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Ma-Minda convexity and starlikeness for certain subclasses of the close-to-starlike functions

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Abstract. An analytic function $f(z) = z + a_2z^2 + \cdots$ defined on the unit disc $\mathbb{D}$ is close-to-starlike if there exists a starlike function $g : \mathbb{D} \to \mathbb{C}$ satisfying the inequality $\operatorname{Re}(f(z)/g(z)) > 0$ for all $z \in \mathbb{D}$. We are particularly interested in the starlikeness of the class $\mathcal{W}$, which consists of all functions that satisfy the close-to-starlike condition with $g(z) \equiv z$ and the subclass $\mathcal{W}_n$ of $\mathcal{W}$, which contains all those functions of the form $f(z) = z + a_{n+1}z^{n+1} + \cdots$. The usual starlikeness of an analytic function f requires that the range of $zf'(z)/f(z)$ is contained in the right half-plane. More generally, a normalized analytic function $f : \mathbb{D} \to \mathbb{C}$ is Ma-Minda starlike if the function zf'/f is subordinate to the function φ and Ma-Minda convex if the function $1 + zf''/f'$ is subordinate to the function φ. We have determined the sharp radius of Ma-Minda convexity/starlikeness of the class $\mathcal{W}$ and $\mathcal{W}_n$ when the range of φ is a nephroid, lune, lemniscate of Bernoulli, cardioid, or, a particular rational function.

1. Introduction and preliminaries

Let $\mathcal{A}_n$ be the class of all analytic functions of the form $f(z) = z + a_{n+1}z^{n+1} + \cdots$ defined on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A} := \mathcal{A}_1$. The subclass of $\mathcal{A}$ consisting of univalent (one-to-one) analytic functions is denoted by $\mathcal{S}$. A function $f \in \mathcal{A}$ is said to be starlike if $f(\mathbb{D})$ is starlike with respect to the origin and the class of all starlike functions is denoted by $\mathcal{ST}$. Similarly, a function is convex if $f(\mathbb{D})$ is convex and the class of all convex functions is denoted by $\mathcal{CV}$. Reade [18] introduced the class of close-to-starlike functions which consists of functions $f \in \mathcal{A}$ that satisfies the condition $\operatorname{Re}(f(z)/s(z)) > 0$ holds in $\mathbb{D}$ where the function s is univalent and starlike with respect to the origin in the unit disc $\mathbb{D}$. The class of close-to-starlike functions is denoted by $\mathcal{CS}$. Reade, Ogawa and Sakaguchi [19] studied the subclass $\mathcal{W}$ of the close to starlike functions $f \in \mathcal{A}$ that satisfies the condition $\operatorname{Re}(f(z)/z) > 0$ for $z \in \mathbb{D}$. We let $\mathcal{W}_n := \mathcal{A}_n \cap \mathcal{W}$. For two subfamilies $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$, the $\mathcal{G}$-radius for the class $\mathcal{F}$, denoted by $\mathcal{R}_{\mathcal{G}}(\mathcal{F})$, is the maximum value $R \in (0, 1]$ such that $r^{-1}f(rz) \in \mathcal{G}$ holds for all $f \in \mathcal{F}$ and for $0 < r < R$. MacGregor [13] has shown that the

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radius of starlikeness of the class $\mathcal{W}$ is $\sqrt{2}-1$. Reade, Ogawa and Sakaguchi [19] proved that the radius of convexity for these functions is $r_0 = 0.179\cdots$. In this article, we find the radii of Ma-Minda convexity and starlikeness of the classes $\mathcal{W}$ and $\mathcal{W}_n$.

To define Ma-Minda convexity/starlikeness, we need the concept of subordination, a notion that generalizes the inequality in the real line to complex plane. Let f and g be two analytic function defined on unit disc $\mathbb{D}$. The function f is subordinate to the function g, denoted by $f \prec g$, if there exists an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ such that $f(z) = g(w(z))$. Clearly, $f \prec g$ implies that $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. If the superordinate function g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Analytically, a normalized analytic function f is starlike if $zf'(z)/f(z) \prec (1+z)/(1-z)$ and it is convex if $1 + zf''(z)/f'(z) \prec (1+z)/(1-z)$. For an analytic function $\varphi : \mathbb{D} \to \mathbb{C}$, define the classes $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$ by

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \quad \text{and} \quad \mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

Ma and Minda [12] investigated growth and distortion inequalities for these general classes of starlike and convex functions when the function $\varphi : \mathbb{D} \to \mathbb{C}$ is an analytic univalent function with a positive real part whose range is symmetric about the real axis and starlike with respect to the origin and $\varphi'(0) > 0$. When $\varphi(z) = (1 + Az)/(1 + Bz)$, where $-1 \leqslant B < A \leqslant 1$, the classes $\mathcal{CV}(\varphi)$ and $\mathcal{ST}(\varphi)$ are respectively denoted as $\mathcal{CV}[A, B]$ and $\mathcal{ST}[A, B]$ and are called as the Janowski convex functions and Janowski starlike functions [9]. For $A = 1 - 2\alpha$ and $B = -1$, the classes $\mathcal{CV}[A, B]$ and $\mathcal{ST}[A, B]$ reduces to the class of convex functions of order α and the class of starlike functions of order α, respectively denoted by $\mathcal{CV}(\alpha)$ and $\mathcal{ST}(\alpha)$.

2. Radius of Convexity

For a function φ with positive real part, we have $\mathcal{CV}(\varphi) \subset \mathcal{CV}$. Reade, Ogawa and Sakaguchi [19] proved that the radius of convexity for these functions is $r_0 = 0.179\cdots$ and, therefore, $\mathcal{CV}(\varphi)$ radius of the class $\mathcal{W}$ is at most $r_0 = 0.179\cdots < \sqrt{2}-1$. In this section, several Ma-Minda convexity radii for the class $\mathcal{W}$ are obtained. We first begin with the class $\mathcal{CV}_C := \mathcal{CV}(\varphi_C)$ where $\varphi_C(z) = 1 + (4/3)z + (2/3)z^2$.

Theorem 2.1. *The $\mathcal{CV}_C$ radius of the class $\mathcal{W}$ is $\rho_1 \approx 0.1347\cdots$, where ρ_1 is the root of the equation:*

$$r^4 + 2r^3 + 4r^2 - 8r + 1 = 0.$$

Proof. We first obtain the disc in which the values of $1 + zf''(z)/f'(z)$ lies for a function $f \in \mathcal{W}$. This was already obtained in [19] and we include it here for the sake of completeness. We first define a function g by

$$g(z) = \left(f\left(\frac{z+a}{1+\bar{\alpha}z}\right) \Big/ \frac{z+\alpha}{1+\bar{\alpha}z} \right) \frac{(z+\alpha)(1+\bar{\alpha}z)}{z},$$

where $|\alpha| < 1$. Since $(z+\alpha)(1+\bar{\alpha}z)/z$ is real and positive for $|z| = 1$, it is easy to see that $\operatorname{Re}(g(z)) > 0$ for $|z| = 1$ and $g(z) = A_{-1}/z + A_0 + A_1 z + \cdots$, where

$$\begin{aligned} A_{-1} &= f(\alpha) \\ A_0 &= (1-|\alpha|^2)f'(\alpha) + 2\bar{\alpha}f(\alpha) \\ A_1 &= \frac{(1-|\alpha|^2)^2}{2}f''(\alpha) + \bar{\alpha}(1-|\alpha|^2)f'(\alpha) + \bar{\alpha}^2 f(\alpha). \end{aligned}$$

Robertson [20] (see [7, Exercise 17, p. 102]) proved that

$$|A_n + \bar{A}_{-n}| \leqslant 2\operatorname{Re}(A_0), \tag{2.1}$$

for a function $M(z) = \sum_{-p}^{\infty} A_k z^k$ analytic in $\mathbb{D}$ with a pole of order not exceeding p at the origin satisfying the condition $\operatorname{Re} M(re^{i\theta}) > 0$, for $0 < 1 - \delta < r < 1$. Using (2.1) for the function g with $n = 1$ and replacing α by z, we obtain

$$\left|\frac{(1-|z|^2)^2}{2} f''(z) + \bar{z}(1-|z|^2)f'(z)\right| \leqslant 2(1-|z|^2)|f'(z)| + (1+4|z|+|z|^2)|f(z)|,$$

or, equivalently, with $|z| = r$,

$$\left|1 + \frac{zf''(z)}{f(z)} - \frac{1-3r^2}{1-r^2}\right| \leqslant \frac{4r}{1-r^2} + \frac{r^2+4r^3+r^4}{(1-r^2)^2}\frac{|f(z)|}{|zf'(z)|}. \tag{2.2}$$

