



Novel sequence spaces based on derangement numbers and their properties

Muhammet Cihat Dağlı^a

^aDepartment of Mathematics, Akdeniz University, 07058-Antalya, Turkey

Abstract. The purpose of this paper is to give new sequence spaces derived from derangement numbers. Also, we investigate their topological properties such as Schauder basis, the β -duals as well as characterization of certain matrix operators. Furthermore, by means of Hausdorff measure of noncompactness, we present the characterization of some compact operators.

1. Introduction

1.1. Sequence spaces and summability

As is known, the theory of sequence spaces is essential for summability. The summability theory covers a wide range of mathematics and occurs in many contexts like approximation theory, in operator theory, the theory of orthogonal series, in numerical analysis to speed up the rate of convergence.

Let $W = (w_{nk})$ be an infinite matrix with real or complex entries and let Λ and Ξ be two sequence spaces. Then, W becomes a matrix transformation $\Lambda \rightarrow \Xi$ if for each sequence $z = (z_k) \in \omega$, $Wz = \{(Wz)_n\} \in \Xi$, i.e., the W -transform of z , where $(Wz)_n = \sum_k w_{nk}z_k$ for $n \in \mathbb{N}$. The notation $(\Lambda \rightarrow \Xi)$ represents the family of all matrices that maps from Λ to Ξ . For brevity, in the sequel, the summation with no limits means from 0 to ∞ . Also, \mathbb{R} denotes the set of all real numbers and $\mathbb{N} = \{0, 1, 2, \dots\}$.

The sequence space

$$\Lambda_W = \{z \in \omega : Wz \in \Lambda\}$$

is called the domain of the matrix W in the space Λ , where ω denotes the space of all sequences. Its being a sequence space leads to surface many studies involving substantial discussions related to domains of provided matrices. For detailed investigation, the reader can consult to the papers [2, 3, 11, 12, 19, 26, 30], and cited references therein.

A Banach space X is named a BK-space if each mapping $p_k : X \rightarrow \mathbb{R}$ defined by $p_k(z) = z_k$ is continuous for all $k \in \mathbb{N}$. It is aware that the spaces of p -absolutely summable sequences and bounded sequences, denoted by ℓ_p and ℓ_∞ , respectively, are BK-spaces. From the continuity of the matrix mappings between

2020 Mathematics Subject Classification. Primary 11B83; Secondary 40C05, 46A45, 47B37, 47B07.

Keywords. derangement numbers, sequence spaces, matrix transformations, compact operators.

Received: 20 May 2025; Revised: 08 January 2026; Accepted: 10 January 2026

Communicated by Pratulananda Das

Email address: mcihatdagli@akdeniz.edu.tr (Muhammet Cihat Dağlı)

ORCID iD: <https://orcid.org/0000-0003-2859-902X> (Muhammet Cihat Dağlı)

BK-spaces, matrix transformations in the sequence spaces and topological properties are widely studied. The reader can consult to the papers [8, 15, 16, 36, 37, 39, 44, 45] in this direction.

Additional features and applications related to summability theory can be found in [4, 28].

The characterization of compact operators can be established via Hausdorff measure of noncompactness. In recent years, lots of remarkable results have been revealed in the theory of sequence spaces in this context. See, for instance [1, 5, 6, 22–24, 29, 31–33], for the relevant literature.

1.2. Motivation and aim

In combinatorial mathematics, a derangement is a permutation of the elements of a set, such that no element appears in its original position. The number of derangements of a set of size n is called the derangement number.

Derangement numbers are usually defined by means of the generating function [9, 40]

$$\frac{e^{-t}}{1-t} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!},$$

and possess the closed-form expression

$$d_n = \sum_{k=0}^n \binom{n}{k} k! (-1)^{n-k} \quad (1)$$

with the inverse

$$\sum_{k=0}^n \binom{n}{k} d_k = n! \quad (2)$$

For further relations such as explicit formulas, closed forms and determinantal representations of derangement numbers, one can consult to the studies [34, 35].

We have concluded from the recent papers [7, 10, 13, 14, 17, 18, 20, 21, 25, 43, 46, 47] that some remarkable relations furnished by special numbers or functions have been applied in order to present new sequence spaces and demonstrate their topological and geometric features.

Our goal in the present paper is to define a new matrix $\tilde{D} = (\tilde{d}_{nk})$ with derangement numbers entries as follows:

$$\tilde{d}_{nk} = \begin{cases} \frac{1}{d_n} \binom{n}{k} k! (-1)^{n-k}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{if } k > n; \end{cases} \quad (3)$$

by inspiring the formula (1). Also, we introduce two Banach spaces, denoted by $\ell_p(\tilde{D})$ and $\ell_\infty(\tilde{D})$, as the domains of this matrix including derangement numbers and we prove that these Banach spaces are linearly isomorphic to the spaces ℓ_p and ℓ_∞ . We study certain topological features like Schauder basis, the β -duals, and the characterization of certain matrix operators. Furthermore, we establish the characterization of the compactness of some matrix operators by means of the Hausdorff measure of noncompactness. In the final section, as the highlights of this context, we mention the plan to consider the domains of the matrix (3) in the spaces of convergent and null sequences and an extension of these papers, known as quantum calculus (or q -calculus).

2. New sequence spaces $\ell_p(\tilde{D})$ and $\ell_\infty(\tilde{D})$

Based on the matrix (3), we will define the following sequence spaces

$$\ell_p(\tilde{D}) = \left\{ z = (z_n) \in \omega : \sum_{n=0}^{\infty} \left| \frac{1}{d_n} \sum_{k=0}^n \binom{n}{k} k! (-1)^{n-k} z_k \right|^p < \infty \right\}$$

and

$$\ell_\infty(\tilde{D}) = \left\{ z = (z_n) \in \omega : \sup_n \left| \frac{1}{d_n} \sum_{k=0}^n \binom{n}{k} k! (-1)^{n-k} z_k \right| < \infty \right\}.$$

Also, let us denote

$$t_n = (\tilde{D}z)_n = \frac{1}{d_n} \sum_{k=0}^n \binom{n}{k} k! (-1)^{n-k} z_k \tag{4}$$

as the \tilde{D} -transform of a sequence $z = (z_n)$.

Firstly, let us present the linearity of the spaces $\ell_p(\tilde{D})$ and $\ell_\infty(\tilde{D})$ without any proof owing to the routine verification.

Theorem 2.1. *The sequence spaces $\ell_p(\tilde{D})$ and $\ell_\infty(\tilde{D})$ are BK-spaces in accordance with the norms*

$$\|z\|_{\ell_p(\tilde{D})} = \left(\sum_{n=0}^{\infty} \left| \frac{1}{d_n} \sum_{k=0}^n \binom{n}{k} k! (-1)^{n-k} z_k \right|^p \right)^{1/p}$$

and

$$\|z\|_{\ell_\infty(\tilde{D})} = \sup_{n \in \mathbb{N}} \left| \frac{1}{d_n} \sum_{k=0}^n \binom{n}{k} k! (-1)^{n-k} z_k \right|,$$

respectively.

