



# Probabilistic derangement numbers and polynomials

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**Abstract.** Let  $Y$  be a random variable such that the moment generating function of  $Y$  exists in a neighborhood of the origin. The aim of this paper is to study probabilistic versions of the derangement polynomials, the derangement polynomials of type 2 and the  $r$ -derangement numbers, namely the probabilistic derangement polynomials associated with  $Y$ , the probabilistic derangement polynomials of type 2 associated with  $Y$  and the probabilistic  $r$ -derangement numbers associated  $Y$ , respectively. We derive some properties, explicit expressions, certain identities and recurrence relations for those polynomials and numbers. In addition, we consider the special case that  $Y$  is the gamma random variable with parameters  $\alpha, \beta > 0$ .

## 1. Introduction

In combinatorics, a derangement is a permutation that has no fixed points. The number of derangements of an  $n$  element set is called the  $n$ th derangement number. For  $0 \leq r \leq n$ , the  $n$ th  $r$ -derangement number is the number of derangements of an  $n + r$  element set under the restriction that the first  $r$  elements are in disjoint cycles.

Let  $Y$  be a random variable satisfying the moment condition (see (11)). The aim of this paper is to study the probabilistic derangement polynomials associated with  $Y$ , the probabilistic derangement polynomials of type 2 associated with  $Y$  and the probabilistic  $r$ -derangement numbers associated  $Y$ , as probabilistic versions of the derangement polynomials, the derangement polynomials of type 2 and the  $r$ -derangement numbers, respectively. We derive some properties, explicit expressions, certain identities and recurrence relations for those polynomials and numbers. In addition, we consider the special case that  $Y$  is the gamma random variable with parameters  $\alpha, \beta > 0$ .

The outline of this paper is as follows. In Section 1, we recall the derangement numbers, the  $r$ -derangement numbers, the derangement polynomials  $D_n(x)$  and the derangement polynomials of type 2. We remind the reader of the Fubini polynomials  $F_n(x)$ , the Stirling numbers of the first kind, the unsigned Stirling numbers of the first kind and the Stirling numbers of the second kind  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ . Assume that  $Y$  is a random variable such that the moment generating function of  $Y$ ,  $E[e^{tY}] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E[Y^n]$ , ( $|t| < r$ ), exists

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2020 *Mathematics Subject Classification.* Primary 11B73; Secondary 11B83.

*Keywords.* probabilistic derangement polynomials; probabilistic Euler numbers; probabilistic  $r$ -derangement numbers; probabilistic derangement polynomials of type 2.

