Asymptotics of systemic expected shortfall for QAI primary variables with heavy tails

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Abstract. The paper considers the asymptotics of systemic expected shortfall in systemic risk. The individual losses we have mainly considered are the products of primary variables and random weights. When the primary variables and random weights are independent, and there is a dependence structure called quasi-asymptotically independence structure among the primary variables, we get the asymptotics of systemic expected shortfall of individual losses for the primary variables having the consistently varying-tailed distributions.

1. Introduction

Systemic risk, a crucial dimension of overall financial risk, refers to the potential that a localized disruption could propagate through the financial or economic system, thereby causing widespread instability or collapse. This concept is particularly relevant in analyzing financial instability within insurance and banking sectors. Characterized by its contagious nature, systemic repercussions, and inherent non-diversifiability, systemic risk demonstrates how relatively minor disturbances can escalate significantly due to external factors such as market-wide panic or loss of confidence. Therefore, scholars have conducted extensive research on the concept of systemic risk. In the mid-1990s, the concept of systemic risk was established and developed in the banking and financial sectors. Since the 2007-2008 global financial crisis, it has become a major research topic, and scholars have proposed various systemic risk measures, including the value at risk (VaR), the systemic expected shortfall (SES), the marginal expected shortfall (MES), and the systemic risk index (SRI), among others.

As Acharya et al. [1] and Chen and Liu [4], we consider a static one-period model, in which a system consists $n(n \geq 2)$ individuals (or sub-portfolios within the investment portfolios or business lines or companies). They generate loss-profit variables $Z_1, \ldots, Z_n$, respectively. Thus, the system has an aggregate loss

$$S_n = \sum_{i=1}^{n} Z_i.$$

Suppose the system follows the inter model approach, using a determined risk measure ρ to determine the solvency capital requirement and allocate A_i to the ith ($i = 1, 2, \ldots, n$) individual. Thus the full allocation holds

$$\rho(S_n) =: A = \sum_{i=1}^{n} A_i.$$

For an individual $k = 1, \ldots, n$, we define its SES as

$$SES_k = E[(Z_k - A_k)^+ | S_n > A],$$

where $a^+ = \max\{a, 0\} = a \vee 0$ and $a^- = -\min\{a, 0\} = -(a \wedge 0)$ are the positive and negative parts of a real number a. Under some capital allocation rules, such as the coherent capital allocation rule introduced by Kalkbrener [10], expected losses are associated with a confidence level $q \in (0, 1)$. This makes the solvency capital A determined by expected losses and the solvency capital $A_i, i = 1, 2, \ldots, n$ of the ith individual being related to the confidence level q. Thus, we assume that A and $A_i, i = 1, 2, \ldots, n$ are functions of the confidence level q and rewrite the SES of the kth ($k = 1, 2, \ldots, n$) individual as

$$SES_k(q) = E[(Z_k - A_k(q))^+ | S_n > A(q)]. \tag{1}$$

As Chen and Liu [4], we will use a popular framework with random weights to model individual losses $Z_1, \ldots, Z_n$. For each $i = 1, 2, \ldots, n$, the loss-profit variable Z_i has the following form

$$Z_i = X_i \theta_i,$$

where X_i is a primary variable, controlling the properties of Z_i, and θ_i is a random weight, capturing the ith individual's exposure to economic factors. In this paper, we suppose that $X_1, \ldots, X_n$ are n real-valued random variables with distributions $F_1, \ldots, F_n$, respectively, and $\theta_1, \ldots, \theta_n$ are n nonnegative and non-degenerate at zero random weights, which are independent of $X_1, \ldots, X_n$ and have distributions $G_1, \ldots, G_n$, respectively. As Asimit and Li [2], Liu and Yang [12], Chen and Liu [4], Li [11], Liu and Shushi [13], Wang and Li [16] and Chen and Cheng [6], we will consider asymptotic behavior of systemic risk measures. This paper will investigate the asymptotics of $SES_k(q), k = 1, 2, \ldots, n$, as $q \uparrow 1$. Throughout the paper, for each $k = 1, 2, \ldots, n$, we assume that $A_k(q) \to \infty$ as $q \uparrow 1$.

For the SES, Chen and Liu [4] considered the primary variables $X_1, \ldots, X_n$ are dependent. They presented the asymptotics of $SES_k(q), k = 1, 2, \ldots, n$ as $q \uparrow 1$ when the primary variables $X_1, \ldots, X_n$ were pairwise strongly quasi-asymptotic independence.

The strongly quasi-asymptotically independent structure was introduced by Geluk and Tang [9].

Definition 1.1. *Let η_1 and η_2 be two real-valued random variables. η_1 and η_2 are said to be strongly quasi-asymptotically independent (SQAI) if for $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$,*

$$\lim_{x_1 \wedge x_2 \to \infty} P(|\eta_i| > x_i | \eta_j > x_j) = 0.$$

In this paper, we will consider the quasi-asymptotically independent structure, which is weaker than the SQAI structure. On the basis of Resnick [14], Chen and Yuen [5] gave the definition of quasi-asymptotically independent structure.