Theorem 2.2. *The spaces $\ell_p(\tilde{D})$ and $\ell_\infty(\tilde{D})$ are linearly isomorphic to ℓ_p and ℓ_∞ , respectively.*

Proof. To prove this, we should show the existence of a linear bijection between the spaces $\ell_p(\tilde{D})$ and ℓ_p . The linearity is clear. It is satisfied $z = 0$ whenever $\tilde{D}z = 0$ yields the injectiveness of \tilde{D} . For any sequence $t = (t_n) \in \ell_p$, we reach the relation for all $k \in \mathbb{N}$,

$$z_k = \sum_{i=0}^k \binom{k}{i} \frac{d_i}{k!} t_i, \tag{5}$$

motivated by (2). So, we have

$$\begin{aligned} (\tilde{D}z)_n &= \frac{1}{d_n} \sum_{k=0}^n \binom{n}{k} k! (-1)^{n-k} z_k \\ &= \frac{1}{d_n} \sum_{k=0}^n \binom{n}{k} k! (-1)^{n-k} \sum_{i=0}^k \binom{k}{i} \frac{d_i}{k!} t_i \\ &= \frac{1}{d_n} \sum_{i=0}^n \sum_{k=i}^n \binom{n}{k} (-1)^{n-k} \binom{k}{i} d_i t_i \\ &= \frac{1}{d_n} \sum_{i=0}^n \sum_{k=0}^{n-i} \binom{n}{k+i} (-1)^{n-k-i} \binom{k+i}{i} d_i t_i \\ &= t_n. \end{aligned}$$

Thus, for $z = (z_k)$ given by (5) and $t = (t_n) \in \ell_p$, we have $(\tilde{D}z)_n = t_n$ for all $n \in \mathbb{N}$. So, the mapping $\tilde{D}z$ is onto from the fact $\tilde{D}z = t$. It follows from Theorem 2.1 that $\|z\|_{\ell_p(\tilde{D})} = \|t\|_{\ell_p}$. Namely, $\ell_p(\tilde{D})$ and ℓ_p are linearly isomorphic, as required. To avoid unnecessary repetitions of similar statements we omit the proof of other assertion. This completes the proof. \square

The β -dual of a subset $\Lambda \subset w$ are defined by

$$\Lambda^\beta = \{x = (x_k) \in w : xz = (x_k z_k) \in cs \text{ for all } z \in \Lambda\},$$

where cs denotes the space of convergent series. The next result points out the characterizations of certain matrix classes and leads to derive the β -duals of the spaces $\ell_p(\tilde{D})$ and $\ell_\infty(\tilde{D})$.

Lemma 2.3. ([41]) $W = (w_{nk}) \in (\ell_1 \rightarrow \ell_\infty)$ iff

$$\sup_{n,k \in \mathbb{N}} |w_{nk}| < \infty. \tag{6}$$

$W = (w_{nk}) \in (\ell_1 \rightarrow c)$ iff (6) holds and

$$\lim_{n \rightarrow \infty} w_{nk} \text{ exists,} \tag{7}$$

for each $k \in \mathbb{N}$. $W = (w_{nk}) \in (\ell_1 \rightarrow c_0)$ iff (6) holds and

$$\lim_{n \rightarrow \infty} w_{nk} = 0, \tag{8}$$

for each $k \in \mathbb{N}$. $W = (w_{nk}) \in (\ell_1 \rightarrow \ell_p)$ iff

$$\sup_k \sum_{n=0}^{\infty} |w_{nk}|^p < \infty,$$

where $1 \leq p < \infty$. $W = (w_{nk}) \in (\ell_p \rightarrow \ell_\infty)$ iff

$$\sup_k \sum_{n=0}^{\infty} |w_{nk}|^q < \infty, \tag{9}$$

where $1 < p < \infty$. $W = (w_{nk}) \in (\ell_p \rightarrow c)$ iff (7) and (9) hold for $1 < p < \infty$. $W = (w_{nk}) \in (\ell_p \rightarrow c_0)$ iff (8) and (9) hold for $1 < p < \infty$. $W = (w_{nk}) \in (\ell_p \rightarrow \ell_1)$ iff

$$\sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} |w_{nk}|^q \right)^q < \infty \text{ for } 1 < p < \infty.$$

$W = (w_{nk}) \in (\ell_\infty \rightarrow \ell_\infty) = (c \rightarrow \ell_\infty) = (c_0 \rightarrow \ell_\infty)$ iff

$$\sup_n \sum_{k=0}^{\infty} |w_{nk}| < \infty.$$

$W = (w_{nk}) \in (\ell_\infty \rightarrow c)$ iff (7) holds and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} |w_{nk}| = \sum_{k=0}^{\infty} \left| \lim_{n \rightarrow \infty} w_{nk} \right|. \tag{10}$$

$W = (w_{nk}) \in (\ell_\infty \rightarrow c_0)$ iff (8) holds and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} |w_{nk}| = 0. \tag{11}$$

$W = (w_{nk}) \in (\ell_\infty \rightarrow \ell_p) = (c \rightarrow \ell_p) = (c_0 \rightarrow \ell_p)$ iff

$$\sup_{K \in F} \sum_{n=0}^{\infty} \left| \sum_{k \in K} w_{nk} \right|^p < \infty, \text{ for } 1 \leq p < \infty,$$

where F denotes any finite subset of \mathbb{N} .

We are ready to offer the duality theorem.

Theorem 2.4. Let the sets $\partial_1, \partial_2, \partial_3$ and ∂_4 be defined by

$$\partial_1 = \left\{ b = (b_k) \in \omega : \lim_{n \rightarrow \infty} \sum_{i=k}^n \binom{i}{k} \frac{d_k}{i!} b_i \text{ exists for each } k \in \mathbb{N} \right\},$$

$$\partial_2 = \left\{ b = (b_k) \in \omega : \sup_n \sum_k \left| \sum_{i=k}^n \binom{i}{k} \frac{d_k}{i!} b_i \right|^q < \infty \right\},$$

$$\partial_3 = \left\{ b = (b_k) \in \omega : \sup_{n,k} \left| \sum_{i=k}^n \binom{i}{k} \frac{d_k}{i!} b_i \right| < \infty \right\},$$

and

$$\partial_4 = \left\{ b = (b_k) \in \omega : \lim_{n \rightarrow \infty} \sum_k \left| \sum_{i=k}^n \binom{i}{k} \frac{d_k}{i!} b_i \right| = \sum_k \left| \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} b_i \right| \right\}.$$

Then, we have $(\ell_p(\tilde{D}))^\beta = \partial_1 \cap \partial_2$, for $1 < p < \infty$. $(\ell_1(\tilde{D}))^\beta = \partial_1 \cap \partial_3$, $(\ell_\infty(\tilde{D}))^\beta = \partial_1 \cap \partial_4$.