Received: 10 January 2024; Accepted: 25 December 2025

Communicated by Hemen Dutta

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for some  $r > 0$ . Let  $(Y_j)_{j \geq 1}$  be a sequence of mutually independent copies of the random variable  $Y$ , and let  $S_k = Y_1 + Y_2 + \dots + Y_k$ , ( $k \geq 1$ ), with  $S_0 = 0$ . Then we recall the probabilistic Stirling numbers of the second kind associated with  $Y$ , the probabilistic Bell polynomials associated with  $Y$ ,  $\phi_n^Y(x)$ , and the gamma random variable with parameters  $\alpha, \beta > 0$ . Section 2 is the main results of this paper. Let  $(Y_j)_{j \geq 1}$ ,  $S_k$ , ( $k = 0, 1, \dots$ ) be as in the above. Then we first define the probabilistic derangement polynomials associated with  $Y$ ,  $D_n^Y(x)$ . In Theorem 2.1, we find two explicit expressions for  $D_n^Y(x)$ . We derive a recurrence relation for  $D_n^Y(x)$  in Theorem 2.2. In Theorem 2.3, we express  $\phi_n^Y(1-x)$  as a finite sum involving  $D_k(x)$ . We define the probabilistic Euler numbers  $E_n^Y$  in a natural manner. In Theorem 2.4, we express the convolution  $\sum_{m=0}^n \binom{n}{m} \phi_m^Y(1-x) E_{n-m}^Y$ , as an infinite sum involving  $D_m(x)$ . We define the probabilistic  $r$ -derangement numbers  $D_n^{(r,Y)}$ . We derive finite sum expressions for  $D_n^{(r,Y)}$  in Theorems 2.5 and 2.6. In Theorem 2.7, the  $n$ th moment of  $Y$  is expressed as a finite sum involving  $D_{k+r}^{(r,Y)}$ . We express  $D_{n+r}^{(r,Y)}$  as a finite sum involving  $D_{n-l}^Y$ , ( $0 \leq l \leq n$ ) in Theorem 2.8. We define the probabilistic derangement polynomials of type 2,  $d_n^Y(x)$ . In Theorem 2.9, we find an explicit expression and a recurrence relation for  $d_n^Y(x)$ . In Theorem 2.10, we derive a finite sum identity which is given by  $\sum_{m=0}^n d_m^Y(x) \binom{n}{m} = \sum_{m=0}^n \sum_{l=0}^m (-1)^l \binom{m}{l} \binom{n}{m} E[Y^l] F_{n-m}(x)$ . We deduce an explicit expression for  $d_n^Y(x)$  in Theorem 2.11. Finally, when  $Y \sim \Gamma(1, 1)$ , we derive an expression of  $D_n^Y(x)$  as a finite sum involving  $D_l(1-x)$ , ( $0 \leq l \leq n$ ), in Theorem 2.12. For the rest of this section, we recall the facts that are needed throughout this paper.

For  $n \geq 0$ , the derangement numbers  $D_n$ , ( $n \geq 0$ ) are given by

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!} = \sum_{k=0}^n \binom{n}{k} (n-k)! (-1)^k, \quad (\text{see [5, 6, 8, 10, 18, 19, 21, 24]}). \tag{1}$$

From (1), we easily derive the following equation

$$\frac{1}{1-t} e^{-t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}, \quad (\text{see [18, 19, 21, 24, 29]}). \tag{2}$$

For  $0 \leq r \leq n$ , the  $r$ -derangement numbers  $D_n^{(r)}$  are given by

$$\sum_{n=0}^{\infty} D_n^{(r)} \frac{t^n}{n!} = \frac{t^r}{(1-t)^{r+1}} e^{-t}, \quad (\text{see [18, 19, 21, 24]}). \tag{3}$$

The derangement polynomials are defined by

$$\frac{e^{xt}}{1-t} e^{-t} = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \quad (\text{see [11, 18, 19, 21, 24, 30]}), \tag{4}$$

and the derangement polynomials of type 2 are given by

$$\frac{1}{1-xt} e^{-t} = \sum_{n=0}^{\infty} d_n(x) \frac{t^n}{n!}, \quad (\text{see [18]}). \tag{5}$$

Note that

$$D_n(0) = D_n \quad \text{and} \quad d_n(1) = D_n, \quad (n \geq 0).$$

It is well known that the Fubini polynomials are defined by

$$F_n(x) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k! x^k, \quad (n \geq 0), \quad (\text{see [6, 16, 17, 20, 23, 30]}), \tag{6}$$

where  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$  are the Stirling numbers of the second kind given by

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see [1 - 32]}). \tag{7}$$

From (6), we have

$$\frac{1}{1 - x(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!}, \quad (\text{see [6, 17, 23]}). \tag{8}$$

The Stirling numbers of the first kind are given by

$$\frac{1}{k!}(\log(1 + t))^k = \sum_{n=k}^{\infty} S_1(n, k) \frac{t^n}{n!}, \quad (\text{see [6 - 30]}). \tag{9}$$

For  $n \geq k \geq 0$ , the unsigned Stirling numbers of the first kind are defined by

$$\left[ \begin{matrix} n \\ k \end{matrix} \right] = (-1)^{n-k} S_1(n, k), \quad (\text{see [6 - 30]}). \tag{10}$$