Definition 1.2. *Let η_1 and η_2 be two nonnegative random variables with distributions V_1 and V_2, respectively. η_1 and η_2 are said to be quasi-asymptotically independent (QAI) if*

$$\lim_{x \to \infty} \frac{P(\eta_1 > x, \eta_2 > x)}{\overline{V_1}(x) + \overline{V_2}(x)} = 0. \tag{2}$$

Generally, two real-valued random variables η_1 and η_2 are still said to be QAI if the relation (2) holds for $\{\eta_1^+, \eta_2^+\}, \{\eta_1^+, \eta_2^-\}$ and $\{\eta_1^-, \eta_2^+\}$. From the definition of SQAI and QAI, we know that the SQAI structure is stronger than the QAI structure.

In the following, we will consider the primary variables $X_1, \ldots, X_n$ are pairwise QAI and investigate the asymptotics of $SES_k(q)$ as $q \uparrow 1$ when the distributions of the primary variables are heavy-tailed.

The remainder of this paper is organized into three sections. Section 2 presents some preliminaries about heavy-tailed distributions. Section 3 gives the main results of this paper and their proofs.

2. Preliminaries

Throughout this paper, the limit relations are either $q \uparrow 1$ or $x \to \infty$, unless otherwise stated. For two positive functions $f(\cdot)$ and $g(\cdot)$, write $f = o(g)$, if $\lim f/g = 0$; write $f = O(g)$, if $\limsup f/g < \infty$; write $f \lesssim g$ or $g \gtrsim f$, if $\limsup f/g \le 1$; write $f \sim g$, if $\lim f/g = 1$ and write $f \asymp g$, if $0 < \liminf f/g \le \limsup f/g < \infty$.

Now we recall the definition of a heavy-tailed distribution and give some important subclasses of the heavy-tailed distribution class. In the following, let V be a distribution on $(-\infty, \infty)$ having a right tail does not vanish in the sense that $\overline{V}(x) = 1 - V(x) > 0$ for all $x \in (-\infty, \infty)$.

Say that a distribution V on $(-\infty, \infty)$ is heavy-tailed , if for any $\lambda > 0$,

$$\int_{-\infty}^{\infty} e^{\lambda x} V(dx) = \infty.$$

One of the important subclasses of the heavy-tailed distribution class is the long-tailed distribution class. Let V be a distribution on $(-\infty, \infty)$, if for any $u > 0$,

$$\lim_{x \to \infty} \frac{\overline{V}(x + u)}{\overline{V}(x)} = 1,$$

then V is said to be long-tailed, denoted by $V \in \mathscr{L}$.

A related distribution class is the dominatedly varying-tailed distribution class. Let V be a distribution on $(-\infty, \infty)$, if for any $0 < u < 1$,

$$\limsup_{x \to \infty} \frac{\overline{V}(xu)}{\overline{V}(x)} < \infty,$$

then V is said to be dominatedly varying-tailed, denoted by $V \in \mathscr{D}$.

A smaller distribution class than the classes $\mathscr{L}$ and $\mathscr{D}$ is the consistently varying-tailed distribution class. Let V be a distribution on $(-\infty, \infty)$, if

$$\lim_{u \uparrow 1} \limsup_{x \to \infty} \frac{\overline{V}(xu)}{\overline{V}(x)} = 1 \quad \text{or} \quad \lim_{u \downarrow 1} \liminf_{x \to \infty} \frac{\overline{V}(xu)}{\overline{V}(x)} = 1,$$

then V is said to be consistently varying-tailed, denoted by $V \in \mathscr{C}$.

An important subclass of the class $\mathscr{C}$ is the regularly varying-tailed distribution class. For a distribution V on $(-\infty, \infty)$, if for any $u > 0$ and some $0 \le \alpha < \infty$, it holds

$$\lim_{x \to \infty} \frac{\overline{V}(xu)}{\overline{V}(x)} = u^{-\alpha},$$

then it is said to be regularly varying-tailed, denoted by $V \in \mathscr{R}_{-\alpha}$. Let $\mathscr{R} = \bigcup_{\alpha \ge 0} \mathscr{R}_{-\alpha}$.

It is well known that

$$\mathscr{R} \subset \mathscr{C} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{L},$$

see, e.g. Embrechts et al. [8] and Cline and Samorodnitsky [7].

Following Tang and Tsitsiashvili [15], we define the lower and upper Matuszewska indices of V, respectively, as

$$J_V^- = \sup \left\{ -\frac{\log \overline{V}^*(y)}{\log y} : y \ge 1 \right\} \quad \text{and} \quad J_V^+ = \inf \left\{ -\frac{\log \overline{V}_*(y)}{\log y} : y \ge 1 \right\},$$

where

$$\overline{V}^*(y) = \limsup_{x \to \infty} \frac{\overline{V}(xy)}{\overline{V}(x)} \quad \text{and} \quad \overline{V}_*(y) = \liminf_{x \to \infty} \frac{\overline{V}(xy)}{\overline{V}(x)}, y \ge 0.$$

3. Main results and their proofs

3.1. Main results

The following main result presents the asymptotics of the SES for pairwise QAI primary variables.