Proof. It is sufficient to prove the first assertion since the others can be obtained analogously. For $1 < p < \infty$, we note that $b = (b_k) \in (\ell_p(\tilde{D}))^\beta$ iff $\sum_{k=0}^{\infty} b_k z_k$ is convergent for all $z = (z_k) \in (\ell_p(\tilde{D}))$. So, it is readily seen that

$$\begin{aligned} \sum_{k=0}^n b_k z_k &= \sum_{k=0}^n b_k \sum_{i=0}^k \binom{k}{i} \frac{d_i}{k!} t_i \\ &= \sum_{k=0}^n \left(\sum_{i=k}^n \binom{i}{k} \frac{d_k}{i!} b_i \right) t_k, \end{aligned}$$

from which we have $b = (b_k) \in (\ell_p(\tilde{D}))^\beta$ iff the following matrix belongs to the class $(\ell_p \rightarrow c)$:

$$d_{nk} = \begin{cases} \sum_{i=k}^n \binom{i}{k} \frac{d_k}{i!} b_i, & \text{if } 0 \leq k \leq n; \\ 0, & \text{if } k > n. \end{cases}$$

Hence, by Lemma 2.3, the limit

$$\lim_{n \rightarrow \infty} \sum_{i=k}^n \binom{i}{k} \frac{d_k}{i!} b_i$$

exists for each $k \in \mathbb{N}$ and the relation

$$\sup_n \sum_{k=0}^n \left| \sum_{i=k}^n \binom{i}{k} \frac{d_k}{i!} b_i \right|^q < \infty$$

holds. Consequently, we have $b = (b_k) \in \partial_1 \cap \partial_2$, as required. \square

3. Certain matrix mappings

This section consists of the characterization of the classes $(\Lambda(\tilde{D}) \rightarrow \Xi)$ for $\Lambda \in \{\ell_p\}$ with $1 \leq p \leq \infty$ and $\Xi \in \{\ell_1, c_0, c, \ell_\infty\}$.

Theorem 3.1. For $n, k \in \mathbb{N}$, define a matrix $R = (r_{nk})$ as

$$r_{nk} = \lim_{m \rightarrow \infty} \sum_{i=k}^m \binom{i}{k} \frac{d_k}{i!} w_{ni}. \tag{12}$$

Then, we have the following facts:

(a) $W \in (\ell_1(\tilde{D}) \rightarrow \ell_\infty)$ iff

$$R = (r_{nk}) \text{ is well defined for all } n, k \in \mathbb{N}, \tag{13}$$

$$\sup_{m,k} \left| \sum_{i=k}^m \binom{i}{k} \frac{d_k}{i!} w_{ni} \right| < \infty, \text{ for each } n \in \mathbb{N} \tag{14}$$

and

$$\sup_{n,k} |r_{nk}| < \infty. \tag{15}$$

(b) $W \in (\ell_1(\tilde{D}) \rightarrow c)$ iff (13), (14) and (15) are satisfied and

$$\lim_{n \rightarrow \infty} r_{nk} \text{ exists for each } k \in \mathbb{N}. \tag{16}$$

(c) $W \in (\ell_1(\tilde{D}) \rightarrow c_0)$ iff (13), (14) and (15) are satisfied and

$$\lim_{n \rightarrow \infty} r_{nk} = 0 \text{ for each } k \in \mathbb{N}. \tag{17}$$

(d) $W \in (\ell_1(\tilde{D}) \rightarrow \ell_1)$ iff (13) and (14) are satisfied and

$$\sup_k \sum_n |r_{nk}| < \infty. \tag{18}$$

Proof. We only prove the assertion (a) since the proofs of the assertions (b), (c) and (d) follow analogously.

It is known that $W \in (\ell_1(\tilde{D}), \ell_\infty)$ iff $Wz \in \ell_\infty$ for all $z \in \ell_1(\tilde{D})$. From the convergence of the series $\sum_{k=0}^\infty w_{nk} z_k$,

we have $w_{nk} \in (\ell_1(\tilde{D}))^\beta$ for each fixed $n \in \mathbb{N}$. Using Theorem 2.4, one arrives that the limit

$$\lim_{m \rightarrow \infty} \sum_{i=k}^m \binom{i}{k} \frac{d_k}{i!} w_{ni}$$

exists for each $n, k \in \mathbb{N}$ and the fact

$$\sup_{m,k} \left| \sum_{i=k}^m \binom{i}{k} \frac{d_k}{i!} w_{ni} \right| < \infty, \text{ for each } n \in \mathbb{N}.$$

So, one can reach that (13) and (14) are satisfied. Consider that

$$\begin{aligned} \sum_{k=0}^m w_{nk} z_k &= \sum_{k=0}^m w_{nk} \sum_{i=0}^k \binom{k}{i} \frac{d_i}{k!} t_i \\ &= \sum_{i=0}^m \left(\sum_{k=i}^m w_{nk} \binom{k}{i} \frac{d_i}{k!} \right) t_i. \end{aligned} \tag{19}$$

Now, by defining a matrix $\tilde{R} = (\tilde{r}_{mi})$ for each $n \in \mathbb{N}$ by

$$\tilde{r}_{mi} = \begin{cases} \sum_{k=i}^m \binom{k}{i} \frac{d_i}{k!} w_{nk}, & \text{if } 0 \leq i \leq m; \\ 0, & \text{if } i > m; \end{cases}$$

one can find the fact that (13) and (14) imply $\tilde{R} = (\tilde{r}_{mi}) \in (\ell_1 \rightarrow c)$. Thus, for all $t \in \ell_1$, the series $\tilde{R}_m(t) = \sum_{i=0}^{\infty} \tilde{r}_{mi}t_i$ converges uniformly in m , from which $\lim_{m \rightarrow \infty} \tilde{R}_m(t) = \sum_{i=0}^{\infty} \lim_{m \rightarrow \infty} \tilde{r}_{mi}t_i$. So, one infers from (19) that

$$W_n(z) = \lim_{m \rightarrow \infty} \tilde{R}_m(t) = \sum_{i=0}^{\infty} (\lim_{m \rightarrow \infty} \tilde{r}_{mi}) t_i = \sum_{i=0}^{\infty} \tilde{r}_{ni}t_i = R_n(t), \tag{20}$$

which yields that for $z \in \ell_1(\tilde{D})$, $W(z) \in \ell_{\infty}$ iff for $t \in \ell_1$, $R(t) \in \ell_{\infty}$. Consequently, $W \in (\ell_1(\tilde{D}) \rightarrow \ell_{\infty})$ iff (13), (14) and (15) hold. So, the proof is complete. \square

Theorem 3.2. *Let $1 < p < \infty$ and let $R = (r_{nk})$ be the matrix in (12). Then,*

(a) $W \in (\ell_p(\tilde{D}) \rightarrow \ell_{\infty})$ iff (13) holds and

$$\sup_m \sum_{k=0}^m \left| \sum_{i=k}^m \binom{i}{k} \frac{d_k}{i!} w_{ni} \right|^q < \infty, \text{ for each } n \in \mathbb{N} \tag{21}$$

and

$$\sup_n \sum_k |r_{nk}|^q < \infty. \tag{22}$$

(b) $W \in (\ell_p(\tilde{D}) \rightarrow c)$ iff equations (13), (16), (21) and (22) are satisfied.

