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Hermite-Hadamard-Mercer and related inequalities for h-convex functions via ψ-Hilfer operators

Noureddine Azzouz[a], Bouharket Benaissa[b], Hüseyin Budak[c,d], Fatih Hezenci[e,*]

[a] *Faculty of Sciences University Center Nour Bachir El Bayadh - Algeria*
[b] *Faculty of Material Sciences University of Tiaret - Algeria*
[c] *Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli 41001, Türkiye*
[d] *Department of Mathematics, Saveetha School of Engineering, SIMATS, Saveetha University, Chennai 602105, Tamil Nadu, India*
[e] *Department of Mathematics, Faculty of Science and Arts, Duzce University, Türkiye*

Abstract. We prove a fractional Hermite-Hadamard-Mercer inequality via ψ-Hilfer integral operators for the class of $h-$convex functions, where h is a B-function. Some novel fractional inequalities related to the left and right sides of the Hermite-Hadamard-Mercer inequalities are established for differentiable mappings whose absolute values of the derivatives are $h-$convex. Moreover, we construct new inequalities for these differentiable functions using Hölder's inequality.

1. Introduction

Convex functions and mathematical inequalities play vital role in the advancement of several fields in pure and applied sciences. The classical convexity is defined as follows

Definition 1.1. *A function $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $\lambda \in (0, 1)$ we have*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1}$$

If the inequality (1) *is reversed, then f is said to be concave.*

Dealing with classical convexity, a vast literature has been reported on inequalities that involve convex functions, such as Bullen inequality [1], Hermite-Hadamard-Fejer inequality [2], Simpson type inequality [3], weighted Newton type inequalities [18] and Ostrowski type inequalities [4].
Likewise, several integral inequality are known, but the most popular is the following well-known Hermite-Hadamard inequality, true for any function f convex and integrable on $[a, b] \subset \mathbb{R}$:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

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* Corresponding author: Fatih Hezenci
Email addresses: n.azzouz@cu-elbayadh.dz (Noureddine Azzouz), bouharket.benaissa@univ-tiaret.dz (Bouharket Benaissa), hsyn.budak@gmail.com (Hüseyin Budak), fatihezenci@gmail.com (Fatih Hezenci)

If f is concave, then inequalities in (2) are reversed. For further details, please refer to [28–30].

Interesting inequalities related to (2) are the trapezoid inequality established in [5], estimating the difference between the right term and the integral of f on $[a,b]$; and the midpoint inequality established in [6], estimating the difference between the left term and the integral of f on $[a,b]$.

We recall the definitions of the left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$, defined for an integrable function $f : [a,b] \to \mathbb{R}$ as

$$\mathcal{RL}^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a,$$
$$\mathcal{RL}^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, x < b, \tag{3}$$

where Γ is the Gamma function. It follows

$$\mathcal{RL}^{\alpha}_{a^+} F(b) + \mathcal{RL}^{\alpha}_{b^-} F(a) = 2\left[\mathcal{RL}^{\alpha}_{a^+} f(b) + \mathcal{RL}^{\alpha}_{b^-} f(a)\right], \tag{4}$$

where $F(s) = f(s) + f(a+b-s)$. For more details, see ([7], [8]).

Fractional versions of the Hermite-Hadamard, the trapezoid and the midpoint inequalities are presented below.

Theorem 1.2 ([9], Theorem 2). *If $f : [a,b] \to \mathbb{R}$ is an integrable convex function, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[\mathcal{RL}^{\alpha}_{a^+} f(b) + \mathcal{RL}^{\alpha}_{b^-} f(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{5}$$

Theorem 1.3 ([9], Theorem 3). *If f is a differentiable mapping on (a,b) such that $|f'|$ is convex, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[\mathcal{RL}^{\alpha}_{a^+} f(b) + \mathcal{RL}^{\alpha}_{b^-} f(a)\right]\right| \le \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^{\alpha}}\right) \left[\left|f'(a)\right| + \left|f'(b)\right|\right].$$

Theorem 1.4 ([10], Theorem 2.3). *If a function f is differentiable on (a,b) and $|f'|$ is convex, then*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[\mathcal{RL}^{\alpha}_{a^+} f(b) + \mathcal{RL}^{\alpha}_{b^-} f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4(\alpha+1)}\left(\alpha + 3 - \frac{1}{2^{\alpha-1}}\right) \left[\, |f'(a)| + |f'(b)| \,\right].$$

A basic result concerning convex functions is the Jensen's inequality [11]. Its formal statement is as follows : if a function $f : [a,b] \to \mathbb{R}$ is convex, then

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \le \sum_{i=1}^{n} \alpha_i f(x_i),$$

for all $x_1, x_2, ..., x_n$ in $[a,b]$ and all scalars $\alpha_i \in [0,1]$ $(i = \overline{1,n})$ such that $\sum_{i=1}^{n} \alpha_i = 1$.

In 2003, a new variant of Jensen's inequality was introduced by Mercer [12]:

Lemma 1.5. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then, we have*

$$f(a+b-x) \le f(a) + f(b) - f(x) \quad \text{for all} \quad x \in [a,b].$$

Theorem 1.6. *For a convex mapping $f : [a,b] \to \mathbb{R}$, the inequality holds:*

$$f\left(a + b - \sum_{i=1}^{n} \alpha_i x_i\right) \le f(a) + f(b) - \sum_{i=1}^{n} \alpha_i f(x_i) \tag{6}$$

for all x_i in $[a,b]$ and all scalars $\alpha_i \in [0,1]$ $(i = \overline{1,n})$ such that $\sum_{i=1}^{n} \alpha_i = 1$.

In particular, for $\alpha_1 = \alpha_2 = \dfrac{1}{2}$ $(n = 2)$, (6) reads

$$f\left(a+b-\frac{x_2+x_1}{2}\right) \leq f(a)+f(b)-\frac{f(x_1)+f(x_2)}{2}. \tag{7}$$

Later, (7) was used to establish the following Hermite-Hadamard-Mercer inequality.

Theorem 1.7. ([13], *Theorem 2.1*) *For a convex mapping* $f : [a,b] \to \mathbb{R}$, *the following inequalities hold for all* $a \leq x < y \leq b$

$$f\left(a+b-\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(a+b-t)dt \leq f(a)+f(b)-\frac{f(x)+f(x_2)}{2}. \tag{8}$$

Setting $a = x$, $b = y$, *inequalities* (8) *yields to inequalities* (2).

Fractional versions are presented bellow.

Theorem 1.8. ([14], *Theorem 2.1*) *For a convex mapping* $f : [a,b] \to \mathbb{R}$, *the following inequalities hold for* $a \leq x < y \leq b$

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\mathcal{RL}^{\alpha}_{(a+b-x)^+}f(a+b-y)+\mathcal{RL}^{\alpha}_{(a+b-y)^-}f(a+b-x)\right] \\ &\leq \frac{f(a+b-x)+f(a+b-y)}{2} \leq f(a)+f(b)-\frac{f(x)+f(y)}{2}. \end{aligned} \tag{9}$$

Theorem 1.9. ([14], *Theorem 3.6*) *If* f *is a differentiable mapping on* (a,b) *such that* $|f'|$ *is convex, then*

$$\begin{aligned} &\left|\frac{f(a+b-x)+f(a+b-y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\mathcal{RL}^{\alpha}_{(a+b-x)^+}f(a+b-y)+\mathcal{RL}^{\alpha}_{(a+b-y)^-}f(a+b-x)\right]\right| \\ &\qquad \leq \frac{y-x}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|-\frac{\left|f'(x)\right|+\left|f'(y)\right|}{2}\right]. \end{aligned} \tag{10}$$

Our main in this paper is to extend the above results for fractional integrals with respect to another function, on the class of h-convex functions. This paper is organized as follow, Section 2 of preliminaries notions and results is presented. In section 3, a first main result is stated and proved, it concerns a Hermit-Hadamard-Mercer inequality for h-convex functions via fractional integral operators with respect to another function. Moreover,trapezoid-Mercer types inequalities and midpoint-Mercer types inequalities are presented in Section 3.

2. Preliminaries

1) Class of h-convex functions (see [15]): given a non-negative function $h \neq 0$, $f : [a,b] \to \mathbb{R}$ is said to be h-convex if for all $x, y \in [a,b]$ and $t \in (0,1)$ we have

$$f(tx+(1-t)y) \leq h(t)f(x)+h(1-t)f(y). \tag{11}$$

This notion generalizes known classes of convex functions : for $h(t) = t$, $h(t) = 1$, $h(t) = t^s$ and $h(t) = \frac{1}{t}$ in (11), h-convex functions reduces respectively to convex functions, P functions [16, 17], s-convex functions [19] and Godunova–Levin functions [20].

2) h-Mercer inequality: in ([21], Lemma.4.1) authors presented the following interesting result.