For $f \in \mathcal{A}$, if $\operatorname{Re}(f(z)/z) > 0$ in $\mathbb{D}$, then the function $p : \mathbb{D} \to \mathbb{C}$ defined by $p(z) = f(z)/z$ is a function with positive real part. Using the inequality $|zp'(z)/p(z)| \leqslant 2r/(1-r^2)$ for a function with positive real part, we get

$$\left|\frac{zf'(z)}{f(z)}\right| = \left|1 + \frac{zp'(z)}{p(z)}\right| \geqslant 1 - \left|\frac{zp'(z)}{p(z)}\right| \geqslant 1 - \frac{2r}{1-r^2} = \frac{1-2r-r^2}{1-r^2}, \qquad |z| = r < \sqrt{2}-1, \tag{2.3}$$

holds in $\mathbb{D}$. Using the inequality (2.3) in (2.2), we get

$$\left|\frac{zf''(z)}{f'(z)} + \frac{2r^2}{1-r^2}\right| \leqslant \frac{4r-6r^2+4r^3+2r^4}{(1-r^2)(1-2r-r^2)} \qquad |z| = r < \sqrt{2}-1,$$

or equivalently,

$$\left|1 + \frac{zf''(z)}{f(z)} - \frac{1-3r^2}{1-r^2}\right| \leqslant \frac{4r-6r^2+4r^3+2r^4}{(1-r^2)(1-2r-r^2)}, \qquad |z| = r < \sqrt{2}-1. \tag{2.4}$$

From (2.4), it obvious that $w = 1 + (zf''(z)/f(z))$ resides within the disc $|w - a(r)| < r_1(r)$, where $a_1(r)$ and $r_1(r)$ denotes the centre and radius respectively:

$$a_1(r) = \frac{1-3r^2}{1-r^2} \quad \text{and} \quad r_1(r) = \frac{4r-6r^2+4r^3+2r^4}{(1-r^2)(1-2r-r^2)}. \tag{2.5}$$

Clearly, the centre $a_1(r)$ is a decreasing function of r and $r_1(r) < 1$. The function $\eta(r)$ defined by

$$\eta(r) = \frac{1-6r+2r^2+2r^3+r^4}{1-2r-2r^2+2r^3+r^4}, \qquad 0 \leqslant r < 1, \tag{2.6}$$

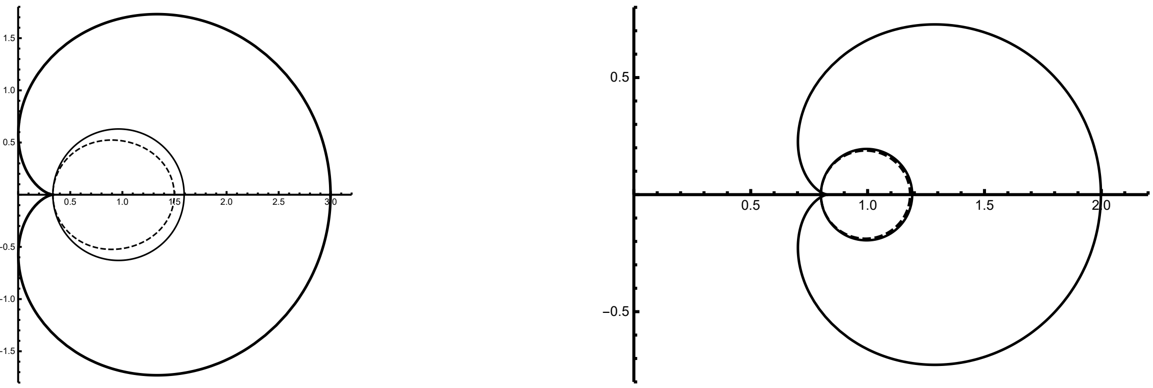
is a decreasing function of r. The number $\rho_1 \in (0,1]$ is a positive root of the equation $\eta(r) = 1/3$. For $0 < r \leqslant \rho_1$, we have $\eta(r) \geqslant 1/3$ or, $r_1(r) \leqslant a_1(r) - 1/3$. Sharma et al. [23] proved that $\{w \in \mathbb{C} : |w - a| < r_a\} \subset \Omega_C := \varphi_C(\mathbb{D})$ holds when

$$r_a = a - \frac{1}{3}, \tag{2.7}$$

for $1/3 < a \leqslant 5/3$. Therefore, the disc given in (2.4) is contained in the region bounded by cardioid using (2.7).

To show sharpness, we consider the function F defined by

$$F(z) = \frac{z(1+z)}{(1-z)}. \tag{2.8}$$



(a). ρ_1 (b). ρ_2

Figure 1: Sharpness of $\mathcal{CV}_C$ and $\mathcal{CV}_R$ radius for the class $\mathcal{W}$

Clearly, the function F belongs to the class $\mathcal{W}$. The function F plays the role of an extremal function for this class $\mathcal{W}$. For the function F given by (2.8), we have

$$1+\frac{zF''(z)}{F(z)}=\frac{1+5z-3z^2+z^3}{1+z-3z^2+z^3}. \tag{2.9}$$

At $z=-\rho_1$, we have

$$1+\frac{zF''(z)}{F(z)}=\frac{1+5(-\rho_1)-3(-\rho_1)^2+(-\rho_1)^3}{1+(-\rho_1)-3(-\rho_1)^2+(-\rho_1)^3}=\frac{1}{3}=\varphi_C(-1),$$

which proves the sharpness of the radius ρ_1 (See Fig.1(a)) (This figure, along with subsequent ones, illustrates the image of the extremal function F, the disc described by equation (2.4), and the boundary of $\varphi(\mathbb{D})$). □

The class $\mathcal{CV}_R := \mathcal{CV}(\varphi_R)$ where $\varphi_R(z)=1+((z^2+kz)/(k^2-kz))$ for $k=\sqrt{2}+1$.

Theorem 2.2. *The $\mathcal{CV}_R$ radius of the class $\mathcal{W}$ is $\rho_2\approx 0.0472\cdots$, where ρ_2 is the root of the equation:*

$$(1+2\sqrt{2})r^4+2(2\sqrt{2}-1)r^3+2(3-2\sqrt{2})r^2+2(2\sqrt{2}-1)r+3-2\sqrt{2}=0.$$

Proof. Consider the function η defined in (2.6) which is a decreasing function of r. The number $\rho_2\in(0,1]$ is a positive root of the equation $\eta(r)=2(\sqrt{2}-1)$. For $0<r\leqslant\rho_2$, we have $\eta(r)\geqslant 2(\sqrt{2}-1)$. That is $r_1(r)\leqslant a_1(r)-2(\sqrt{2}-1)$. Kumar and Ravichandran [11] proved that $\{w:|w-a|<r_a\}\subset\varphi_R(\mathbb{D})=:\Omega_R$ holds when

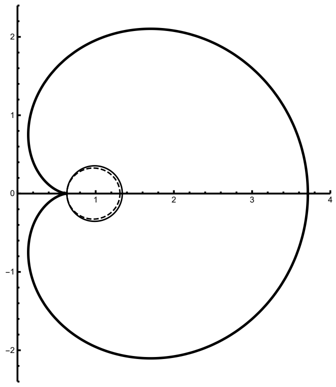
$$r_a=a-2(\sqrt{2}-1),\quad \text{for}\quad 2(\sqrt{2}-1)<a\leqslant\sqrt{2}. \tag{2.10}$$

Therefore, the disc given in (2.4) is contained in the region bounded by cardioid using (2.10). Consider the function F defined in (2.8). For $z=-\rho_2$ in (2.9), we have

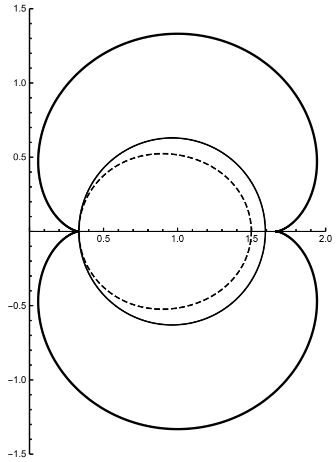
$$1+\frac{zF''(z)}{F(z)}=\frac{1+5(-\rho_2)-3(-\rho_2)^2+(-\rho_2)^3}{1+(-\rho_2)-3(-\rho_2)^2+(-\rho_2)^3}=2(\sqrt{2}-1)=\varphi_R(-1),$$

which proves the sharpness of the radius ρ_2 (See Fig.1(b)).
□

The class $\mathcal{CV}_\wp := \mathcal{CV}(\varphi_\wp)$ where $\varphi_\wp(z)=1+ze^z$ and the boundary of $\varphi_\wp(\mathbb{D})$ is a cardioid.



