



## $\alpha$ -Well posedness by perturbations of split quasi-equilibrium problems in Banach spaces

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**Abstract.** Zolezzi introduced the notion of well-posedness by perturbations for the minimization problem. In this paper, we extend this concept to Levitin-Polyak  $\alpha$ -well-posedness by perturbations for split quasi-equilibrium problems in real Banach spaces. We establish some metric characterization results between the (generalized)  $\alpha$ -well-posedness by perturbations for split quasi-equilibrium problems and their solution set with the help of Kuratowski's measure of non-compactness. Moreover, we derive some conditions under which the  $\alpha$ -well-posedness by perturbations of a split quasi-equilibrium problem is equivalent to the existence and uniqueness of its solution.

### 1. Introduction

Equilibrium problems were first introduced by Fan [16] and have been studied deeply since then (see [20, 28–31]). Equilibrium problems play a significant role in different branches of mathematics, such as dynamical systems, partial differential equations, and calculus of variation. The framework of equilibrium problems covers a broad area of mathematics, such as non-cooperative games [14], linear complementary problems [12], variational inequalities [15, 20] and vector optimization problems [33].

Let  $X$  be a real Banach space with norm  $\|\cdot\|$ . Let  $X^*$  be the dual of  $X$ , and let  $\langle \cdot, \cdot \rangle$  denote the duality pairing between  $X^*$  and  $X$ . Suppose that  $C$  is a non-empty closed and convex subset of  $X$ , and that  $f : X \times X \rightarrow \mathbb{R} := \mathbb{R} \cup \{\infty\}$  is a bifunction. The *equilibrium problem* is to find

$$x \in C \text{ such that } f(x, y) \geq 0 \quad \forall y \in C. \quad (1)$$

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The problem (1) is denoted by  $EP(f, C)$ . The classical example of the equilibrium problems is the well-known Stampacchia variational inequality problem [40].

Another class of problems is the quasi-equilibrium problem that presents mathematical framework such as equilibrium problems, variational inequality problems and fixed point problems. Solving quasi-equilibrium problems typically involves optimization problems, focusing on identifying the most effective optimal state. This expertise is crucial in sectors whose resource utilization and operational efficiency are paramount. A prime example of such a scenario is the Nash equilibrium problem [14], which is applicable to various real-world situations, including economics and non-cooperative games (see [27] and references therein).

The *quasi-equilibrium problem* (QEP) is to find:

$$x \in K(x) \text{ such that } f(x, y) \geq 0, \forall y \in K(x),$$

where  $K : C \rightarrow 2^C$  is a set-valued map. If we take

$$f(x, y) = \langle F(x), y - x \rangle, \quad (2)$$

where  $F$  is a single-valued mapping  $F : C \rightarrow X$ , then QEP becomes the *quasi-variational inequality problem* [1] of finding  $x^* \in C$  such that

$$x^* \in K(x^*) \text{ with } \langle F(x^*), y - x^* \rangle \geq 0, \forall y \in K(x^*). \quad (3)$$

The problem (3) is denoted by  $QVI(F, K)$ .

The concept of well-posedness for variational inequalities was first given by Lucchetti and Patrone [26]. Subsequently, the notions of Levitin-Polyak well-posedness and  $\alpha$ -well posedness by perturbations were investigated in [11, 18, 37]. These notions were extended for (mixed) quasi-variational inequality problems in [5, 23, 35, 36, 41] and in hemivariational inequalities [2–4, 6, 7].

In the last few years, split inverse problems have gained interests in its relevance to various domains like radiation therapy, sensor networks, cancer treatment planning, and image reconstruction (see [9, 39]). We need two vector spaces,  $U_1$  and  $U_2$ , and a bounded linear operator  $A : U_1 \rightarrow U_2$  for these mathematical formulations and the problem is to find  $x^* \in U_1$  such that

$$x^* \text{ solves } I_1$$

and  $y^* = Ax^*$  satisfies:

$$y^* \text{ solves } I_2,$$

where  $I_1$  and  $I_2$  represent inverse problems in  $U_1$  and  $U_2$ , respectively. Hu and Fang [19] analyzed perturbation-based Levitin-Polyak well-posedness and established its equivalence with the non-emptiness and boundedness of the solution set for algorithmic convergence of these problems. By choice of suitable conditions, several well-known problems can be viewed as special cases of this general framework that includes split convex feasibility problem [10] and split variational inclusion problem [34].

Suppose that  $X_1$  and  $X_2$  are real Banach spaces with their dual  $X_1^*$  and  $X_2^*$ , respectively, and that  $C_1$  and  $C_2$  are non-empty closed and convex subsets of  $X_1$  and  $X_2$ , respectively. Let  $f_1 : X_1 \times X_1 \rightarrow \mathbb{R}$  and  $f_2 : X_2 \times X_2 \rightarrow \mathbb{R}$  be two bifunctions and  $A : X_1 \rightarrow X_2$  be a bounded linear operator. In 2011, Moudafi [34] introduced the *split equilibrium problem* (SEP) as follows: Find  $x^* \in C_1$  such that

$$\begin{cases} f_1(x^*, u) \geq 0, \forall u \in C_1, \\ y^* = Ax^* \in C_2 \text{ solves } f_2(y^*, v) \geq 0, \forall v \in C_2. \end{cases} \quad (4)$$

Motivated by the concept of split inverse problems, Mittal et al. [32] considered the *split quasi-equilibrium problem* (SQEP) which is given as follows: Find  $(x^*, y^*) \in X_1 \times X_2$  such that

$$\begin{cases} x^* \in K_1(x^*) \text{ and } y^* = Ax^* \in K_2(y^*); \\ \text{with } f_1(x^*, z) \geq 0, \forall z \in K_1(x^*); \\ f_2(y^*, w) \geq 0, \forall w \in K_2(y^*), \end{cases} \quad (\text{SQEP})$$

where  $K_1 : C_1 \rightarrow 2^{C_1}$  and  $K_2 : C_2 \rightarrow 2^{C_2}$  are constrained set-valued maps. We denote the solution set of problem (SQEP) by  $S$ .

Considering that perturbations can enhance the stability of a problem [42] and inspired by the notion of well-posedness in variational inequalities and equilibrium problems [13, 30], in this work, we present and study this notion for split quasi-equilibrium problems in an extended manner.

The paper is organized as follows. In Section 2, we provide several notations, definitions and technical lemmas that used throughout the paper. Section 3 introduces the perturbed split quasi-equilibrium problems and the concept of the (Levitin-Polyak)  $\alpha$ -approximating sequence and generalized (Levitin-Polyak)  $\alpha$ -approximating sequence for (SQEP). This section also presents various notions of  $\alpha$ -well-posedness for (SQEP) and establishes a result connecting the solutions of (SQEP) with the concept of  $\alpha$ -well-posedness. Section 4 provides several metric criteria for characterizing (generalized) Levitin-Polyak  $\alpha$ -well-posedness by perturbations for (SQEP), illustrated with non-trivial examples. Moreover, we establish the equivalence of the  $\alpha$ -well-posedness by perturbations for (SQEP) with the existence and uniqueness of its solution. Finally, some conclusions are given in Section 5.

## 2. Notations and preliminaries

This section collects some definitions and technical results that will be used later in the paper. Throughout, the set  $C$  is a non-empty closed and convex subset of  $X$ . The strong convergence and the weak convergence of sequences in  $X$  are denoted by  $\rightarrow$  and  $\rightharpoonup$ , respectively. For any  $x \in X$ , define

$$\text{dist}(x, C) := \inf\{\|x - y\| : y \in C\}.$$

Let  $A$  and  $B$  be two non-empty subsets of a real Banach space  $X$ . We denote  $\text{int } A, \text{Cl}(A)$  by the interior and closure of set  $A$ , respectively. The Hausdorff metric  $H(\cdot, \cdot)$  between  $A$  and  $B$  is defined by

$$H(A, B) = \max\{e(A, B), e(B, A)\},$$

where  $e(A, B) = \sup_{a \in A} \text{dist}(a, B)$ . Let  $\{A_n\}$  be a sequence of subsets of  $X$ . We say that  $\{A_n\}$  converges to  $A$  if and only if  $H(A_n, A) \rightarrow 0$  as  $n \rightarrow \infty$ . The interior of  $B$ , closure of  $B$  and diameter of  $B$  is denoted by  $\text{int } B, \text{Cl}(B)$  and

$$\text{diam}(B) = \sup\{\|x - y\| : x, y \in B\},$$

respectively.

Next we recall the notion of *Mosco convergence* [22]. A sequence  $\{A_n\}$  of subsets of  $X$  *Mosco converges* to a set  $A$  if

$$A = \liminf_n A_n \subseteq \text{w-lim sup}_n A_n,$$

where

$$\begin{aligned} \liminf_n A_n &= \{y \in X : \exists y_n \in A_n \text{ such that } y_n \rightarrow y\}, \\ \text{w-lim sup}_n A_n &= \{y \in X : \exists \text{ a subsequence } y_{n_k} \in A_{n_k}, k \in \mathbb{N}, \text{ such that } y_{n_k} \rightharpoonup y\}. \end{aligned}$$

Also,  $\liminf_n A_n$  and  $\text{w-lim sup}_n A_n$  are, respectively, called the Painlevé-Kuratowski strong limit inferior and weak limit superior of the sequence  $\{A_n\}$ . If

$$A = \liminf_n A_n,$$

then the sequence  $\{A_n\}$  of subsets of  $X$  *Lower Semi-Mosco converges* to the set  $A$ . Note that if  $\{A_n\}$  Mosco converges to  $A$ , then the sequence is also Lower Semi-Mosco convergent to  $A$ . However, the converse is not generally true.

**Definition 2.1.** [28] Let  $C, D \subset X$ . A bifunction  $f : X \times X \rightarrow \bar{\mathbb{R}}$  is

(i) convex on  $C$  with respect to the second argument if for  $x \in X$  and  $t \in [0, 1]$

$$f(x, ty_1 + (1-t)y_2) \leq tf(x, y_1) + (1-t)f(x, y_2), \forall y_1, y_2 \in C.$$

(ii) jointly u.s.c on  $C \times D$  if for any two sequences  $\{x_n\}$  and  $\{y_n\}$  in  $C$  and  $D$ , respectively with  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , we have  $f(x, y) \geq \limsup_{n \rightarrow \infty} f(x_n, y_n)$ .

**Definition 2.2.** [38] The map  $K : C \rightarrow 2^C$  is called  $M$ -continuous, if the following conditions hold:

(M1) For any sequence  $\{x_n\} \subset C$  with  $x_n \rightarrow x$ , and for each  $y \in K(x)$ , there exists a sequence  $\{y_n\}$  such that  $y_n \in K(x_n)$  and  $y_n \rightarrow y$ .

(M2) For  $y_n \in K(x_n)$  with  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , we have  $y \in K(x)$ , that is, the graph of  $K$  is closed.

**Definition 2.3.** [21] Let  $G$  be a nonempty subset of a real Banach space  $X$ . The Kuratowski measure of non-compactness of the set  $G$  is defined by

$$\beta(G) = \inf \left\{ \varepsilon > 0 : G \subset \bigcup_{i=1}^n G_i, \text{diam}(G_i) < \varepsilon, i = 1, \dots, n \right\}.$$

**Lemma 2.4.** [8] Let  $A$  and  $B$  be two nonempty subsets of a real Banach space  $X$ . Then

$$|\beta(A) - \beta(B)| \leq 2H(A, B),$$

where  $\beta$  is the Kuratowski measure of non-compactness.

We recall a lemma that will be needed in the rest of this paper.

**Lemma 2.5.** [8] Let  $(Y, \|\cdot\|)$  be a normed space and  $A, B$  be two nonempty, bounded and closed subsets of  $Y$ . Then for any  $\varepsilon > 0$  and  $x \in A$ , there exists  $y \in B$  such that

$$\|x - y\| \leq (1 + \varepsilon)H(A, B).$$

In particular, if  $A$  and  $B$  are compact, then  $\|x - y\| \leq H(A, B)$ .

