}[\mathcal{S}_n(\delta_1, \delta_2)]}. \quad (37)$$

#### 4.3. The beta approximation

Johannesson and Giri [8] provided an approximation for the CDF of a linear combination of a finite number of beta-distributed random variables. Building on this work, Noughabi [12] utilized their approximation to estimate the percentage critical points of survival cumulative entropy for a finite sample size  $n$ . Following a similar approach, one can derive an approximation for  $\mathcal{S}_n(\delta_1, \delta_2)$  for finite  $n$  as follows:

$$\mathcal{S}_n(\delta_1, \delta_2) \approx \left( \sum_{k=1}^{n-1} D_k \right) \cdot \chi,$$

with noting that the random variable  $\chi$  is distributed according to the beta distribution  $\text{Beta}_d(\cdot; \alpha, \beta)$ . The parameters  $\alpha$  and  $\beta$  are defined as:

$$\alpha = \frac{(n+2) \left( \sum_{k=1}^{n-1} D_k \right)^2}{(n+1) \left( \sum_{k=1}^{n-1} D_k^2 \right)} - \frac{1}{n+1}, \quad \text{and} \quad \beta = \frac{n}{n+1} \left( \frac{(n+2) \left( \sum_{k=1}^{n-1} D_k \right)^2}{\sum_{k=1}^{n-1} D_k^2} - 1 \right). \quad (38)$$

Using this formulation, the expected value and variance of  $S_n(\delta_1, \delta_2)$  can be calculated as:

$$E(S_n(\delta_1, \delta_2)) = \left( \sum_{k=1}^{n-1} D_k \right) \cdot \frac{\alpha}{\alpha + \beta},$$

and

$$\text{Var}(S_n(\delta_1, \delta_2)) = \left( \sum_{k=1}^{n-1} D_k \right)^2 \cdot \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

This approximation value of  $S_n(\delta_1, \delta_2)$  can then be used to determine the quantiles at  $\frac{\gamma}{2}$  and  $1 - \frac{\gamma}{2}$  of the approximated CDF for the test statistic under  $H_0$ . These quantiles are expressed as:

$$\text{lower} := \left( \sum_{k=1}^{n-1} A_k \right) \cdot \text{Beta}_d^{-1}\left(\frac{\gamma}{2}; \alpha, \beta\right), \quad \text{and} \quad \text{upper} := \left( \sum_{k=1}^{n-1} A_k \right) \cdot \text{Beta}_d^{-1}\left(1 - \frac{\gamma}{2}; \alpha, \beta\right), \tag{39}$$

as  $\text{Beta}_d^{-1}(\cdot; \alpha, \beta)$  is the beta distribution’s quantile function of  $\text{Beta}_d(\cdot; \alpha, \beta)$ , with parameters  $\alpha$  and  $\beta$  as specified in equation (38).

#### 4.4. Percentage points

50,000 samples are generated with  $n = 10, 20, 30, 40, 50, 70, 100$  sample sizes from  $U(0, 1)$  distribution. The test statistic  $S_n(\delta_1, \delta_2)$  is estimated using the empirical survival cumulative Sharma–Taneja–Mittal entropy for each sample, as described in (25). Additionally, it is observed that  $SCSMn_{1.1,1.8}(U) = -0.170068$ ,  $SCSMn_{1.7,1.2}(U) = -0.16835$ ,  $SCSMn_{2,3}(U) = -0.0833333$ ,  $SCSMn_{4,2}(U) = -0.0666667$ ,  $SCSMn_{3,5,5}(U) = -0.037037$ , and  $SCSMn_{5,5,2}(U) = -0.0512821$ , where  $SCSMn_{\delta_1, \delta_2}(U)$  refers to the survival cumulative Sharma–Taneja–Mittal entropy of the CDF  $U(0, 1)$  distribution. Therefore, for  $S_n(\delta_1, \delta_2)$ , we provide the percentage critical points of the Monte Carlo approach, asymptotic normality, and beta approximation by applying (36), (37), and (39), respectively. Table 2 displays this information. Table 2 illustrates that as  $n$  grows, the difference in percentage critical points narrows. Moreover, the Monte Carlo method shows greater accuracy compared to the other two methods for  $S_n(\delta_1, \delta_2)$ .

The empirical PDFs of the test statistics acquired using Monte Carlo samples for  $n$  values of 10, 20, 30, 40, and 50 are shown in Figure 6. The test statistics are shown to approach the precise values more closely as  $n$  grows, indicating that bigger sample sizes may reduce bias and variation.