Let  $Y$  be a random variable such that the moment generating function of  $Y$

$$E[e^{tY}] = \sum_{n=0}^{\infty} E[Y^n] \frac{t^n}{n!}, \quad (|t| < r) \quad \text{exists for some } r > 0. \tag{11}$$

Assume that  $(Y_j)_{j \geq 1}$  is a sequence of mutually independent copies of  $Y$ , and that  $S_k = Y_1 + Y_2 + \dots + Y_k$ , ( $k \geq 1$ ),  $S_0 = 0$ , (see [2,3,14,31]). The probabilistic Stirling numbers of the second kind associated with random variable  $Y$  are defined by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_Y = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} E[S_j^n], \quad (n \geq 0), \quad (\text{see [2, 3, 14, 22, 32, 33, 34]}).$$

Equivalently, they are given by

$$\frac{1}{k!}(E[e^{tY}] - 1)^k = \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_Y \frac{t^n}{n!}. \tag{12}$$

When  $Y = 1$ ,  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_Y = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ , ( $n \geq k \geq 0$ ). The probabilistic Bell polynomials associated with random variable  $Y$  are defined by

$$\phi_n^Y(x) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_Y x^k, \quad (n \geq 0), \quad (\text{see [3, 14, 32]}). \tag{13}$$

From (13), we have

$$e^{x(E[e^{Yt}]-1)} = \sum_{n=0}^{\infty} \phi_n^Y(x) \frac{t^n}{n!}. \tag{14}$$

A continuous random variable  $X$  whose density function is given by

$$f(x) = \begin{cases} \frac{\beta}{\Gamma(\alpha)} e^{-\beta x} (\beta x)^{\alpha-1}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

for some  $\alpha, \beta > 0$  is said to be the gamma random variable with parameters  $\alpha, \beta$ , which is denoted by  $X \sim \Gamma(\alpha, \beta)$ , (see [28,31]).

In particular, if  $X \sim \Gamma(1, 1)$ , then we have

$$\begin{aligned} E[e^{(X-1+p)t}] &= \int_0^\infty e^{(x-1+p)t} f(x) dx = \int_0^\infty e^{(x-1+p)t} e^{-x} dx \\ &= \frac{e^{-t}}{1-t} e^{pt} = \sum_{n=0}^\infty D_n(p) \frac{t^n}{n!}, \quad (t < 1). \end{aligned}$$

Thus, we get

$$E[(X - 1 + p)^n] = D_n(p), \quad (n \geq 0).$$

### 2. Probabilistic derangement numbers and polynomials associated with random variables

Let  $(Y_k)_{k \geq 1}$  be a sequence of mutually independent copies of the random variable  $Y$ , and let

$$S_0 = 0, \quad S_k = Y_1 + Y_2 + \dots + Y_k, \quad (k \geq 1).$$

Now, we define the *probabilistic derangement polynomials associated with  $Y$*  by

$$\frac{e^{xt}}{1-t} E[e^{-tY}] = \sum_{n=0}^\infty D_n^Y(x) \frac{t^n}{n!}. \tag{15}$$

In particular, for  $x = 0$ ,  $D_n^Y = D_n^Y(0)$  are called the *probabilistic derangement numbers associated with  $Y$* .

When  $Y = 1$ , we have  $D_n^Y(x) = D_n(x)$ , ( $n \geq 0$ ). From (15), we have

$$\begin{aligned} \sum_{n=0}^\infty D_n^Y(x) \frac{t^n}{n!} &= \frac{E[e^{-tY}]}{1-t} e^{xt} = \sum_{l=0}^\infty D_l^Y \frac{t^l}{l!} \sum_{m=0}^\infty x^m \frac{t^m}{m!} \\ &= \sum_{n=0}^\infty \sum_{l=0}^n \binom{n}{l} D_l^Y x^{n-l} \frac{t^n}{n!}. \end{aligned} \tag{16}$$