Throughout the paper, we assume the following: for each  $i = 1, 2$ ,  $X_i$  is a Banach space,  $C_i$  is a non-empty closed and convex set of  $X_i$ ,  $f_i : X_i \times X_i \rightarrow \bar{\mathbb{R}}$  is a bifunction and  $K_i : C_i \rightarrow 2^{C_i}$  is a non-empty set-valued map;  $A : X_1 \rightarrow X_2$  is a bounded linear operator.

### 3. Well-posedness by perturbations of split quasi-equilibrium problems

Let  $\Lambda$  and  $\Gamma$  be parametric normed spaces. Further, let  $\Lambda_1 \subset \Lambda$  and  $\Gamma_1 \subset \Gamma$  be closed balls with positive radius centered at  $\lambda^* \in \Lambda_1$  and  $\gamma^* \in \Gamma_1$ , respectively.

We define the perturbed split quasi-equilibrium problem of (SQEP) as follows: Find  $(x^*, y^*) \in C_1 \times C_2$  such that

$$\begin{cases} x^* \in K_1(x^*), y^* \in K_2(y^*), y^* = Ax^*; \\ \tilde{f}_1(\lambda, x^*, z) \geq 0, \forall z \in K_1(x^*), \lambda \in \Lambda_1; \\ \tilde{f}_2(\gamma, y^*, w) \geq 0, \forall w \in K_2(y^*), \gamma \in \Gamma_1, \end{cases} \quad (\text{SQEP}_{\lambda^*, \gamma^*})$$

where  $\tilde{f}_1 : \Lambda_1 \times X_1 \times X_1 \rightarrow \bar{\mathbb{R}}$ ,  $\tilde{f}_2 : \Gamma_1 \times X_2 \times X_2 \rightarrow \bar{\mathbb{R}}$  are functions such that  $\tilde{f}_1(\lambda^*, \cdot, \cdot) = f_1(\cdot, \cdot)$  and  $\tilde{f}_2(\gamma^*, \cdot, \cdot) = f_2(\cdot, \cdot)$ . Here (SQEP) is the original problem and (SQEP) $_{\lambda^*, \gamma^*}$  is its perturbed model corresponding to the parameters  $\lambda^* \in \Lambda_1$  and  $\gamma^* \in \Gamma_1$ .

Next, we consider some particular assumptions on the bifunctions  $f_1$  and  $f_2$ .

- (F1)  $f_1(x, x) = 0$  and  $f_2(y, y) = 0$  for each  $x \in C_1, y \in C_2$ ;
- (F2)  $f_1$  and  $f_2$  are convex on  $C_1$  and  $C_2$  respectively, with respect to the second argument.

In the sequel, we study the approximate sequences for perturbed (SQEP $_{\lambda^*, \gamma^*}$ ) under small perturbations around  $(\lambda^*, \gamma^*)$ . Let  $\alpha > 0$  be a fixed number.

**Definition 3.1.** Let  $\{(\lambda_n, \gamma_n)\} \subset \Lambda_1 \times \Gamma_1$  with  $(\lambda_n, \gamma_n) \rightarrow (\lambda^*, \gamma^*)$ . A sequence  $\{(x_n, y_n)\} \subseteq C_1 \times C_2$  is called

- (i)  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP) if there exists a sequence  $\{\varepsilon_n\}$  of non-negative integers with  $\varepsilon_n \rightarrow 0$  such that for each  $n \in \mathbb{N}$ ,

$$\left\{ \begin{array}{l} (x_n, y_n) \in K_1(x_n) \times K_2(y_n), \|y_n - Ax_n\| \leq \varepsilon_n; \\ \tilde{f}_1(\lambda_n, x_n, z) + \varepsilon_n \geq -\frac{\alpha}{2}\|x_n - z\|^2, \forall z \in K_1(x_n); \\ \tilde{f}_2(\gamma_n, y_n, w) + \varepsilon_n \geq -\frac{\alpha}{2}\|y_n - w\|^2, \forall w \in K_2(y_n). \end{array} \right.$$

- (ii) Levitin-Polyak (LP)  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP) if there exists a sequence  $\{\varepsilon_n\}$  of non-negative integers with  $\varepsilon_n \rightarrow 0$  such that for each  $n \in \mathbb{N}$ ,

$$\left\{ \begin{array}{l} d(x_n, K_1(x_n)) \leq \varepsilon_n, d(y_n, K_2(y_n)) \leq \varepsilon_n, \|y_n - Ax_n\| \leq \varepsilon_n; \\ \tilde{f}_1(\lambda_n, x_n, z) + \varepsilon_n \geq -\frac{\alpha}{2}\|x_n - z\|^2, \forall z \in K_1(x_n); \\ \tilde{f}_2(\gamma_n, y_n, w) + \varepsilon_n \geq -\frac{\alpha}{2}\|y_n - w\|^2, \forall w \in K_2(y_n). \end{array} \right. \tag{5}$$

**Remark 3.2.** (i) When  $\alpha = 0$ , we say the sequence  $\{(x_n, y_n)\}$  is an approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP). Clearly, every  $\alpha_2$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  is an  $\alpha_1$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  whenever  $0 \leq \alpha_2 < \alpha_1$ .

- (ii) Similarly, for  $\alpha = 0$ ,  $\{(x_n, y_n)\}$  is a LP approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP).
- (iii) Every  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP) is a LP  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP).

**Definition 3.3.** The split quasi-equilibrium problem (SQEP) is said to be

- (i)  $\alpha$ -well-posed by perturbations, if its solution set  $S$  is a singleton set and for any  $\{(\lambda_n, \gamma_n)\} \subset \Lambda_1 \times \Gamma_1$  with  $(\lambda_n, \gamma_n) \rightarrow (\lambda^*, \gamma^*)$ , every  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP) converges uniquely in the solution set  $S$ .
- (ii) generalized  $\alpha$ -well-posed by perturbations, if its solution set  $S$  is non-empty, and for any  $\{(\lambda_n, \gamma_n)\} \subset \Lambda_1 \times \Gamma_1$  with  $(\lambda_n, \gamma_n) \rightarrow (\lambda^*, \gamma^*)$ , every  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP) has a subsequence that converges to some element of  $S$ .

**Remark 3.4.** (i) In the sequel, 0-well-posedness by perturbations for (SQEP) is always referred to as well-posedness by perturbations. If  $0 \leq \alpha_2 < \alpha_1$ , then (generalized)  $\alpha_1$ -well-posedness by perturbations implies (generalized)  $\alpha_2$ -well-posedness by perturbations for (SQEP).

- (ii) Let  $C_1 = C_2 = C, X_1 = X_2 = X, A = I$  and  $\lambda_n \equiv \lambda^*, \gamma_n \equiv \gamma^*$ . Also,  $f_1 = f_2 = f$  is given by (2). Then Definition 3.3 coincides with [36, Definition 2.2] of  $\alpha$ -well-posed and generalized  $\alpha$ -well-posed for the classical quasi-variational inequality problem QVI(F, K).

**Definition 3.5.** The split quasi-equilibrium problem (SQEP) is said to be

- (i) LP  $\alpha$ -well-posed by perturbations, if its solution set  $S$  is a singleton set and for  $\{(\lambda_n, \gamma_n)\} \subset \Lambda_1 \times \Gamma_1$  with  $(\lambda_n, \gamma_n) \rightarrow (\lambda^*, \gamma^*)$ , every LP  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP) converges uniquely in the solution set  $S$ .
- (ii) generalized LP  $\alpha$ -well-posed by perturbations, if its solution set  $S$  is non-empty, and for any  $\{(\lambda_n, \gamma_n)\} \subset \Lambda_1 \times \Gamma_1$  with  $(\lambda_n, \gamma_n) \rightarrow (\lambda^*, \gamma^*)$ , every LP  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP) has a subsequence that converges to some element of  $S$ .

- Remark 3.6.** (i) Remark 3.4(i) is also true for (generalized) LP  $\alpha$ -well-posedness by perturbations for (SQEP).  
 (ii) Let  $K_1(x) = C_1, \forall x \in C_1$  and  $K_2(y) = C_2, \forall y \in C_2$ . Then, Definitions 3.3 and 3.5 coincide with Definitions 5 and 6 of [13].  
 (iii) Let  $C_1 = C_2 = C, X_1 = X_2 = X, A = I$  and  $\lambda_n \equiv \lambda^*, \gamma_n \equiv \gamma^*$ . Moreover,  $f_1 = f_2 = f$  is given by (2) and  $\alpha = 0$ . Then Definitions 3.3 (i) and 3.5 (i) coincide with [23, Definition 2.4] of well-posed and LP-well-posed for the classical quasi-variational inequality problem QVI(F, K).

**Remark 3.7.** If (SQEP) is (generalized) LP  $\alpha$ -well-posed by perturbations, then it is also (generalized)  $\alpha$ -well-posed by perturbations.

Next, we consider an example to understand the above definitions.

**Example 3.8.** Take  $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}, C_1 = C_2 = [0, \infty), \Lambda_1 = \Gamma_1 = [0, 2]$  and  $\lambda^* = \gamma^* = 1$ . Define  $\tilde{f}_1, \tilde{f}_2 : \Lambda_1 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  as:

$$\tilde{f}_1(\lambda, x, z) = \lambda e^{-x}(x - z), \text{ and } \tilde{f}_2(\gamma, y, w) = \gamma e^{-y}(y - w),$$

for all  $x, z, y, w \in \mathbb{R}, \lambda \in \Lambda_1$  and  $\gamma \in \Gamma_1$ . Also,

$$K_1(x) = K_2(x) = \begin{cases} [x, 1], & \text{if } x \in [0, 1], \\ [1, 2x - 1], & \text{if } x \in (1, \infty). \end{cases}$$

Define  $A := I$ , the identity operator. Also,

$$f_1(x, z) = \tilde{f}_1(\lambda^*, x, z) = e^{-x}(x - z);$$

$$f_2(y, w) = \tilde{f}_2(\gamma^*, y, w) = e^{-y}(y - w),$$

for  $x, y, z, w \in \mathbb{R}$ . Then (1, 1) is the unique solution of the split quasi-equilibrium problem (SQEP). Define

$$x_n = y_n = n + 1, \forall n \in \mathbb{N}.$$

Fix an  $\alpha \geq \frac{2}{e^2}$ . Then,  $\{(x_n, y_n)\}$  is a  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n) = (1 + \frac{1}{n}, 1 + \frac{1}{n})\}$  for (SQEP) as there exists a sequence  $\{\frac{1}{e^{n+1}}\}$  converges to 0 such that for each  $n \in \mathbb{N}$ ,

$$e^{-x_n}(x_n - z) + \frac{\alpha n^2}{2} \geq -\frac{1}{e^{n+1}}, \forall z \in [1, 2x_n - 1];$$

$$e^{-y_n}(y_n - w) + \frac{\alpha n^2}{2} \geq -\frac{1}{e^{n+1}}, \forall w \in [1, 2y_n - 1].$$

Moreover,  $\{(x_n, y_n)\}$  is also a LP  $\alpha$ -approximating sequence corresponding to  $\{(1 + \frac{1}{n}, 1 + \frac{1}{n})\}$  for (SQEP) with  $\{\varepsilon_n = \frac{1}{e^{n+1}}\}$ . Since  $\{(x_n, y_n)\}$  is an unbounded sequence and monotonically increasing, it does not have any convergent subsequence. This implies that (SQEP) is neither  $\alpha$ -well-posed by perturbation nor LP  $\alpha$ -well-posed by perturbation for any  $\alpha \geq \frac{2}{e^2}$ .