(a). ρ_3



(b). ρ_4

Figure 2: Sharpness of $\mathcal{CV}_\wp$ and $\mathcal{CV}_{Ne}$ radius for the class $\mathcal{W}$

Theorem 2.3. *The $\mathcal{CV}_\wp$ radius of the class $\mathcal{W}$ is $\rho_3 \approx 0.0825\cdots$, where ρ_3 is the root of the equation:*

$$r^4 + 2r^3 + (4e-2)r^2 - (4e+2)r + 1 = 0.$$

Proof. Consider the function η defined in (2.6) which is a decreasing function of r. The number $\rho_3 \in (0,1]$ is a positive root of the equation $\eta(r) = 1-(1/e)$. For $0 < r \leqslant \rho_3$, we have $\eta(r) \geqslant 1-(1/e)$. That is $r_1(r) \leqslant a_1(r) - 1 + (1/e)$. Recall that Kumar and Kamaljeet [10] proved that $\{w : |w-a| < r_a\} \subset \varphi_\wp(\mathbb{D}) =: \Omega_\wp$ holds when

$$r_a = (a-1) + \frac{1}{e}, \quad \text{for} \quad 1 - \frac{1}{e} < a \leqslant 1 + \frac{e-e^{-1}}{2}. \tag{2.11}$$

Therefore, the disc given in (2.4) is contained in the region $\Omega_\wp$ using (2.11). Consider the function F defined in (2.8). For $z = -\rho_3$ in (2.9), we have

$$1 + \frac{zF''(z)}{F(z)} = \frac{1+5(-\rho_3)-3(-\rho_3)^2+(-\rho_3)^3}{1+(-\rho_3)-3(-\rho_3)^2+(-\rho_3)^3} = 1 - \frac{1}{e} = \varphi_\wp(-1),$$

which proves the sharpness of the radius ρ_3 (See Fig.2(a)). □

The class $\mathcal{CV}_{Ne} := \mathcal{CV}(\varphi_{Ne})$ where $\varphi_{Ne}(z) = 1 + z + (z^3/3)$, and the boundary of $\varphi_{Ne}(\mathbb{D})$ is a nephroid.

Theorem 2.4. *The $\mathcal{CV}_{Ne}$ radius of the class $\mathcal{W}$ is $\rho_4 \approx 0.1347\cdots$, where ρ_4 is the root of the equation in r:*

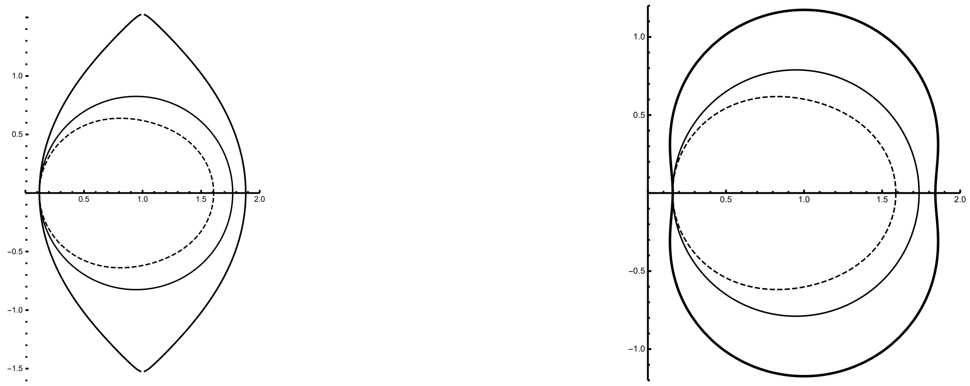
$$r^4 + 2r^3 + 4r^2 - 8r + 1 = 0.$$

Proof. Consider the function η defined in (2.6) which is a decreasing function of r. The number $\rho_4 \in (0,1]$ is a positive root of the equation $\eta(r) = 1/3$. For $0 < r \leqslant \rho_4$, we have $\eta(r) \geqslant 1/3$. That is $r_1(r) \leqslant a_1(r) - 1/3$. Wani and Swaminathan [26] studied the class $\mathcal{CV}_{Ne} = \mathcal{CV}(\varphi_{Ne})$, In [25], it has been proved that the inclusion $\{w : |w-a| < r_a\} \subset \varphi_{Ne}(\mathbb{D}) =: \Omega_{Ne}$ holds when

$$r_a = a - \frac{1}{3}, \quad \text{for} \quad \frac{1}{3} < a \leqslant 1. \tag{2.12}$$

Therefore, the disc given in (2.4) is contained in the region bounded by a nephroid using (2.12). Consider the function F defined in (2.8). For $z = -\rho_4$ in (2.9), we have

$$1 + \frac{zF''(z)}{F(z)} = \frac{1+5(-\rho_4)-3(-\rho_4)^2+(-\rho_4)^3}{1+(-\rho_4)-3(-\rho_4)^2+(-\rho_4)^3} = \frac{1}{3} = \varphi_{Ne}(-1),$$



(a). ρ_5 (b). ρ_6

Figure 3: Sharpness of $\mathcal{CV}_h$ and $\mathcal{CV}_{\sin}$ radius for the class $\mathcal{W}$

which proves the sharpness of the radius ρ_4 (See Fig.2(b)).
□

The class $\mathcal{CV}_h := \mathcal{CV}(\varphi_h)$ where $\varphi_h(z) = 1 + \sinh^{-1} z$ and the boundary of $\varphi_h(\mathbb{D})$ is a petal shaped domain.

Theorem 2.5. *The $\mathcal{CV}_h$ radius of the class $\mathcal{W}$ is $\rho_5 \approx 0.1650\cdots$, where ρ_5 is the root of the equation:*

$$(\sinh^{-1}(1))r^4 + 2(\sinh^1(1))r^3 - 2(\sinh^{-1}(1) - 2)r^2 - 2(\sinh^{-1}(1) + 2)r + \sinh^{-1}(1) = 0.$$

Proof. Consider the function η defined in (2.6) which is a decreasing function of r. The number $\rho_5 \in (0, 1]$ is a positive root of the equation $\eta(r) = 1 - \sinh^{-1}(1)$. For $0 < r \leqslant \rho_5$, we have $\eta(r) \geqslant 1 - \sinh^{-1}(1)$. That is $r_1(r) \leqslant a_1(r) - (1 - \sinh^{-1}(1))$. Kumar and Arora [2] proved that the inclusion $\{w : |w - a| < r_a\} \subset \varphi_h(\mathbb{D}) =: \Omega_h$ holds when

$$r_a = a - (1 - \sinh^{-1}(1)), \quad \text{for} \quad 1 - \sinh^{-1}(1) < a \leqslant 1. \tag{2.13}$$

Therefore, the disc given in (2.4) is contained in a petal shaped region using (2.13).

Consider the function F defined in (2.8). For $z = -\rho_5$ in (2.9), we have

$$1 + \frac{zF''(z)}{F(z)} = \frac{1 + 5(-\rho_5) - 3(-\rho_5)^2 + (-\rho_5)^3}{1 + (-\rho_5) - 3(-\rho_5)^2 + (-\rho_5)^3} = 1 - \sinh^{-1}(1) = \varphi_h(-1),$$

which proves the sharpness of the radius ρ_5 (See Fig.3(a)). □

The class $\mathcal{CV}_{\sin} := \mathcal{CV}(\varphi_{\sin})$ where $\varphi_{\sin}(z) = 1 + \sin z$.