Lemma 2.1. *If f is an h-convex function, then for any $z \in [a,b]$, there exists $\lambda \in [0,1]$ such that*

$$f(a+b-z) \leq [h(\lambda)+h(1-\lambda)]\,[f(a)+f(b)] - f(z). \tag{12}$$

***N.B.**: $\lambda \in [0,1]$ is determined by the relation $z = \lambda a + (1-\lambda)b$ or the relation $z = (1-\lambda)a + \lambda b$.*

3) B-function: this notion was introduced by Benaissa et al. in [22] and [23] as follows: a non-negative function $h : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be a B-function, if for all $x \in (a,b)$, we have

$$h(x-a)+h(b-x) \leq 2\,h\left(\frac{a+b}{2}\right). \tag{13}$$

In particular, a non-negative function $h : [0,1] \to \mathbb{R}$ is a B-function if and only if

$$h(t)+h(1-t) \leq 2\,h\Big(\frac{1}{2}\Big), \quad \text{for all } t \in (0,1). \tag{14}$$

For examples, $h_1(t)=1$, $h_2(t)=t$ and $h_3(t)=t^s,\ s\in(0,1)$ are B-function.

4) ψ-Hilfer fractional integrals ([7], [24]): left and right sided fractional integrals of a function f with respect to another function ψ on $[a,b]$ are defined, for $\alpha>0$ and a positive increasing differentiable function ψ such that $\psi'(t) \neq 0$ for all $t \in [a,b]$, by respectively

$$\begin{aligned} {}^{\psi}\mathfrak{I}^{\alpha}_{a^+} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (\psi(x)-\psi(t))^{\alpha-1}\,\psi'(t) f(t)dt, \quad a<x\leq b,\\ {}^{\psi}\mathfrak{I}^{\alpha}_{b^-} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (\psi(t)-\psi(x))^{\alpha-1}\,\psi'(t) f(t)dt, \quad a\leq x< b\,. \end{aligned}$$

Considered on $X([a,b]) = \left\{ f : \| f \|_X = \int_a^b | f(t) |\, \psi'(t)\, dt < \infty \right\}$, ${}^{\psi}\mathfrak{I}^{\alpha}_{a^+} f \in X([a,b])$ and ${}^{\psi}\mathfrak{I}^{\alpha}_{b^-} f \in X([a,b])$ for all $\alpha>0$ and the operators ${}^{\psi}\mathfrak{I}^{\alpha}_{a^+}$ and ${}^{\psi}\mathfrak{I}^{\alpha}_{b^-}$ are bounded on $X([a,b])$.

Depending on the function ψ, particular type of fractional integrals are obtained.
1) Taking $\psi(t)=t$, we get Riemann-Liouville fractional operators (3),
2) Using $\psi(t)=\ln t$, we deduce Hadamard fractional operators of order $\alpha>0$,
3) Putting $\psi(\tau)=\frac{\tau^{\rho}}{\rho}$ $(\rho>0)$, we obtain Katugompola fractional operators of order $\alpha>0$.

5) Notations: in all what follow, we consider $[a,b]\subset\mathbb{R}$, $\alpha>0$, $h:[a,b]\to\mathbb{R}$ a non-negative integrable function and ψ a positive differentiable function such that $\psi'(t)>0$ for all $t\in[a,b]$.
We will also denote in this paper

$$\Delta_{\psi}(x,y) := \psi(a+b-x)-\psi(a+b-y). \tag{15}$$

6) Useful results:

Lemma 2.2. *Consider $a \leq x < y \leq b$ and $\tau \in [0,1]$*
$A(\tau) := \dfrac{(\psi(a+b-x)-\psi(a+b-y+(y-x)\tau))^{\alpha}}{\alpha}$, $B(\tau) := \dfrac{(\psi(a+b-x)-\psi(a+b-x-(y-x)\tau))^{\alpha}}{\alpha}$, $C(\tau) := \dfrac{(\psi(a+b-x-(y-x)\tau)-\psi(a+b-y))^{\alpha}}{\alpha}$, $D(\tau) := \dfrac{(\psi(a+b-y+(y-x)\tau)-\psi(a+b-y))^{\alpha}}{\alpha}$,
and

$$\Lambda(x,y,\tau) = A(\tau)-B(\tau)+C(\tau)-D(\tau). \tag{16}$$

Then, we have the following result

$$\int_0^1 \Big|\Lambda(x,y,\tau)\Big| d\tau \leq \int_0^{\frac{1}{2}} \Big[A(\tau)-B(\tau)+C(\tau)-D(\tau)\Big] d\tau + \int_{\frac{1}{2}}^1 \Big[B(\tau)-A(\tau)+D(\tau)-C(\tau)\Big] d\tau. \tag{17}$$

Proof. Putting $\Delta = \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}$, for τ varying in $[0,1]$, values of $A(\tau)$ are decreasing from Δ to zero and conversely values of $B(\tau)$ are increasing from zero to Δ with equality $A(\frac{1}{2}) = B(\frac{1}{2})$, this gives

$$\int_0^1 \left|A(\tau) - B(\tau)\right| d\tau = \int_0^{\frac{1}{2}} \left[A(\tau) - B(\tau)\right] d\tau + \int_{\frac{1}{2}}^1 \left[B(\tau) - A(\tau)\right] d\tau. \tag{18}$$

Similarly, values of $C(\tau)$ are decreasing from Δ to zero and values of $D(\tau)$ are increasing from zero to Δ with equality $C(\frac{1}{2}) = D(\frac{1}{2})$, which gives

$$\int_0^1 \left|C(\tau) - D(\tau)\right| d\tau = \int_0^{\frac{1}{2}} \left[C(\tau) - D(\tau)\right] d\tau + \int_{\frac{1}{2}}^1 \left[D(\tau) - C(\tau)\right] d\tau. \tag{19}$$

One can deduce the desired result (17) by applying (18) and (19) in the following inequality

$$\int_0^1 \left|\Lambda(x,y,\tau)\right| d\tau \leq \int_0^1 \left[\left|A(\tau) - B(\tau)\right| + \left|C(\tau) - D(\tau)\right|\right] d\tau. \tag{20}$$

□

Remark 2.3. *Setting in Lemma 2.2 $\psi(t) = t$, it results*

$$\int_0^1 \left|\Lambda(x,y,\tau)\right| d\tau \leq 4\,\frac{(y-x)^\alpha}{\alpha(\alpha+1)}\left[1 - \frac{1}{2^\alpha}\right]. \tag{21}$$

3. Results

3.1. Hermite-Hadamard-Mercer type inequality on the class of h-convex functions

Theorem 3.1. *If $f \in X[a,b]$ is h-convex and h is a B-function, then for all $a \leq x < y \leq b$, we have*

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ & \leq \frac{1}{2} h\left(\tfrac{1}{2}\right) \frac{\Gamma(\alpha+1)}{\left(\Delta_\psi(x,y)\right)^\alpha} \left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+} F(a+b-x) + {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-} F(a+b-y)\right] \\ & \leq 2h^2\left(\tfrac{1}{2}\right)\left[f(a+b-x) + f(a+b-y)\right] \leq 8\,h^3\left(\tfrac{1}{2}\right)\left[f(a)+f(b)\right] - 2h^2\left(\tfrac{1}{2}\right)\left[f(x)+f(y)\right], \end{aligned} \tag{22}$$

where $F(s) = f(s) + f\left(2(a+b) - x - y - s\right)$ *and* $\Delta_\psi(x,y)$ *is defined in* (15).

Proof. For any $a \leq x < y \leq b$, considering $X = tx + (1-t)y$ and $Y = (1-t)x + ty$ with $t \in (0,1)$, then using the h-convexity of the function f, we get for all $t \in (0,1)$

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &= f\left(a+b-\frac{X+Y}{2}\right) = f\left(\frac{a+b-X}{2} + \frac{a+b-Y}{2}\right) \\ &\leq h\left(\tfrac{1}{2}\right)\left[f(a+b-tx-(1-t)y) + f(a+b-(1-t)x-ty)\right] \\ &= h\left(\tfrac{1}{2}\right)\left[f\left(t(a+b-x) + (1-t)(a+b-y)\right) + f\left((1-t)(a+b-x) + t(a+b-y)\right)\right] \\ &\leq h\left(\tfrac{1}{2}\right)\left(h(t) + h(1-t)\right)\left[f(a+b-x) + f(a+b-y)\right]. \end{aligned}$$

Using the B-function property (14) of h and the h-Mercer inequality (12) we obtain

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq h\left(\tfrac{1}{2}\right)\left[f(a+b-tx-(1-t)y) + f(a+b-(1-t)x-ty)\right] \\ &\leq 2h^2\left(\tfrac{1}{2}\right)\left[f(a+b-x) + f(a+b-y)\right] \\ &\leq 8h^3\left(\tfrac{1}{2}\right)\left[f(a)+f(b)\right] - 2h^2\left(\tfrac{1}{2}\right)\left[f(x)+f(y)\right]. \end{aligned} \tag{23}$$