(c) $W \in (\ell_p(\tilde{D}) \rightarrow c_0)$ iff equations (13), (17), (21) and (22) are satisfied.

(d) $W \in (\ell_p(\tilde{D}) \rightarrow \ell_1)$ iff equations (13) and (21) are satisfied and

$$\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} r_{nk} \right|^q < \infty. \tag{23}$$

Proof. Let us only offer the proof of (a) since the others can be reached by proceeding the similar arguments. First we note that $W \in (\ell_p(\tilde{D}) \rightarrow \ell_{\infty})$ iff $Wz \in \ell_{\infty}$ for all $z \in \ell_p(\tilde{D})$. From the convergence of the series $\sum_{k=0}^{\infty} w_{nk}z_k$, we have $w_{nk} \in (\ell_p(\tilde{D}))^{\beta}$ for each fixed $n \in \mathbb{N}$. It yields from Theorem 2.4 that the relations (13) and

$$\sup_m \sum_{k=0}^m \left| \sum_{i=k}^m \binom{i}{k} \frac{d_k}{i!} w_{ni} \right|^q < \infty, \text{ for each } n \in \mathbb{N}$$

hold. In analogy, from equation (20), we have $Wz \in \ell_{\infty}$ for all $z \in \ell_p(\tilde{D})$ iff $R(t) \in \ell_{\infty}$ for $t \in \ell_p$. Thus, it is satisfied that $W \in (\ell_p(\tilde{D}) \rightarrow \ell_{\infty})$ iff (13) and (21) are valid and $R \in (\ell_p \rightarrow \ell_{\infty})$ i.e. (22) is satisfied. So, the proof is complete. \square

Theorem 3.3. *Let $R = (r_{nk})$ be the matrix given by (12). Then,*

(a) $W \in (\ell_{\infty}(\tilde{D}) \rightarrow \ell_{\infty})$ iff (13) is valid and

$$\lim_{m \rightarrow \infty} \sum_k \left| \sum_{i=k}^m \binom{i}{k} \frac{d_k}{i!} w_{ni} \right| = \sum_k \left| \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \right| \tag{24}$$

and

$$\sup_n \sum_k |r_{nk}| < \infty. \tag{25}$$

(b) $W \in (\ell_\infty(\tilde{D}) \rightarrow c)$ iff equations (13), (16) and (24) are satisfied and

$$\lim_{n \rightarrow \infty} \sum_k |r_{nk}| = \sum_k \left| \lim_{n \rightarrow \infty} r_{nk} \right|. \tag{26}$$

(c) $W \in (\ell_\infty(\tilde{D}) \rightarrow c_0)$ iff equations (13), (17) and (24) are satisfied and

$$\lim_{n \rightarrow \infty} \sum_k |r_{nk}| = 0. \tag{27}$$

(d) $W \in (\ell_\infty(\tilde{D}) \rightarrow \ell_1)$ iff equations (13) and (24) are satisfied and

$$\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} r_{nk} \right| < \infty. \tag{28}$$

Proof. The proof of (a) follows by using the fact $W \in (\ell_\infty(\tilde{D}) \rightarrow \ell_\infty)$ iff $Wz \in \ell_\infty$ for all $z \in \ell_\infty(\tilde{D})$ and by applying the manipulations in the proofs of previous theorems in this section. To avoid unnecessary repetitions, the proofs of the assertions (b), (c) and (d) are omitted. \square

This part of the paper will be concluded with additional characterization of matrix classes $(\Lambda \rightarrow \ell_p(\tilde{D}))$ with $1 \leq p \leq \infty$, where $\Lambda \in \{\ell_\infty, c, c_0, \ell_1\}$.

Theorem 3.4. *Let $1 \leq p \leq \infty$. Then,*

(a) $W \in (\ell_\infty \rightarrow \ell_p(\tilde{D})) = (c \rightarrow \ell_p(\tilde{D})) = (c_0 \rightarrow \ell_p(\tilde{D}))$ iff

$$\sup_{K \in \mathcal{F}} \sum_{n=0}^{\infty} \left| \sum_{k \in K, i=0}^n \binom{n}{i} \frac{i!}{d_n} (-1)^{n-i} w_{ik} \right|^p < \infty.$$

(b) $W \in (\ell_1 \rightarrow \ell_p(\tilde{D}))$ iff

$$\sup_k \sum_{n=0}^{\infty} \left| \sum_{i=0}^n \binom{n}{i} \frac{i!}{d_n} (-1)^{n-i} w_{ik} \right|^p < \infty.$$

(c) $W \in (\ell_\infty \rightarrow \ell_\infty(\tilde{D})) = (c \rightarrow \ell_\infty(\tilde{D})) = (c_0 \rightarrow \ell_\infty(\tilde{D}))$ iff

$$\sup_n \sum_{k=0}^{\infty} \left| \sum_{i=0}^n \binom{n}{i} \frac{i!}{d_n} (-1)^{n-i} w_{ik} \right| < \infty.$$

(d) $W \in (\ell_1 \rightarrow \ell_\infty(\tilde{D}))$ iff

$$\sup_{n,k} \left| \sum_{i=0}^n \binom{n}{i} \frac{i!}{d_n} (-1)^{n-i} w_{ik} \right| < \infty.$$

Proof. We prove the case $W \in (\ell_\infty \rightarrow \ell_p(\tilde{D}))$ for $p \geq 1$. Consider the matrix $\tilde{W} = (\tilde{w}_{nk})$ as

$$\tilde{w}_{nk} = \sum_{i=0}^n \binom{n}{i} \frac{i!}{d_n} (-1)^{n-i} w_{ik}, \quad \text{for all } n, k \in \mathbb{N}.$$

Then, one has

$$\sum_{k=0}^{\infty} \tilde{w}_{nk} z_k = \sum_{i=0}^n \binom{n}{i} \frac{i!}{d_n} (-1)^{n-i} \sum_{k=0}^{\infty} w_{ik} z_k, \quad \text{for any } z = (z_k) \in \ell_\infty,$$

which means that $\tilde{W}_n(z) = \tilde{D}_n(Wz)$ for all $n \in \mathbb{N}$ and $Wz \in \ell_p(\tilde{D})$ for $z = (z_k) \in \ell_\infty$ iff $\tilde{W}z \in \ell_p$ for $z = (z_k) \in \ell_\infty$. So, we have

$$\sup_{K \in \mathcal{F}} \sum_{n=0}^{\infty} \left| \sum_{k \in Ki=0}^n \binom{n}{i} \frac{i!}{d_n} (-1)^{n-i} w_{ik} \right|^p < \infty.$$

The remainder assertions follow in a similar way. \square

4. Compact operators on the spaces $\ell_p(\tilde{D})$ and $\ell_\infty(\tilde{D})$

In this section, we offer the characterizing the compactness of certain matrix operators helped by Hausdorff measure of noncompactness.