#### 4.5. Power analysis

In this subsection, we analyze the power test analysis of the Monte Carlo procedure under different alternative distributions. Using the Monte Carlo approach with 50,000 produced samples for each sample size of  $n = 10, 20, 30$ , the power of  $S_n(\delta_1, \delta_2)$  is found and compared with the alternative CDFs provided by Stephens [19] in his power analysis of uniformity tests as follows

$$\begin{aligned} V_1^{(\omega)} : F(t) &= 1 - (1 - t)^\omega, \quad 0 \leq t \leq 1, \omega = 1.5, 2, \\ V_2^{(\omega)} : F(t) &= \begin{cases} 2^{\omega-1}t^\omega, & 0 \leq t \leq 0.5, \\ 1 - 2^{\omega-1}(1 - t)^\omega, & 0.5 \leq t \leq 1, \omega = 1.5, 2, 3, \end{cases} \\ V_3^{(\omega)} : F(t) &= \begin{cases} 0.5 - 2^{\omega-1}(0.5 - t)^\omega, & 0 \leq t \leq 0.5, \\ 0.5 + 2^{\omega-1}(t - 0.5)^\omega, & 0.5 \leq t \leq 1, \omega = 1.5, 2. \end{cases} \end{aligned} \tag{40}$$

Table 2: Critical points methods for the test statistic  $S_n(\delta_1, \delta_2)$  at  $\gamma = 0.05$ .

$n$	$\delta_1, \delta_2$	$S_n(\delta_1, \delta_2)$					
		Normal technique		Beta technique		Monte Carlo technique	
		lower	upper	lower	upper	lower	upper
10	1.1, 1.8	-0.249171	-0.0571567	-0.26169	-0.0714618	-0.200113	-0.0986391
	1.7, 1.2	-0.246942	-0.0564676	-0.259378	-0.070681	-0.198591	-0.0974269
	2, 3	-0.125188	-0.0248117	-0.132146	-0.0328312	-0.105282	-0.0437768
	4, 2	-0.100822	-0.0188776	-0.106625	-0.0255862	-0.0860055	-0.0338412
	3.5, 5	-0.058043	-0.00778994	-0.0619815	-0.0124153	-0.0525649	-0.0157646
	5.5, 2	-0.0780547	-0.0136764	-0.0827206	-0.0190894	-0.0676043	-0.0249781
20	1.1, 1.8	-0.235885	-0.0873197	-0.243512	-0.0956751	-0.194406	-0.124286
	1.7, 1.2	-0.233683	-0.0863147	-0.241263	-0.0946191	-0.192823	-0.122844
	2, 3	-0.118005	-0.0403287	-0.122247	-0.0450038	-0.100737	-0.0569786
	4, 2	-0.0950045	-0.031583	-0.098538	-0.0354848	-0.0818799	-0.0445931
	3.5, 5	-0.0545392	-0.0156112	-0.0569279	-0.0182762	-0.0489127	-0.0221725
	5.5, 2	-0.0735629	-0.0237208	-0.0763993	-0.0268596	-0.0640443	-0.0335033
30	1.1, 1.8	-0.227004	-0.101834	-0.232465	-0.107714	-0.190925	-0.134612
	1.7, 1.2	-0.224854	-0.100692	-0.230281	-0.106537	-0.189331	-0.133006
	2, 3	-0.113278	-0.0478332	-0.116317	-0.0511212	-0.0983	-0.0623443
	4, 2	-0.0911442	-0.0377089	-0.093675	-0.0404516	-0.0797515	-0.0490437
	3.5, 5	-0.0521528	-0.0193527	-0.0538631	-0.0212217	-0.0473482	-0.0250782
	5.5, 2	-0.0705357	-0.0285408	-0.0725666	-0.0307456	-0.0623431	-0.0370651