On the other hand, by (15), we also have

$$\begin{aligned} \sum_{n=0}^\infty D_n^Y(x) \frac{t^n}{n!} &= \frac{E[e^{-tY}]}{1-t} e^{xt} = \frac{1}{1-t} E[e^{t(x-Y)}] \\ &= \sum_{l=0}^\infty t^l \sum_{m=0}^\infty E[(x-Y)^m] \frac{t^m}{m!} \\ &= \sum_{n=0}^\infty n! \sum_{m=0}^n \frac{E[(x-Y)^m]}{m!} \frac{t^n}{n!}. \end{aligned} \tag{17}$$

Thus, by comparing the coefficients on both sides of (16) and (17), we obtain the following theorem.

**Theorem 2.1.** For  $n \geq 0$ , we have

$$D_n^Y(x) = n! \sum_{m=0}^n \frac{E[(x-Y)^m]}{m!} = \sum_{l=0}^n \binom{n}{l} D_l^Y x^{n-l}.$$

From (15), we note that

$$\begin{aligned}
 E[e^{t(x-Y)}] &= (1-t) \sum_{n=0}^{\infty} D_n^Y(x) \frac{t^n}{n!} \\
 &= \sum_{n=1}^{\infty} (D_n^Y(x) - nD_{n-1}^Y(x)) \frac{t^n}{n!} + 1.
 \end{aligned}
 \tag{18}$$

On the other hand, by Taylor expansion, we also get

$$\begin{aligned}
 E[e^{t(x-Y)}] &= \sum_{n=0}^{\infty} E[(x-Y)^n] \frac{t^n}{n!} \\
 &= 1 + \sum_{n=1}^{\infty} E[(x-Y)^n] \frac{t^n}{n!}.
 \end{aligned}
 \tag{19}$$

Therefore, by (18) and (19), we obtain the following theorem.

**Theorem 2.2.** For  $n \geq 1$ , we have

$$D_n^Y(x) - nD_{n-1}^Y(x) = E[(x-Y)^n].$$

In particular, for  $x = 0$ , we get

$$D_n^Y - nD_{n-1}^Y = (-1)^n E[Y^n].$$

Replacing  $t$  by  $1 - E[e^{Yt}]$  in (4), we get

$$\begin{aligned}
 e^{(1-x)(E[e^{Yt}]-1)} &= \sum_{k=0}^{\infty} D_k(x) \frac{1}{k!} (1 - E[e^{Yt}])^k E[e^{Yt}] \\
 &= \sum_{k=0}^{\infty} D_k(x) (-1)^k \sum_{l=k}^{\infty} \left\{ \begin{matrix} l \\ k \end{matrix} \right\}_Y \frac{t^l}{l!} \sum_{m=0}^{\infty} E[Y^m] \frac{t^m}{m!} \\
 &= \sum_{l=0}^{\infty} \sum_{k=0}^l D_k(x) (-1)^l \left\{ \begin{matrix} l \\ k \end{matrix} \right\}_Y \frac{t^l}{l!} \sum_{m=0}^{\infty} E[Y^m] \frac{t^m}{m!} \\
 &= \sum_{n=0}^{\infty} \sum_{l=0}^n \sum_{k=0}^l \binom{n}{l} \left\{ \begin{matrix} l \\ k \end{matrix} \right\}_Y D_k(x) (-1)^l E[Y^{n-l}] \frac{t^n}{n!}.
 \end{aligned}
 \tag{20}$$

Therefore, by (14) and (20), we obtain the following theorem.