We conclude this section with a crucial result regarding the solution set of (SQEP).

**Proposition 3.9.** Assume that the set-valued maps  $K_1$  and  $K_2$  are nonempty and convex-valued, and  $f_1$  and  $f_2$  satisfy conditions (F1) and (F2). For any  $\alpha \in \mathbb{R}_+, (x^*, y^*) \in S$  if and only if the following conditions hold:

$$(x^*, y^*) \in K_1(x^*) \times K_2(y^*) \text{ such that } y^* = Ax^*,$$

$$f_1(x^*, z) \geq -\frac{\alpha}{2} \|x^* - z\|^2, \forall z \in K_1(x^*); \tag{6}$$

$$f_2(y^*, w) \geq -\frac{\alpha}{2} \|y^* - w\|^2, \forall w \in K_2(y^*). \tag{7}$$

*Proof.* The necessary part of the proof clearly holds. Conversely, let  $x^* \in K_1(x^*)$  and  $y^* \in K_2(y^*)$  with  $y^* = Ax^*$  such that (6) and (7) hold, respectively. For  $t \in (0, 1), z \in K_1(x^*)$  and  $w \in K_2(y^*)$ , define

$$z_t = tz + (1 - t)x^* \text{ and } w_t = tw + (1 - t)y^*.$$

Since  $K_1$  and  $K_2$  are convex-valued, we have  $z_t \in K_1(x^*)$  and  $w_t \in K_2(y^*)$ . Then, for all  $t \in (0, 1)$

$$tf_1(x^*, z) + (1 - t)f_1(x^*, x^*) \geq f_1(x^*, z_t) \geq -\frac{\alpha t^2}{2}\|x^* - z\|^2,$$

$$tf_2(y^*, w) + (1 - t)f_2(y^*, y^*) \geq f_2(y^*, w_t) \geq -\frac{\alpha t^2}{2}\|y^* - w\|^2.$$

Therefore,

$$f_1(x^*, z) \geq -t\frac{\alpha}{2}\|x^* - z\|^2, \forall z \in K_1(x^*);$$

$$f_2(y^*, w) \geq -t\frac{\alpha}{2}\|y^* - w\|^2, \forall w \in K_2(y^*).$$

As  $t \rightarrow 0$ , we have  $(x^*, y^*)$  is a solution of (SQEP). This completes the proof.  $\square$

#### 4. Metric characterization of $\alpha$ -well-posedness by perturbations for split quasi-equilibrium problems

In this section, we investigate some metric characterizations of  $\alpha$ -well-posedness and LP  $\alpha$ -well-posedness by perturbations for (SQEP). We start by introducing the following approximating solution set of (SQEP).

Let  $\alpha \geq 0$  be a fixed number. For a given  $\varepsilon \geq 0$ , define

$$S_\alpha(\varepsilon) := \bigcup_{\substack{\lambda \in \mathcal{B}(\lambda^*, \varepsilon), \\ \gamma \in \mathcal{B}(\gamma^*, \varepsilon)}} \left\{ (x, y) \in C_1 \times C_2 : (x, y) \in K_1(x) \times K_2(y), \|y - Ax\| \leq \varepsilon; \right.$$

$$\tilde{f}_1(\lambda, x, z) + \varepsilon \geq -\frac{\alpha}{2}\|x - z\|^2, \forall z \in K_1(x) \text{ and}$$

$$\left. \tilde{f}_2(\gamma, y, w) + \varepsilon \geq -\frac{\alpha}{2}\|y - w\|^2, \forall w \in K_2(y) \right\},$$

$$L_\alpha(\varepsilon) := \bigcup_{\substack{\lambda \in \mathcal{B}(\lambda^*, \varepsilon), \\ \gamma \in \mathcal{B}(\gamma^*, \varepsilon)}} \left\{ (x, y) \in C_1 \times C_2 : d(x, K_1(x)) \leq \varepsilon, d(y, K_2(y)) \leq \varepsilon, \|y - Ax\| \leq \varepsilon; \right.$$

$$\tilde{f}_1(\lambda, x, z) + \varepsilon \geq -\frac{\alpha}{2}\|x - z\|^2, \forall z \in K_1(x) \text{ and}$$

$$\left. \tilde{f}_2(\gamma, y, w) + \varepsilon \geq -\frac{\alpha}{2}\|y - w\|^2, \forall w \in K_2(y) \right\},$$

where  $\mathcal{B}(a, \varepsilon)$  denote the closed ball of radius  $\varepsilon$  centered at  $a$ . The set  $S_\alpha(\varepsilon)$  and  $L_\alpha(\varepsilon)$  are called  $\alpha$ -approximate solution set and LP  $\alpha$ -approximate solution set of (SQEP), respectively. For  $0 \leq \varepsilon_1 < \varepsilon_2$ , we have  $S_\alpha(\varepsilon_1) \subseteq S_\alpha(\varepsilon_2)$  and  $L_\alpha(\varepsilon_1) \subseteq L_\alpha(\varepsilon_2)$ . Also,  $S \subseteq L_\alpha(\varepsilon) \subseteq S_\alpha(\varepsilon)$  for all  $\varepsilon > 0$ .

The next two theorems provide an equivalent condition to determine the (LP)  $\alpha$ -well-posedness by perturbations for (SQEP) via approximate solution sets.

**Theorem 4.1.** *The (SQEP) is  $\alpha$ -well-posed by perturbations if and only if*

$$S \neq \emptyset \text{ and } \lim_{\varepsilon \rightarrow 0} \text{diam}(S_\alpha(\varepsilon)) = 0.$$

*Proof.* First, we assume that (SQEP) is  $\alpha$ -well-posed by perturbations. By definition,  $S$  is non-empty. We need to show that

$$\lim_{\varepsilon \rightarrow 0} \text{diam}(S_\alpha(\varepsilon)) = 0.$$

On the contrary, suppose that there exist  $\delta > 0$ , a sequence of positive numbers  $\{\varepsilon_n\}$  with  $\varepsilon_n \rightarrow 0$ ,  $(x_n, y_n) \in S_\alpha(\varepsilon_n)$  and  $(x'_n, y'_n) \in S_\alpha(\varepsilon_n)$  such that

$$\|(x_n, y_n) - (x'_n, y'_n)\| > \delta, \forall n \in \mathbb{N}. \tag{8}$$

Then, there exist  $(\lambda_n, \gamma_n), (\lambda'_n, \gamma'_n) \in \mathcal{B}(\lambda^*, \varepsilon_n) \times \mathcal{B}(\gamma^*, \varepsilon_n)$  such that

$$\begin{cases} (x_n, y_n) \in K_1(x_n) \times K_2(y_n), \|y_n - Ax_n\| \leq \varepsilon_n; \\ \tilde{f}_1(\lambda_n, x_n, z) + \varepsilon_n \geq -\frac{\alpha}{2}\|x_n - z\|^2, \forall z \in K_1(x_n); \\ \tilde{f}_2(\gamma_n, y_n, w) + \varepsilon_n \geq -\frac{\alpha}{2}\|y_n - w\|^2, \forall w \in K_2(y_n), \end{cases}$$

and

$$\begin{cases} (x'_n, y'_n) \in K_1(x'_n) \times K_2(y'_n), \|y'_n - Ax'_n\| \leq \varepsilon_n; \\ \tilde{f}_1(\lambda'_n, x'_n, z) + \varepsilon_n \geq -\frac{\alpha}{2}\|x'_n - z\|^2, \forall z \in K_1(x'_n); \\ \tilde{f}_2(\gamma'_n, y'_n, w) + \varepsilon_n \geq -\frac{\alpha}{2}\|y'_n - w\|^2, \forall w \in K_2(y'_n). \end{cases}$$

Since the sequences  $(\lambda_n, \gamma_n) \rightarrow (\lambda^*, \gamma^*)$  and  $(\lambda'_n, \gamma'_n) \rightarrow (\lambda^*, \gamma^*)$ , we have that  $\{(x_n, y_n)\}$  and  $\{(x'_n, y'_n)\}$  are  $\alpha$ -approximating sequences corresponding to  $\{(\lambda_n, \gamma_n)\}$  and  $\{(\lambda'_n, \gamma'_n)\}$  for (SQEP), respectively. By the  $\alpha$ -well-posedness of (SQEP) by perturbations, we deduce that both the sequences converge to the same point, that lies in the solution set of (SQEP). Therefore,

$$\|(x_n, y_n) - (x'_n, y'_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which is a contradiction to (8).

Conversely, suppose that  $S \neq \emptyset$  and  $\lim_{\varepsilon \rightarrow 0} \text{diam}(S_\alpha(\varepsilon)) = 0$ . Then

$$\emptyset \neq S \subseteq S_\alpha(\varepsilon), \forall \varepsilon > 0.$$

This implies that  $S$  is a singleton set. Let  $\{(\lambda_n, \gamma_n)\} \subset \Lambda_1 \times \Gamma_1$  with  $(\lambda_n, \gamma_n) \rightarrow (\lambda^*, \gamma^*)$  and  $\{(x_n, y_n)\}$  be any arbitrary  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP). There exists a sequence  $\{\varepsilon_n\} \subset (0, \infty)$  with  $\varepsilon_n \rightarrow 0$  such that

$$(x_n, y_n) \in S_\alpha(\varepsilon_n) \text{ for all } n \in \mathbb{N}.$$

Let  $(x^*, y^*) \in S$ . It easily follows that  $(x^*, y^*) \in S_\alpha(\varepsilon_n)$  for all  $n \in \mathbb{N}$ . Therefore, we deduce that

$$\|(x_n, y_n) - (x^*, y^*)\| \leq \text{diam}(S_\alpha(\varepsilon_n)), \forall n \in \mathbb{N}.$$

As  $\varepsilon_n \rightarrow 0$ , which implies that  $\text{diam}(S_\alpha(\varepsilon_n)) \rightarrow 0$ . Therefore, the sequence  $\{(x_n, y_n)\}$  converges to  $(x^*, y^*) \in S$ . This completes the proof.  $\square$

**Remark 4.2.** Theorem 4.1 is a generalization of [13, Theorem 4] for split equilibrium problem, where for  $i = 1, 2$ ,  $K_i$  is a constant map, that is,  $K_i(x) = C_i$  for all  $x \in C_i$ , and  $\alpha = 0$ .

**Theorem 4.3.** The (SQEP) is LP  $\alpha$ -well-posed by perturbations if and only if

$$S \neq \emptyset \text{ and } \lim_{\varepsilon \rightarrow 0} \text{diam}(L_\alpha(\varepsilon)) = 0.$$

*Proof.* The proof follows on the same lines as that of Theorem 4.1.  $\square$

**Remark 4.4.** (i) To show the necessary part of Theorem 4.3, we can also make use of the fact that  $\text{diam}(L_\alpha(\varepsilon)) \leq \text{diam}(S_\alpha(\varepsilon))$ ,  $\forall \varepsilon > 0$  and Remark 3.7.

(ii) If  $C_1 = C_2 = C$ ,  $X_1 = X_2 = X$ ,  $K_1 = K_2 = K$ ,  $f_1 = f_2 = f$  is given by (2) and  $A = I$ , then Theorem 4.3 reduces to [35, Theorem 3.1(a)] for QVI(F, K).

Next, we consider an example that illustrates Theorem 4.1 and Theorem 4.3.