Theorem 3.1. *Recall the SES introduced in (1). Assume that the primary variables $X_1, \ldots, X_n$ are pairwise QAI. Suppose that for all $i = 1, 2, \ldots, n$, $F_i \in \mathscr{C}$, $J_{F_i}^- > 1$, and $E\left[\theta_i^p\right] < \infty$ for some $p > \vee_{i=1}^n J_{F_i}^+$. If for any $1 \le i \ne j \le n$, $\overline{F}_i(x) \asymp \overline{F}_j(x)$ and for some individual k $(k = 1, 2, \ldots, n)$, $A(q) = O(A_k(q))$, then*

$$SES_k(q) \sim \frac{P(Z_k > A(q)) \sum\limits_{i=1, i \ne k}^n A_i(q) + E[(Z_k - A(q))^+]}{\sum\limits_{i=1}^n P(Z_i > A(q))}.$$

Remark 3.2. *Chen and Liu [4] investigated the primary variables $X_1, \ldots, X_n$ are pairwise SQAI and studied the asymptotics of SES when the distributions of the primary variables belonged to the class $\mathscr{L} \cap \mathscr{D}$. Theorem 3.1 considers the primary variables $X_1, \ldots, X_n$ having the QAI structure, which is weaker than the SQAI structure. When the distributions of $X_1, \ldots, X_n$ belong to the class $\mathscr{C}$, Theorem 3.1 presents the asymptotics of the SES.*

As Chen and Liu [4], in the following we consider a coherent capital allocation rule, which is introduced by Kalkbrener [10], to give a more explicit result for the regularly varying-tailed primary variables. For a risk variable ξ with a distribution V, its VaR and expected shortfall (ES) at level $0 < q < 1$ are defined by, respectively,

$$VaR_q(\xi) = V^{\leftarrow}(q) = \inf\{x \in (-\infty, \infty) : V(x) \ge q\}$$

and

$$ES_q(\xi) = \frac{1}{1-q} \int_q^1 VaR_p(\xi) dp.$$

According to Section 5.2 of Kalkbrener [10], under $\rho(S_n) = ES_q(S_n)$, the coherent capital allocation rule allocates for each $k = 1, 2, \ldots, n$,

$$A_k(q) = \frac{P\left(S_n > H_n^{\leftarrow}(q)\right)}{1-q} E\left[Z_k | S_n > H_n^{\leftarrow}(q)\right] + \frac{P\left(S_n \le H_n^{\leftarrow}(q)\right) - q}{1-q} E\left[Z_k | S_n = H_n^{\leftarrow}(q)\right],$$

where H_n is the distribution of S_n.

Corollary 3.3. *Consider SES in (1) under the coherent capital allocation rule. Assume that the primary variables $X_1, \ldots, X_n$ are pairwise QAI. Suppose that there is a distribution F on $(-\infty, \infty)$ satisfying $F \in \mathscr{R}_{-\alpha}$ for some $\alpha > 1$ and $\overline{F}_i(x) \sim b_i \overline{F}(x)$ for all $i = 1, 2, \ldots, n$ and some constants $b_i > 0$. If for all $i = 1, 2, \ldots, n$, $E\left[\theta_i^p\right] < \infty$ for some $p > \alpha$, then for each $k = 1, 2, \ldots, n$,*

$$
\begin{aligned}
&SES_k(q) \\
&\sim \; d_n^{\frac{1}{\alpha}-1} \frac{\alpha}{\alpha-1} b_k E\left[\theta_k^\alpha\right] \left[d_n^{-1} \sum_{i=1, i \ne k}^n b_i E\left[\theta_i^\alpha\right] + \frac{1}{\alpha-1} \right] F^{\leftarrow}(q),
\end{aligned}
$$

where $d_n = \sum_{i=1}^n b_i E\left[\theta_i^\alpha\right]$.

In the following, we will give the proofs of main results.

3.2. Some lemmas

Before giving the proofs of main results, we first present some lemmas. The first lemma can be obtained from Proposition 2.2.1 of Bingham et al. [3] and Lemma 3.5 of Tang and Tsitsashvili [15].

Lemma 3.4. *Let V be a distribution on $(-\infty, \infty)$. If $V \in \mathscr{D}$, then for each $0 < \widetilde{p} < J_V^- \leq J_V^+ < p$, there exist positive constants C_i and D_i, $i = 1, 2$, such that*

$$\frac{\overline{V}(y)}{\overline{V}(x)} \leq C_1 \left(\frac{y}{x}\right)^{-p}$$

for all $x \geq y \geq D_1$ and

$$\frac{\overline{V}(y)}{\overline{V}(x)} \geq C_2 \left(\frac{y}{x}\right)^{-\widetilde{p}}$$

for all $x \geq y \geq D_2$. Furthermore, for any $p > J_V^+$, it holds that

$$x^{-p} = o(\overline{V}(x)).$$

The next lemma is Theorem 3.3 (ii) and (iv) of Cline and Samorodnitsky [7].