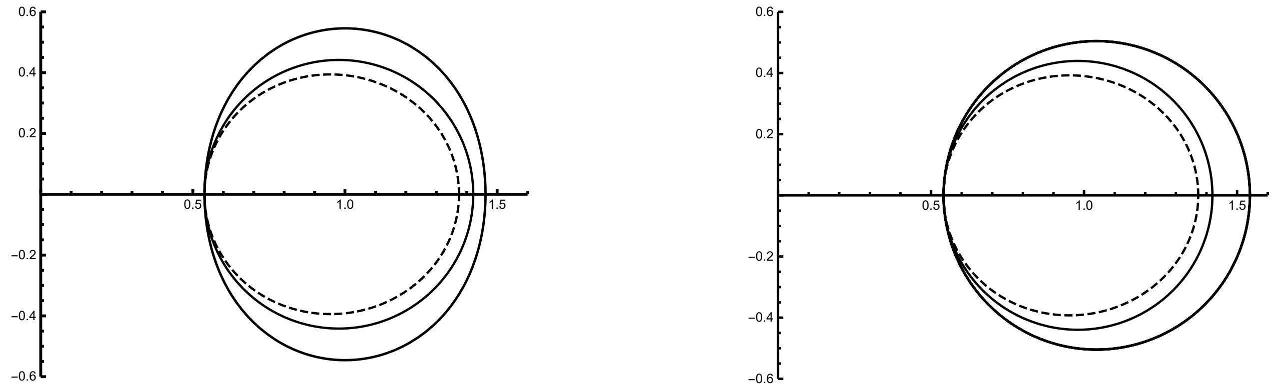
Theorem 2.6. *The $\mathcal{ST}_{\sin}$ radius of the class $\mathcal{W}$ is $\rho_6 \approx 0.1597\cdots$, where ρ_6 is the root of the equation in r:*

$$(\sin 1)r^4 + 2(\sin 1)r^3 - 2((\sin 1) - 2)r^2 - 2(\sin 1) + 2)r + \sin 1 = 0.$$

Proof. Consider the function η defined in (2.6) which is a decreasing function of r. The number $\rho_6 \in (0, 1]$ is a positive root of the equation $\eta(r) = 1 - \sin 1$. For $0 < r \leqslant \rho_6$, we have $\eta(r) \geqslant 1 - \sin 1$. That is $r_1(r) \leqslant a_1(r) - (1 - \sin 1)$. Cho et al. [3] proved that the inclusion $\{w : |w - a| < r_a\} \subset \varphi_{\sin}(\mathbb{D}) =: \Omega_{\sin}$ holds when

$$r_a = a - (1 - \sin 1), \quad \text{for} \quad 1 - \sin 1 < a \leqslant 1. \tag{2.14}$$

Therefore, the disc given in (2.4) is contained in the region $\Omega_{\sin}$ using (2.14).



(a). ρ_7 (b). ρ_8

Figure 4: Sharpness of $\mathcal{CV}_{SG}$ and $\mathcal{CV}_{\varrho}$ radius for the class $\mathcal{W}$

Consider the function F defined in (2.8). For $z = -\rho_6$ in (2.9), we have

$$1 + \frac{zF''(z)}{F(z)} = \frac{1 + 5(-\rho_6) - 3(-\rho_6)^2 + (-\rho_6)^3}{1 + (-\rho_6) - 3(-\rho_6)^2 + (-\rho_6)^3} = 1 - \sin 1 = \varphi_{sin}(-1),$$

which proves the sharpness of the radius ρ_6 (See Fig.3(b)). □

The class $\mathcal{CV}_{SG} := \mathcal{CV}(\varphi_{SG})$ where $\varphi_{SG}(z) = 2/(1 + e^{-z})$ and the boundary of $\varphi_{SG}(\mathbb{D})$ is a modified sigmoid.

Theorem 2.7. *The $\mathcal{CV}_{SG}$ radius of the class $\mathcal{W}$ is $\rho_7 \approx 0.1003\cdots$, where ρ_7 is the root of the equation:*

$$(1 - e)r^4 + 2(1 - e)r^3 - 2(3 + e)r^2 + 2(1 + 3e)r - e + 1 = 0.$$

Proof. Consider the function η defined in (2.6) which is a decreasing function of r. The number $\rho_7 \in (0, 1]$ is a positive root of the equation $\eta(r) = 2/(e + 1))$. For $0 < r \leqslant \rho_7$, we have $\eta(r) \geqslant 2/(e + 1))$. That is $r_1(r) \leqslant a_1(r) - (2/(e + 1))$. Goel and Kumar [6] proved that $\{w : |w - a| < r_a\} \subset \varphi_{SG}(\mathbb{D}) =: \Omega_{SG}$ holds when

$$r_a = a - \frac{2}{e + 1}, \quad \text{for} \quad \frac{2}{1 + e} < a \leqslant 1. \tag{2.15}$$

Therefore, the disc given in (2.4) is contained in the region bounded by a modified sigmoid using (2.15).

Consider the function F defined in (2.8). For $z = -\rho_7$ in (2.9), we have

$$1 + \frac{zF''(z)}{F(z)} = \frac{1 + 5(-\rho_7) - 3(-\rho_7)^2 + (-\rho_7)^3}{1 + (-\rho_7) - 3(-\rho_7)^2 + (-\rho_7)^3} = \frac{2}{e + 1} = \varphi_{SG}(-1),$$

which proves the sharpness of the radius ρ_7 (See Fig.4(a)).

□

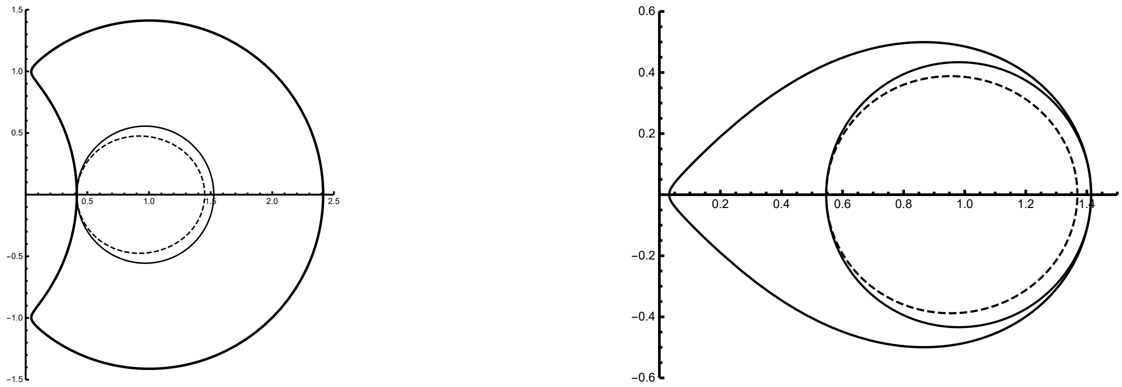
The class $\mathcal{CV}_{\varrho} := \mathcal{CV}(\varphi_{\varrho})$ where $\varphi_{\varrho}(z) = \cosh\sqrt{z}$.

Theorem 2.8. *The $\mathcal{CV}_{\varrho}$ radius of the class $\mathcal{W}$ is $\rho_8 \approx 0.0998\cdots$, where ρ_8 is the root of the equation:*

$$(1 - \cos 1)r^4 + 2(1 - \cos 1)r^3 - 2(1 + \cos 1)r^2 + 2(3 - \cos 1)r - (1 - \cos 1) = 0.$$

Proof. Consider the function η defined in (2.6) which is a decreasing function of r. The number $\rho_8 \in (0, 1]$ is a positive root of the equation $\eta(r) = \cos 1$. For $0 < r \leqslant \rho_8$, we have $\eta(r) \geqslant \cos 1$. That is $r_1(r) \leqslant a_1(r) - \cos 1$. Mridula and Sivaprasad [16] proved that $\{w : |w - a| < r_a\} \subset \varphi_{\varrho}(\mathbb{D}) =: \Omega_{\varrho}$ holds when

$$r_a = a - \cos 1, \quad \text{for} \quad \cos 1 < a \leqslant \frac{\cosh 1 + \cos 1}{2}. \tag{2.16}$$



(a). ρ_9 (b). ρ_{10}

Figure 5: Sharpness of $\mathcal{CV}_{\leftmoon}$ and $\mathcal{CV}_L$ radius for the class $\mathcal{W}$

Therefore, the disc given in (2.4) is contained in the region Ω_ϱ using (2.16).