**Example 4.5.** Let  $X_1 = X_2 = \mathbb{R}$ ,  $C_1 = C_2 = [0, \infty) \subseteq \mathbb{R}$  and  $\Lambda_1 = \Gamma_1 = [-1, 1]$ . Define the bifunctions  $\tilde{f}_1 : \Lambda_1 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\tilde{f}_2 : \Gamma_1 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  as follows:

$$\tilde{f}_1(\lambda, x, z) = (\lambda^2 + 5)(z^2 - x^2) \text{ and } \tilde{f}_2(\gamma, y, w) = w - 2y^2e^{\gamma^2},$$

for all  $x, y, z, w \in \mathbb{R}$ ,  $\lambda \in \Lambda_1$  and  $\gamma \in \Gamma_1$ . The constraint maps are given by

$$K_1(x) = K_2(x) := \left[0, \frac{x}{2}\right], \forall x \in [0, \infty).$$

Define the bounded linear operator  $A : \mathbb{R} \rightarrow \mathbb{R}$  by  $Ax = x$  for all  $x \in \mathbb{R}$  and  $\lambda^* = \gamma^* = 0$ . Then,

$$f_1(x, z) = \tilde{f}_1(\lambda^*, x, z) = 5(z^2 - x^2);$$

$$f_2(y, w) = \tilde{f}_2(\gamma^*, y, w) = w - 2y^2,$$

for  $x, y, z, w \in \mathbb{R}$ . The point  $(x^*, y^*)$  is a solution of the split quasi-equilibrium problem (SQEP) if there exists  $(x^*, y^*) \in [0, x^*/2] \times [0, y^*/2]$  with  $y^* = x^*$  such that

$$z^2 \geq x^{*2}, \forall z \in [0, x^*/2] \text{ and } w \geq 2y^{*2}, \forall w \in [0, y^*/2].$$

Then,  $S = \{(0, 0)\}$ . In particular, the solution set of (SQEP) is non-empty.

Fix  $\alpha = 2$ . For a given  $\varepsilon > 0$ ,

$$L_2(\varepsilon) := \bigcup_{\lambda, \gamma \in \mathcal{B}(0, \varepsilon)} \left\{ (x, y) \in [0, \infty) \times [0, \infty) : d(x, [0, x/2]) \leq \varepsilon, d(y, [0, y/2]) \leq \varepsilon, |y - x| \leq \varepsilon; \right. \\ \left. (\lambda^2 + 5)(z^2 - x^2) \geq -\varepsilon - (x - z)^2, \forall z \in [0, x/2] \text{ and } \right. \\ \left. w - 2y^2e^{\gamma^2} \geq -\varepsilon - (y - w)^2, \forall w \in [0, y/2] \right\}.$$

For  $\lambda \in \mathcal{B}(0, \varepsilon)$ , we will solve for  $x \in [0, \infty)$  such that

$$\begin{cases} d(x, [0, x/2]) \leq \varepsilon \text{ and} \\ (\lambda^2 + 5)(z^2 - x^2) \geq -\varepsilon - (x - z)^2, \forall z \in [0, x/2]. \end{cases} \tag{9}$$

Clearly,  $x = 0$  satisfies (9). For any  $x > 0$  satisfies (9), we have  $x \in (0, 2\varepsilon]$  and

$$\frac{x^2}{\lambda^2 + 6} + x^2(\lambda^2 + 4) \leq \varepsilon + \left( z\sqrt{\lambda^2 + 6} - \frac{x}{\sqrt{\lambda^2 + 6}} \right)^2, \forall z \in [0, x/2].$$

On solving for all  $z \in [0, x/2]$ , we get  $x \in (0, 2\varepsilon]$  and

$$\frac{x^2}{\lambda^2 + 6} + x^2(\lambda^2 + 4) \leq \varepsilon.$$

Therefore,  $x \in \left[0, \sqrt{\frac{\varepsilon(\lambda^2 + 6)}{1 + (\lambda^2 + 4)(\lambda^2 + 6)}}\right] \subseteq \left[0, \sqrt{\frac{\varepsilon}{\lambda^2 + 4}}\right]$ . Similarly, for  $\gamma \in \mathcal{B}(0, \varepsilon)$ , we need to find  $y \geq 0$  such that

$$\begin{cases} d(y, [0, y/2]) \leq \varepsilon \text{ and} \\ w - 2y^2e^{\gamma^2} \geq -\varepsilon - (y - w)^2, \forall w \in [0, y/2]. \end{cases} \tag{10}$$

Now the second inequality in (10) can be written as follows:

$$y^2(2e^{y^2} - 1) - w + 2yw - w^2 \leq \varepsilon, \forall w \in [0, y/2].$$

Note that  $2yw - w^2 \geq 0$  for all  $w \in [0, y/2]$ . Thus, for  $y$  satisfying (10), we have

$$\begin{cases} y \in [0, 2\varepsilon] \text{ and} \\ y^2(2e^{y^2} - 1) - w \leq \varepsilon, \forall w \in [0, y/2]. \end{cases}$$

This implies  $y \in \left[0, \sqrt{\frac{\varepsilon}{2e^{y^2} - 1}}\right]$ . Therefore, for sufficiently small  $\varepsilon > 0$ , we have

$$\begin{aligned} L_2(\varepsilon) &\subseteq \bigcup_{\lambda, \gamma \in \mathcal{B}(0, \varepsilon)} \left\{ (x, y) \in \left[0, \sqrt{\frac{\varepsilon}{\lambda^2 + 4}}\right] \times \left[0, \sqrt{\frac{\varepsilon}{2e^{y^2} - 1}}\right] : |y - x| \leq \varepsilon \right\} \\ &\subseteq \left[0, \sqrt{\frac{\varepsilon}{4}}\right] \times [0, \sqrt{\varepsilon}]. \end{aligned}$$

Thus,

$$\text{diam}(L_2(\varepsilon)) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

By Theorem 4.3, the (SQEP) for the above bifunctions  $\tilde{f}_1$  and  $\tilde{f}_2$  and constrained maps  $K_1$  and  $K_2$  is LP 2-well-posed by perturbations. By Remark 3.7, (SQEP) is also 2-well-posed by perturbations.

**Definition 4.6.** For any normed space  $L$  and  $C, D \subseteq X$ , we say a function  $\tilde{f} : L \times X \times X \rightarrow \mathbb{R}$  is jointly u.s.c on  $L \times C \times D$  if for any sequences  $\{\lambda_n\}, \{x_n\}$  and  $\{y_n\}$  in  $L, C$  and  $D$ , respectively with  $\lambda_n \rightarrow \lambda^*, x_n \rightarrow x$  and  $y_n \rightarrow y$ , we have  $f(\lambda^*, x, y) \geq \limsup_{n \rightarrow \infty} f(\lambda_n, x_n, y_n)$ .

The next two theorems establish  $\alpha$ -well-posedness and LP  $\alpha$ -well-posedness of (SQEP) by perturbations and characterize its equivalency to the existence and uniqueness of its solution.

**Theorem 4.7.** Let  $K_1$  and  $K_2$  be convex-valued and  $M$ -continuous. Let  $\tilde{f}_1 : \Lambda_1 \times X_1 \times X_1 \rightarrow \mathbb{R}$  and  $\tilde{f}_2 : \Gamma_1 \times X_2 \times X_2 \rightarrow \mathbb{R}$ . Assume that the following conditions hold:

- (i)  $\tilde{f}_1$  and  $\tilde{f}_2$  satisfy the conditions (F1) and (F2);
- (ii)  $\tilde{f}_1(\cdot, \cdot, \cdot)$  and  $\tilde{f}_2(\cdot, \cdot, \cdot)$  are jointly u.s.c on  $\Lambda_1 \times C_1 \times C_1$  and  $\Gamma_1 \times C_2 \times C_2$ , respectively.

Then, (SQEP) is  $\alpha$ -well-posed by perturbations if and only if

$$\forall \varepsilon > 0, S_\alpha(\varepsilon) \neq \emptyset \text{ and } \lim_{\varepsilon \rightarrow 0} \text{diam}(S_\alpha(\varepsilon)) = 0. \tag{11}$$

*Proof.* If (SQEP) is  $\alpha$ -well-posed by perturbations, then clearly

$$\emptyset \neq S \subseteq S_\alpha(\varepsilon)$$

for all  $\varepsilon > 0$ . From Theorem 4.1, it directly follows that  $\lim_{\varepsilon \rightarrow 0} \text{diam}(S_\alpha(\varepsilon)) = 0$ .

Conversely, suppose that (11) holds. Since  $S \subseteq S_\alpha(\varepsilon)$  for all  $\varepsilon > 0$ ,  $S$  can have almost one element. We only need to show that  $S$  is non-empty. Let  $\{(\lambda_n, \gamma_n)\} \subset \Lambda_1 \times \Gamma_1$  with  $(\lambda_n, \gamma_n) \rightarrow (\lambda^*, \gamma^*)$  and  $\{(x_n, y_n)\}$  be any arbitrary  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP). Then there exists a sequence  $\{\varepsilon_n\}$  of non-negative integers with  $\varepsilon_n \rightarrow 0$  such that

$$\begin{aligned} (x_n, y_n) &\in K_1(x_n) \times K_2(y_n), \|y_n - Ax_n\| \leq \varepsilon_n; \\ \tilde{f}_1(\lambda_n, x_n, z) + \varepsilon_n &\geq -\frac{\alpha}{2} \|x_n - z\|^2, \forall z \in K_1(x_n); \end{aligned} \tag{12}$$

$$\tilde{f}_2(\gamma_n, y_n, w) + \varepsilon_n \geq -\frac{\alpha}{2} \|y_n - w\|^2, \forall w \in K_2(y_n). \tag{13}$$

Set  $\delta_n = \max\{\|\lambda_n - \lambda^*\|, \|\gamma_n - \gamma^*\|, \varepsilon_n\}$  for each  $n \in \mathbb{N}$ . Then,

$$\lim_{n \rightarrow \infty} \delta_n = 0 \text{ and } (x_n, y_n) \in S_\alpha(\delta_n) \forall n \in \mathbb{N}.$$

As  $\text{diam}(S_\alpha(\delta_n)) \rightarrow 0$  as  $n \rightarrow \infty$ , we have

$$\|(x_n, y_n) - (x_m, y_m)\| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Thus,  $\{(x_n, y_n)\}$  is a Cauchy sequence in  $C_1 \times C_2$  and hence a convergent sequence. Let  $(x_n, y_n) \rightarrow (x^*, y^*) \in X_1 \times X_2$ . Our claim is to show that  $(x^*, y^*) \in S$ . As  $K_1$  and  $K_2$  are  $M$ -continuous and  $x_n \in K_1(x_n), y_n \in K_2(y_n)$ , therefore  $x^* \in K_1(x^*)$  and  $y^* \in K_2(y^*)$ . Let  $(z, w) \in K_1(x^*) \times K_2(y^*)$ . Again by the  $M$ -continuity of the maps  $K_1$  and  $K_2$ , there exists a sequence  $\{(z_n, w_n)\} \rightarrow (z, w)$  with  $z_n \in K_1(x_n)$  and  $w_n \in K_2(y_n)$ . Together inequalities (12) and (13) with assumption (ii), we have

$$\begin{aligned} f_1(x^*, z) &= \tilde{f}_1(\lambda^*, x^*, z) \geq \limsup_{n \rightarrow \infty} \tilde{f}_1(\lambda_n, x_n, z_n) \geq -\lim_{n \rightarrow \infty} \frac{\alpha}{2} \|x_n - z_n\|^2 = -\frac{\alpha}{2} \|x^* - z\|^2; \\ f_2(y^*, w) &= \tilde{f}_2(\gamma^*, y^*, w) \geq \limsup_{n \rightarrow \infty} \tilde{f}_2(\gamma_n, y_n, w_n) \geq -\lim_{n \rightarrow \infty} \frac{\alpha}{2} \|y_n - w_n\|^2 = -\frac{\alpha}{2} \|y^* - w\|^2. \end{aligned}$$

The above inequalities are true for all  $z \in K_1(x^*)$  and  $w \in K_2(y^*)$ . Moreover, letting  $n \rightarrow \infty$  in  $\|y_n - Ax_n\| \leq \varepsilon_n$ , it follows that  $y^* = Ax^*$ . Therefore, we get

$$\begin{aligned} x^* &\in K_1(x^*), y^* \in K_2(y^*) \text{ and } y^* = Ax^*; \\ f_1(x^*, z) &\geq -\frac{\alpha}{2} \|x^* - z\|^2, \forall z \in K_1(x^*) \text{ and } f_2(y^*, w) \geq -\frac{\alpha}{2} \|y^* - w\|^2, \forall w \in K_2(y^*). \end{aligned}$$

Thus, by Proposition 3.9, we have that  $(x^*, y^*) \in S$  which implies that every  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP) converges uniquely to a point in  $S$ . Henceforth, (SQEP) is  $\alpha$ -well-posed by perturbations, and this completes the proof.  $\square$

**Remark 4.8.** Theorem 4.7 is a generalization of [13, Theorem 5] for split equilibrium problem (4), where the constrained maps  $K_1$  and  $K_2$  are constants, that is,  $K_1(\cdot) = C_1$  and  $K_2(\cdot) = C_2$ , and  $\alpha = 0$ . Moreover, all the conditions on  $K_1$  and  $K_2$  hold trivially, and only upper semicontinuity in the first and second argument is required for  $\tilde{f}_1$  and  $\tilde{f}_2$ .