Multiplying (23) by $\Psi(t) := \left(\psi(a+b-x) - \psi(a+b-tx-(1-t)y)\right)^{\alpha-1} \psi'(a+b-tx-(1-t)y)$ and integrating over $t \in [0,1]$, using the change of variable $s = a+b-tx-(1-t)y$, it results

$$\begin{aligned} \frac{1}{y-x} \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} f\left(a+b-\frac{x+y}{2}\right) &\leq \frac{1}{y-x} h\left(\tfrac{1}{2}\right) \Gamma(\alpha)\, {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+} F(a+b-x) \\ &\leq 2 \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha(y-x)} h^2\left(\tfrac{1}{2}\right) \left[f(a+b-x) + f(a+b-y)\right] \\ &\leq \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha(y-x)} \left(8h^3\left(\tfrac{1}{2}\right)[f(a)+f(b)] - 2h^2\left(\tfrac{1}{2}\right)[f(x)+f(y)]\right). \end{aligned} \tag{24}$$

Similarly, multiplying (23) by $\tilde{\Psi}(t) := \left(\psi(a+b-tx-(1-t)y) - \psi(a+b-y)\right)^{\alpha-1} \psi'(a+b-tx-(1-t)y)$ and integrating over $t \in [0,1]$ leads to

$$\begin{aligned} \frac{1}{y-x} \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} f\left(a+b-\frac{x+y}{2}\right) &\leq \frac{1}{y-x} h\left(\tfrac{1}{2}\right) \Gamma(\alpha)\, {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-} F(a+b-y) \\ &\leq 2 \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha(y-x)} h^2\left(\tfrac{1}{2}\right) \left[f(a+b-x) + f(a+b-y)\right] \\ &\leq \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha(y-x)} \left(8h^3\left(\tfrac{1}{2}\right)[f(a)+f(b)] - 2h^2\left(\tfrac{1}{2}\right)[f(x)+f(y)]\right). \end{aligned} \tag{25}$$

Adding (24) and (25) gives the desired result (22). □

Remark 3.2. *Setting $x = a$ and $y = b$ in Theorem 3.1 reduces to the Hermit-Hadamard inequality for the class of h-convex functions where h is a B-function, proved in [22, Theorem 4.1].*

Remark 3.3. *We cite here some special types of convexity.*
1) For s-convex functions :

- *Take $h(t) = t^s$, $s \in (0,1)$ in (22) to get Hermite-Hadamard-Mercer type inequalities involving the ψ-Hilfer fractional integral operators*

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ & \leq \frac{1}{2^{s+1}} \frac{\Gamma(\alpha+1)}{\left(\Delta_\psi(x,y)\right)^\alpha} \left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+} F(a+b-x) + {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-} F(a+b-y)\right] \\ & \leq \frac{1}{2^{2s-1}} \left[f(a+b-x) + f(a+b-y)\right] \leq \frac{1}{2^{3s-3}} \left[f(a)+f(b)\right] - \frac{1}{2^{2s-1}} \left[f(x)+f(y)\right], \end{aligned} \tag{26}$$

  for $x = a$ and $y = b$ we get the result of [22, Corollary 5].

- *With $\psi(t) = t$ in (26) we have inequalities via the Riemann-Liouville fractional integral operators*

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ & \leq \frac{1}{2^{s}} \frac{\Gamma(\alpha+1)}{\left(y-x\right)^\alpha} \left[\mathcal{RL}^{\alpha}_{(a+b-y)^+} f(a+b-x) + \mathcal{RL}^{\alpha}_{(a+b-x)^-} f(a+b-y)\right] \\ & \leq \frac{1}{2^{2s-1}} \left[f(a+b-x) + f(a+b-y)\right] \leq \frac{1}{2^{3s-3}} \left[f(a)+f(b)\right] - \frac{1}{2^{2s-1}} \left[f(x)+f(y)\right], \end{aligned} \tag{27}$$

- *Setting* $\alpha = 1$ *in* (27)*, we obtain inequalities via the classical Riemann integral operator*

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq & \frac{1}{2^{s-2}} \frac{1}{y-x} \int_{a+b-y}^{a+b-x} f(s)\, ds = \frac{1}{2^{s-2}} \frac{1}{y-x} \int_{x}^{y} f(a+b-t)\, dt \\ \leq & \frac{1}{2^{2s-1}} \left[f(a+b-x)+f(a+b-y)\right] \leq \frac{1}{2^{3s-3}} \left[f(a)+f(b)\right] - \frac{1}{2^{2s-1}} \left[f(x)+f(y)\right]. \end{aligned} \tag{28}$$

2) For convex functions:

- *Put* $s = 1$ *in* (26) *to get the Hermite-Hadamard-Mercer type inequality for convex functions, involving* ψ*-Hilfer fractional integral operators:*

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq & \frac{1}{4} \frac{\Gamma(\alpha+1)}{\left(\Delta_{\psi}(x,y)\right)^{\alpha}} \left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+} F(a+b-x) + {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-} F(a+b-y)\right] \\ \leq & \frac{1}{2}\left[f(a+b-x)+f(a+b-y)\right] \leq \left[f(a)+f(b)\right] - \frac{1}{2}\left[f(x)+f(y)\right], \end{aligned}$$

  for $x = a$ *and* $y = b$ *we get the result of [22, Corollary 3].*

- *Taking* $s = 1$ *in* (27) *gives the Theorem 1.8 proved in [14] and for* $x = a$ *and* $y = b$ *we get a result of Theorem 1.2 proved in [9].*

- *Setting* $s = 1$ *in* (28) *we get Theorem 1.7 proved in [13] and for* $x = a$ *and* $y = b$ *it is inequality* (2).

3) For P-convex functions : just put $s = 0$ *in* (26)*, in* (27) *and in* (28) *to get analog results for P-convex functions.*

Remark 3.4.
1) Choosing $\psi(t) = \ln t$ *in* (22) *or* (26) *leads to the Hermite-Hadamard-Mercer type inequalities via the Hadamard fractional integral operators for* $h-$*convex functions or* $s-$*convex functions, respectively.*
In the last choice, putting $s = 1$ *or* $s = 0$ *implies results for convex or P-convex functions.*
2) Similarly, doing with $\psi(\tau) = \frac{\tau^{\rho}}{\rho}$ $(\rho > 0)$*, we obtain analog results via the Katugompola fractional operators.*

3.2. Trapezoid-Mercer Type Inequality

Lemma 3.5. *If* $f \in X[a,b]$ *is a differentiable mapping on* (a,b) *such that* $a \leq x < y \leq b$*, then the following identity holds:*

$$\begin{aligned} & -2\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha} \left(f(a+b-x)+f(a+b-y)\right) + \Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+} F(a+b-x) + {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-} F(a+b-y)\right] \\ = & \frac{y-x}{2} \int_0^1 \Lambda(x,y,\tau)\Big[f'\left((a+b-y)(1-\tau)+(a+b-x)\tau\right) - f'\left((a+b-y)\tau+(a+b-x)(1-\tau)\right)\Big] d\tau. \end{aligned} \tag{29}$$

Here, $F(s) = f(s) + f\left(2(a+b)-x-y-s\right)$*,* $\Lambda(x,y,\tau)$ *is defined in* (16) *and* $\Delta_{\psi}(x,y)$ *in* (15).

Proof. Let $a \leq x < y \leq b$ and define

$$J_1 = \int_{a+b-y}^{a+b-x} \left[\frac{\left(\psi(a+b-x)-\psi(t)\right)^{\alpha}}{\alpha} - \frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\right] F'(t)dt. \tag{30}$$

Integrating by parts (30), we get

$$J_1 = -\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} F(a+b-x) + \Gamma(\alpha)\, {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+} F(a+b-x). \tag{31}$$

Similarly, let

$$J_2 = \int_{a+b-y}^{a+b-x} \left[\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} - \frac{(\psi(t)-\psi(a+b-y))^\alpha}{\alpha} \right] F'(t)dt. \tag{32}$$

Integrating by parts (32) , we obtain

$$J_2 = -\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} F(a+b-y) + \Gamma(\alpha)\, {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-} F(a+b-y). \tag{33}$$

Since $F(a+b-x) = F(a+b-y) = f(a+b-x) + f(a+b-y)$, from (31) and (33), we obtain

$$\begin{aligned} J_1 + J_2 &= -2\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} \left(f(a+b-x) + f(a+b-y) \right) \\ &+\Gamma(\alpha) \left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+} F(a+b-x) + {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-} F(a+b-y) \right]. \end{aligned} \tag{34}$$