Lemma 3.5. *Let ξ be a real-valued random variable with a distribution V_1 and η be a nonnegative nondegenerate at zero random variable with a distribution V_2. Assume that ξ and η are independent and W is the distribution of $\xi\eta$. If $V_1 \in \mathscr{D}$ and $E[\eta^p] < \infty$ for some $p > J_{V_1}^+$, then for each $y > 1$, $\overline{V_{1*}}(y) \leq \overline{W_*}(y) \leq \overline{W^*}(y) \leq \overline{V_1^*}(y)$, $W \in \mathscr{D}$ and*

$$0 < E\left[\overline{V_{1*}}\left(\frac{1}{\eta}\right)\mathbf{1}_{\{\eta>0\}}\right] \leq \liminf_{x\to\infty} \frac{P(\xi\eta > x)}{\overline{V_1}(x)} \leq \limsup_{x\to\infty} \frac{P(\xi\eta > x)}{\overline{V_1}(x)} \leq E\left[\overline{V_1^*}\left(\frac{1}{\eta}\right)\mathbf{1}_{\{\eta>0\}}\right] < \infty.$$

The following two lemmas are Theorem 3.2 and Lemma 3.1 of Chen and Yuen [5], respectively.

Lemma 3.6. *Let $\xi_i, i = 1, 2, \ldots, n$ be n QAI real-valued random variables with distributions $V_i, i = 1, 2, \ldots, n$, respectively. Let $\eta_i, i = 1, 2, \ldots, n$ be another n nonnegative random variables independent of $\xi_i, i = 1, 2, \ldots, n$. If $V_i \in \mathscr{C}$ and $E[\eta_i^p] < \infty$ for some $p > \vee_{i=1}^n J_{V_i}^+$ and for all $i = 1, 2, \ldots, n$, then*

$$P\left(\sum_{i=1}^n \xi_i\eta_i > x\right) \sim \sum_{i=1}^n P(\xi_i\eta_i > x).$$

Lemma 3.7. *Let ξ_1 and ξ_2 be two QAI real-valued random variables with distributions $V_1 \in \mathscr{C}$ and $V_2 \in \mathscr{C}$, respectively. Let η_1 and η_2 be two nonnegative random variables independent of ξ_1 and ξ_2. If $E[\eta_i^p] < \infty$ for some $p > J_{V_1}^+ \vee J_{V_2}^+$ and for all $i = 1, 2$, then $\xi_1\eta_1$ and $\xi_2\eta_2$ are QAI with distributions both belonging to the class $\mathscr{C}$.*

Lemma 3.8. *Under the conditions of Theorem 3.1, for any $u > 0$, it holds that*

$$P\left(X_k\theta_k > ux, \sum_{i=1}^n X_i\theta_i > x\right) \sim P(X_k\theta_k > (u \vee 1)x).$$

Proof. We will follow the line of the proof of Lemma A.3 in Chen and Liu [4] to prove this lemma. Without loss of generality, we only consider the case of $k = 1$. Let $Z_i = X_i\theta_i, i = 1, 2, \ldots, n$.

When $0 < u \leq 1$, we need to prove

$$P\left(Z_1 > ux, \sum_{i=1}^n Z_i > x\right) \sim P(Z_1 > x). \tag{3}$$

For any $x > 0$, it holds that

$$
\begin{aligned}
P\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) &\leq P\left(Z_1 > ux, \sum_{i=1}^{n} Z_i^+ > x\right) \\
&= P\left(\sum_{i=1}^{n} Z_i^+ > x\right) - P\left(Z_1 \leq ux, \sum_{i=1}^{n} Z_i^+ > x\right) \\
&=: I_1(x) - I_2(x).
\end{aligned}
\tag{4}
$$

By Lemma 3.6 it holds that

$$
I_1(x) \sim \sum_{i=1}^{n} P(Z_i > x).
\tag{5}
$$

For $I_2(x)$, since $0 < u \leq 1$, it holds for any $x > 0$ that

$$
\begin{aligned}
I_2(x) &\geq P\left(Z_1 \leq ux, \bigcup_{i=1}^{n}\{Z_i > x\}\right) \\
&\geq \sum_{i=2}^{n} P(Z_i > x) - \sum_{i=2}^{n} P(Z_i > ux, Z_1 > ux) - \sum_{2 \leq i \neq j \leq n} P(Z_i > x, Z_j > x) \\
&=: \sum_{i=2}^{n} P(Z_i > x) - I_{21}(x) - I_{22}(x).
\end{aligned}
\tag{6}
$$