**Theorem 4.9.** Let  $K_1$  and  $K_2$  be non-empty convex-valued and  $M$ -continuous. Let  $\tilde{f}_1 : \Lambda_1 \times X_1 \times X_1 \rightarrow \bar{\mathbb{R}}$  and  $\tilde{f}_2 : \Gamma_1 \times X_2 \times X_2 \rightarrow \bar{\mathbb{R}}$ . Assume that the following conditions hold:

- (i)  $f_1$  and  $f_2$  satisfy the conditions (F1) and (F2);
- (ii)  $\tilde{f}_1$  and  $\tilde{f}_2$  are jointly u.s.c on  $\Lambda_1 \times C_1 \times C_1$  and  $\Gamma_1 \times C_2 \times C_2$ , respectively.

Then, the (SQEP) is LP  $\alpha$ -well-posed by perturbations if and only if

$$\forall \varepsilon > 0, L_\alpha(\varepsilon) \neq \emptyset \text{ and } \lim_{\varepsilon \rightarrow 0} \text{diam}(L_\alpha(\varepsilon)) = 0. \tag{14}$$

*Proof.* The necessary part follows directly from Theorem 4.3. For the sufficient part, suppose that (14) holds. We will proceed in the same way as we did in Theorem 4.7. We need to show that  $S$  is a non-empty set. Let  $\{(\lambda_n, \gamma_n)\} \subset \Lambda_1 \times \Gamma_1$  with  $(\lambda_n, \gamma_n) \rightarrow (\lambda^*, \gamma^*)$  and  $\{(x_n, y_n)\}$  be any arbitrary LP  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP). Then there exists a sequence  $\{\varepsilon_n\}$  of non-negative integers with  $\varepsilon_n \rightarrow 0$  such that (5) holds. Set  $\delta_n = \max\{\|\lambda_n - \lambda^*\|, \|\gamma_n - \gamma^*\|, \varepsilon_n\}$  for each  $n \in \mathbb{N}$ . Then,

$$\lim_{n \rightarrow \infty} \delta_n = 0 \text{ and } (x_n, y_n) \in L_\alpha(\delta_n) \forall n \in \mathbb{N}.$$

From  $\text{diam}(L_\alpha(\delta_n)) \rightarrow 0$  as  $n \rightarrow \infty$ , we can deduce that  $\{(x_n, y_n)\}$  is a Cauchy sequence and hence, a convergent sequence that converges to some point  $(x^*, y^*) \in C_1 \times C_2$ , say. Since  $A$  is a bounded linear

operator, it follows from  $\|y_n - Ax_n\| \leq \varepsilon_n$  in (5) that  $y^* = Ax^*$ . We now prove that  $x^* \in K_1(x^*)$  and  $y^* \in K_1(y^*)$ . For each  $n \in \mathbb{N}$ , select  $(x'_n, y'_n) \in K_1(x_n) \times K_2(y_n)$  such that

$$\|x_n - x'_n\| < d(x_n, K_1(x_n)) + \varepsilon_n \leq 2\varepsilon_n,$$

$$\|y_n - y'_n\| < d(y_n, K_2(y_n)) + \varepsilon_n \leq 2\varepsilon_n.$$

Since  $(x_n, y_n) \rightarrow (x^*, y^*)$ , we can easily deduce that  $(x'_n, y'_n) \rightarrow (x^*, y^*)$ . By the  $M$ -continuity of the maps  $K_1$  and  $K_2$ , we get  $x^* \in K_1(x^*)$  and  $y^* \in K_2(y^*)$ .

To complete the proof, consider an arbitrary point  $(z, w) \in K_1(x^*) \times K_2(y^*)$ . Again by the  $M$ -continuity of the maps  $K_1$  and  $K_2$ , there exists a sequence  $\{(z_n, w_n)\}$  such that  $(z_n, w_n) \in K_1(x_n) \times K_2(y_n)$  for all  $n \in \mathbb{N}$  and  $(z_n, w_n) \rightarrow (z, w)$ . As  $\{(x_n, y_n)\}$  is a LP  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP), we have

$$\tilde{f}_1(\lambda_n, x_n, z_n) + \varepsilon_n \geq -\frac{\alpha}{2}\|x_n - z_n\|^2;$$

$$\tilde{f}_2(\gamma_n, y_n, w_n) + \varepsilon_n \geq -\frac{\alpha}{2}\|y_n - w_n\|^2.$$

Taking  $\limsup$  on both sides of the above inequalities, then by applying condition (ii), we get

$$f_1(x^*, z) = \tilde{f}_1(\lambda^*, x^*, z) \geq \limsup_{n \rightarrow \infty} \tilde{f}_1(\lambda_n, x_n, z_n) + \lim_{n \rightarrow \infty} \varepsilon_n \geq -\frac{\alpha}{2}\|x^* - z\|^2;$$

$$f_2(y^*, w) = \tilde{f}_2(\gamma^*, y^*, w) \geq \limsup_{n \rightarrow \infty} \tilde{f}_2(\gamma_n, y_n, w_n) + \lim_{n \rightarrow \infty} \varepsilon_n \geq -\frac{\alpha}{2}\|y^* - w\|^2.$$

As  $(z, w) \in K_1(x^*) \times K_2(y^*)$  is arbitrary, it follows from Proposition 3.9 that  $(x^*, y^*) \in S$ , and this completes the proof.  $\square$

From the proof of Theorem 4.9, one can obtain the following Corollary 4.10. That is, the conclusion of Theorem 4.9 still holds if we replace the  $M$ -continuity of the maps  $K_1$  and  $K_2$  that assumed in Theorem 4.9 by the condition (ii) in Corollary 4.10.

**Corollary 4.10.** For  $i = 1, 2$ , suppose that

- (i)  $K_i$  is non-empty and convex-valued;
- (ii) For any sequence  $\{x_{i,n}\} \subseteq C_i$  converges to  $x_i$ , the sequence  $\{K(x_{i,n})\}_n$  Lower Semi-Mosco converges to  $K(x_i)$ ;
- (iii)  $f_i$  satisfies the conditions (F1) and (F2).

Moreover, suppose that  $\tilde{f}_1$  and  $\tilde{f}_2$  are jointly u.s.c on  $\Lambda_1 \times C_1 \times C_1$  and  $\Gamma_1 \times C_2 \times C_2$ , respectively. Then, the (SQEP) is LP  $\alpha$ -well-posed by perturbations if and only if

$$\forall \varepsilon > 0, L_\alpha(\varepsilon) \neq \emptyset \text{ and } \lim_{\varepsilon \rightarrow 0} \text{diam}(L_\alpha(\varepsilon)) = 0.$$

**Remark 4.11.** (i) If  $K_1$  and  $K_2$  are constant maps, that is,  $K_1(\cdot) = C_1$  and  $K_2(\cdot) = C_2$ , then Theorem 4.9 provides metric characterization of LP  $\alpha$ -well-posedness by perturbations for split equilibrium problems (4).

- (ii) Let  $C_1 = C_2 = C, X_1 = X_2 = X, K_1 = K_2 = K, f_1 = f_2 = f$  be given by (2) and  $A = I$ . Then,
  - (a) Theorem 4.3 and Theorem 4.9 establish metric characterization of LP  $\alpha$ -well posedness by perturbations for quasi-variational inequality problem QVI(F, K).
  - (b) Corollary 4.10 is a generalization of Theorem 3.2 in [35] for quasi-variational inequality problem QVI(F, K). Moreover, we can replace (ii) in [35, Theorem 3.2] with a weaker condition:  $\text{int}(K(x)) \neq \emptyset, \forall x \in C$ .

The notion of generalized (LP)  $\alpha$ -well-posedness by perturbations for split quasi-equilibrium problems (SQEP) allows us to relax the assumption that the solution set  $S$  of the (SQEP) is a singleton. Next, we characterize this notion of well-posedness with the help of Hausdorff metric.

**Theorem 4.12.** Let  $\tilde{f}_1 : \Lambda_1 \times X_1 \times X_1 \rightarrow \bar{\mathbb{R}}$  and  $\tilde{f}_2 : \Gamma_1 \times X_2 \times X_2 \rightarrow \bar{\mathbb{R}}$ . Then, (SQEP) is generalized  $\alpha$ -well-posed by perturbations if and only if  $S$  is non-empty compact and

$$\lim_{\varepsilon \rightarrow 0} H(S_\alpha(\varepsilon), S) = 0.$$

*Proof.* Suppose that (SQEP) is generalized  $\alpha$ -well-posed by perturbations. By definition  $S$  is non-empty and  $S \subseteq S_\alpha(\varepsilon)$  for all  $\varepsilon > 0$ . We claim that  $S$  is compact. For this, let  $\{(x_n, y_n)\}$  be any arbitrary sequence in  $S$ . Let  $\lambda_n \equiv \lambda^*$  and  $\gamma_n \equiv \gamma^*$  for all  $n \in \mathbb{N}$ . Then  $\{(x_n, y_n)\}$  is a  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP). By generalized  $\alpha$ -well-posedness by perturbations of (SQEP), the sequence  $\{(x_n, y_n)\}$  has a convergent subsequence that converges to some point in  $S$ . Hence, the claim holds. We are left to show that  $\lim_{\varepsilon \rightarrow 0} H(S_\alpha(\varepsilon), S) = 0$ . As  $S \subseteq S_\alpha(\varepsilon)$  for all  $\varepsilon > 0$ , we can write

$$H(S_\alpha(\varepsilon), S) = \max \{e(S, S_\alpha(\varepsilon)), e(S_\alpha(\varepsilon), S)\} = e(S_\alpha(\varepsilon), S) = \sup_{a \in S_\alpha(\varepsilon)} \text{dist}(a, S).$$

Also, for  $0 < \varepsilon \leq \varepsilon'$ ,  $S_\alpha(\varepsilon) \subseteq S_\alpha(\varepsilon')$ . This implies that

$$H(S_\alpha(\varepsilon), S) \leq H(S_\alpha(\varepsilon'), S).$$

Therefore,  $H(S_\alpha(\varepsilon), S)$  is decreasing in  $\varepsilon > 0$ . Denote  $l = \lim_{\varepsilon \rightarrow 0} H(S_\alpha(\varepsilon), S)$ . On the contrary, suppose that  $l > 0$ . Then for a sequence  $\{\varepsilon_n\} \subseteq (0, \infty)$  with  $\varepsilon_n \downarrow 0$ , we can find  $(x_n, y_n) \in S_\alpha(\varepsilon_n)$  such that

$$\text{dist}((x_n, y_n), S) > \frac{l}{2}, \text{ for all } n \in \mathbb{N}. \tag{15}$$

Since  $(x_n, y_n) \in S_\alpha(\varepsilon_n)$  for all  $n \in \mathbb{N}$ , there exist  $\lambda_n \in \mathcal{B}(\lambda^*, \varepsilon_n)$  and  $\gamma_n \in \mathcal{B}(\gamma^*, \varepsilon_n)$  such that

$$\begin{cases} (x_n, y_n) \in K_1(x_n) \times K_2(y_n), \|y_n - Ax_n\| \leq \varepsilon_n; \\ \tilde{f}_1(\lambda_n, x_n, z) + \varepsilon_n \geq -\frac{\alpha}{2}\|x_n - z\|^2, \forall z \in K_1(x_n); \\ \tilde{f}_2(\gamma_n, y_n, w) + \varepsilon_n \geq -\frac{\alpha}{2}\|y_n - w\|^2, \forall w \in K_2(y_n). \end{cases}$$

As  $\varepsilon_n \downarrow 0$  and  $(\lambda_n, \gamma_n) \rightarrow (\lambda^*, \gamma^*)$ ,  $\{(x_n, y_n)\}$  is a  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP). Then, by generalized  $\alpha$ -well-posedness by perturbations of (SQEP), the sequence  $\{(x_n, y_n)\}$  has a subsequence  $\{(x_{n_k}, y_{n_k})\}_k$  that converges to some point  $(x^*, y^*)$  in  $S$ . But this is a contradiction to (15). Therefore, we must have  $l = 0$ , and this proves the necessary part.