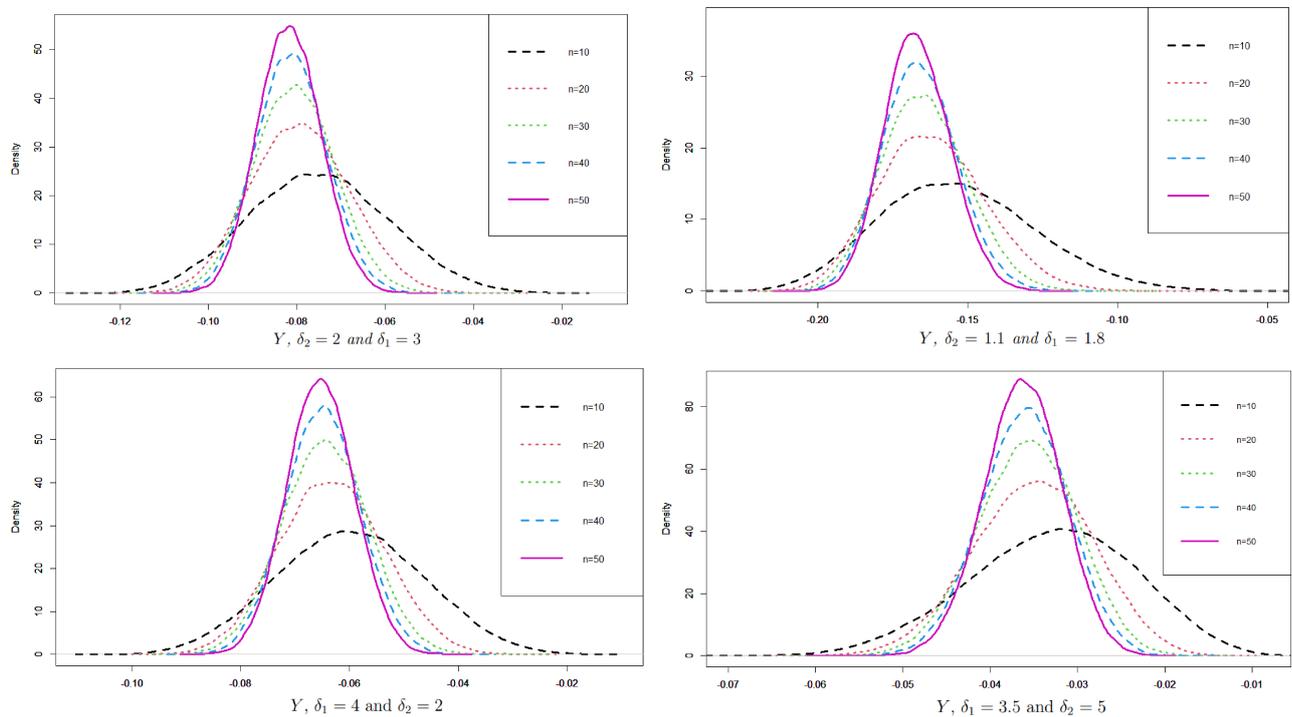


Figure 5: The estimated PDFs of  $S_n(\delta_1, \delta_2)$  for the distribution  $U(0, 1)$  with  $\delta_1 = 1.1, \delta_2 = 1.8$  (Left) and  $\delta_1 = 2, \delta_2 = 3$  (Right).

The power values for the proposed test statistics  $S_n(\delta_1, \delta_2)$ , along with the Kolmogorov-Smirnov ( $T_{KS}$ ),

Kuiper ( $T_{Kp}$ ), Cramer-von Mises ( $T_{CM}$ ), Watson ( $T_W$ ), and Anderson-Darling ( $T_{AD}$ ) tests, were recorded in Table 3 for sample sizes  $n = 10, 20, 30$  and significance level  $\gamma = 0.05$ . Based on Table 3, the following can be concluded conclusions:

1. For fixed both  $\delta_1, \delta_2$  and increasing  $n$ , we see that  $\mathcal{S}_n(\delta_1, \delta_2)$  is shown to rise in power.
2. For the alternatives  $V_2^{(\omega)}$  and  $V_3^{(\omega)}$ , the power analysis of  $\mathcal{S}_n(\delta_1, \delta_2)$  increases and give the best performance against the other tests especially when  $\delta_1, \delta_2$  has small values.

Table 3: Power analysis estimates at level  $\gamma = 0.05$  of the tests.