By Lemma 3.7, we can know that $Z_1, \ldots, Z_n$ are pairwise QAI, and for all $i = 1, 2, \ldots, n$, the distribution of Z_i belongs to class $\mathscr{C}$. Therefore, the distributions of $Z_i, i = 1, 2, \ldots, n$ also belong to class $\mathscr{D}$. Since $F_i \in \mathscr{C} \subset \mathscr{D}$ and $E[\eta_i^p] < \infty$ for some $p > \vee_{i=1}^{n} J_{V_i}^+$ and for all $i = 1, 2, \ldots, n$, by Lemma 3.5, it holds for all $i = 1, 2, \ldots, n$ that

$$
P(Z_i > x) \asymp \overline{F_i}(x).
\tag{7}
$$

Since for any $1 \leq i \neq j \leq n$, $\overline{F_i}(x) \asymp \overline{F_j}(x)$, by (7) it holds that

$$
P(Z_i > x) \asymp P(Z_j > x).
\tag{8}
$$

Therefore, since $Z_1, \ldots, Z_n$ are QAI, by (8) and the distributions of $Z_i, i = 1, 2, \ldots, n$ belong to the class $\mathscr{D}$, it holds that

$$
I_{21}(x) = o\left(\sum_{i=2}^{n} P(Z_i > ux)\right) = o\left(\sum_{i=2}^{n} P(Z_i > x)\right)
$$

and

$$
I_{22}(x) = o\left(\sum_{i=2}^{n} P(Z_i > x)\right),
$$

which, combining with (6), yields that

$$
I_2(x) \gtrsim \sum_{i=2}^{n} P(Z_i > x).
\tag{9}
$$

By (5) and (9), for any $\varepsilon > 0$, there exists a constant $x_0 > 0$ such that for any $x \geq x_0$, it holds that

$$
I_1(x) \leq (1 + \varepsilon) \sum_{i=1}^{n} P(Z_i > x) \quad \text{and} \quad I_2(x) \geq (1 - \varepsilon) \sum_{i=2}^{n} P(Z_i > x).
$$

Thus, combining with (4), for any $x \geq x_0$, it holds that

$$P\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) \leq (1 + \varepsilon)P(Z_1 > x) + 2\varepsilon \sum_{i=2}^{n} P(Z_i > x). \tag{10}$$

Since for any $i = 1, 2, \ldots, n$, $\overline{F_i}(x) = O\left(\overline{F_1}(x)\right)$, it follows from (7) that

$$\sum_{i=2}^{n} P(Z_i > x) = O\left(P(Z_1 > x)\right).$$

Therefore, by (10) and the arbitrariness of ε, it holds that

$$P\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) \lesssim P(Z_1 > x). \tag{11}$$

In the following we will show that for any $0 < u \leq 1$,

$$P\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) \gtrsim P(Z_1 > x).$$

For any $0 < u \leq 1$, $0 < \varepsilon < 1$ and $x > 0$,

$$
\begin{aligned}
P\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) & \geq P\left(Z_1 > x, \sum_{i=1}^{n} Z_i > x\right) \\
& \geq P\left(Z_1 > (1 + \varepsilon)x, \sum_{i=1}^{n} Z_i > x\right) \\
& = P(Z_1 > (1 + \varepsilon)x) - P\left(Z_1 > (1 + \varepsilon)x, \sum_{i=1}^{n} Z_i \leq x\right) \\
& =: J_1(x) - J_2(x).
\end{aligned}
\tag{12}
$$

Since the distribution of Z_1 belongs to the class $\mathscr{C}$, it holds that

$$\lim_{\varepsilon \downarrow 0} \liminf_{x \to \infty} \frac{J_1(x)}{P(Z_1 > x)} = \lim_{\varepsilon \downarrow 0} \liminf_{x \to \infty} \frac{P(Z_1 > (1 + \varepsilon)x)}{P(Z_1 > x)} = 1. \tag{13}$$

For $J_2(x)$, since $Z_1, \ldots, Z_n$ are pairwise QAI, by (8), we can get that

$$
\begin{aligned}
J_2(x) & \leq P\left(Z_1 > (1 + \varepsilon)x, \sum_{i=2}^{n} Z_i \leq -\varepsilon x\right) \\
& \leq \sum_{i=2}^{n} P\left(Z_1 > x, Z_i \leq -\frac{\varepsilon x}{n-1}\right) \\
& \leq \sum_{i=2}^{n} P\left(Z_1 > \frac{\varepsilon x}{n-1}, Z_i \leq -\frac{\varepsilon x}{n-1}\right) \\
& = o\left(P\left(Z_1 > \frac{\varepsilon x}{n-1}\right)\right) \\
& = o(P(Z_1 > x)),
\end{aligned}
\tag{14}
$$

where the last step is due to the fact that the distribution of Z_1 belongs to class $\mathscr{D}$. By (12)-(14), it holds that

$$P\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) \gtrsim P(Z_1 > x). \tag{15}$$

Thus, by (11) and (15) we can get (3) holds when $0 < u \le 1$.