Conversely, suppose that  $S$  is non-empty and compact, and  $\lim_{\varepsilon \rightarrow 0} H(S_\alpha(\varepsilon), S) = 0$ . Let  $\{(\lambda_n, \gamma_n)\} \subset \Lambda_1 \times \Gamma_1$  with  $(\lambda_n, \gamma_n) \rightarrow (\lambda^*, \gamma^*)$  and  $\{(x_n, y_n)\}$  be any arbitrary  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP). Then, there exists a sequence  $\{\varepsilon_n\} \subseteq (0, \infty)$  with  $\varepsilon_n \rightarrow 0$  such that  $(x_n, y_n) \in S_\alpha(\varepsilon_n)$  for all  $n \in \mathbb{N}$ . Thus,

$$\text{dist}((x_n, y_n), S) \leq e(S_\alpha(\varepsilon_n), S) = H(S_\alpha(\varepsilon_n), S).$$

As  $n \rightarrow \infty$ , we have  $\text{dist}((x_n, y_n), S) \rightarrow 0$ . The compactness of  $S$  implies that  $\{(x_n, y_n)\}$  has a subsequence converging to some point in  $S$ . This shows that (SQEP) is generalized  $\alpha$ -well-posed by perturbations. This completes the proof.  $\square$

**Theorem 4.13.** Let  $\tilde{f}_1 : \Lambda_1 \times X_1 \times X_1 \rightarrow \bar{\mathbb{R}}$  and  $\tilde{f}_2 : \Gamma_1 \times X_2 \times X_2 \rightarrow \bar{\mathbb{R}}$ . Then, (SQEP) is generalized LP  $\alpha$ -well-posed by perturbations if and only if  $S$  is non-empty compact and

$$\lim_{\varepsilon \rightarrow 0} H(L_\alpha(\varepsilon), S) = 0.$$

*Proof.* The proof follows the same idea as that of Theorem 4.12.  $\square$

The following example illustrates the applicability of Theorem 4.12 and Theorem 4.13.

**Example 4.14.** Let  $X_1 = X_2 = \mathbb{R}$ ,  $C_1 = [0, \infty)$ ,  $C_2 = [1, \infty)$ ,  $\Lambda_1 = \Gamma_1 = [-1, 1]$  and  $\alpha = 0$ . Define the bifunctions  $\tilde{f}_1, \tilde{f}_2$  as follows:

$$\begin{aligned} \tilde{f}_1(\lambda, x, z) &= (1 - x)(z - x)^2 + \lambda^2, \lambda \in \Lambda_1, x, z \in \mathbb{R}, \\ \tilde{f}_2(\gamma, y, w) &= (w - \gamma)^2 - (y - \gamma)^2, \forall \gamma \in \Gamma_1, y, w \in \mathbb{R}. \end{aligned}$$

For  $i = 1, 2$ , the constraint maps  $K_i : C_i \rightarrow 2^{C_i}$  are given by

$$\begin{aligned} K_1(x) &= \begin{cases} [x, 1], & \text{if } x \in [0, 1] \\ [1, 2x - 1], & \text{if } x \in (1, \infty). \end{cases} \\ K_2(x) &= [x, x + 1], \forall x \in [1, \infty). \end{aligned}$$

Note that the fixed point set of the map  $K_1$  and  $K_2$  is the whole set  $C_1$  and  $C_2$ , respectively. Define  $A : \mathbb{R} \rightarrow \mathbb{R}$  by  $Ax = 3x$  for all  $x \in \mathbb{R}$ .

Take  $\lambda^* = \gamma^* = 0$ . Then,

$$\begin{aligned} f_1(x, z) &= \tilde{f}_1(\lambda^*, x, z) = (1 - x)(z - x)^2, \\ f_2(y, w) &= \tilde{f}_2(\gamma^*, y, w) = w^2 - y^2, \end{aligned}$$

for  $x, y, z, w \in \mathbb{R}$ . The solution set of (SQEP) is

$$S = \{(x, 3x) : x \in [1/3, 1]\}.$$

Clearly,  $S$  is non-empty and compact. For sufficiently small  $\varepsilon > 0$ , the set

$$\begin{aligned} L_0(\varepsilon) &= \bigcup_{\lambda, \gamma \in \mathcal{B}(0, \varepsilon)} \left\{ (x, y) \in [0, 1] \times [1, \infty) : |y - 3x| \leq \varepsilon, (1 - x)(z - x)^2 + \lambda^2 \geq -\varepsilon, \forall z \in [x, 1]; \right. \\ &\quad \left. (w - \gamma)^2 - (y - \gamma)^2 \geq -\varepsilon, \forall w \in [y, y + 1] \right\} \\ &\quad \bigcup_{\lambda, \gamma \in \mathcal{B}(0, \varepsilon)} \left\{ (x, y) \in (1, \infty) \times [1, \infty) : |y - 3x| \leq \varepsilon, (1 - x)(z - x)^2 + \lambda^2 \geq -\varepsilon, \right. \\ &\quad \left. \forall z \in [1, 2x - 1]; (w - \gamma)^2 - (y - \gamma)^2 \geq -\varepsilon, \forall w \in [y, y + 1] \right\}. \end{aligned}$$

Note that for  $y \in [1, \infty)$ ,  $\gamma \in \mathcal{B}(0, \varepsilon)$

$$(w - \gamma)^2 - (y - \gamma)^2 \geq 0 \geq -\varepsilon, \forall w \in [y, y + 1]. \tag{16}$$

So the last inequality (16) follows for all  $y \in [1, \infty)$ . Therefore,

$$\begin{aligned} L_0(\varepsilon) &\subseteq \{(x, y) \in [0, 1] \times [1, \infty) : \frac{1}{3} - \frac{\varepsilon}{3} \leq x \leq 1; 1 \leq y \leq 3 + \varepsilon\} \\ &\quad \bigcup_{\lambda \in \mathcal{B}(0, \varepsilon)} \left\{ (x, y) \in (1, \infty) \times [1, \infty) : 1 < x \leq 1 + (\varepsilon + \lambda^2)^{1/3}, \right. \\ &\quad \left. 3 - \varepsilon < y \leq 3 + 3(\varepsilon + \lambda^2)^{1/3} + \varepsilon \right\} \\ &\subseteq \{(x, y) \in [0, 1] \times [1, \infty) : \frac{1}{3} - \frac{\varepsilon}{3} \leq x \leq 1; 1 \leq y \leq 3 + \varepsilon\} \\ &\quad \bigcup \{(x, y) \in (1, \infty) \times [1, \infty) : 1 < x \leq 1 + (\varepsilon + \varepsilon^2)^{1/3}; 3 - \varepsilon < y \leq 3 + 3(\varepsilon + \varepsilon^2)^{1/3} + \varepsilon\}. \end{aligned}$$

Therefore,  $\lim_{\varepsilon \rightarrow 0} H(L_0(\varepsilon), S) = 0$ . Since  $S$  is compact, by Theorem 4.13, (SQEP) is generalized LP well-posed by perturbations.

The next theorem relaxes the condition on the solution set  $S$  of the (SQEP) by employing Kuratowski’s measure of non-compactness.

**Theorem 4.15.** *Let  $\Lambda$  and  $\Gamma$  be finite-dimensional spaces and  $\tilde{f}_1$  and  $\tilde{f}_2$  be jointly u.s.c on  $\Lambda_1 \times C_1 \times C_1$  and  $\Gamma_1 \times C_2 \times C_2$ , respectively, and  $f_1$  and  $f_2$  satisfy the conditions (F1) and (F2). Also,  $K_1$  and  $K_2$  are convex-valued and  $M$ -continuous. Then, (SQEP) is generalized  $\alpha$ -well-posed by perturbations if and only if*

$$\forall \varepsilon > 0, S_\alpha(\varepsilon) \neq \emptyset \text{ and } \lim_{\varepsilon \rightarrow 0} \beta(S_\alpha(\varepsilon)) = 0,$$

where  $\beta$  is the Kuratowski’s measure of non-compactness.

*Proof.* Suppose that (SQEP) is generalized  $\alpha$ -well-posed by perturbations. By definition,  $S \subseteq S_\alpha(\varepsilon)$  for all  $\varepsilon > 0$ . By Theorem 4.12,  $S$  is a non-empty compact set and  $\lim_{\varepsilon \rightarrow 0} H(S_\alpha(\varepsilon), S) = 0$ . By Lemma 2.4, we have

$$\beta(S_\alpha(\varepsilon)) \leq 2H(S_\alpha(\varepsilon), S) + \beta(S) = 2H(S_\alpha(\varepsilon), S) \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ . The last inequality follows as  $S$  is a compact set which implies that  $\beta(S) = 0$ . This proves the necessary part.

Conversely, suppose that  $S_\alpha(\varepsilon) \neq \emptyset$  for all  $\varepsilon > 0$  and  $\lim_{\varepsilon \rightarrow 0} \beta(S_\alpha(\varepsilon)) = 0$ . First, we show that for all  $\varepsilon > 0$ ,  $S_\alpha(\varepsilon)$  is a closed subset of  $C_1 \times C_2$ . Fix  $\varepsilon > 0$ . Let  $\{(x_n, y_n)\}$  be a sequence in  $S_\alpha(\varepsilon)$  such that  $(x_n, y_n) \rightarrow (x, y) \in X_1 \times X_2$ . We need to show that  $(x, y) \in S_\alpha(\varepsilon)$ . Clearly,  $(x, y) \in C_1 \times C_2$  as both the sets  $C_1$  and  $C_2$  are closed. Since  $\{(x_n, y_n)\} \subseteq S_\alpha(\varepsilon)$ , there exists a sequence  $\{(\lambda_n, \gamma_n)\} \subset \Lambda_1 \times \Gamma_1$  with  $(\lambda_n, \gamma_n) \in \mathcal{B}(\lambda^*, \varepsilon) \times \mathcal{B}(\gamma^*, \varepsilon)$  such that

$$\begin{cases} (x_n, y_n) \in K_1(x_n) \times K_2(y_n), \|y_n - Ax_n\| \leq \varepsilon; \\ \tilde{f}_1(\lambda_n, x_n, z) + \varepsilon \geq -\frac{\alpha}{2}\|x_n - z\|^2, \forall z \in K_1(x_n); \\ \tilde{f}_2(\gamma_n, y_n, w) + \varepsilon \geq -\frac{\alpha}{2}\|y_n - w\|^2, \forall w \in K_2(y_n). \end{cases} \tag{17}$$

As  $K_1$  and  $K_2$  are  $M$ -continuous and  $x_n \in K_1(x_n), y_n \in K_2(y_n)$ , therefore  $x \in K_1(x)$  and  $y \in K_2(y)$ . Since  $\Lambda$  and  $\Gamma$  are finite dimensional, without loss of generality, we can assume that  $\{(\lambda_n, \gamma_n)\}$  converges to some  $(\bar{\lambda}, \bar{\gamma}) \in \mathcal{B}(\lambda^*, \varepsilon) \times \mathcal{B}(\gamma^*, \varepsilon)$ .