By Ψ , we denote the set of all finite sequences. Let $z = (z_n) \in \omega$ and S_Λ be unit sphere in the BK-space $\Lambda \supset \Psi$, then, we need the relation

$$\|z\|_\Lambda^* = \sup_{s \in S_\Lambda} \left| \sum_k z_k s_k \right| < \infty,$$

which will be frequently used in the sequel subject to condition that the supremum is finite. In this case, we have $z \in \Lambda^\beta$. As well, by $C(\Lambda \rightarrow \Xi)$, we denote the set of all bounded (continuous) linear operators from Λ to Ξ .

We are in position to state some lemmas, given in [27].

Lemma 4.1. *We have*

- (a) $\ell_\infty^\beta = \ell_1$ and $\|z\|_{\ell_\infty}^* = \|z\|_{\ell_1}$ for all $z \in \ell_1$.
- (b) $\ell_1^\beta = \ell_\infty$ and $\|z\|_{\ell_1}^* = \|z\|_{\ell_\infty}$ for all $z \in \ell_\infty$.
- (c) $\ell_p^\beta = \ell_q$ and $\|z\|_{\ell_p}^* = \|z\|_{\ell_q}$ for all $z \in \ell_q$ with $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 4.2. *For every $W \in (\Lambda \rightarrow \Xi)$ with the BK-spaces Λ and Ξ , there exists a linear operator $\mathcal{L}_W \in C(\Lambda \rightarrow \Xi)$ such that $\mathcal{L}_W(z) = W(z)$ for all $z \in \Lambda$.*

Lemma 4.3. *Let $\Lambda \supset \Psi$ be a BK-space and $\Xi = \{c_0, c, \ell_\infty\}$. If $W \in (\Lambda \rightarrow \Xi)$, then, we have*

$$\|\mathcal{L}_W\| = \|W\|_{(\Lambda \rightarrow \Xi)} = \sup_n \|W_n\|_\Lambda^* < \infty.$$

The notion of Hausdorff measure of noncompactness of a bounded set Z , denoted by $\chi(Z)$ is defined by

$$\chi(Z) = \inf \left\{ \delta > 0 : Z \subset \cup_{k=1}^n B(x_k, r_k), x_k \in \Lambda, r_k < \delta, n \in \mathbb{N} \setminus \{0\} \right\}.$$

Here, $B(x_k, r_k)$ is the open ball centered at x_k and radius r_k for each $k = 1, 2, \dots, n$. Further results concerning Hausdorff measure of noncompactness can be found [27] and references cited therein.

Theorem 4.4. ([38, Theorem 2.8]) *The Hausdorff measure of noncompactness of a bounded set Z in ℓ_p can be evaluated as*

$$\chi(Z) = \lim_k \left(\sup_{z \in Z} \|I - \mathcal{P}_k(z)\|_{\ell_p} \right), \text{ for } 1 \leq p < \infty,$$

where $\mathcal{P}_k : \ell_p \rightarrow \ell_p$ denotes the operator defined by $\mathcal{P}_k(z) = (z_0, z_1, \dots, z_k, 0, 0, \dots)$ for each $k \in \mathbb{N}$ and $I : \ell_p \rightarrow \ell_p$ is the identity operator defined by $I(z) = (z_0, z_1, \dots, z_k, z_{k+1}, \dots)$.

A linear operator $\mathcal{L} : \Lambda \rightarrow \Xi$ is said to be compact operator if the domain of \mathcal{L} is all of Λ and the image under \mathcal{L} of any bounded subset of Λ is a totally bounded subset of Ξ . In other words, the sequence $(\mathcal{L}(z_n))$ has a convergent subsequence in Ξ for every bounded sequence $z = (z_n) \in \Lambda$.

There is a close relationship between Hausdorff measure of noncompactness and compact operators. Concretely speaking, the necessary and sufficient condition for an operator to be compact is that $\|\mathcal{L}_\chi\| = 0$, where $\|\mathcal{L}_\chi\|$ denotes the Hausdorff measure of noncompactness of \mathcal{L} and it is defined as $\|\mathcal{L}_\chi\| = \chi(\mathcal{L}(\mathcal{S}_\Lambda))$.

Let $x = (x_k) \in (\ell_p(\tilde{D}))^\beta$ be a sequence, define a sequence $y = (y_k) \in \ell_q$ as

$$y_k = \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} x_i \tag{29}$$

for all $k \in \mathbb{N}$. The following lemmas will be needed in the proofs.

Lemma 4.5. *Let $1 \leq p \leq \infty$ and let $x = (x_k) \in (\ell_p(\tilde{D}))^\beta$. Then, $y = (y_k) \in \ell_q$ and*

$$\sum_k x_k z_k = \sum_k y_k t_k$$

for all $z = (z_k) \in \ell_p(\tilde{D})$.

Lemma 4.6. *Let $y = (y_k)$ be defined as (29), then,*

- (a) $\|x\|_{\ell_1(\tilde{D})}^* = \sup_k |y_k| < \infty$ for all $x = (x_k) \in (\ell_1(\tilde{D}))^\beta$.
- (b) $\|x\|_{\ell_p(\tilde{D})}^* = \left(\sum_k |y_k|^q\right)^{1/q} < \infty$ for all $x = (x_k) \in (\ell_p(\tilde{D}))^\beta$ and for $1 < p < \infty$.
- (c) $\|x\|_{\ell_\infty(\tilde{D})}^* = \sum_k |y_k| < \infty$ for all $x = (x_k) \in (\ell_\infty(\tilde{D}))^\beta$.