On the other hand, from $F'(t) = f'(t) - f'(2(a+b)-x-y-t)$ we get by (30) and (32)

$$\begin{aligned} &J_1 + J_2 \\ &= \int_{a+b-y}^{a+b-x} \left[\frac{(\psi(a+b-x)-\psi(t))^\alpha}{\alpha} - \frac{(\psi(t)-\psi(a+b-y))^\alpha}{\alpha} \right] \left[f'(t) - f'(2(a+b)-x-y-t) \right] dt. \end{aligned} \tag{35}$$

By changing the variable $t = a+b-x-(y-x)\tau$ for $\tau \in [0,1]$ in (35), we obtain

$$\begin{aligned} &\frac{J_1 + J_2}{y-x} \\ &= \int_0^1 \left[\frac{(\psi(a+b-x)-\psi(a+b-x-(y-x)\tau))^\alpha}{\alpha} - \frac{(\psi(a+b-x-(y-x)\tau)-\psi(a+b-y))^\alpha}{\alpha} \right] \\ &\times \left[f'(a+b-x-(y-x)\tau) - f'(a+b-y+(y-x)\tau) \right] d\tau, \end{aligned} \tag{36}$$

and by changing the variable $t = a+b-y+(y-x)\tau$ in (35), we get

$$\begin{aligned} &\frac{J_1 + J_2}{y-x} \\ &= \int_0^1 \left[\frac{(\psi(a+b-x)-\psi(a+b-y+(y-x)\tau))^\alpha}{\alpha} - \frac{(\psi(a+b-y+(y-x)\tau)-\psi(a+b-y))^\alpha}{\alpha} \right] \\ &\times \left[f'(a+b-y+(y-x)\tau) - f'(a+b-x-(y-x)\tau) \right] d\tau. \end{aligned} \tag{37}$$

Adding (36) and (37), it results

$$\begin{aligned} J_1 + J_2 &= \frac{y-x}{2} \\ \times \int_0^1 &\left[\frac{(\psi(a+b-x)-\psi(a+b-y+(y-x)\tau))^\alpha}{\alpha} - \frac{(\psi(a+b-x)-\psi(a+b-x-(y-x)\tau))^\alpha}{\alpha} \right. \\ &\left. + \frac{(\psi(a+b-x-(y-x)\tau)-\psi(a+b-y))^\alpha}{\alpha} - \frac{(\psi(a+b-y+(y-x)\tau)-\psi(a+b-y))^\alpha}{\alpha} \right] \\ &\times \left[f'(a+b-y+(y-x)\tau) - f'(a+b-x-(y-x)\tau) \right] d\tau. \end{aligned} \tag{38}$$

The desired equality (29) is obtained by replacing (34) in (38). □

Theorem 3.6. *If $f \in X[a,b]$ is a differentiable mapping on (a,b) such that $|f'|$ is h-convex where h is a B-function then, for all $a \le x < y \le b$, the following trapezoid-Mercer type inequality is obtained as*

$$\begin{aligned}
&\left|2\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\left[f(a+b-x)+f(a+b-y)\right]-\Gamma(\alpha)\,\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}F(a+b-y)\right]\right|\\
&\le (y-x)\,h\left(\tfrac{1}{2}\right)\left[\left|f'\left(a+b-x\right)\right|+\left|f'\left(a+b-y\right)\right|\right]\\
&\times\left[\int_0^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^{1}\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right] \qquad (39)\\
&\le (y-x)\,h\left(\tfrac{1}{2}\right)\left[4\,h\left(\tfrac{1}{2}\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right]\\
&\times\left[\int_0^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^{1}\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right],
\end{aligned}$$

where $A(\tau)$, $B(\tau)$, $C(\tau)$ and $D(\tau)$ are defined in Lemma 2.2 and $\Delta_{\psi}(x,y)$ in (15).

Proof. Using the absolute value of identity (29), we get

$$\begin{aligned}
&\left|2\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\left[f(a+b-x)+f(a+b-y)\right]-\Gamma(\alpha)\,\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}F(a+b-y)\right]\right|\\
&\le \frac{y-x}{2}\\
&\times\int_0^1\left|\Lambda(x,y,\tau)\right|\left[\left|f'\left((a+b-y)(1-\tau)+(a+b-x)\tau\right)\right|+\left|f'\left((a+b-y)\tau+(a+b-x)(1-\tau)\right)\right|\right]d\tau.
\end{aligned}$$

By the h-convexity of the function $|f'|$, the B-function property (14) of h and then inequality (17), it results

$$\begin{aligned}
&\left|2\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\left[f(a+b-x)+f(a+b-y)\right]-\Gamma(\alpha)\,\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}F(a+b-y)\right]\right|\\
&\le (y-x)\,h\left(\tfrac{1}{2}\right)\left[\left|f'\left(a+b-x\right)\right|+\left|f'\left(a+b-y\right)\right|\right] \qquad (40)\\
&\times\left[\int_0^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^{1}\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right].
\end{aligned}$$

Then, using the $h-$Mercer inequality (12), one can deduce

$$\begin{aligned}
&(y-x)\,h\left(\tfrac{1}{2}\right)\left[\left|f'\left(a+b-x\right)\right|+\left|f'\left(a+b-y\right)\right|\right]\\
&\times\left[\int_0^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^{1}\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right]\\
&\le (y-x)\,h\left(\tfrac{1}{2}\right)\left[4\,h\left(\tfrac{1}{2}\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right] \qquad (41)\\
&\times\left[\int_0^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^{1}\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right].
\end{aligned}$$

Mombining the results (40) and (41), we obtain (39). □

Remark 3.7. *Considering some special types of convexity we have :*
1) For s-convex functions :

- *Take $h(t) = t^{s}$, $s \in (0,1)$ in* (39) *to get trapezoid-Mercer inequalities, involving the ψ-Hilfer fractional integral operators*

$$\begin{aligned}
&\left|2\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\Big[f(a+b-x)+f(a+b-y)\Big]-\Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}F(a+b-y)\right]\right|\\
&\leq (y-x)\,\frac{1}{2^{s}}\left[\left|f'\,(a+b-x)\right|+\left|f'\,(a+b-y)\right|\right]\\
&\times\left[\int_{0}^{\frac{1}{2}}\Big[A(\tau)-B(\tau)+C(\tau)-D(\tau)\Big]d\tau+\int_{\frac{1}{2}}^{1}\Big[B(\tau)-A(\tau)+D(\tau)-C(\tau)\Big]d\tau\right]\\
&\leq (y-x)\,\frac{1}{2^{s}}\left[\frac{1}{2^{s-2}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right]\\
&\times\left[\int_{0}^{\frac{1}{2}}\Big[A(\tau)-B(\tau)+C(\tau)-D(\tau)\Big]d\tau+\int_{\frac{1}{2}}^{1}\Big[B(\tau)-A(\tau)+D(\tau)-C(\tau)\Big]d\tau\right].
\end{aligned}\tag{42}$$

- *With $\psi(t)=t$ in* (42)*, we have the following inequalities via the Riemann-Liouville fractional integral operators*

$$\begin{aligned}
&\left|\frac{f(a+b-x)+f(a+b-y)}{2}-\frac{\Gamma(\alpha+1)}{2\,(y-x)^{\alpha}}\left[\mathcal{RL}^{\alpha}_{(a+b-y)^{+}}f(a+b-x)+\mathcal{RL}^{\alpha}_{(a+b-x)^{-}}f(a+b-y)\right]\right|\\
&\leq \frac{1}{2^{s}}\,\frac{(y-x)}{(\alpha+1)}\left[1-\frac{1}{2^{\alpha}}\right]\left[\left|f'\,(a+b-x)\right|+\left|f'\,(a+b-y)\right|\right]\\
&\leq \frac{1}{2^{s}}\,\frac{(y-x)}{(\alpha+1)}\left[1-\frac{1}{2^{\alpha}}\right]\left[\frac{1}{2^{s-2}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right].
\end{aligned}\tag{43}$$

- *Setting $\alpha=1$ in* (43)*, we obtain the following inequalities via the classical Riemann integral operator*

$$\begin{aligned}
&\left|\frac{f(a+b-x)+f(a+b-y)}{2}-\frac{1}{(y-x)}\int_{x}^{y}f(a+b-t)\,dt\right|\\
&\leq \frac{y-x}{2^{s+2}}\left[\left|f'\,(a+b-x)\right|+\left|f'\,(a+b-y)\right|\right]\leq\frac{y-x}{2^{s+2}}\left[\frac{1}{2^{s-2}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right].
\end{aligned}\tag{44}$$

2) For convex functions:

- *Put $s=1$ in* (42) *to get the trapezoid-Mercer inequality for convex functions involving ψ-Hilfer fractional integral operators*

$$\begin{aligned}
&\left|2\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\Big[f(a+b-x)+f(a+b-y)\Big]-\Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}F(a+b-y)\right]\right|\\
&\leq \frac{y-x}{2}\left[\left|f'\,(a+b-x)\right|+\left|f'\,(a+b-y)\right|\right]\\
&\times\left[\int_{0}^{\frac{1}{2}}\Big[A(\tau)-B(\tau)+C(\tau)-D(\tau)\Big]d\tau+\int_{\frac{1}{2}}^{1}\Big[B(\tau)-A(\tau)+D(\tau)-C(\tau)\Big]d\tau\right]\\
&\leq \frac{y-x}{2}\left[2\left(\left|f'(a)\right|+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right]\\
&\times\left[\int_{0}^{\frac{1}{2}}\Big[A(\tau)-B(\tau)+C(\tau)-D(\tau)\Big]d\tau+\int_{\frac{1}{2}}^{1}\Big[B(\tau)-A(\tau)+D(\tau)-C(\tau)\Big]d\tau\right].
\end{aligned}$$

- *setting $s=1$ in* (43)*, we obtain the trapezoid-Mercer inequality proved in [14].*

3) For P-convex functions: just put $s=0$ in (42), (43) *and* (44) *to get analog results for P-convex functions.*

Remark 3.8.
1) Choosing $\psi(t) = \ln t$ *in* (39) *or* (42) *leads to the trapezoid-Mercer inequalities via the Hadamard fractional integral operators for h-convex functions or s-convex functions, respectively.*
In the last choice, putting $s = 1$ *or* $s = 0$ *implies results for convex or P-convex functions.*
2) Similarly, doing with $\psi(\tau) = \frac{\tau^{\rho}}{\rho}$ $(\rho > 0)$*, we obtain analog results for Katugompola fractional operators.*

Setting $x = a$ and $y = b$ in (39), we get the following trapezoid inequality for the class of h-convex functions where h is a B-function, via the ψ-Hilfer fractional integral operators.

Corollary 3.9. *Under hypothesis of Theorem 39, we have*

$$\begin{aligned}
&\left|2\,\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}\,(f(b)+f(a)) - \Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a)^+}F(b) + {}^{\psi}\mathfrak{I}^{\alpha}_{(b)^-}F(a)\right]\right| \\
&\le (b-a)\,h\left(\tfrac{1}{2}\right)\left[\left|f'(b)\right| + \left|f'(a)\right|\right] \\
&\times\left[\int_0^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau + \int_{\frac{1}{2}}^{1}\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right] \\
&\le (b-a)\,h\left(\tfrac{1}{2}\right)\left[4\,h\left(\tfrac{1}{2}\right)-1\right]\left[\left|f'(a)\right| + \left|f'(b)\right|\right] \\
&\times\left[\int_0^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau + \int_{\frac{1}{2}}^{1}\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right],
\end{aligned} \tag{45}$$

where $F(s) = f(s) + f(a+b-s)$ *and* $A(\tau)$, $B(\tau)$, $C(\tau)$ *and* $D(\tau)$ *are defined in Lemma 2.2.*

Remark 3.10. *Considering different values of* $h(t)$ *and* $\psi(t)$ *we get specific trapezoid inequalities. For their expressions, just replace* $x = a$ *and* $y = b$ *in in Remark 3.7 and Remark 3.8.*
Specially, for $h(t) = t$ *and* $\psi(t) = t$ *we get Theorem 1.3 proved in [9] and if furthermore* $\alpha = 1$*, then it reduces to Theorem 2.2 proved in [5].*

Theorem 3.11. *Let* $p > 1$ *and* q *its dual number* $\left(\frac{1}{p}+\frac{1}{q}=1\right)$*. If* $f \in X[a,b]$ *is a differentiable mapping on* (a,b) *such that* $|f'|^p$ *is h-convex, where h is a B-function, then for all* $a \le x < y \le b$ *the following trapezoid-Mercer type inequalities hold:*

$$\begin{aligned}
&\left|2\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\left[f(a+b-x)+f(a+b-y)\right] - \Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+}F(a+b-x) + {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-}F(a+b-y)\right]\right| \\
&\le (y-x)\left(\int_0^1\left|\Lambda(x,y,\tau)\right|^q d\tau\right)^{\frac{1}{q}} h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left(\left|f'(a+b-y)\right|^p + \left|f'(a+b-x)\right|^p\right)^{\frac{1}{p}} \\
&\le (y-x)\left(\int_0^1\left|\Lambda(x,y,\tau)\right|^q d\tau\right)^{\frac{1}{q}} h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left[4\,h\left(\tfrac{1}{2}\right)\left(\left|f'(a)\right|^p + \left|f'(b)\right|^p\right) - \left|f'(x)\right|^p - \left|f'(y)\right|^p\right]^{\frac{1}{p}}.
\end{aligned} \tag{46}$$

Here, $\Lambda(x,y,\tau)$ *is defined in Lemma 2.2 and* $\Delta_{\psi}(x,y)$ *is defined in* (15).

Proof. Using the absolute value of identity (29) and Holder's inequality, we get

$$\begin{aligned}
&\left|2\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\left[f(a+b-x)+f(a+b-y)\right] - \Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+}F(a+b-x) + {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-}F(a+b-y)\right]\right| \\
&\qquad \le \frac{y-x}{2}\left(\int_0^1\left|\Lambda(x,y,\tau)\right|^q d\tau\right)^{\frac{1}{q}}\left(\int_0^1\left|f'\left((a+b-y)(1-\tau)+(a+b-x)\tau\right)\right|^p d\tau\right)^{\frac{1}{p}} \\
&\qquad + \frac{y-x}{2}\left(\int_0^1\left|\Lambda(x,y,\tau)\right|^q d\tau\right)^{\frac{1}{q}}\left(\int_0^1\left|f'\left((a+b-y)\tau+(a+b-x)(1-\tau)\right)\right|^p d\tau\right)^{\frac{1}{p}}.
\end{aligned}$$

By the h-convexity of the function $|f'|^p$, the property $A^{\frac{1}{p}}+B^{\frac{1}{p}}\leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$ with $A,B\geq 0$ and the B-function property (14) of h, it results

$$\left|2\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\left[f(a+b-x)+f(a+b-y)\right]-\Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-}F(a+b-y)\right]\right|$$
$$\leq (y-x)\left(\int_0^1\left|\Lambda(x,y,\tau)\right|^q d\tau\right)^{\frac{1}{q}} h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left(\left|f'(a+b-y)\right|^p+\left|f'(a+b-x)\right|^p\right)^{\frac{1}{p}}. \tag{47}$$

Using the h–Mercer inequality (12) applied to the function $|f'|^p$, we get

$$(y-x)\left(\int_0^1\left|\Lambda(x,y,\tau)\right|^q d\tau\right)^{\frac{1}{q}} h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left(\left|f'(a+b-y)\right|^p+\left|f'(a+b-x)\right|^p\right)^{\frac{1}{p}}$$
$$\leq (y-x)\left(\int_0^1\left|\Lambda(x,y,\tau)\right|^q d\tau\right)^{\frac{1}{q}} h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left[4h\left(\tfrac{1}{2}\right)\left(\left|f'(a)\right|^p+\left|f'(b)\right|^p\right)-\left|f'(x)\right|^p-\left|f'(y)\right|^p\right]^{\frac{1}{p}}. \tag{48}$$

Thus, the required result (46) is obtained by combining (47) and (48). □

Remark 3.12. *Depending on the choice of the functions h and ψ we obtain various results analogous to those in Remark 3.7. Especially, for $\psi(t)=t$ we get a trapezoid-Mercer type inequality involving the Riemann-Liouville integral operators on the class of h-convex functions :*

$$\left|\frac{f(a+b-x)+f(a+b-y)}{2}-\frac{\Gamma(\alpha+1)}{2\,(y-x)^\alpha}\left[\mathcal{RL}^{\alpha}_{(a+b-y)^+}f(a+b-x)+\mathcal{RL}^{\alpha}_{(a+b-x)^-}f(a+b-y)\right]\right|$$
$$\leq\frac{y-x}{2^{\frac{1}{p}}\,(\alpha q+1)^{\frac{1}{q}}}\left[1-\frac{1}{2^{\alpha q}}\right]^{\frac{1}{q}} h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left[\left|f'(a+b-y)\right|^p+\left|f'(a+b-x)\right|^p\right]^{\frac{1}{p}}$$
$$\leq\frac{y-x}{2^{\frac{1}{p}}\,(\alpha q+1)^{\frac{1}{q}}}\left[1-\frac{1}{2^{\alpha q}}\right]^{\frac{1}{q}} h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left[4h\left(\tfrac{1}{2}\right)\left(\left|f'(a)\right|^p+\left|f'(b)\right|^p\right)-\left|f'(x)\right|^p-\left|f'(y)\right|^p\right]^{\frac{1}{p}}.$$

Choosing $h(t)=t^s$ with $s\in(0,1)$, $s=1$ and $s=0$, we'll get a trapezoid-Mercer type inequality involving the Riemann-Liouville integral operators on the class of s-convex, convex and P-convex functions, respectively.