Consider the function F defined in (2.8). For $z = -\rho_8$ in (2.9), we have

$$1 + \frac{zF''(z)}{F(z)} = \frac{1 + 5(-\rho_8) - 3(-\rho_8)^2 + (-\rho_8)^3}{1 + (-\rho_8) - 3(-\rho_8)^2 + (-\rho_8)^3} = \cos 1 = \varphi_\varrho(-1),$$

which proves the sharpness of the radius ρ_8 (See Fig.4(b)).

□

Raina and Sokól [17] introduced the class $\mathcal{CV}_{\leftmoon} := \mathcal{CV}(\varphi_{\leftmoon})$ where $\varphi_{\leftmoon}(z) = z + \sqrt{1+z^2}$ and the boundary of $\varphi_{\leftmoon}(\mathbb{D})$ is a lune.

Theorem 2.9. *The $\mathcal{CV}_{\leftmoon}$ radius of the class $\mathcal{W}$ is $\rho_9 \approx 0.1218\cdots$, where ρ_9 is the root of the equation:*

$$(\sqrt{2}-2)r^4 + 2(2\sqrt{2}-4)r^3 - 2\sqrt{2}r^2 + (8-2\sqrt{2})r + \sqrt{2} - 2 = 0.$$

Proof. Consider the function η defined in (2.6) which is a decreasing function of r. The number $\rho_9 \in (0,1]$ is a positive root of the equation $\eta(r) = \sqrt{2}-1$. For $0 < r \leqslant \rho_9$, we have $\eta(r) \geqslant \sqrt{2}-1$. That is $r_1(r) \leqslant a_1(r) - (\sqrt{2}-1)$. Gandhi and Ravichandran [5] proved that $\{w : |w-a| < r_a\} \subset \varphi_{\leftmoon}(\mathbb{D}) =: \Omega_{\leftmoon}$ holds when

$$r_a = a - (\sqrt{2}-1), \quad \text{for} \quad \sqrt{2}-1 < a \leqslant \sqrt{2}. \tag{2.17}$$

Therefore, the disc given in (2.4) is contained in the region bounded by a lune using (2.17). Consider the function F defined in (2.8). For $z = -\rho_9$ in (2.9), we have

$$1 + \frac{zF''(z)}{F(z)} = \frac{1 + 5(-\rho_9) - 3(-\rho_9)^2 + (-\rho_9)^3}{1 + (-\rho_9) - 3(-\rho_9)^2 + (-\rho_9)^3} = \sqrt{2} - 1 = \varphi_{\leftmoon}(-1),$$

which proves the sharpness of the radius ρ_9 (See Fig.5(a)).

□

The class $\mathcal{CV}_L := \mathcal{CV}(\varphi_L)$ where $\varphi_L(z) = \sqrt{1+z}$ and the boundary of $\varphi_L(\mathbb{D})$ is the right half of lemniscate of Bernoulli.

Theorem 2.10. *The $\mathcal{CV}_L$ radius of the class $\mathcal{W}$ is $\rho_{10} = 0.0987\cdots$, where ρ_{10} is the root of the equation:*

$$(\sqrt{2}-5)r^4 + 2(\sqrt{2}-5)r^3 - 2(\sqrt{2}-5)r^2 - 2(\sqrt{2}+1)r + \sqrt{2} - 1 = 0.$$

Proof. Consider the function $\tilde{\eta}$ defined by

$$\tilde{\eta}(r) := \frac{1 + 2r - 10r^2 + 10r^3 + 5r^4}{1 - 2r - 2r^2 + 2r^3 + r^4}, \qquad 0 \leqslant r < 1,$$

which is a decreasing function of r. The number $\rho_{10} \in (0,1]$ is a positive root of the equation $\tilde{\eta}(r) = \sqrt{2}$. For $0 < r \leqslant \rho_{10}$, we have $\tilde{\eta}(r) \leqslant \sqrt{2}$. That is $r_1(r) \leqslant \sqrt{2} - a_1(r)$. Sokół and Stankiewicz [24] proved that $\{w : |w - a| < r_a\} \subset \varphi_L(\mathbb{D}) := \Omega_L$ holds when

$$r_a = \sqrt{2} - a, \quad \text{for} \quad \frac{2\sqrt{2}}{3} < a \leqslant \sqrt{2}. \tag{2.18}$$

Therefore, the disc given in (2.4) is contained in the region bounded by a lemniscate using (2.18). Consider the function F defined in (2.8). For $z = \rho_{10}$ in (2.9), we have

$$1 + \frac{zF''(z)}{F(z)} = \frac{1 + 5(\rho_{10}) - 3(\rho_{10})^2 + (\rho_{10})^3}{1 + (\rho_{10}) - 3(\rho_{10})^2 + (\rho_{10})^3} = \sqrt{2} = \varphi_L(1),$$

which proves the sharpness of the radius ρ_{10} (See Fig.5(b)).
□

The class $\mathcal{CV}_{3L} := \mathcal{CV}(\varphi_{3L})$ where $\varphi_{3L}(z) = 1 + (4z/5) + (z^4/5)$ and the boundary of $\varphi_{3L}(\mathbb{D})$ is a three leaf domain.

Theorem 2.11. *The $\mathcal{CV}_{3L}$ radius of the class $\mathcal{W}$ is $\rho_{11} \approx 0.1241\cdots$, where ρ_{11} is the root of the equation:*

$$3r^4 + 6r^3 + 14r^2 - 26r + 3 = 0.$$

Proof. Consider the function η defined in (2.6) which is a decreasing function of r. The number $\rho_{11} \in (0,1]$ is a positive root of the equation $\eta(r) = (2/5)$. For $0 < r \leqslant \rho_{11}$, we have $\eta(r) \geqslant (2/5)$. That is $r_1(r) \leqslant a_1(r) - (2/5)$. Gandhi [4] proved that the inclusion $\{w : |w - a| < r_a\} \subset \varphi_{3L}(\mathbb{D}) =: \Omega_{3L}$ holds when

$$r_a = a - \frac{2}{5}, \quad \text{for} \quad \frac{2}{5} < a \leqslant 1. \tag{2.19}$$

Therefore, the disc given in (2.4) is contained in a three leaf shaped region using (2.19).
Consider the function F defined in (2.8). For $z = -\rho_{11}$ in (2.9), we have

$$1 + \frac{zF''(z)}{F(z)} = \frac{1 + 5(-\rho_{11}) - 3(-\rho_{11})^2 + (-\rho_{11})^3}{1 + (-\rho_{11}) - 3(-\rho_{11})^2 + (-\rho_{11})^3} = \frac{2}{5} = \varphi_{3L}(-1),$$

which proves the sharpness of the radius ρ_{11} (See Fig.6(a)).
□

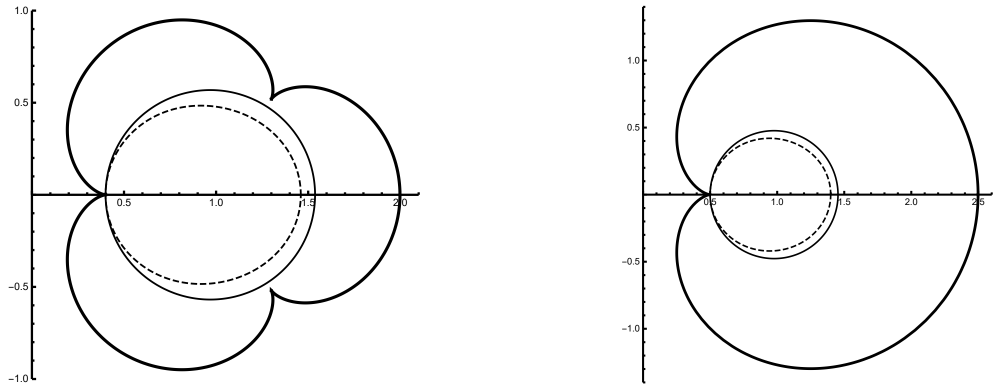
The class $\mathcal{CV}_{car} := \mathcal{CV}(\varphi_{car})$ where $\varphi_{car}(z) = 1 + z + (z^2/2)$, and the boundary of $\varphi_{car}(\mathbb{D})$ is a cardioid.