Proof. Let us present the proof of assertion (c). The other results can be revealed similarly. For $x = (x_k) \in (\ell_\infty(\tilde{D}))^\beta$, using Lemma 4.5, we have $y = (y_k) \in \ell_1$ and $\sum_k x_k z_k = \sum_k y_k t_k$ for all $z = (z_k) \in \ell_\infty(\tilde{D})$. Since

$\ell_\infty(\tilde{D})$ and ℓ_∞ are isomorphic, we arrive at $z \in \mathcal{S}_{\ell_\infty(\tilde{D})}$ iff $t \in \mathcal{S}_{\ell_\infty}$. So, $\|x\|_{\ell_\infty(\tilde{D})}^* = \sup_{z \in \mathcal{S}_{\ell_\infty(\tilde{D})}} \left| \sum_k x_k z_k \right| = \sup_{z \in \mathcal{S}_{\ell_\infty}} \left| \sum_k y_k t_k \right| = \|y\|_{\ell_\infty}^*$. Employing Lemma 4.1 gives

$$\|x\|_{\ell_\infty(\tilde{D})}^* = \|y\|_{\ell_\infty}^* = \|x\|_{\ell_1} = \sum_k |y_k| < \infty,$$

which completes the proof. \square

Lemma 4.7. ([31, Theorem 3.7 and Theorem 3.11]) *For any BK-space $\Lambda \supset \Psi$, we have*

- (a) *If $W \in (\Lambda \rightarrow \ell_\infty)$, then $0 \leq \|\mathcal{L}_W\|_\chi \leq \limsup_n \|W_n\|_\Lambda^*$ and \mathcal{L}_W is compact iff $\lim_n \|W_n\|_\Lambda^* = 0$.*
- (b) *If $W \in (\Lambda \rightarrow c_0)$, then $\|\mathcal{L}_W\|_\chi \leq \limsup_n \|W_n\|_\Lambda^*$ and \mathcal{L}_W is compact iff $\lim_n \|W_n\|_\Lambda^* = 0$.*
- (c) *If $W \in (\Lambda \rightarrow \ell_1)$, then*

$$\lim_r \left(\sup_{N \in F_r} \left\| \sum_{n \in N} W_n \right\|_\Lambda^* \right) \leq \|\mathcal{L}_W\|_\chi \leq 4 \lim_r \left(\sup_{N \in F_r} \left\| \sum_{n \in N} W_n \right\|_\Lambda^* \right)$$

and \mathcal{L}_W is compact iff $\lim_r \left(\sup_{N \in F_r} \left\| \sum_{n \in N} W_n \right\|_\Lambda^* \right) = 0$. Here, F_r denotes the subcollection of with elements that are greater than r .

Next, let's mention a lemma without any proof due to the similarities to Lemma 4.5. For this purpose, we define an infinite matrix $\hat{W} = (\hat{w}_{nk})$ as

$$\hat{w}_{nk} = \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \tag{30}$$

for all $n, k \in \mathbb{N}$, provided that the series is convergent.

Lemma 4.8. *Let Λ be any sequence space and $W = (w_{ni})$ be an infinite matrix. If $W \in (\ell_p(\tilde{D}) \rightarrow \Lambda)$, then $\hat{W} \in (\ell_p \rightarrow \Lambda)$ and $Wz = \hat{W}t$ hold for all $z \in \ell_p(\tilde{D})$ for $1 \leq p \leq \infty$.*

Theorem 4.9. *Let $1 < p < \infty$. Then, we have*

(a) If $W \in (\ell_p(\tilde{D}) \rightarrow \ell_\infty)$, then

$$0 \leq \|\mathcal{L}_W\|_\chi \leq \limsup_n \left(\sum_k \left| \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \right|^q \right)^{1/q}$$

and \mathcal{L}_W is compact if

$$\lim_n \left(\sum_k \left| \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \right|^q \right)^{1/q} = 0.$$

(b) If $W \in (\ell_p(\tilde{D}) \rightarrow c_0)$, then

$$\|\mathcal{L}_W\|_\chi = \limsup_n \left(\sum_k \left| \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \right|^q \right)^{1/q}$$

and \mathcal{L}_W is compact iff

$$\lim_n \left(\sum_k \left| \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \right|^q \right)^{1/q} = 0.$$

(c) If $W \in (\ell_p(\tilde{D}) \rightarrow \ell_1)$, then

$$\lim_r \|W\|_{(\ell_p(\tilde{D}) \rightarrow \ell_1)}^r \leq \|\mathcal{L}_W\|_\chi \leq 4 \lim_r \|W\|_{(\ell_p(\tilde{D}) \rightarrow \ell_1)}^r$$

and \mathcal{L}_W is compact iff

$$\lim_r \|W\|_{(\ell_p(\tilde{D}) \rightarrow \ell_1)}^r = 0,$$

where

$$\|W\|_{(\ell_p(\tilde{D}) \rightarrow \ell_1)}^r = \sup_{N \in \mathcal{F}_r} \left(\sum_k \left| \sum_{n \in N} \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \right|^q \right)^{1/q}.$$

Proof. For the proof of (a) let $W \in (\ell_p(\tilde{D}) \rightarrow \ell_\infty)$ and $z \in \ell_p(\tilde{D})$, then from the convergence of the series $\sum_{k=0}^{\infty} w_{nk}z_k$ that $W_n \in (\ell_p(\tilde{D}))^\beta$ for each $n \in \mathbb{N}$. Employing Lemma 4.6 (b) to find for each $n \in \mathbb{N}$ that

$\|W_n\|_{\ell_p(\tilde{D})}^* = \left(\sum_k |\hat{w}_{nk}|^q \right)^{1/q}$, where \hat{w}_{nk} is the representation in (30). So, using Lemma 4.7 (a) gives that

$$0 \leq \|\mathcal{L}_W\|_\chi \leq \limsup_n \left(\sum_k |\hat{w}_{nk}|^q \right)^{1/q}$$

and \mathcal{L}_W is compact if

$$\lim_n \left(\sum_k |\hat{w}_{nk}|^q \right)^{1/q} = 0.$$

For the proof of (b), let $W \in (\ell_p(\tilde{D}) \rightarrow c_0)$, then, from $\|W_n\|_{\ell_p(\tilde{D})}^* = \left(\sum_k |\hat{w}_{nk}|^q \right)^{1/q}$ and Lemma 4.7 (b), one finds that

$$\|\mathcal{L}_W\|_\chi = \limsup_n \left(\sum_k |\hat{w}_{nk}|^q \right)^{1/q}$$

and \mathcal{L}_W is compact iff

$$\lim_n \left(\sum_k |\hat{w}_{nk}|^q \right)^{1/q} = 0.$$

For the proof of (c), let $W \in (\ell_p(\tilde{D}) \rightarrow \ell_1)$, then, it follows from Lemma 4.6 that $\left\| \sum_{n \in \mathbb{N}} W_n \right\|_{\ell_p(\tilde{D})}^* = \left\| \sum_{n \in \mathbb{N}} \hat{W}_n \right\|_{\ell_q}^*$. So, use Lemma 4.7 (c) to write that

$$\lim_r \left(\sup_{N \in \mathcal{F}_r} \sum_k \left| \sum_{n \in N} \hat{w}_{nk} \right|^q \right)^{1/q} \leq \|\mathcal{L}_W\|_\chi \leq 4 \lim_r \left(\sup_{N \in \mathcal{F}_r} \sum_k \left| \sum_{n \in N} \hat{w}_{nk} \right|^q \right)^{1/q}$$

and \mathcal{L}_W is compact iff

$$\lim_r \left(\sup_{N \in \mathcal{F}_r} \sum_k \left| \sum_{n \in N} \hat{w}_{nk} \right|^q \right)^{1/q} = 0,$$

which completes the proof. \square

Finally, we give two results whose proofs are omitted owing to the similar manipulations to the proof of Theorem 4.9.