**Theorem 2.3.** For  $n \geq 0$ , we have

$$\phi_n^Y(1-x) = \sum_{l=0}^n \sum_{k=0}^l \binom{n}{l} \left\{ \begin{matrix} l \\ k \end{matrix} \right\}_Y D_k(x) (-1)^l E[Y^{n-l}].$$

The probabilistic Euler numbers are given by

$$\frac{2}{E[e^{tY}] + 1} = \sum_{n=0}^{\infty} E_n^Y \frac{t^n}{n!}. \tag{21}$$

When  $Y = 1$ ,  $E_n^Y = E_n$ , ( $n \geq 0$ ), where  $E_n$  are the ordinary Euler numbers given by

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (\text{see [30]}).$$

Replacing  $t$  by  $-E[e^{Yt}]$  in (4), we have

$$\begin{aligned} \frac{1}{1 + E[e^{Yt}]} e^{(1-x)E[e^{Yt}]} &= \sum_{m=0}^{\infty} D_m(x) \frac{(-1)^m}{m!} (E[e^{Yt}])^m \\ &= \sum_{m=0}^{\infty} D_m(x) \frac{(-1)^m}{m!} E[e^{(Y_1+\dots+Y_m)t}] \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} D_m(x) \frac{(-1)^m}{m!} E[S_m^n] \frac{t^n}{n!}. \end{aligned} \tag{22}$$

On the other hand, by (21), we get

$$\begin{aligned} \frac{1}{1 + E[e^{Yt}]} e^{(1-x)E[e^{Yt}]} &= \frac{e^{1-x}}{2} \frac{2}{1 + E[e^{Yt}]} e^{(1-x)(E[e^{Yt}]-1)} \\ &= \frac{e^{1-x}}{2} \sum_{l=0}^{\infty} E_l^Y \frac{t^l}{l!} \sum_{m=0}^{\infty} \phi_m^Y (1-x) \frac{t^m}{m!} \\ &= \frac{e^{1-x}}{2} \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \phi_m^Y (1-x) E_{n-m}^Y \frac{t^n}{n!}. \end{aligned} \tag{23}$$

Therefore, by (22) and (23), we obtain the following theorem.

**Theorem 2.4.** For  $n \geq 0$ , we have

$$\sum_{m=0}^n \binom{n}{m} \phi_m^Y (1-x) E_{n-m}^Y = 2e^{x-1} \sum_{m=0}^{\infty} \frac{D_m(x)}{m!} (-1)^m E[S_m^n].$$

For  $0 \leq r \leq n$ , we consider the *probabilistic  $r$ -derangement numbers associated with  $Y$* , which are defined by

$$\frac{t^r}{(1-t)^{r+1}} E[e^{-Yt}] = \sum_{n=r}^{\infty} D_n^{(r,Y)} \frac{t^n}{n!}. \tag{24}$$

When  $Y = 1$ ,  $D_n^{(r,Y)} = D_n^{(r)}$ , ( $n \geq 0$ ). From (24), we have

$$\begin{aligned} \frac{t^r}{(1-t)^{r+1}} E[e^{-Yt}] &= \sum_{k=0}^{\infty} \binom{k+r}{r} t^{k+r} \sum_{m=0}^{\infty} (-1)^m E[Y^m] \frac{t^m}{m!} \\ &= \sum_{k=r}^{\infty} \binom{k}{r} t^k \sum_{m=0}^{\infty} (-1)^m E[Y^m] \frac{t^m}{m!} \\ &= \sum_{n=r}^{\infty} n! \sum_{k=r}^n \binom{k}{r} (-1)^{n-k} \frac{E[Y^{n-k}]}{(n-k)!} \frac{t^n}{n!}. \end{aligned} \tag{25}$$

Therefore, by (24) and (25), we obtain the following theorem.

**Theorem 2.5.** For  $n \geq r \geq 0$ , we have

$$D_n^{(r,Y)} = n! \sum_{k=r}^n \binom{k}{r} (-1)^{n-k} \frac{E[Y^{n-k}]}{(n-k)!}.$$

From (24), we note that

$$\begin{aligned} \sum_{n=r}^{\infty} D_n^{(r,Y)} \frac{t^n}{n!} &= \left(\frac{t}{1-t}\right)^r \frac{1}{1-t} E[e^{-Yt}] \\ &= \sum_{l=r}^{\infty} \binom{l-1}{r-1} t^l \sum_{m=0}^{\infty} D_m^Y \frac{t^m}{m!} \\ &= \sum_{n=r}^{\infty} n! \sum_{l=r}^n \binom{l-1}{r-1} \frac{D_{n-l}^Y}{(n-l)!} \frac{t^n}{n!}. \end{aligned} \tag{26}$$

By comparing the coefficients on both sides of (26), we obtain the following theorem.