Theorem 2.12. *The $\mathcal{CV}_{car}$ radius of the class $\mathcal{W}$ is $\rho_{12} \approx 0.1071\cdots$, where ρ_{12} is the root of the equation:*

$$r^4 + 2r^3 + 6r^2 - 10r + 1 = 0.$$

Proof. Consider the function η defined in (2.6) which is a decreasing function of r. The number $\rho_{12} \in (0,1]$ is a positive root of the equation $\eta(r) = 1/2$. For $0 < r \leqslant \rho_{12}$, we have $\eta(r) \geqslant 1/2$. That is $r_1(r) \leqslant a_1(r) - (1/2)$. Gupta et.al [8] proved that $\{w : |w - a| < r_a\} \subset \varphi_{car}(\mathbb{D}) =: \Omega_{car}$ holds when

$$r_a = a - \frac{1}{2}, \quad \text{for} \quad \frac{1}{2} < a \leqslant \frac{3}{2}. \tag{2.20}$$



(a). ρ_{11} (b). ρ_{12}

Figure 6: Sharpness of $\mathcal{CV}_{3L}$ and $\mathcal{CV}_{car}$ radius for the class $\mathcal{W}$

Therefore, the disc given in (2.4) is contained in the region Ω_{car} using (2.20).

Consider the function F defined in (2.8). For $z = -\rho_{12}$ in (2.9), we have

$$1 + \frac{zF''(z)}{F(z)} = \frac{1 + 5(-\rho_{12}) - 3(-\rho_{12})^2 + (-\rho_{12})^3}{1 + (-\rho_{12}) - 3(-\rho_{12})^2 + (-\rho_{12})^3} = \frac{1}{2} = \varphi_{car}(-1),$$

which proves the sharpness of the radius ρ_{12} (See Fig.6(b)).

□

Ronning [21] defined the class $\mathcal{CV}_{par} = \mathcal{CV}(\varphi_{par})$, where $\varphi_{par}(z) = 1 + (2/\pi^2)(\log((1+\sqrt{z})/(1-\sqrt{z}))^2)$, and the boundary of $\varphi_{par}(\mathbb{D})$ is a parabolic symmetric with respect to the real axis and with vertex at $(1/2, 0)$.

Theorem 2.13. *The $\mathcal{CV}_{par}$ radius of the class $\mathcal{W}$ is $\rho_{13} \approx 0.1071\cdots$, where ρ_{13} is the root of the equation:*

$$r^4 + 2r^3 + 6r^2 - 10r + 1 = 0.$$

Proof. Consider the function η defined in (2.6) which is a decreasing function of r. The number $\rho_{13} \in (0, 1]$ is a positive root of the equation $\eta(r) = 1/2$. For $0 < r \leqslant \rho_{13}$, we have $\eta(r) \geqslant 1/2$. That is $r_1(r) \leqslant a_1(r) - (1/2)$. Shanmugam et.al [22] proved that the inclusion

$$\{w : |w - a| < r_a\} \subset \varphi_{par}(\mathbb{D}) =: \Omega_{par}$$

holds when

$$r_a = a - \frac{1}{2}, \quad \text{for} \quad \frac{1}{2} < a \leqslant \frac{3}{2}. \tag{2.21}$$

Therefore, the disc given in (2.4) is contained in the region bounded by a parabola using (2.21).

Consider the function F defined in (2.8). For $z = -\rho_{13}$ in (2.9), we have

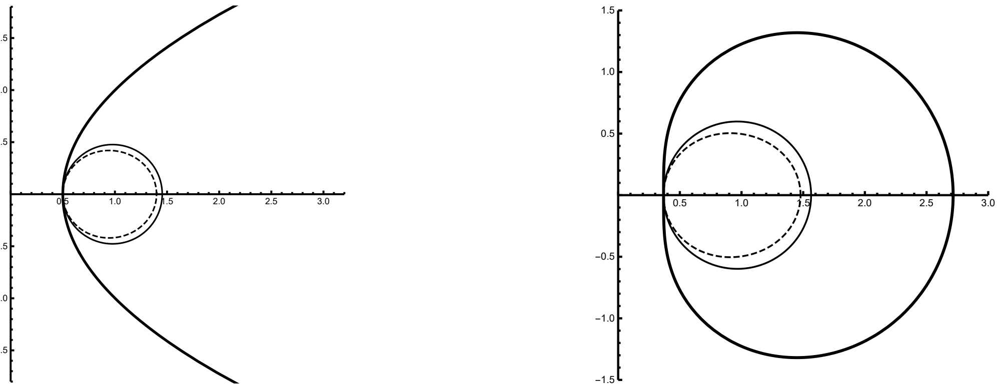
$$1 + \frac{zF''(z)}{F(z)} = \frac{1 + 5(-\rho_{13}) - 3(-\rho_{13})^2 + (-\rho_{13})^3}{1 + (-\rho_{13}) - 3(-\rho_{13})^2 + (-\rho_{13})^3} = \frac{1}{2} = \varphi_{par}(-1),$$

which proves the sharpness of the radius ρ_{13} (See Fig.7(a)). □

The class $\mathcal{CV}_e := \mathcal{CV}(\varphi_e)$ where $\varphi_e(z) = e^z$.

Theorem 2.14. *The $\mathcal{CV}_e$ radius of the class $\mathcal{W}$ is $\rho_{14} \approx 0.1293\cdots$, where ρ_{14} is the root of the equation:*

$$(e-1)r^4 + 2(e-1)r^3 + 2(e+1)r^2 - 2(3e-1)r + (e-1) = 0.$$



(a). ρ_{13} (b). ρ_{14}

Figure 7: Sharpness of $\mathcal{CV}_{par}$ and $\mathcal{CV}_e$ radius for the class $\mathcal{W}$

Proof. Consider the function η defined in (2.6) which is a decreasing function of r. The number $\rho_{13} \in (0, 1]$ is a positive root of the equation $\eta(r) = 1/e$. For $0 < r \leqslant \rho_{14}$, we have $\eta(r) \geqslant 1/e$. That is $r_1(r) \leqslant a_1(r) - (1/e)$. Mendiratta et al. [15] proved that $\{w : |w - a| < r_a\} \subset \varphi_e(\mathbb{D}) =: \Omega_e$ holds when

$$r_a = a - \frac{1}{e}, \quad \text{for} \quad \frac{1}{e} < a \leqslant \frac{e + e^{-1}}{2}, \tag{2.22}$$

Therefore, the disc given in (2.4) is contained in the region bounded by cardioid using (2.22).

Consider the function F defined in (2.8). For $z = -\rho_1 4$ in (2.9), we have

$$1 + \frac{zF''(z)}{F(z)} = \frac{1 + 5(-\rho_{14}) - 3(-\rho_{14})^2 + (-\rho_{14})^3}{1 + (-\rho_{14}) - 3(-\rho_{14})^2 + (-\rho_{14})^3} = \frac{1}{e} = \varphi_e(-1),$$

which proves the sharpness of the radius ρ_{14} (See Fig.7(b)).

□

3. Radius of Starlikeness

For a function φ with positive real part, we have $\mathcal{ST}(\varphi) \subset \mathcal{ST}$. MacGregor [13] has shown that the radius of starlikeness of the class $\mathcal{W}$ is $\sqrt{2} - 1$ and, therefore, $\mathcal{ST}(\varphi)$ radius of the class $\mathcal{W}$ is at most $\sqrt{2} - 1$. The MacGregor class $\mathcal{W}_n$ consists of function $f \in \mathcal{A}_n$ such that $\mathrm{Re}(f(z)/z) > 0$. This class has been studied for several radius problems by the authors in [1, 3, 14, 15, 25]. In this section we find the radius of starlikeness associated with Ma-Minda starlike classes such as $\mathcal{ST}_R, \mathcal{ST}_{\wp}, \mathcal{ST}_{SG}, \mathcal{ST}_h$ and $\mathcal{ST}_{\varrho}$.

The class $\mathcal{ST}_R := \mathcal{ST}(\varphi_R)$ where $\varphi_R(z) = 1 + ((z^2 + kz)/(k^2 - kz))$ for $k = \sqrt{2} + 1$.