n	Alternative	$\mathcal{S}_n(\delta_1, \delta_2)$				$T_{KS}$	$T_{Kp}$	$T_{CM}$	$T_W$	$T_{AD}$
		$\delta_1 = 1.1, \delta_2 = 1.8$	$\delta_1 = 2, \delta_2 = 3$	$\delta_1 = 4, \delta_2 = 2$	$\delta_1 = 3.5, \delta_2 = 5$					
10	$V_1^{(1.5)}$	0.08132	0.10312	0.10666	0.10098	0.12516	0.0746	0.1456	0.07476	0.1857
	$V_1^{(2)}$	0.18746	0.2572	0.26258	0.23606	0.31298	0.1631	0.3551	0.16308	0.4761
	$V_2^{(1.5)}$	0.139	0.10398	0.09222	0.05944	0.07352	0.0971	0.0741	0.1017	0.1349
	$V_2^{(2)}$	0.36	0.25718	0.22302	0.12264	0.1184	0.2307	0.1104	0.2481	0.3269
	$V_2^{(3)}$	0.80164	0.63718	0.56382	0.3195	0.2424	0.5394	0.2154	0.5699	0.72308
	$V_3^{(1.5)}$	0.12798	0.13134	0.12908	0.1215	0.0352	0.0964	0.0239	0.1031	0.0222
	$V_3^{(2)}$	0.27792	0.25878	0.2416	0.19878	0.0422	0.2333	0.01114	0.2475	0.00924
20	$V_1^{(1.5)}$	0.13092	0.21852	0.22614	0.20734	0.2179	0.1226	0.25208	0.1225	0.3235
	$V_1^{(2)}$	0.39082	0.5798	0.58992	0.53514	0.5616	0.3486	0.6241	0.3358	0.7538
	$V_2^{(1.5)}$	0.3026	0.20456	0.16778	0.0755	0.0869	0.1634	0.0781	0.1786	0.1774
	$V_2^{(2)}$	0.75318	0.56988	0.47548	0.20446	0.1849	0.4647	0.162	0.5067	0.52802
	$V_2^{(3)}$	0.9953	0.96376	0.924	0.5891	0.4588	0.8711	0.4615	0.8978	0.93998
	$V_3^{(1.5)}$	0.2266	0.19836	0.1816	0.1482	0.0509	0.1621	0.02406	0.1791	0.0213
	$V_3^{(2)}$	0.5436	0.40476	0.3448	0.23868	0.1162	0.4633	0.0462	0.5048	0.0338
30	$V_1^{(1.5)}$	0.18808	0.32584	0.33884	0.31824	0.3154	0.18202	0.3665	0.1731	0.4498
	$V_1^{(2)}$	0.5732	0.787	0.79972	0.75618	0.7522	0.5447	0.8105	0.5071	0.8973
	$V_2^{(1.5)}$	0.47032	0.31102	0.24742	0.09676	0.1021	0.2477	0.0873	0.2667	0.2281
	$V_2^{(2)}$	0.92974	0.78382	0.68612	0.29866	0.2706	0.6695	0.25108	0.7076	0.7002
	$V_2^{(3)}$	1	0.99828	0.99174	0.79194	0.6701	0.97506	0.7237	0.9819	0.99104
	$V_3^{(1.5)}$	0.34126	0.2554	0.21826	0.15296	0.07	0.2492	0.0323	0.2678	0.0271
	$V_3^{(2)}$	0.74646	0.50164	0.40376	0.24102	0.2067	0.6711	0.1258	0.7131	0.1105

### Conclusion

In this consideration, some of the properties and features of the survival cumulative Sharma–Taneja–Mittal entropy were discussed. The most important of these properties is knowing the non-positivity of this model and its generalization to the Tsallis entropy. It is worth noting that employing this measure is computationally more convenient. Moreover, the bounds, convergence, and its relationship with differential entropy have been discussed. Given that the mean residual lifespan of a PHz model may be used to describe the surviving cumulative Sharma–Taneja–Mittal entropy, several outcomes, including comparable formulations and preservation qualities under specific well-known stochastic orders, have been investigated. Additionally, the relationship between the cumulative Sharma–Taneja–Mittal function of entropy of survival and the excess wealth order, along with its normalized form, has been explored. We explore the issue of estimating the survival cumulative Sharma–Taneja–Mittal entropy using the empirical CDF. To do so, we utilize two different empirical estimators of the CDF. Using random samples taken from an exponential distribution, we prove a central limit theorem for the empirical measure for the first estimator. We give a central limit theorem for the empirical model based on random samples from an unknown distribution for the second estimator. We then apply the estimator of this measure in a test for uniformity. A power comparison was conducted between our measure and other tests, showing when the test had greater or lesser power compared to others by varying the values of  $\delta_1$  and  $\delta_2$ .

**Author contributions**

Conceptualization, Formal Analysis, Investigation, Methodology, Software, Visualization, Resources, Writing—Original Draft, Validation, and Writing—Review & Editing: M.S.M.

**Competing Interests**

The author declares that he has no financial or non-financial conflicts of interest to disclose.

**Data Availability**

All datasets analyzed or generated during this study are included within the article.

**Ethics Approval and Consent to Participate**

The author approved any ethics issues and participate.

**AI Usage Declaration**

The author confirms that no artificial intelligence tools were employed in the preparation of this manuscript.

**Conflict of Interest**

The author states that there are no conflicts of interest associated with this work.

**Consent for Publication**

The author read and approved the publication of this work.

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