**Theorem 2.6.** For  $n \geq r \geq 0$ , we have

$$D_n^{(r,Y)} = n! \sum_{l=r}^n \binom{l-1}{r-1} \frac{D_{n-l}^Y}{(n-l)!}.$$

By (24), we get

$$\begin{aligned} E[e^{-Yt}] &= (1-t)^{r+1} \sum_{k=r}^{\infty} D_k^{(r,Y)} \frac{t^{k-r}}{k!} \\ &= (1-t)^{r+1} \sum_{k=0}^{\infty} D_{k+r}^{(r,Y)} \frac{t^k}{(k+r)!} \\ &= \sum_{l=0}^{\infty} (-1)^l \binom{r+1}{l} t^l \sum_{k=0}^{\infty} D_{k+r}^{(r,Y)} \frac{t^k}{(k+r)!} \\ &= \sum_{n=0}^{\infty} n! \sum_{k=0}^n \frac{D_{k+r}^{(r,Y)}}{(k+r)!} (-1)^{n-k} \binom{r+1}{n-k} \frac{t^n}{n!}. \end{aligned} \tag{27}$$

Therefore, by (27), we obtain the following theorem.

**Theorem 2.7.** For  $n \geq 0$ , we have

$$E[Y^n] = n! \sum_{k=0}^n \frac{D_{k+r}^{(r,Y)}}{(k+r)!} (-1)^k \binom{r+1}{n-k}.$$

Now, we observe that

$$\begin{aligned} \frac{1}{(1-t)^{r+1}} E[e^{-Yt}] &= \frac{1}{t^r} \frac{t^r}{(1-t)^{r+1}} E[e^{-tY}] \\ &= \sum_{n=0}^{\infty} \frac{D_{n+r}^{(r,Y)}}{(n+r)!} t^n = \sum_{n=0}^{\infty} \frac{D_{n+r}^{(r,Y)}}{\binom{n+r}{r} r!} \frac{t^n}{n!}. \end{aligned} \tag{28}$$

On the other hand, by binomial expansion, we get

$$\begin{aligned} \frac{1}{(1-t)^{r+1}} E[e^{-Yt}] &= \left(\frac{1}{1-t}\right)^r \frac{1}{1-t} E[e^{-Yt}] \\ &= \sum_{l=0}^{\infty} \binom{r+l-1}{l} t^l \sum_{m=0}^{\infty} D_m^Y \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} n! \sum_{l=0}^n \binom{r+l-1}{l} \frac{D_{n-l}^Y}{(n-l)!} \frac{t^n}{n!}. \end{aligned} \tag{29}$$

Thus, by (28) and (29), we get the following theorem.

**Theorem 2.8.** For  $n \geq 0$ , we have

$$D_{n+r}^{(r,Y)} = (n+r)! \sum_{l=0}^n \binom{r+l-1}{l} \frac{D_{n-l}^Y}{(n-l)!}.$$

Now, we define the *probabilistic derangement polynomials of type 2 associated with Y* as

$$\frac{1}{1-xt} E[e^{-Yt}] = \sum_{n=0}^{\infty} d_n^Y(x) \frac{t^n}{n!}. \tag{30}$$