Let  $(z, w) \in K_1(x) \times K_2(y)$ . Again by the  $M$ -continuity of the maps  $K_1$  and  $K_2$ , there exists a sequence  $\{(z_n, w_n)\} \rightarrow (z, w)$  with  $z_n \in K_1(x_n)$  and  $w_n \in K_2(y_n)$ . By inequality (17) and from the joint upper semicontinuity of  $\tilde{f}_1$  and  $\tilde{f}_2$ , we have

$$\begin{aligned} x \in K_1(x), y \in K_2(y), \|y - Ax\| \leq \varepsilon; \\ \tilde{f}_1(\bar{\lambda}, x, z) + \varepsilon \geq \limsup_{n \rightarrow \infty} \tilde{f}_1(\lambda_n, x_n, z_n) + \varepsilon \geq -\lim_{n \rightarrow \infty} \frac{\alpha}{2}\|x_n - z_n\|^2 = -\frac{\alpha}{2}\|x - z\|^2; \\ \tilde{f}_2(\bar{\gamma}, y, w) + \varepsilon \geq \limsup_{n \rightarrow \infty} \tilde{f}_2(\gamma_n, y_n, w_n) + \varepsilon \geq -\lim_{n \rightarrow \infty} \frac{\alpha}{2}\|y_n - w_n\|^2 = -\frac{\alpha}{2}\|y - w\|^2. \end{aligned}$$

Thus,  $(x, y) \in S_\alpha(\varepsilon)$ . Therefore, for all  $\varepsilon > 0$ ,  $S_\alpha(\varepsilon)$  is a closed set.

Next we show that

$$S = \bigcap_{\varepsilon > 0} S_\alpha(\varepsilon).$$

It is trivial that  $S \subseteq \bigcap_{\varepsilon > 0} S_\alpha(\varepsilon)$ . For the other side inclusion, let  $(x^*, y^*) \in \bigcap_{\varepsilon > 0} S_\alpha(\varepsilon)$ . Consider a sequence  $\{\varepsilon_n\} \subseteq (0, \infty)$  such that  $\varepsilon_n \rightarrow 0$ . So for all  $n \in \mathbb{N}$ , there exists  $(\lambda_n, \gamma_n) \in \mathcal{B}(\lambda^*, \varepsilon_n) \times \mathcal{B}(\gamma^*, \varepsilon_n)$  such that

$$\begin{cases} (x^*, y^*) \in K_1(x^*) \times K_2(y^*), \|y^* - Ax^*\| \leq \varepsilon_n; \\ \tilde{f}_1(\lambda_n, x^*, z) + \varepsilon_n \geq -\frac{\alpha}{2}\|x^* - z\|^2, \forall z \in K_1(x^*); \\ \tilde{f}_2(\gamma_n, y^*, w) + \varepsilon_n \geq -\frac{\alpha}{2}\|y^* - w\|^2, \forall w \in K_2(y^*). \end{cases}$$

Letting  $n \rightarrow \infty$ , we get

$$\begin{cases} (x^*, y^*) \in K_1(x^*) \times K_2(y^*), y^* = Ax^*; \\ f_1(x^*, z) = \tilde{f}_1(\lambda^*, x^*, z) \geq \limsup_{n \rightarrow \infty} \tilde{f}_1(\lambda_n, x^*, z) + \lim_{n \rightarrow \infty} \varepsilon_n \geq -\frac{\alpha}{2} \|x^* - z\|^2, \forall z \in K_1(x^*); \\ f_2(y^*, w) = \tilde{f}_2(\gamma^*, y^*, w) \geq \limsup_{n \rightarrow \infty} \tilde{f}_2(\gamma_n, y^*, w) + \lim_{n \rightarrow \infty} \varepsilon_n \geq -\frac{\alpha}{2} \|y^* - w\|^2, \forall w \in K_2(y^*). \end{cases}$$

The last inequalities follow as  $\tilde{f}_1$  and  $\tilde{f}_2$  are upper semicontinuous in the first argument. By Proposition 3.9, we have  $(x^*, y^*) \in S$ . Hence,  $\bigcap_{\varepsilon > 0} S_\alpha(\varepsilon) = S$ . As  $\lim_{\varepsilon \rightarrow 0} \beta(S_\alpha(\varepsilon)) = 0$ , by [17, Theorem 1], we can show that  $S$  is non-empty compact and

$$\lim_{\varepsilon \rightarrow 0} H(S_\alpha(\varepsilon), S) = 0.$$

By Theorem 4.12, we conclude that (SQEP) is generalized  $\alpha$ -well-posed by perturbations. This completes the proof.  $\square$

**Corollary 4.16.** *Let  $\Lambda$  and  $\Gamma$  be finite-dimensional spaces. For  $i = 1, 2$ , suppose that*

- (i)  $K_i$  is non-empty and convex-valued;
- (ii) For any sequence  $\{x_{i,n}\} \subseteq C_i$  converges to  $x_i$ , the sequence  $\{K(x_{i,n})\}_n$  Lower Semi-Mosco converges to  $K(x_i)$ ;
- (iii)  $f_i$  satisfies the conditions (F1) and (F2).

Moreover, suppose that  $\tilde{f}_1$  and  $\tilde{f}_2$  are jointly u.s.c on  $\Lambda_1 \times C_1 \times C_1$  and  $\Gamma_1 \times C_2 \times C_2$ , respectively. Then, the (SQEP) is generalized  $\alpha$ -well-posed by perturbations if and only if

$$\forall \varepsilon > 0, S_\alpha(\varepsilon) \neq \emptyset \text{ and } \lim_{\varepsilon \rightarrow 0} \beta(S_\alpha(\varepsilon)) = 0,$$

where  $\beta$  is Kuratowski's measure of non-compactness.

**Remark 4.17.** *Theorem 4.12 and Theorem 4.15 hold true for split equilibrium problem (4) and generalize the results of [13].*

**Theorem 4.18.** *Let  $\Lambda$  and  $\Gamma$  be finite-dimensional spaces, and  $\tilde{f}_1$  and  $\tilde{f}_2$  be jointly u.s.c on  $\Lambda_1 \times C_1 \times C_1$  and  $\Gamma_1 \times C_2 \times C_2$ , respectively, and  $f_1$  and  $f_2$  satisfy the conditions (F1) and (F2). Also,  $K_1$  and  $K_2$  are convex-valued and  $M$ -continuous. Then, (SQEP) is generalized LP  $\alpha$ -well-posed by perturbations if and only if*

$$\forall \varepsilon > 0, L_\alpha(\varepsilon) \neq \emptyset \text{ and } \lim_{\varepsilon \rightarrow 0} \beta(L_\alpha(\varepsilon)) = 0, \tag{18}$$

where  $\beta$  is the Kuratowski's measure of non-compactness.

*Proof.* The necessary part follows the same path as the proof of Theorem 4.15. For the sufficient part, suppose that (18) holds. We prove that

$$S = \bigcap_{\varepsilon > 0} Cl(L_\alpha(\varepsilon)).$$

It is trivial that  $S \subseteq \bigcap_{\varepsilon > 0} Cl(L_\alpha(\varepsilon))$ . For the other side inclusion, let  $(x^*, y^*) \in \bigcap_{\varepsilon > 0} Cl(L_\alpha(\varepsilon))$ . This implies that

$$d((x^*, y^*), L_\alpha(\varepsilon)) = 0, \forall \varepsilon > 0.$$

Given  $\{\varepsilon_n\} \subseteq (0, \infty)$  with  $\varepsilon_n \rightarrow 0$ , then for every  $n \in \mathbb{N}$ , there exists  $(x_n, y_n) \in L_\alpha(\varepsilon_n)$  such that

$$d((x^*, y^*), (x_n, y_n)) < \varepsilon_n.$$

Thus,  $(x_n, y_n) \rightarrow (x^*, y^*)$  and there exists a sequence  $\{(\lambda_n, \gamma_n)\} \subset \mathcal{B}(\lambda^*, \varepsilon_n) \times \mathcal{B}(\gamma^*, \varepsilon_n)$  such that

$$d(x_n, K_1(x_n)) \leq \varepsilon_n, d(y_n, K_2(y_n)) \leq \varepsilon_n, \|y_n - Ax_n\| \leq \varepsilon_n; \tag{19}$$

$$\tilde{f}_1(\lambda_n, x_n, z) + \varepsilon_n \geq -\frac{\alpha}{2}\|x_n - z\|^2, \forall z \in K_1(x_n); \tag{20}$$

$$\tilde{f}_2(\gamma_n, y_n, w) + \varepsilon_n \geq -\frac{\alpha}{2}\|y_n - w\|^2, \forall w \in K_2(y_n). \tag{21}$$

Since  $A$  is bounded linear, it follows from  $\|y_n - Ax_n\| \leq \varepsilon_n$  in (19) that  $y^* = Ax^*$ . We show that  $x^* \in K_1(x^*)$  and  $y^* \in K_2(y^*)$ . It follows from (19) that for each  $n \in \mathbb{N}$ , we can select  $(x'_n, y'_n) \in K_1(x_n) \times K_2(y_n)$  such that

$$\|x_n - x'_n\| < d(x_n, K_1(x_n)) + \varepsilon_n \leq 2\varepsilon_n,$$

$$\|y_n - y'_n\| < d(y_n, K_2(y_n)) + \varepsilon_n \leq 2\varepsilon_n.$$

Then, we deduce that  $(x'_n, y'_n) \rightarrow (x^*, y^*)$ . By the  $M$ -continuity of the maps  $K_1$  and  $K_2$ , we get  $x^* \in K_1(x^*)$  and  $y^* \in K_2(y^*)$ .

Let  $(z, w) \in K_1(x^*) \times K_2(y^*)$ . By the  $M$ -continuity of the maps  $K_1$  and  $K_2$  again, there exists a sequence  $\{(z_n, w_n)\} \rightarrow (z, w)$  with  $z_n \in K_1(x_n)$  and  $w_n \in K_2(y_n)$ . Together inequalities (20) and (21) with joint upper semicontinuity of  $\tilde{f}_1$  and  $\tilde{f}_2$ , we have

$$f_1(x^*, z) = \tilde{f}_1(\lambda^*, x^*, z) \geq \limsup_{n \rightarrow \infty} \tilde{f}_1(\lambda_n, x_n, z_n) \geq -\lim_{n \rightarrow \infty} \frac{\alpha}{2}\|x_n - z_n\|^2 = -\frac{\alpha}{2}\|x^* - z\|^2;$$

$$f_2(y^*, w) = \tilde{f}_2(\gamma^*, y^*, w) \geq \limsup_{n \rightarrow \infty} \tilde{f}_2(\gamma_n, y_n, w_n) \geq -\lim_{n \rightarrow \infty} \frac{\alpha}{2}\|y_n - w_n\|^2 = -\frac{\alpha}{2}\|y^* - w\|^2.$$

The above inequalities hold for all  $z \in K_1(x^*)$  and  $w \in K_2(y^*)$ . Therefore, we get from Proposition 3.9 that  $(x^*, y^*) \in S$ . Thus,  $\bigcap_{\varepsilon > 0} Cl(L_\alpha(\varepsilon)) \subseteq S$ . By (18), we have

$$\lim_{\varepsilon \rightarrow 0} \beta(Cl(L_\alpha(\varepsilon))) = \lim_{\varepsilon \rightarrow 0} \beta(L_\alpha(\varepsilon)) = 0.$$

By the generalized Cantor theorem [17, Theorem 1], one can show that

$$\lim_{\varepsilon \rightarrow 0} H(L_\alpha(\varepsilon), S) = 0$$

and  $S$  is non-empty compact. By Theorem 4.13, we conclude that (SQEP) is generalized LP  $\alpha$ -well-posed by perturbations. This completes the proof.  $\square$

The concept of well-posedness for equilibrium problems or split equilibrium problems was introduced in references [24, 25]. It was demonstrated that, for finite-dimensional spaces  $X_1$  and  $X_2$ , and under appropriate conditions, a split equilibrium problem is well-posed if and only if it admits a unique solution. However, this equivalence does not generally hold even for the simplest case of split quasi-equilibrium problem illustrated by the following example that is motivated by [23, Example 3.1].