3.3. Midpoint Type Inequality

Lemma 3.13. *If $f:[a,b]\to\mathbb{R}$ is a differentiable mapping on (a,b), then the following identity holds for all $a\leq x<y\leq b$*

$$\begin{aligned}
&-4\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\,f\left(a+b-\frac{x+y}{2}\right)+\Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-}F(a+b-y)\right]\\
&=\frac{y-x}{2}\int_0^1\Lambda(x,y,\tau)\left[f'\left((a+b-y)(1-\tau)+(a+b-x)\tau\right)-f'\left((a+b-y)\tau+(a+b-x)(1-\tau)\right)\right]d\tau\\
&+(y-x)\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\\
&\times\left[-\int_0^{\frac{1}{2}}\left(f'\left((a+b-y)(1-\tau)+(a+b-x)\tau\right)-f'\left((a+b-y)\tau+(a+b-x)(1-\tau)\right)\right)dt\right.\\
&\left.+\int_{\frac{1}{2}}^1\left(f'\left((a+b-y)(1-\tau)+(a+b-x)\tau\right)-f'\left((a+b-y)\tau+(a+b-x)(1-\tau)\right)\right)dt\right],
\end{aligned}\tag{49}$$

where $F(s)=f(s)+f\left(2(a+b)-x-y-s\right)$, $\Lambda(x,y,\tau)$ is defined in Lemma 2.2 and $\Delta_\psi(x,y)$ is defined in (15).

Proof. Define

$$K_1 = \int_{a+b-y}^{a+b-x} \left[\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} - \frac{\left(\psi(a+b-x) - \psi(t)\right)^\alpha}{\alpha} \right] F'(t)\, dt - \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} \int_{a+b-\frac{x+y}{2}}^{a+b-x} F'(t)\, dt$$

for $a \le x < y \le b$. Integrating by parts, we get

$$K_1 = \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} F\left(a+b-\frac{x+y}{2}\right) - \Gamma(\alpha)\, {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+} F(a+b-x). \tag{50}$$

Similarly, let

$$K_2 = \int_{a+b-y}^{a+b-x} \left[\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} - \frac{\left(\psi(t) - \psi(a+b-y)\right)^\alpha}{\alpha} \right] F'(t)\, dt - \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} \int_{a+b-y}^{a+b-\frac{x+y}{2}} F'(t)\, dt.$$

Integrating by parts, we obtain

$$K_2 = -\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} F\left(a+b-\tfrac{x+y}{2}\right) + \Gamma(\alpha)\, {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-} F(a+b-y). \tag{51}$$

From (50) and (51) and noting that $F\left(a+b-\frac{x+y}{2}\right) = 2f\left(a+b-\frac{x+y}{2}\right)$, we deduce

$$K_2 - K_1 = -4\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} f\left(a+b-\tfrac{x+y}{2}\right) + \Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+} F(a+b-x) + {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-} F(a+b-y) \right]. \tag{52}$$

On the other hand, since $F'(t) = f'(t) - f'(2(a+b)-x-y-t)$, we have

$$\begin{aligned}
&K_2 - K_1 \\
&= \int_{a+b-y}^{a+b-x} \left[\frac{\left(\psi(a+b-x) - \psi(t)\right)^\alpha}{\alpha} - \frac{\left(\psi(t) - \psi(a+b-y)\right)^\alpha}{\alpha} \right] [f'(t) - f'(2(a+b)-x-y-t)]\, dt \\
&+ \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} \\
&\times \left[\int_{a+b-\frac{x+y}{2}}^{a+b-x} (f'(t) - f'(2(a+b)-x-y-t))\, dt - \int_{a+b-y}^{a+b-\frac{x+y}{2}} (f'(t) - f'(2(a+b)-x-y-t))\, dt \right].
\end{aligned} \tag{53}$$

By changing the variable $t = a+b-x-(y-x)\tau$ for $\tau \in [0,1]$ in (53), we obtain

$$\begin{aligned}
&\frac{K_2 - K_1}{y-x} \\
&= \int_0^1 \left[\frac{\left(\psi(a+b-x) - \psi(a+b-x-(y-x)\tau)\right)^\alpha}{\alpha} - \frac{\left(\psi(a+b-x-(y-x)\tau) - \psi(a+b-y)\right)^\alpha}{\alpha} \right] \\
&\times [f'(a+b-x-(y-x)\tau) - f'(a+b-y+(y-x)\tau)]\, d\tau \\
&+ \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} \left[\int_0^{\frac{1}{2}} (f'(a+b-x-(y-x)\tau) - f'(a+b-y+(y-x)\tau))\, d\tau \right. \\
&\left. - \int_{\frac{1}{2}}^1 (f'(a+b-x-(y-x)\tau) - f'(a+b-y+(y-x)\tau))\, d\tau \right].
\end{aligned} \tag{54}$$

Similarly, by changing the variable $t = a + b - y + (y - x)\tau$ for $\tau \in [0, 1]$ in (53), we obtain

$$
\begin{aligned}
&\frac{K_2 - K_1}{y - x} \\
&= \int_0^1 \left[\frac{\left(\psi(a+b-x) - \psi(a+b-y+(y-x)\tau)\right)^\alpha}{\alpha} - \frac{\left(\psi(a+b-y+(y-x)\tau) - \psi(a+b-y)\right)^\alpha}{\alpha} \right] \\
&\times \left[f'(a+b-y+(y-x)\tau) - f'(a+b-x-(y-x)\tau) \right] d\tau \\
&+ \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} \left[\int_{\frac{1}{2}}^1 \left(f'(a+b-y+(y-x)\tau) - f'(a+b-x-(y-x)\tau) \right) dt \right. \\
&\left. - \int_0^{\frac{1}{2}} \left(f'(a+b-y+(y-x)\tau) - f'(a+b-x-(y-x)\tau) \right) dt \right].
\end{aligned}
\tag{55}
$$

From (54) and (55), we get

$$
\begin{aligned}
K_2 - K_1 &= \frac{y-x}{2} \int_0^1 \Lambda(x,y,\tau) \left[f'(a+b-y+(y-x)\tau) - f'(a+b-x-(y-x)\tau) \right] d\tau \\
&+ (y-x) \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} \left[- \int_0^{\frac{1}{2}} \left(f'(a+b-y+(y-x)\tau) - f'(a+b-x-(y-x)\tau) \right) dt \right. \\
&\left. + \int_{\frac{1}{2}}^1 \left(f'(a+b-y+(y-x)\tau) - f'(a+b-x-(y-x)\tau) \right) dt \right].
\end{aligned}
\tag{56}
$$

Combining (56) and (52) it follows

$$
\begin{aligned}
&-4 \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} f\left(a+b-\tfrac{x+y}{2}\right) + \Gamma(\alpha) \left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+} F(a+b-x) + {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-} F(a+b-y) \right] \\
&= \frac{y-x}{2} \int_0^1 \Lambda(x,y,\tau) \left[f'(a+b-y+(y-x)\tau) - f'(a+b-x-(y-x)\tau) \right] d\tau \\
&+ (y-x) \frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha} \left[- \int_0^{\frac{1}{2}} \left(f'(a+b-y+(y-x)\tau) - f'(a+b-x-(y-x)\tau) \right) dt \right. \\
&\left. + \int_{\frac{1}{2}}^1 \left(f'(a+b-y+(y-x)\tau) - f'(a+b-x-(y-x)\tau) \right) dt \right],
\end{aligned}
$$

which gives the identity (49). □

Theorem 3.14. *If $f : [a, b] \to \mathbb{R}$ is a differentiable mapping on (a, b) and $|f'|$ is h-convex, then the midpoint-Mercer* *type inequality is obtained for $a \leq x < y \leq b$ as*

$$\begin{aligned}
&\left|4\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\, f\left(a+b-\frac{x+y}{2}\right)-\Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-}F(a+b-y)\right]\right|\\
&\leq (y-x)\,h\left(\tfrac{1}{2}\right)\left[\left|f'(a+b-x)\right|+\left|f'(a+b-y)\right|\right]\\
&\times\left[\int_0^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^1\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right]\\
&+(y-x)\,2h\left(\tfrac{1}{2}\right)\frac{\Delta_\psi(x,y)^\alpha}{\alpha}\left[\left|f'(a+b-x)\right|+\left|f'(a+b-y)\right|\right] \qquad (57)\\
&\leq (y-x)\,h\left(\tfrac{1}{2}\right)\left[4\,h\left(\tfrac{1}{2}\right)\left(\left|f'(a)+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right]\\
&\times\left[\int_0^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^1\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right]\\
&+(y-x)\,2h\left(\tfrac{1}{2}\right)\frac{\Delta_\psi(x,y)^\alpha}{\alpha}\left[4\,h\left(\tfrac{1}{2}\right)\left(\left|f'(a)+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right].
\end{aligned}$$

Here, $A(\tau)$, $B(\tau)$, $C(\tau)$ and $D(\tau)$ are defined in Lemma 2.2 and $\Delta_\psi(x,y)$ is defined in (15).