Theorem 3.1. *The $\mathcal{ST}_R$ radius of the class $\mathcal{W}_n$ is given by*

$$\mu_1 = \left(-(3 + 2\sqrt{2})n + \sqrt{1 + (17 + 12\sqrt{2})n^2}\right)^{1/n},$$

where μ_1 is the root of the equation:

$$(3 - 2\sqrt{2})r^{2n} + 2nr^n - (3 - 2\sqrt{2}) = 0.$$

Proof. For $f \in \mathcal{A}_n$, let the function p be defined by $p(z) = f(z)/z$. From [13], we know that

$$\left|\frac{zp'(z)}{p(z)}\right| \leqslant \frac{2n|z|^n}{1 - |z|^{2n}}. \tag{3.1}$$

Hence, using the definition of the function p and (3.1), we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{zp'(z)}{p(z)}\right| \leqslant \frac{2nr^n}{1-r^{2n}} \qquad |z| = r < 1. \tag{3.2}$$

Clearly, the function $w = zf'(z)/f(z)$ lies within the disc centered at $a_2(r)$ with radius $r_2(r)$, where

$$a_2(r) = 1 \quad \text{and} \quad r_2(r) = \frac{2nr^n}{1-r^{2n}}. \tag{3.3}$$

Our aim is to show that the disc $\mathbb{D}(a_2(r); r_2(r))$ given in (3.2) is contained in Ω_φ, for all $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_\varphi}(\mathcal{W}_n)$ in the following theorems and all the figures in this section illustrates the image of the extremal function G, the disc described by equation (3.2), and the boundary of $\varphi(\mathbb{D})$.

Define the function $\zeta(r) := (1-2nr^n-r^{2n})/(1-r^{2n})$ which is a decreasing function of r. Let $\mu_1 := \mathcal{R}_{\mathcal{ST}_R}(\mathcal{W}_n)$. The number μ_1 be a positive root of the equation $\zeta(r) = 2(\sqrt{2}-1)$ that is less than 1. For $0 < r \leqslant \mu_1$, we have $\zeta(r) \geqslant 2(\sqrt{2}-1)$. That is $r_2(r) \leqslant a_2(r) - 2(\sqrt{2}-1)$. Therefore, the disc given in (3.2) is contained in the region bounded by cardioid using the inclusion result in (2.10).

In order to prove the sharpness of the radius obtained, we consider the function G defined by

$$G(z) = \frac{z(1+z^n)}{(1-z^n)}. \tag{3.4}$$

A simple calculation yields

$$\frac{zG'(z)}{G(z)} = \frac{1+2nz^n-r^{2n}}{1-z^{2n}}. \tag{3.5}$$

For $z = -\mu_1$ in (3.5), we have

$$\frac{zG'(z)}{G(z)} = \frac{1+2n(-\mu_1)^n-(-\mu_1)^{2n}}{1-(-\mu_1)^{2n}} = 2(\sqrt{2}-1) = \varphi_R(-1),$$

which proves the sharpness of the radius μ_1 (See Fig.8(a)) .
□

The class $\mathcal{ST}_\wp := \mathcal{ST}(\varphi_\wp)$ where $\varphi_\wp(z) = 1 + ze^z$.

Theorem 3.2. *The $\mathcal{ST}_\wp$ radius of the class $\mathcal{W}_n$ is given by*

$$\mu_2 = \left(-en + \sqrt{1+e^2n^2}\right)^{1/n},$$

where μ_2 is the root of the equation:

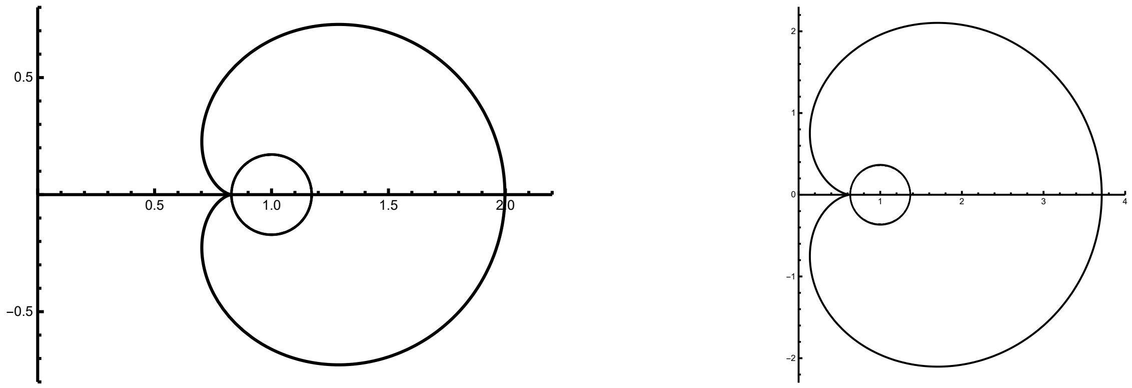
$$r^{2n} + 2ner^n - 1 = 0.$$

Proof. Consider the function $\zeta(r)$ which is a decreasing function of r. The number $\mu_2 < 1$ be a positive root of the equation $\zeta(r) = 1 - (1/e)$. For $0 < r \leqslant \mu_2$, we have $\zeta(r) \geqslant 1 - (1/e)$. That is $r_2(r) \leqslant a_2(r) - (1 - (1/e))$. Therefore, the disc given in (3.2) is contained inside the region $\Omega_\wp$ using the inclusion condition in (2.11).

For the function G defined in (3.4), at $z = -\mu_2$ in (3.5), we have

$$\frac{zG'(z)}{G(z)} = \frac{1+2n(-\mu_2)^n-(-\mu_2)^{2n}}{1-(-\mu_2)^{2n}} = 1 - \frac{1}{e} = \varphi_\wp(-1),$$

which proves the sharpness of the radius μ_2 (See Fig.8(b)). □



(a). μ_1 (b). μ_2

Figure 8: Sharpness of $\mathcal{ST}_R$ and $\mathcal{ST}_\wp$ radius for the class $\mathcal{W}_n$

The class $\mathcal{ST}_{SG} := \mathcal{ST}(\varphi_{SG})$ where $\varphi_{SG}(z) = 2/(1 + e^{-z})$.

Theorem 3.3. *The $\mathcal{ST}_{SG}$ radius of the class $\mathcal{W}_n$ is given by*

$$\mu_3 = \left(\frac{e-1}{(e+1)n + \sqrt{(e-1)^2 + (e+1)^2 n^2}} \right)^{1/n},$$

where μ_3 is the root of the equation:

$$(e-1)r^{2n} + 2n(e+1)r^n - (e-1) = 0.$$

Proof. Consider the function $\zeta(r)$ which is a decreasing function of r. The number $\mu_3 < 1$ be a positive root of the equation $\zeta(r) = 2/(1+e)$. For $0 < r \leqslant \mu_3$, we have $\zeta(r) \geqslant 2/(1+e)$. That is $r_2(r) \leqslant a_2(r) - 2/(1+e)$. Therefore, the disc given in (3.2) is contained in the region bounded by a modified sigmoid using the inclusion condition in (2.15).

For the function G defined in (3.4), at $z = -\mu_3$ in (3.5), we have

$$\frac{zG'(z)}{G(z)} = \frac{1 + 2n(-\mu_3)^n - (-\mu_3)^{2n}}{1 - (-\mu_3)^{2n}} = \frac{2}{1+e} = \varphi_{SG}(-1),$$

which proves the sharpness of the radius μ_3 (See Fig.9(a)). □

The class $\mathcal{ST}_h := \mathcal{ST}(\varphi_h)$ where $\varphi_h(z) = 1 + \sinh^{-1} z$.

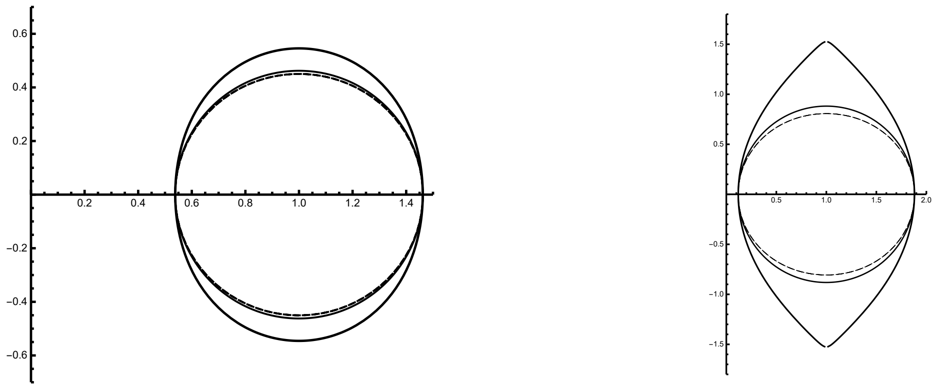
Theorem 3.4. *The $\mathcal{ST}_h$ radius of the class $\mathcal{W}_n$ is given by*

$$\mu_4 = \left(\frac{\sinh^{-1}(1)}{n + \sqrt{n^2 + (\sinh^{-1}(1))^2}} \right)^{1/n},$$

where μ_4 is the root of the equation:

$$\sinh^{-1}(1) r^{2n} + 2nr^n - \sinh^{-1}(1) = 0.$$

Proof. Consider the function $\zeta(r)$ which is a decreasing function of r. The number $\mu_4 < 1$ be a positive root of the equation $\zeta(r) = 1 - \sinh^{-1}(1)$. For $0 < r \leqslant \mu_4$, we have $\zeta(r) \geqslant 1 - \sinh^{-1}(1)$. That is $r_2(r) \leqslant$



(a). μ_3 (b). μ_4

Figure 9: Sharpness of $\mathcal{ST}_{SG}$ and $\mathcal{ST}_h$ radius for the class $\mathcal{W}_n$

$a_2(r) - (1 - \sinh^{-1}(1))$. Therefore, the disc given in (3.2) is contained in a petal shaped region using the inclusion condition in (2.13).