**Example 4.19.** Take  $X_1 = X_2 = \mathbb{R}, C_1 = C_2 = [0, \infty), \Lambda_1 = \Gamma_1 = [-1, 1], \alpha = 0$ . Define  $\tilde{f}_1 = \tilde{f}_2$  as:

$$\tilde{f}_1(\lambda, x, z) = e^{-x}(x - z) + \lambda^4, \lambda \in \Lambda_1, x, z \in \mathbb{R},$$

and

$$K_1(x) = K_2(x) = \begin{cases} [x, 1], & \text{if } x \in [0, 1], \\ [1, 2x - 1], & \text{if } x \in (1, \infty). \end{cases}$$

For  $\lambda^* = \gamma^* = 0$ ,  $f_1(x, z) = f_2(x, z) = e^{-x}(x - z)$ . Also, define  $A := I$ , the identity operator. Then  $(1, 1)$  is the unique solution of the split quasi-equilibrium problem (SQEP). For  $\varepsilon > 0$ ,

$$L_0(\varepsilon) = \bigcup_{\lambda, \gamma \in \mathcal{B}(0, \varepsilon)} \left\{ (x, y) \in [0, 1] \times [0, 1] : |y - x| \leq \varepsilon, e^{-x}(1 - x) - \lambda^4 \leq \varepsilon, e^{-y}(1 - y) - \gamma^4 \leq \varepsilon \right\} \\ \bigcup_{\lambda, \gamma \in \mathcal{B}(0, \varepsilon)} \left\{ (x, y) \in [0, 1] \times (1, \infty) : |y - x| \leq \varepsilon, e^{-x}(1 - x) - \lambda^4 \leq \varepsilon, e^{-y}(y - 1) - \gamma^4 \leq \varepsilon \right\} \\ \bigcup_{\lambda, \gamma \in \mathcal{B}(0, \varepsilon)} \left\{ (x, y) \in (1, \infty) \times [0, 1] : |y - x| \leq \varepsilon, e^{-x}(x - 1) - \lambda^4 \leq \varepsilon, e^{-y}(1 - y) - \gamma^4 \leq \varepsilon \right\} \\ \bigcup_{\lambda, \gamma \in \mathcal{B}(0, \varepsilon)} \left\{ (x, y) \in (1, \infty) \times (1, \infty) : |y - x| \leq \varepsilon, e^{-x}(x - 1) - \lambda^4 \leq \varepsilon, e^{-y}(y - 1) - \gamma^4 \leq \varepsilon \right\}.$$

Define

$$x_n = y_n = n + 1 \text{ and } \lambda_n = \gamma_n = \frac{1}{n^{1/4}}, \forall n \in \mathbb{N}.$$

Then,  $\{(x_n, y_n)\}$  is an approximating sequence corresponding to  $\left\{\left(\frac{1}{n^{1/4}}, \frac{1}{n^{1/4}}\right)\right\}$  for (SQEP) as there exists a sequence  $\left\{\frac{1}{n} + \frac{n}{e^{n+1}}\right\}$  converges to 0 such that for each  $n \in \mathbb{N}$ ,

$$e^{-x_n}(x_n - z) + \frac{1}{n^{1/4}} \geq -\frac{1}{n} - \frac{n}{e^{n+1}}, \forall z \in [1, 2x_n - 1]; \\ e^{-y_n}(y_n - w) + \frac{1}{n^{1/4}} \geq -\frac{1}{n} - \frac{n}{e^{n+1}}, \forall w \in [1, 2y_n - 1].$$

Also,  $\{(x_n, y_n)\}$  is a diverging sequence, which implies that (SQEP) is not LP well-posed by perturbations.

However, if the constraint  $C_1, C_2 \subseteq \mathbb{R}^n$  are compact and the split quasi-equilibrium problem (SQEP) admits a unique solution, then under appropriate conditions, it can be shown that the (SQEP) is both  $\alpha$ -well-posed and LP  $\alpha$ -well-posed by perturbations.

**Theorem 4.20.** Let  $X_1 = \mathbb{R}^{n_1}$  and  $X_2 = \mathbb{R}^{n_2}$ . For  $i = 1, 2$ , let  $\tilde{f}_1 : \Lambda_1 \times X_1 \times X_1 \rightarrow \bar{\mathbb{R}}, \tilde{f}_2 : \Gamma_1 \times X_2 \times X_2 \rightarrow \bar{\mathbb{R}}, K_1 : C_1 \rightarrow 2^{C_1}$  and  $K_2 : C_2 \rightarrow 2^{C_2}$  satisfy the following conditions:

- (i) For  $i = 1, 2, K_i$  is non-empty convex-valued,  $M$ -continuous and  $K_i(C_i)$  is bounded;
- (ii)  $\tilde{f}_1$  and  $\tilde{f}_2$  satisfy the conditions (F1) and (F2);
- (iii)  $\tilde{f}_1$  and  $\tilde{f}_2$  are jointly u.s.c on  $\Lambda_1 \times C_1 \times C_1$  and  $\Gamma_1 \times C_2 \times C_2$ , respectively.

Then, (SQEP) is  $\alpha$ -well-posed by perturbations if and only if (SQEP) has a unique solution.

*Proof.* The necessary part follows from the definition. Conversely, assume that (SQEP) has a unique solution, say  $(\bar{x}, \bar{y})$ . Let  $\{(\lambda_n, \gamma_n)\} \subset \Lambda_1 \times \Gamma_1$  with  $(\lambda_n, \gamma_n) \rightarrow (\lambda^*, \gamma^*)$  and  $\{(x_n, y_n)\}$  be any arbitrary  $\alpha$ -approximating sequence corresponding to  $\{(\lambda_n, \gamma_n)\}$  for (SQEP). Then there exists a sequence  $\{\varepsilon_n\}$  of non-negative integers with  $\varepsilon_n \rightarrow 0$  such that for all  $n \in \mathbb{N}$ ,

$$\begin{cases} (x_n, y_n) \in K_1(x_n) \times K_2(y_n), \|y_n - Ax_n\| \leq \varepsilon_n; \\ \tilde{f}_1(\lambda_n, x_n, z) + \varepsilon_n \geq -\frac{\alpha}{2}\|x_n - z\|^2, \forall z \in K_1(x_n); \\ \tilde{f}_2(\gamma_n, y_n, w) + \varepsilon_n \geq -\frac{\alpha}{2}\|y_n - w\|^2, \forall w \in K_2(y_n). \end{cases} \tag{22}$$

As  $K_1(C_1)$  and  $K_2(C_2)$  are bounded sets, the sequence  $\{(x_n, y_n)\}$  is bounded. Then, there exists a point  $(x^*, y^*)$  and a subsequence  $\{(x_{n_k}, y_{n_k})\}$  of  $\{(x_n, y_n)\}$  such that

$$(x_{n_k}, y_{n_k}) \rightarrow (x^*, y^*).$$

Take any  $(z, w) \in K_1(x^*) \times K_2(y^*)$ . Then, by the  $M$ -continuity of the maps  $K_1$  and  $K_2$ , there exists a sequence  $\{(z_{n_k}, w_{n_k})\} \rightarrow (z, w)$  with  $z_{n_k} \in K_1(x_{n_k})$  and  $w_{n_k} \in K_2(y_{n_k})$ . Applying (22) to the subsequences  $\{x_{n_k}\}$ ,  $\{y_{n_k}\}$ ,  $\{z_{n_k}\}$ ,  $\{w_{n_k}\}$ ,  $\{\lambda_{n_k}\}$ ,  $\{\gamma_{n_k}\}$  and taking  $\limsup_{k \rightarrow \infty}$ , one can obtain from the  $M$ -continuity of the maps  $K_1$  and  $K_2$  and the

joint upper semicontinuity of  $\tilde{f}_1$  and  $\tilde{f}_2$  that

$$\begin{aligned} x^* &\in K_1(x^*), \quad y^* \in K_2(y^*) \text{ and } y^* = Ax^*; \\ f_1(x^*, z) &\geq -\frac{\alpha}{2} \|x^* - z\|^2, \quad \forall z \in K_1(x^*) \text{ and } f_2(y^*, w) \geq -\frac{\alpha}{2} \|y^* - w\|^2, \quad \forall w \in K_2(y^*). \end{aligned}$$

Thus, Proposition 3.9 implies that  $(x^*, y^*) \in S$ . By the uniqueness of solution of (SQEP), the whole sequence  $\{(x_n, y_n)\}$  converges to  $(x^*, y^*)$ . Therefore, every  $\alpha$ -approximating sequence converges in  $S$ . Hence, (SQEP) is  $\alpha$ -well-posed by perturbations and this completes the proof.  $\square$

**Remark 4.21.** When the constraint mappings  $K_1$  and  $K_2$  are constant self-maps, then (SQEP) reduces to a split equilibrium problem. In this situation, instead of imposing condition (i) from Theorem 4.20, it is sufficient to assume that the sets  $C_1$  and  $C_2$  are non-empty, convex and bounded. Moreover, condition (iii) in Theorem 4.20 can be weakened to upper semicontinuity with respect to only the first and second argument.

Let  $F_1 : X_1 \rightarrow X_1$  and  $F_2 : X_2 \rightarrow X_2$  be two vector-valued maps, then split quasi-variational inequality problem is: Find  $(x^*, y^*) \in C_1 \times C_2$  such that

$$\begin{cases} x^* \in K_1(x^*) \text{ and } y^* = Ax^* \in K_2(y^*); \\ x^* \text{ solves QVI}(F_1, K_1) \text{ and } y^* \text{ solves QVI}(F_2, K_2). \end{cases} \quad (\text{SQVI})$$

**Remark 4.22.** If the bifunctions  $f_1$  and  $f_2$  are given as follows:

$$\begin{aligned} f_1(x, z) &= \langle F_1(x), z - x \rangle, \quad \forall x, z \in X_1, \\ f_2(y, w) &= \langle F_2(y), w - y \rangle, \quad \forall y, w \in X_2, \end{aligned}$$

then the results of (LP)  $\alpha$ -well-posedness by perturbations proved for (SQEP) hold true for (SQVI).

## 5. Conclusions

In this paper, we introduced the concept of the perturbed split quasi-equilibrium problem (SQEP) and extended Levitin-Polyak  $\alpha$ -well-posedness by perturbation to (SQEP) in infinite-dimensional real Banach spaces. By establishing metric characterizations, we provided equivalence results that connect LP  $\alpha$ -well-posedness by perturbations for (SQEP) and its solution set. Utilizing Kuratowski's measure of compactness, we investigated generalized LP  $\alpha$ -well-posedness in a broader sense, offering new insights into its structural properties. Furthermore, we established that, in finite-dimensional real Banach spaces,  $\alpha$ -well-posedness by perturbations is equivalent to the existence and uniqueness of a solution. To support our theoretical findings, we also presented several nontrivial examples.

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