Proof. Using the absolute value of identity (49), we get

$$\begin{aligned}
&\left|4\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\, f\left(a+b-\frac{x+y}{2}\right)-\Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-}F(a+b-y)\right]\right|\\
&\leq \frac{y-x}{2}\int_0^1\left|\Lambda(x,y,\tau)\right|\\
&\times\left[\left|f'((a+b-y)(1-\tau)+(a+b-x)\tau)\right|+\left|f'((a+b-y)\tau+(a+b-x)(1-\tau))\right|\right]d\tau\\
&+(y-x)\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\times\\
&\times\left[\int_0^{\frac{1}{2}}\left(\left|f'((a+b-y)(1-\tau)+(a+b-x)\tau)\right|+\left|f'((a+b-y)\tau+(a+b-x)(1-\tau))\right|\right)dt\right.\\
&\left.+\int_{\frac{1}{2}}^1\left(\left|f'((a+b-y)(1-\tau)+(a+b-x)\tau)\right|+\left|f'((a+b-y)\tau+(a+b-x)(1-\tau))\right|\right)dt\right].
\end{aligned}$$

By the h-convexity of the function $|f'|$, the B-function property (14) of h and the inequality (17), we have

$$\begin{aligned}
&\left|4\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\, f\left(a+b-\frac{x+y}{2}\right)-\Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-}F(a+b-y)\right]\right|\\
&\leq (y-x)\,h\left(\tfrac{1}{2}\right)\left[\left|f'(a+b-x)\right|+\left|f'(a+b-y)\right|\right]\\
&\left[\int_0^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^1\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right]\\
&+(y-x)\,2h\left(\tfrac{1}{2}\right)\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\left[\left|f'(a+b-x)\right|+\left|f'(a+b-y)\right|\right].
\end{aligned}$$

The result (57) is obtained by using the h-Mercer inequality (12). □

Remark 3.15. *Consider some special types of convexity :*
1) For s-convex functions:

- *Take* $h(t) = t^s$, $s \in (0,1)$ *in* (57) *to get midpoint-Mercer inequalities involving the* ψ*-Hilfer fractional integral operators*

$$
\begin{aligned}
&\left|4\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\,f\left(a+b-\frac{x+y}{2}\right)-\Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\,\alpha}_{(a+b-y)^{+}}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\,\alpha}_{(a+b-x)^{-}}F(a+b-y)\right]\right|\\
&\le \frac{y-x}{2^{s}}\left[\left|f'\left(a+b-x\right)\right|+\left|f'\left(a+b-y\right)\right|\right]\\
&\times\left[\int_{0}^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^{1}\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right]\\
&+\frac{y-x}{2^{s-1}}\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\left[\left|f'\left(a+b-x\right)\right|+\left|f'\left(a+b-y\right)\right|\right]\\
&\le \frac{y-x}{2^{s}}\left[\frac{1}{2^{s-2}}\left(\left|f'(a)+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right]\\
&\times\left[\int_{0}^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^{1}\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right]\\
&+\frac{y-x}{2^{s-1}}\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\left[\frac{1}{2^{s-2}}\left(\left|f'(a)+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right].
\end{aligned}
\tag{58}
$$

- *With* $\psi(t) = t$ *in* (58) *we have the following inequalities via the Riemann-Liouville fractional integral operators*

$$
\begin{aligned}
&\left|f\left(a+b-\frac{x+y}{2}\right)-\frac{\Gamma(\alpha+1)}{2\left(y-x\right)^{\alpha}}\left[\mathcal{RL}^{\,\alpha}_{(a+b-y)^{+}}f(a+b-x)+\mathcal{RL}^{\,\alpha}_{(a+b-x)^{-}}f(a+b-y)\right]\right|\\
&\le \frac{y-x}{2^{s+1}(\alpha+1)}\left[\alpha\,+3-\frac{1}{2^{\alpha}-1}\right]\left[\left|f'\left(a+b-x\right)\right|+\left|f'\left(a+b-y\right)\right|\right]\\
&\le \frac{y-x}{2^{s+1}(\alpha+1)}\left[\alpha\,+3-\frac{1}{2^{\alpha}-1}\right]\left[\frac{1}{2^{s-2}}\left(\left|f'(a)+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right].
\end{aligned}
\tag{59}
$$

- *Setting* $\alpha = 1$ *in* (59), *we obtain the following inequalities via the classical Riemann integral*

$$
\begin{aligned}
&\left|f\left(a+b-\frac{x+y}{2}\right)-\frac{1}{y-x}\int_{x}^{y}f(a+b-t)\,dt\right|\\
&\le 3\,\frac{y-x}{2^{s+2}}\left[\left|f'\left(a+b-x\right)\right|+\left|f'\left(a+b-y\right)\right|\right]\\
&\le 3\,\frac{y-x}{2^{s+2}}\left[\frac{1}{2^{s-2}}\left(\left|f'(a)+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right].
\end{aligned}
\tag{60}
$$

2) For convex functions:

- *Put* $s = 1$ *in* (58) *to get the midpoint-Mercer inequality for convex functions via the* ψ*-Hilfer fractional integral* *operators :*

$$\left|4\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\,f\left(a+b-\frac{x+y}{2}\right)-\Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\,\alpha}_{(a+b-y)^+}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\,\alpha}_{(a+b-x)^-}F(a+b-y)\right]\right|$$
$$\leq \frac{y-x}{2}\left[\left|f'\left(a+b-x\right)\right|+\left|f'\left(a+b-y\right)\right|\right]$$
$$\times\left[\int_0^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^1\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right]$$
$$+(y-x)\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\left[\left|f'\left(a+b-x\right)\right|+\left|f'\left(a+b-y\right)\right|\right]$$
$$\leq \frac{y-x}{2}\left[2\left(\left|f'(a)+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right]$$
$$\times\left[\int_0^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^1\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right]$$
$$+(y-x)\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\left[2\left(\left|f'(a)+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right].$$

- *Taking $s=1$ in* (59) *gives a midpoint-Mercer inequality using the Riemann-Liouville operators*

$$\begin{aligned}&\left|f\left(a+b-\frac{x+y}{2}\right)-\frac{\Gamma(\alpha+1)}{2\left(y-x\right)^\alpha}\left[\mathcal{RL}^{\,\alpha}_{(a+b-y)^+}f(a+b-x)+\mathcal{RL}^{\,\alpha}_{(a+b-x)^-}f(a+b-y)\right]\right|\\&\leq \frac{y-x}{4\,(\alpha+1)}\left[\alpha+3-\frac{1}{2^{\alpha-1}}\right]\left[\left|f'\left(a+b-x\right)\right|+\left|f'\left(a+b-y\right)\right|\right]\\&\leq \frac{y-x}{4\,(\alpha+1)}\left[\alpha+3-\frac{1}{2^{\alpha-1}}\right]\left[2\left(\left|f'(a)+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right].\end{aligned}\tag{61}$$

- *Setting $s=1$ in* (60)*, it results a midpoint-Mercer inequality using the classical Riemann integral*

$$\begin{aligned}&\left|f\left(a+b-\frac{x+y}{2}\right)-\frac{1}{y-x}\int_x^y f(a+b-t)\,dt\right|\\&\leq 3\,\frac{y-x}{8}\left[\left|f'\left(a+b-x\right)\right|+\left|f'\left(a+b-y\right)\right|\right]\\&\leq 3\,\frac{y-x}{8}\left[2\left(\left|f'(a)+\left|f'(b)\right|\right)-\left|f'(x)\right|-\left|f'(y)\right|\right].\end{aligned}\tag{62}$$

3) For P-convex functions : just put $s=0$ in (58)*, in* (59) *and in* (60) *to get results for P-convex functions.*

Remark 3.16.
1) Choosing $\psi(t)=\ln t$ in (57) *or* (58)*, leads to the midpoint-Mercer inequalities via the ψ-Hilfer fractional integral operators for h-convex functions or s-convex functions, respectively.*
In the last choice, putting $s=1$ or $s=0$ implies results for convex or P-convex functions.
2) Similarly, doing with $\psi(\tau)=\frac{\tau^\rho}{\rho}$ $(\rho>0)$, we obtain midpoint-Mercer inequalities for Katugompola fractional operators.

Setting $x=a$ and $y=b$ in Theorem 3.14, we get the following midpoint inequality for the class of h-convex functions where h is a B-function.