Next, we consider the condition of $u > 1$. That is to say, we need to prove

$$P\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) \sim P(Z_1 > ux). \tag{16}$$

It is obvious that for any $x > 0$,

$$P\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) \le P(Z_1 > ux). \tag{17}$$

For the lower bound, for any $x > 0$,

$$
\begin{aligned}
P\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) &\ge\quad P(Z_1 > ux) - P\left(Z_1 > ux, \sum_{i=1}^{n} Z_i \le (1-u)x\right) \\
&=:\quad P(Z_1 > ux) - K_1(x).
\end{aligned}
\tag{18}
$$

Since $Z_1, \ldots, Z_n$ are pairwise QAI, by (8), it holds that

$$
\begin{aligned}
K_1(x) &\le\quad \sum_{i=2}^{n} P\left(Z_1 > ux, Z_i \le -\frac{(u-1)x}{n-1}\right) \\
&\le\quad \sum_{i=2}^{n} P\left(Z_1 > \frac{(u-1)x}{n-1}, Z_i \le -\frac{(u-1)x}{n-1}\right) \\
&=\quad o\left(P\left(Z_1 > \frac{(u-1)x}{n-1}\right)\right) \\
&=\quad o(P(Z_1 > ux)),
\end{aligned}
$$

where in the last step the fact that the distribution of Z_1 belongs to class $\mathscr{D}$ is used. Combining with (18), it holds that

$$P\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) \gtrsim P(Z_1 > ux). \tag{19}$$

Thus, by (17) and (19), we can get (16) holds for the case of $u > 1$. This completes the proof of Lemma 3.8.

3.3. Proof of Theorem 3.1

We will use the way of the proof of Theorem 3.1 of Chen and Liu [4] to prove Theorem 3.1. Recall (1) in Section 1, for any $0 < q < 1$,

$$
\begin{aligned}
SES_k(q) &=\quad \frac{E\left[(Z_k - A_k(q))^+ \mathbf{1}_{\{S_n > A(q)\}}\right]}{P(S_n > A(q))} \\
&=\quad \frac{\int_{A_k(q)}^{\infty} P(Z_k > z, S_n > A(q)) dz}{P(S_n > A(q))} \\
&=\quad \frac{A(q)}{P(S_n > A(q))} \left(\int_{\frac{A_k(q)}{A(q)}}^{1} + \int_{1}^{\infty}\right) P(Z_k > uA(q), S_n > A(q)) du \\
&=:\quad \frac{A(q)}{P(S_n > A(q))} (H_1(q) + H_2(q)).
\end{aligned}
\tag{20}
$$

We first deal with $H_1(q)$. Since $\liminf\limits_{q\uparrow 1}\frac{A_k(q)}{A(q)} =: a_k > 0$, for any $0 < \delta < 1$, there exists a constant $q_0 \in (0,1)$ such that for any $q \in (q_0, 1)$,

$$\frac{A_k(q)}{A(q)} \geq a_k\delta. \tag{21}$$

We will show that

$$P(Z_k > uA(q), S_n > A(q)) \sim P(Z_k > A(q)) \tag{22}$$

holds uniformly for $u \in [a_k\delta, 1]$. It follows from Lemma 3.8 that

$$\begin{aligned}
P(Z_k > uA(q), S_n > A(q)) &\geq& P(Z_k > A(q), S_n > A(q)) \\
&\sim& P(Z_k > A(q))
\end{aligned}$$

and

$$\begin{aligned}
P(Z_k > uA(q), S_n > A(q)) &\leq& P(Z_k > a_k\delta A(q), S_n > A(q)) \\
&\sim& P(Z_k > A(q)).
\end{aligned}$$

Therefore, by (21), we get that (22) holds uniformly for $u \in \left[\frac{A_k(q)}{A(q)}, 1\right]$, which means that

$$H_1(q) \sim \int_{\frac{A_k(q)}{A(q)}}^{1} P(Z_k > A(q))du = \left(1 - \frac{A_k(q)}{A(q)}\right)P(Z_k > A(q)). \tag{23}$$

In the following, we are going to prove

$$H_2(q) \sim \int_1^{\infty} P(Z_k > uA(q))du. \tag{24}$$

For any $0 < \delta, \Delta < \infty$ satisfying $1 + \delta \leq \Delta$, dividing $H_2(q)$ into three parts

$$\begin{aligned}
H_2(q) &=& \left(\int_1^{1+\delta} + \int_{1+\delta}^{\Delta} + \int_{\Delta}^{\infty}\right)P(Z_k > uA(q), S_n > A(q))du \\
&=:& H_{21}(q) + H_{22}(q) + H_{23}(q).
\end{aligned} \tag{25}$$

It is obvious that

$$H_{21}(q) \leq \int_1^{1+\delta} P(Z_k > uA(q))du \leq \delta P(Z_k > A(q)). \tag{26}$$