Theorem 4.10. *The following statements hold:*

(a) If $W \in (\ell_\infty(\tilde{D}) \rightarrow \ell_\infty)$, then

$$0 \leq \|\mathcal{L}_W\|_\chi \leq \limsup_n \sum_k \left| \sum_{i=k}^\infty \binom{i}{k} \frac{d_k}{i!} w_{ni} \right|$$

and \mathcal{L}_W is compact if

$$\limsup_n \sum_k \left| \sum_{i=k}^\infty \binom{i}{k} \frac{d_k}{i!} w_{ni} \right| = 0.$$

(b) If $W \in (\ell_\infty(\tilde{D}) \rightarrow c_0)$, then

$$\|\mathcal{L}_W\|_\chi = \limsup_n \sum_k \left| \sum_{i=k}^\infty \binom{i}{k} \frac{d_k}{i!} w_{ni} \right|$$

and \mathcal{L}_W is compact iff

$$\lim_n \sum_k \left| \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \right| = 0.$$

(c) If $W \in (\ell_{\infty}(\tilde{D}) \rightarrow \ell_1)$, then

$$\lim_r \|W\|_{(\ell_{\infty}(\tilde{D}) \rightarrow \ell_1)}^r \leq \|\mathcal{L}_W\|_{\chi} \leq 4 \lim_r \|W\|_{(\ell_{\infty}(\tilde{D}) \rightarrow \ell_1)}^r$$

and \mathcal{L}_W is compact iff

$$\lim_r \|W\|_{(\ell_{\infty}(\tilde{D}) \rightarrow \ell_1)}^r = 0,$$

where

$$\|W\|_{(\ell_{\infty}(\tilde{D}) \rightarrow \ell_1)}^r = \sup_{N \in F_r} \sum_k \left| \sum_{n \in N, i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \right|.$$

Theorem 4.11. *The following statements hold:*

(a) If $W \in (\ell_1(\tilde{D}) \rightarrow \ell_{\infty})$, then

$$0 \leq \|\mathcal{L}_W\|_{\chi} \leq \limsup_n \left(\sup_k \left| \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \right| \right)$$

and \mathcal{L}_W is compact if

$$\lim_n \left(\sup_k \left| \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \right| \right) = 0.$$

(b) If $W \in (\ell_1(\tilde{D}) \rightarrow c_0)$, then

$$\|\mathcal{L}_W\|_{\chi} = \limsup_n \left(\sup_k \left| \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \right| \right)$$

and \mathcal{L}_W is compact iff

$$\lim_n \left(\sup_k \left| \sum_{i=k}^{\infty} \binom{i}{k} \frac{d_k}{i!} w_{ni} \right| \right) = 0.$$

5. Conclusions and future studies

Since the most general linear mapping from a sequence space into another sequence space can be represented by an infinite matrix, matrix transformations between sequence spaces has aroused interest over years due to its significant implications in summability theory. In recent years, by inspiring some remarkable relations for fascinating numbers, an infinite matrix has been introduced and then its domain in some famous sequence spaces has been discussed. Also, a number of important properties containing special duals and matrix transformations, characterizing its compact operators helped by Hausdorff measure of noncompactness have been investigated.

In this paper, we define two sequence spaces $\ell_p(\tilde{D})$ and $\ell_{\infty}(\tilde{D})$ as the domains of the triangular matrix including derangement numbers, playing an important role in combinatorial mathematics, and study their topological properties.

By proceeding the same motivation aforementioned, it is planned to study the domains of the matrix, defined here, in the spaces of convergent and null sequences.

As a final note, the q -calculus or q -analog plays a key role in generalizing various concepts in the several branches of mathematics specially in combinatorics, approximation theory, algebra, differential-integro equations, special functions, hypergeometric functions, etc. In the literature, it is available the following formula for q -derangement numbers [42]:

$$d_q(n) = [n]! \sum_{k=0}^n q^{\binom{k}{2}} \frac{(-1)^k}{[k]}.$$

For further studies, we will define a novel matrix involving q -derangement numbers and study its domain in the sequence spaces as well as topological properties.