Corollary 3.17. *Under hypothesis of Theorem 3.14, we have*

$$\begin{aligned}
&\left|4\,\frac{\left(\psi(b)-\psi(a)\right)^{\alpha}}{\alpha}\,f\left(\frac{a+b}{2}\right)-\Gamma(\alpha)\,\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a)^{+}}F(b)+{}^{\psi}\mathfrak{I}^{\alpha}_{(b)^{-}}F(a)\right]\right| \\
&\le (b-a)\,h\left(\tfrac{1}{2}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \\
&\times\left[\int_{0}^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^{1}\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right] \\
&+(b-a)\,2h\left(\tfrac{1}{2}\right)\,\frac{\psi(b)-\psi(a))^{\alpha}}{\alpha}\,\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \qquad (63)\\
&\le (b-a)\,h\left(\tfrac{1}{2}\right)\left(4\,h\left(\tfrac{1}{2}\right)-1\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \\
&\times\left[\int_{0}^{\frac{1}{2}}\left[A(\tau)-B(\tau)+C(\tau)-D(\tau)\right]d\tau+\int_{\frac{1}{2}}^{1}\left[B(\tau)-A(\tau)+D(\tau)-C(\tau)\right]d\tau\right] \\
&+(b-a)\,2h\left(\tfrac{1}{2}\right)\,\frac{\psi(b)-\psi(a))^{\alpha}}{\alpha}\,\left(4\,h\left(\tfrac{1}{2}\right)-1\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right],
\end{aligned}$$

where $F(s)=f(s)+f(a+b-s)$ *and* $A(\tau)$, $B(\tau)$, $C(\tau)$ *and* $D(\tau)$ *are defined in Lemma 2.2.*

Remark 3.18. *Considering specific values of* $h(t)$ *and* $\psi(t)$*, we get variant midpoint inequalities. For their expressions, replace* $x=a$ *and* $y=b$ *in Remark 3.15 and Remark 3.16. Specially, taking* $h(t)=t$ *and* $\psi(t)=t$ *we get Theorem 1.4 proved in [10].*

Theorem 3.19. *Let* $p>1$ *and* q *its dual number* $\left(\frac{1}{p}+\frac{1}{q}=1\right)$. *If* $f:[a,b]\to\mathbb{R}$ *is a differentiable mapping on* (a,b) *and* $|f'|^{p}$ *is h-convex, then we have the the following midpoint-Mercer type inequalities*

$$\begin{aligned}
&\left|4\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\,f\left(a+b-\frac{x+y}{2}\right)-\Gamma(\alpha)\,\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}F(a+b-y)\right]\right| \\
&\le (y-x)\left[\left(\int_{0}^{1}\left|\Lambda(x,y,\tau)\right|^{q}d\tau\right)^{\frac{1}{q}}+2\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\right]h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left(\left|f'(a+b-y)\right|^{p}+\left|f'(a+b-x)\right|^{p}\right)^{\frac{1}{p}} \qquad (64)\\
&\le (y-x)\left[\left(\int_{0}^{1}\left|\Lambda(x,y,\tau)\right|^{q}d\tau\right)^{\frac{1}{q}}+2\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\right]h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left[4h\left(\tfrac{1}{2}\right)\left(\left|f'(a)\right|^{p}+\left|f'(b)\right|^{p}\right)-\left|f'(x)\right|^{p}-\left|f'(y)\right|^{p}\right]^{\frac{1}{p}}
\end{aligned}$$

for $a\le x<y\le b$. *Here,* $A(\tau)$, $B(\tau)$, $C(\tau)$ *and* $D(\tau)$ *are defined in Lemma 2.2 and* $\Delta_{\psi}(x,y)$ *in* (15).

Proof. Using the absolute value of identity (49) and Holder's inequality, we get

$$\begin{aligned}
&\left|4\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\,f\left(a+b-\frac{x+y}{2}\right)-\Gamma(\alpha)\,\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}F(a+b-x)+{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}F(a+b-y)\right]\right| \\
&\le \frac{y-x}{2}\left(\int_{0}^{1}\left|\Lambda(x,y,\tau)\right|^{q}d\tau\right)^{\frac{1}{q}}\left(\int_{0}^{1}\left|f'\left((a+b-y)(1-\tau)+(a+b-x)\tau\right)\right|^{p}d\tau\right)^{\frac{1}{p}} \\
&+\frac{y-x}{2}\left(\int_{0}^{1}\left|\Lambda(x,y,\tau)\right|^{q}d\tau\right)^{\frac{1}{q}}\left(\int_{0}^{1}\left|f'\left((a+b-y)\tau+(a+b-x)(1-\tau)\right)\right|^{p}d\tau\right)^{\frac{1}{p}} \\
&+(y-x)\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\left(\int_{0}^{1}\left|f'\left((a+b-y)(1-\tau)+(a+b-x)\tau\right)\right|^{p}d\tau\right)^{\frac{1}{p}} \\
&+(y-x)\,\frac{\left(\Delta_{\psi}(x,y)\right)^{\alpha}}{\alpha}\left(\int_{0}^{1}\left|f'\left((a+b-y)\tau+(a+b-x)(1-\tau)\right)\right|^{p}d\tau\right)^{\frac{1}{p}},
\end{aligned}$$

by the h-convexity of the function $|f'|^p$, the property $A^{\frac{1}{p}} + B^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$ and the B-function property (14) of h, it results

$$\begin{aligned} &\left|4\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\, f\left(a+b-\frac{x+y}{2}\right) - \Gamma(\alpha)\left[{}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-y)^+}F(a+b-x) + {}^{\psi}\mathfrak{I}^{\alpha}_{(a+b-x)^-}F(a+b-y)\right]\right| \\ &\leq (y-x)\left[\left(\int_0^1 \left|\Lambda(x,y,\tau)\right|^q d\tau\right)^{\frac{1}{q}} + 2\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\right] h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left(\left|f'(a+b-y)\right|^p + \left|f'(a+b-x)\right|^p\right)^{\frac{1}{p}}. \end{aligned} \tag{65}$$

Using the $h-$Mercer inequality (12) applied to the function $|f'|^p$, we have

$$\begin{aligned} &(y-x)\left[\left(\int_0^1 \left|\Lambda(x,y,\tau)\right|^q d\tau\right)^{\frac{1}{q}} + 2\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\right] h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left(\left|f'(a+b-y)\right|^p + \left|f'(a+b-x)\right|^p\right)^{\frac{1}{p}} \\ &\leq (y-x)\left[\left(\int_0^1 \left|\Lambda(x,y,\tau)\right|^q d\tau\right)^{\frac{1}{q}} + 2\,\frac{\left(\Delta_\psi(x,y)\right)^\alpha}{\alpha}\right] h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left[4h\left(\tfrac{1}{2}\right)\left(\left|f'(a)\right|^p + \left|f'(b)\right|^p\right) - \left|f'(x)\right|^p - \left|f'(y)\right|^p\right]^{\frac{1}{p}}. \end{aligned} \tag{66}$$

Combining (65) and (66), we obtain (64). □

Remark 3.20. *Depending on the choice of the functions h and ψ we obtain various results analogous to those in Remark 3.15 and Remark 3.16.*
Especially, for $\psi(t) = t$ we get a midpoint-Mercer type inequality involving the Riemann-Liouville integral operators on the class of $h-$convex functions

$$\begin{aligned} &\left|f\left(a+b-\frac{x+y}{2}\right) - \frac{\Gamma(\alpha+1)}{2\left(y-x\right)^\alpha}\left[\mathcal{RL}^{\alpha}_{(a+b-y)^+}f(a+b-x) + \mathcal{RL}^{\alpha}_{(a+b-x)^-}f(a+b-y)\right]\right| \\ &\leq \frac{y-x}{2}\left[\left(\frac{2}{\alpha q+1}\right)^{\frac{1}{q}}\left(1-\frac{1}{2^{\alpha q}}\right)^{\frac{1}{q}} + 1\right] h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left(\left|f'(a+b-y)\right|^p + \left|f'(a+b-x)\right|^p\right)^{\frac{1}{p}} \\ &\leq \frac{y-x}{2}\left[\left(\frac{2}{\alpha q+1}\right)^{\frac{1}{q}}\left(1-\frac{1}{2^{\alpha q}}\right)^{\frac{1}{q}} + 1\right] h^{\frac{1}{p}}\left(\tfrac{1}{2}\right)\left[4h\left(\tfrac{1}{2}\right)\left(\left|f'(a)\right|^p + \left|f'(b)\right|^p\right) - \left|f'(x)\right|^p - \left|f'(y)\right|^p\right]^{\frac{1}{p}}, \end{aligned}$$

Choosing $h(t) = t^s$ with $s \in (0,1)$, $s = 1$ and $s = 0$ we'll get a midpoint-Mercer type inequality via the Riemann-Liouville integral operators on the class of respectively s-convex functions, convex functions and P-convex functions, respectively.

Taking $p = q = 1$, $x = a$, $y = b$, $h(t) = t$ and $\psi(t) = t$ we get Theorem 1.4 proved in [10].

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