When  $Y = 1$ , we have  $d_n^Y(x) = d_n(x)$ , ( $n \geq 0$ ). In particular, for  $x = 1$ ,  $d_n^Y(1) = D_n^Y$ , ( $n \geq 0$ ). From (30), we have

$$\begin{aligned} \sum_{n=0}^{\infty} d_n^Y(x) \frac{t^n}{n!} &= \frac{1}{1-xt} E[e^{-Yt}] \\ &= \sum_{m=0}^{\infty} x^m t^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} E[Y^k] t^k \\ &= \sum_{n=0}^{\infty} n! \sum_{k=0}^n \frac{(-1)^k}{k!} E[Y^k] x^{n-k} \frac{t^n}{n!}, \end{aligned} \tag{31}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n E[Y^n] \frac{t^n}{n!} &= E[e^{-Yt}] = (1-xt) \sum_{n=0}^{\infty} d_n^Y(x) \frac{t^n}{n!} \\ &= d_0^Y(x) + \sum_{n=1}^{\infty} \left( d_n^Y(x) - n x d_{n-1}^Y(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{32}$$

Thus, by (32), we get

$$d_0^Y(x) = 1, \quad d_n^Y(x) = n x d_{n-1}^Y(x) + (-1)^n E[Y^n], \quad (n \geq 1). \tag{33}$$

Therefore, by (31) and (33), we obtain the following theorem.

**Theorem 2.9.** For  $n \geq 0$ , we have

$$d_n^Y(x) = n! \sum_{k=0}^n \frac{(-1)^k}{k!} E[Y^k] x^{n-k}.$$

Moreover, we have

$$d_0^Y(x) = 1, \quad d_n^Y(x) = n x d_{n-1}^Y(x) + (-1)^n E[Y^n], \quad (n \geq 1).$$

Replacing  $t$  by  $e^t - 1$  in (30), we get

$$\begin{aligned} \frac{1}{1-x(e^t-1)} E[e^{-Y(e^t-1)}] &= \sum_{m=0}^{\infty} d_m^Y(x) \frac{1}{m!} (e^t-1)^m \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n d_m^Y(x) \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \frac{t^n}{n!}. \end{aligned} \tag{34}$$

On the other hand, by (8), we obtain

$$\begin{aligned}
 \frac{1}{1-x(e^t-1)}E[e^{-Y(e^t-1)}] &= \sum_{k=0}^{\infty} F_k(x) \frac{t^k}{k!} \sum_{l=0}^{\infty} (-1)^l E[Y^l] \frac{(e^t-1)^l}{l!} \\
 &= \sum_{k=0}^{\infty} F_k(x) \frac{t^k}{k!} \sum_{l=0}^{\infty} (-1)^l E[Y^l] \sum_{m=l}^{\infty} \begin{Bmatrix} m \\ l \end{Bmatrix} \frac{t^m}{m!} \\
 &= \sum_{k=0}^{\infty} F_k(x) \frac{t^k}{k!} \sum_{m=0}^{\infty} \sum_{l=0}^m (-1)^l E[Y^l] \begin{Bmatrix} m \\ l \end{Bmatrix} \frac{t^m}{m!} \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{l=0}^m (-1)^l \begin{Bmatrix} n \\ m \end{Bmatrix} \begin{Bmatrix} m \\ l \end{Bmatrix} E[Y^l] F_{n-m}(x) \frac{t^n}{n!}.
 \end{aligned}
 \tag{35}$$

Therefore, by (34) and (35), we obtain the following theorem.

**Theorem 2.10.** For  $n \geq 0$ , we have

$$\sum_{m=0}^n d_m^Y(x) \begin{Bmatrix} n \\ m \end{Bmatrix} = \sum_{m=0}^n \sum_{l=0}^m (-1)^l \begin{Bmatrix} m \\ l \end{Bmatrix} \begin{Bmatrix} n \\ m \end{Bmatrix} E[Y^l] F_{n-m}(x).$$

From (30), we note that

$$\begin{aligned}
 \sum_{n=0}^{\infty} d_n^Y(x) \frac{t^n}{n!} &= e^{-\log(1-xt)} E[e^{-Yt}] = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} (\log(1-xt))^l E[e^{-Yt}] \\
 &= \sum_{j=0}^{\infty} \sum_{l=0}^j \begin{Bmatrix} j \\ l \end{Bmatrix} x^j \frac{t^j}{j!} \sum_{m=0}^{\infty} (-1)^m E[Y^m] \frac{t^m}{m!} \\
 &= \sum_{n=0}^{\infty} \sum_{j=0}^n \sum_{l=0}^j \begin{Bmatrix} j \\ l \end{Bmatrix} \begin{Bmatrix} n \\ j \end{Bmatrix} x^j (-1)^{n-j} E[Y^{n-j}] \frac{t^n}{n!}.
 \end{aligned}
 \tag{36}$$

Therefore, by comparing the coefficients on both sides of (36), we obtain the following theorem.