Assume that W_k is the distribution of Z_k. Since $F_k \in \mathscr{C} \subset \mathscr{D}$ and $E\left[\theta_k^p\right] < \infty$ for some $p > J_{F_k}^+$, by Lemma 3.5, we can get $J_{W_k}^- \geq J_{F_k}^- > 1$. Since $W_k \in \mathscr{D}$, for each $1 < \widetilde{p} < J_{W_k}^- \leq J_{W_k}^+ < p$, it follows from Lemma 3.4 that there exist positive constants C_i and $D_i, i = 1, 2$, when $x \geq y \geq D_1$, it holds

$$\frac{\overline{W_k}(y)}{\overline{W_k}(x)} \leq C_1\left(\frac{y}{x}\right)^{-p},$$

and when $x \geq y \geq D_2$

$$\frac{\overline{W_k}(y)}{\overline{W_k}(x)} \geq C_2\left(\frac{y}{x}\right)^{-\widetilde{p}}. \tag{27}$$

Therefore, when $q \uparrow 1$, for some $p > J_{W_k}^+ \geq J_{W_k}^- > 1$,

$$\frac{\int_1^{\infty} P\left(Z_k > uA(q)\right)du}{P\left(Z_k > A(q)\right)} \geq \int_1^{\infty} C_1^{-1}u^{-p}du = \frac{C_1^{-1}}{p-1}. \tag{28}$$

By (26) and (28) it holds that

$$
\begin{aligned}
&\limsup_{\delta\downarrow 0}\limsup_{q\uparrow 1}\frac{H_{21}(q)+\int_{1}^{1+\delta}P\left(Z_k > uA(q)\right)du}{\int_{1}^{\infty}P\left(Z_k > uA(q)\right)du}\\
&\leq \limsup_{\delta\downarrow 0}\limsup_{q\uparrow 1}\frac{2\delta P\left(Z_k > A(q)\right)}{\int_{1}^{\infty}P\left(Z_k > uA(q)\right)du}\\
&= 0.
\end{aligned}
\tag{29}
$$

For $H_{23}(q)$, when $q\uparrow 1$, by (27), it holds that

$$
\begin{aligned}
H_{23}(q) &\leq \int_{\Delta}^{\infty}P(Z_k > uA(q))du\\
&= \Delta\int_{1}^{\infty}P(Z_k > \Delta uA(q))du\\
&\leq C_2^{-1}\Delta^{1-\widetilde{p}}\int_{1}^{\infty}P(Z_k > uA(q))du.
\end{aligned}
\tag{30}
$$

Since $\widetilde{p} > 1$, by (30),

$$
\limsup_{\Delta\uparrow\infty}\limsup_{q\uparrow 1}\frac{H_{23}(q)+\int_{\Delta}^{\infty}P\left(Z_k > uA(q)\right)du}{\int_{1}^{\infty}P\left(Z_k > uA(q)\right)du}\leq \limsup_{\Delta\uparrow\infty}2C_2^{-1}\Delta^{1-\widetilde{p}}=0.
\tag{31}
$$

In the following, we deal with $H_{22}(q)$. Since $Z_1,\ldots,Z_n$ are pairwise QAI, and by (8), we can get

$$
\begin{aligned}
&\left|H_{22}(q)-\int_{1+\delta}^{\Delta}P(Z_k > uA(q))du\right|\\
&\leq \int_{1+\delta}^{\Delta}P\left(Z_k > uA(q),\sum_{i=1}^{n}Z_i \leq A(q)\right)du\\
&\leq \int_{1+\delta}^{\Delta}P\left(Z_k > uA(q),\sum_{i=1,i\neq k}^{n}Z_i \leq (1-u)A(q)\right)du\\
&\leq \int_{1+\delta}^{\Delta}\sum_{i=1,i\neq k}^{n}P\left(|Z_i|\geq\frac{(u-1)A(q)}{n-1},Z_k > uA(q)\right)du\\
&\leq (\Delta-1-\delta)\sum_{i=1,i\neq k}^{n}P\left(|Z_i|\geq\frac{\delta A(q)}{n-1},Z_k > (1+\delta)A(q)\right)\\
&\leq (\Delta-1-\delta)\sum_{i=1,i\neq k}^{n}P\left(|Z_i|\geq\frac{\delta A(q)}{n-1}\wedge(1+\delta)A(q),Z_k > \frac{\delta A(q)}{n-1}\wedge(1+\delta)A(q)\right)\\
&= o\left(P\left(Z_k > \frac{\delta A(q)}{n-1}\wedge(1+\delta)A(q)\right)\right)\\
&= o(P(Z_k > A(q))),
\end{aligned}
\tag{32}
$$

where the last step is due to the fact that the distribution of Z_k belongs to class $\mathscr{D}$. By (28) and (32), it holds

that

$$\limsup_{q\uparrow 1} \frac{\left|H_{22}(q) - \int_{1+\delta}^{\Delta} P\left(Z_k > uA(q)\right)du\right|}{\int_1^{\infty} P\left(Z_k > uA(q)\right)du}$$