References

- [1] A. Alotaibi, M. Mursaleen, B.A.S. Alamri, S.A. Mohiuddine, Compact operators on some Fibonacci difference sequence spaces, *J. Inequal. Appl.* 2015 (2015) 203.
- [2] B. Altay, F. Başar, The matrix domain and the fine spectrum of the difference operator Δ on the sequence space ℓ_p , ($0 < p < 1$), *Commun. Math. Anal.* 2(2) (2007) 1–11.
- [3] B. Altay, F. Başar, Certain topological properties and duals of the matrix domain of a triangle matrix in a sequence space, *J. Math. Anal. Appl.* 336 (2007) 632–645.
- [4] F. Başar, *Summability Theory and its Applications*, 2nd edn. CRC Press/Taylor and Francis Group, Boca Raton, London, New York; 2022.
- [5] F. Başar, E. Malkowsky, The characterization of compact operators on spaces of strongly summable and bounded sequences, *Appl. Math. Comput.* 217 (2011) 5199–5207.
- [6] M. Başarı, E.E. Kara, On the B -difference sequence space derived by generalized weighted mean and compact operators, *J. Math. Anal. Appl.* 391 (2012) 67–81.
- [7] N.L. Braha, T. Mansour, Some properties of new sequence spaces based on Riordan numbers, *J. Math. Anal. Appl.* 543(2), Part 1, (2025) Article 128902.
- [8] M. Candan, A new sequence space isomorphic to the space $\ell(p)$ and compact operators, *J. Math. Comput. Sci.* 4(2) (2014) 306–334.
- [9] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition. D. Reidel Publishing Co., Dordrecht; 1974.
- [10] M.C. Dağlı, A novel conservative matrix arising from Schröder numbers and its properties, *Linear Multilinear Algebra*, 71(8) (2023) 1338–1351.
- [11] M.C. Dağlı, A new almost convergent sequence space defined by Schröder matrix, *Linear Multilinear Algebra*, 71(11) (2023) 1863–1874.
- [12] M.C. Dağlı, Matrix mappings and compact operators for Schröder sequence spaces, *Turkish J. Math.* 46(6) (2022) 2304–2320.
- [13] S. Erdem, Schröder–Catalan Matrix and Compactness of Matrix Operators on Its Associated Sequence Spaces, *Symmetry*, 16(10) (2024) 1317. <https://doi.org/10.3390/sym16101317>
- [14] S. Erdem, S. Demiriz, A. Şahin, Motzkin sequence spaces and Motzkin core, *Numer. Funct. Anal. Optim.* 45 (4-6) (2024) 283–303.
- [15] F. Gökçe, Characterizations of matrix and compact operators on BK spaces, *Univ. J. Math. Appl.* 6 (2023) 76–85.
- [16] F. Gökçe, M.A. Sarıgöl, Generalization of the space $\ell(p)$ derived by absolute Euler summability and matrix operators, *J. Inequal. Appl.* 2018;2018:133.
- [17] M. İlkhani, Matrix domain of a regular matrix derived by Euler totient function in the spaces c_0 and c , *Mediterr. J. Math.* 17(1):27 (2020).
- [18] M. İlkhani, A new conservative matrix derived by Catalan numbers and its matrix domain in the spaces c and c_0 , *Linear Multilinear Algebra*, 68(2) (2020) 417–434.
- [19] M. İlkhani, E.E. Kara, A new Banach space defined by Euler totient matrix operator, *Oper. Matrices*, 13(2) (2019) 527–544.
- [20] M. İlkhani, N. Simsek, E.E. Kara, A new regular infinite matrix defined by Jordan totient function and its matrix domain in ℓ_p , *Math. Meth. Appl. Sci.* 44(9) (2020) 7622–7633.
- [21] E.E. Kara, Some topological and geometrical properties of new Banach sequence spaces, *J. Inequal. Appl.* 2013(38) (2013), 15 pages.
- [22] E.E. Kara, M. Başarı, On some Euler $B^{(m)}$ difference sequence spaces and compact operators, *J. Math. Anal. Appl.* 379 (2011) 499–511.
- [23] E.E. Kara, M. İlkhani, Some properties of generalized Fibonacci sequence spaces, *Linear Multilinear Algebra*, 64(11) (2016) 2208–2223.
- [24] M.İ. Kara, E.E. Kara, Matrix transformations and compact operators on Catalan sequence spaces, *J. Math. Anal. Appl.* 498(1) (2021) Article no: 124925.
- [25] M. Karakaş, On the sequence spaces involving bell numbers, *Linear Multilinear Algebra*, 71(14) (2023) 2298–2309.
- [26] M. Kirişçi, F. Başar, Some new sequence spaces derived by the domain of generalized difference matrix, *Comput. Math. Appl.* 60 (2010) 1299–1309.

- [27] E. Malkowsky, V. Rakocevic, An introduction into the theory of sequence spaces and measure of noncompactness, *Zb. Rad. (Matematički Inst. SANU, Belgrade)* 9(17) (2000) 143–234.
- [28] M. Mursaleen, F. Başar, *Sequence Spaces: Topics in Modern Summability Theory. Series: Mathematics and Its Applications*, CRC Press, Taylor and Francis Group, Boca Raton, London, New York; 2020.
- [29] M. Mursaleen, S.A. Mohiuddine, Applications of measures of non-compactness to the infinite system of differential equations in ℓ_p spaces, *Nonlinear Anal.* 75(4) (2012) 2111–2115.
- [30] M. Mursaleen, A.K. Noman, On some new difference sequence spaces of non-absolute type, *Math. Comput. Model.* 52(3–4) (2010) 603–617.
- [31] M. Mursaleen, A.K. Noman, Compactness by the Hausdorff measure of noncompactness, *Nonlinear Anal.* 73(8) (2010) 2541–2557.
- [32] M. Mursaleen, A.K. Noman, Applications of the Hausdorff measure of noncompactness in some sequence spaces of weighted means, *Comput. Math. Appl.* 60(5) (2010) 1245–1258.
- [33] M. Mursaleen, A.K. Noman, The Hausdorff measure of noncompactness of matrix operators on some BK spaces, *Oper. Matrices*, 5 (2011) 473–486.
- [34] F. Qi, B.-N. Guo, Explicit formulas for derangement numbers and their generating function, *Journal of Nonlinear Functional Analysis*, 2016 (2016), Article ID 45, 10 pages.
- [35] F. Qi, J.-L. Zhao, B.-N. Guo, Closed forms for derangement numbers in terms of the Hessenberg determinants, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales-Serie A: Matemáticas*, 112(4) (2018) 933–944.
- [36] K. Raj, J. Aljedani, D. Narrania, S.A. Mohiuddine, Almost convergent Motzkin sequence spaces and core theorems, *Filomat*, 39(17) (2025) 5751–5764.
- [37] K. Raj, S.A. Mohiuddine, S. Jasrotia, Characterization of summing operators in multiplier spaces of deferred Nörlund summability, *Positivity* 27 (2023), Article 9.
- [38] V. Rakocevic, Measures of noncompactness and some applications, *Filomat*, 12 (1998) 87–120.
- [39] M.A. Sarıgöl, Spaces of series summable by absolute Cesàro and matrix operators, *Comm. Math. Appl.* 7(1) (2016) 11–22.
- [40] N.J.A. Sloane, The on-line encyclopedia of integer sequences. <http://www.research.att.com/njas/sequences/>
- [41] M. Stieglitz, H. Tietz, Matrix transformationen von folgenräumen eine ergebnisübersicht, *Math Z.* 154 (1977) 1–16.
- [42] M.L. Wachs, On q -derangement numbers, *Proc. Amer. Math. Soc.* 106 (1989) 273–278.
- [43] T. Yaying, B. Hazarika, On sequence spaces defined by the domain of a regular Tribonacci matrix, *Math. Slovaca*, 70(3) (2020) 697–706.
- [44] T. Yaying, B. Hazarika, A. Esi, Geometric properties and compact operator on fractional Riesz difference space, *Kragujevac J. Math.* 47 (2023) 545–566.
- [45] T. Yaying, S.A. Mohiuddine, J. Aljedani, Exploring the q -analogue of Fibonacci sequence spaces associated with c and c_0 , *AIMS Math.* 10(1) (2025) 634–653.
- [46] T. Yaying, N. Saikia, On sequence spaces defined by arithmetic function and Hausdorff measure of noncompactness, *Rocky Mountain J. Math.* 52 (2022) 1867–1885.
- [47] T. Yaying, N. Saikia, M. Mursaleen, New sequence spaces derived by using generalized arithmetic divisor sum function and compact operators, *Forum Math.* 37 (2025) 205–223.