**Theorem 2.11.** For  $n \geq 0$ , we have

$$d_n^Y(x) = \sum_{j=0}^n \sum_{l=0}^j \begin{Bmatrix} j \\ l \end{Bmatrix} \begin{Bmatrix} n \\ j \end{Bmatrix} x^j (-1)^{n-j} E[Y^{n-j}].$$

Let  $Y \sim \Gamma(1, 1)$ . Then, for  $t > -1$ , we have

$$\begin{aligned}
 E[e^{(x-Y)t}] &= e^{xt} \int_0^{\infty} e^{-yt} e^{-y} dy = \frac{1}{1+t} e^t e^{(x-1)t} \\
 &= \sum_{l=0}^{\infty} (-1)^l D_l(1-x) \frac{t^l}{l!}.
 \end{aligned}
 \tag{37}$$

From (37), we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} D_n^Y(x) \frac{t^n}{n!} &= \frac{e^{xt}}{1-t} E[e^{-Yt}] = \frac{1}{1-t} \frac{1}{1+t} e^t e^{(x-1)t} \\
 &= \frac{1}{1-t} \sum_{l=0}^{\infty} (-1)^l D_l(1-x) \frac{t^l}{l!} = \sum_{n=0}^{\infty} n! \sum_{l=0}^n \frac{(-1)^l}{l!} D_l(1-x) \frac{t^n}{n!}.
 \end{aligned}
 \tag{38}$$

Therefore, by comparing the coefficients on both sides of (38), we obtain the following theorem.

**Theorem 2.12.** Let  $Y \sim \Gamma(1, 1)$ . For  $n \geq 0$ , we have

$$D_n^Y(x) = n! \sum_{l=0}^n \frac{(-1)^l}{l!} D_l(1-x).$$

### 3. Conclusion

By means of generating functions we studied probabilistic versions of the derangement polynomials, the  $r$ -derangement numbers and the derangement polynomials of type 2, namely the probabilistic derangement polynomials  $D_n^Y(x)$  associated with  $Y$ , the probabilistic  $r$ -derangement numbers  $D_n^{(r,Y)}$  associated  $Y$  and the probabilistic derangement polynomials of type 2  $d_n^Y(x)$  associated with  $Y$ . Here  $Y$  is a random variable such that the moment generating function of  $Y$  exists in a neighborhood of the origin. In more detail, we derived an explicit expression for  $D_n^Y(x)$  (see Theorem 2.1) and that in terms of derangement polynomials  $D_l(x)$  for the special case of  $Y \sim \Gamma(1, 1)$  (see Theorem 2.12). We deduced a recurrence relation for  $D_n^Y(x)$  (see Theorem 2.2). We expressed  $\phi_n^Y(1-x)$  and the convolution of that with the probabilistic Euler numbers  $E_n^Y$  in terms of  $D_k(x)$  (see Theorems 2.3, 2.4). We found explicit expressions for  $D_n^{(r,Y)}$  (see Theorems 2.5, 2.6) and two identities involving those numbers (see Theorems 2.7, 2.8). We obtained explicit expressions for  $d_n^Y(x)$  (see Theorems 2.9, 2.11), a recurrence relation for  $d_n^Y(x)$  (see Theorem 2.9) and an identity involving those polynomials (see Theorem 2.10).

As one of our future projects, we would like to continue to study probabilistic versions of many special polynomials and numbers and to find their applications to physics, science and engineering as well as to mathematics.

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