$$\leq \limsup_{q\uparrow 1} \frac{\left|H_{22}(q) - \int_{1+\delta}^{\Delta} P\left(Z_k > uA(q)\right)du\right|}{P\left(Z_k > A(q)\right)} \limsup_{q\uparrow 1} \frac{P\left(Z_k > A(q)\right)}{\int_1^{\infty} P\left(Z_k > uA(q)\right)du}$$

$$= 0. \tag{33}$$

By (25), for any $0 < \delta, \Delta < \infty$ satisfying $1 + \delta < \Delta$ and any $0 < q < 1$,

$$\left|H_2(q) - \int_1^{\infty} P(Z_k > uA(q))du\right|$$

$$= \left|H_{21}(q) + H_{22}(q) + H_{23}(q) - \int_1^{\infty} P(Z_k > uA(q))du\right|$$

$$\leq H_{21}(q) + \int_1^{1+\delta} P(Z_k > uA(q))du + \left|H_{22}(q) - \int_{1+\delta}^{\Delta} P(Z_k > uA(q))du\right|$$

$$+H_{23}(q) + \int_1^{\infty} P(Z_k > uA(q))du,$$

which combining with (29), (31) and (33) yields that

$$\left|H_2(q) - \int_1^{\infty} P(Z_k > uA(q))du\right| = o\left(\int_1^{\infty} P(Z_k > uA(q))du\right),$$

i.e., (24) holds. By Lemma 3.6, it holds that

$$P(S_n > A(q)) \sim \sum_{i=1}^{n} P(Z_i > A(q)). \tag{34}$$

By (20), (23), (24) and (34), it holds that

$$SES_k(q) \sim \frac{A(q)}{\sum\limits_{i=1}^{n} P(Z_i > A(q))}\left(\left(1 - \frac{A_k(q)}{A(q)}\right)P(Z_k > A(q)) + \int_1^{\infty} P(Z_k > uA(q))du\right)$$

$$\sim \frac{P(Z_k > A(q))\sum\limits_{i=1,i\neq k}^{n} A_i(q) + E\left[(Z_k - A(q))^+\right]}{\sum\limits_{i=1}^{n} P(Z_i > A(q))}.$$

This completes the proof of Theorem 3.1.   □

3.4. Proof of Corollary 3.3

By following the line of the proof of Corollary 3.1 in Chen and Liu [4] and applying Lemmas 3.4 - 3.8, Corollary 3.3 can be shown with appropriate minor modifications. The detailed argument is omitted here. □

References

[1] V.V. Acharya, L.H. Pedersen, T. Philippon, M. Richardson, *Measuring systemic risk*, Rev. Financ. Stud., **30** (2017), 2–47.
[2] A.V. Asimit, J. Li, *Systemic risk: an asymptotic evaluation*, ASTIN Bulletin, **48** (2018), 673–698.
[3] N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, 1987.
[4] Y. Chen, J. Liu, *An asymptotic study of systemic expected shortfall and marginal expected shortfall*, Insur. Math. Econ., **105** (2022), 238–251.
[5] Y. Chen, K.C. Yuen, *Sums of pairwise quasi-asymptotically independent random variables with consistent variation*, Stoch. Models, **25** (2009), 76–89.
[6] Z. Chen, D. Cheng, *On the tail behavior for randomly weighted sums of dependent random variables with its applications to risk measures*, Methodol. Comput. Appl., **26** (2024), 50.
[7] D.B.H. Cline, G. Samorodnitsky, *Subexponentiality of the product of independent random variables*, Stoch. Proc. Appl., **49** (1994), 75–98.
[8] P. Embrechts, C. Klüppelberg, T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, Springer, Berlin, 1997.
[9] J. Geluk, Q. Tang, *Asymptotic tail probabilities of sums of dependent subexponential random variables*, J. Theor. Probab., **22** (2009), 871–882.
[10] M. Kalkbrener, *An axiomatic approach to capital allocation*, Math. Financ., **15** (2005), 425–437.
[11] J. Li, *Asymptotic results on marginal expected shortfalls for dependent risks*, Insur. Math. Econ., **102** (2022), 146–168.
[12] J. Liu, Y. Yang, *Aaymptotics for systemic risk with dependent heavy-tailed losses*, ASTIN Bulletin, **51** (2021), 571–605.
[13] J. Liu, T. Shushi, *Asymptotics of the loss-based tail risk measures in the presence of extreme risks*, Eur. Actuar. J., **14** (2024), 205–224.
[14] S.I. Resnick, *Extreme Values, Regular Variation and Point Processes*, Springer, New York, 1987.
[15] Q. Tang, G. Tsitsiashvili, P*recise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks*, Stoch. Proc. Appl., **108** (2003), 299–325.
[16] B. Wang, J. Li, *Asymptotic results on tail moment for light-tailed risks*, Insur. Math. Econ., **114** (2024), 43–55.