For the function G defined in (3.4), at $z = -\mu_4$ in (3.5), we have

$$\frac{zG'(z)}{G(z)} = \frac{1 + 2n(-\mu_4)^n - (-\mu_4)^{2n}}{1 - (-\mu_4)^{2n}} = 1 - \sinh^{-1}(1) = \varphi_h(-1),$$

which proves the sharpness of the radius μ_4 (See Fig.9(b)). □

The class $\mathcal{ST}_\varrho := \mathcal{ST}(\varphi_\varrho)$ where $\varphi_\varrho(z) = \cosh\sqrt{z}$.

Theorem 3.5. *The $\mathcal{ST}_\varrho$ radius of the class $\mathcal{W}_n$ is given by*

$$\mu_5 = \left(\frac{1 - \cos 1}{n + \sqrt{n^2 + (1 - \cos 1)^2}}\right)^{1/n},$$

where μ_5 is the root of the equation:

$$(1 - \cos 1)r^{2n} - 2nr^n - (1 - \cos 1) = 0.$$

Proof. Consider the function $\zeta(r)$ which is a decreasing function of r. The number $\mu_5 < 1$ be a positive root of the equation $\zeta(r) = \cos 1$. For $0 < r \leqslant \mu_5$, we have $\zeta(r) \geqslant \cos 1$. That is $r_2(r) \leqslant a_2(r) - \cos 1$. Therefore, the disc given in (3.2) is contained inside the region Ω_ϱ using the inclusion condition in (2.16).

For the function G defined in (3.4), at $z = -\mu_5$ in (3.5), we have

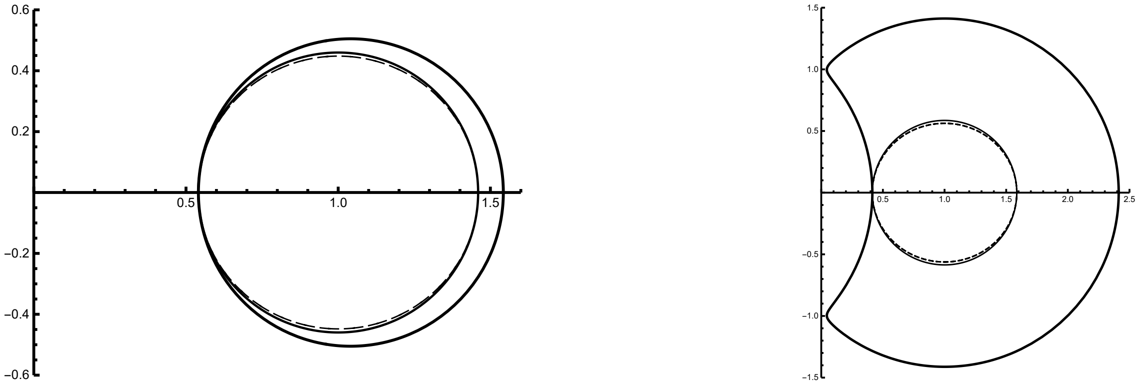
$$\frac{zG'(z)}{G(z)} = \frac{1 + 2n(-\mu_5)^n - (-\mu_5)^{2n}}{1 - (-\mu_5)^{2n}} = \cos 1 = \varphi_\varrho(-1),$$

which proves the sharpness of the radius μ_5 (See Fig.10(a)). □

The class $\mathcal{ST}_{\leftmoon} := \mathcal{ST}(\varphi_{\leftmoon})$ where $\varphi_{\leftmoon}(z) = z + \sqrt{1 + z^2}$.

Theorem 3.6. *The $\mathcal{ST}_{\leftmoon}$ radius of the class $\mathcal{W}_n$ is given by*

$$\mu_6 = \left(\frac{2 - \sqrt{2}}{n + \sqrt{n^2 + (2 - \sqrt{2})^2}}\right)^{1/n},$$



(a). μ_5 (b). μ_6

Figure 10: Sharpness of $\mathcal{ST}_{\varrho}$ and $\mathcal{ST}_{\leftmoon}$ radius for the class $\mathcal{W}_n$

where μ_6 is the root of the equation:

$$(2-\sqrt{2})r62n - 2nr^n - (2-\sqrt{2}) = 0.$$

Proof. Consider the function $\zeta(r)$ which is a decreasing function of r. The number $\mu_6 < 1$ be a positive root of the equation $\zeta(r) = \sqrt{2}-1$. For $0 < r \leqslant \mu_6$, we have $\zeta(r) \geqslant \sqrt{2}-1$. That is $r_2(r) \leqslant a_2(r) - (\sqrt{2}-1)$. Therefore, the disc given in (3.2) is contained in a lune shaped region using the inclusion condition in (2.17).

For the function G defined in (3.4), at $z = -\mu_6$ in (3.5), we have

$$\frac{zG'(z)}{G(z)} = \frac{1+2n(-\mu_6)^n - (-\mu_6)^{2n}}{1-(-\mu_6)^{2n}} = \sqrt{2}-1 = \varphi_{\leftmoon}(-1),$$

which proves the sharpness of the radius μ_6 (See Fig.10(b)). □

The class $\mathcal{ST}_{3L} := \mathcal{ST}(\varphi_{3L})$ where $\varphi_{3L}(z) = 1 + (4z/5) + (z^4/5)$.

Theorem 3.7. *The $\mathcal{ST}_{3L}$ radius of the class $\mathcal{W}_n$ is given by*

$$\mu_7 = \left(\frac{3}{5n+\sqrt{25n^2+9}}\right)^{1/n},$$

where μ_7 is the root of the equation:

$$3r^{2n} + 10nr^n - 3 = 0.$$

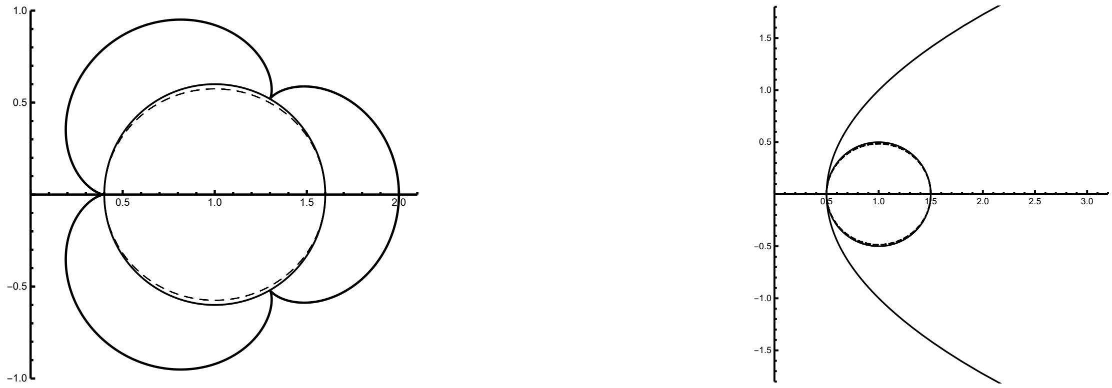
Proof. Consider the function $\zeta(r)$ which is a decreasing function of r. The number μ_7 be a positive root of the equation $\zeta(r) = 2/5$ in the interval $(0,1]$. For $0 < r \leqslant \mu_7$, we have $\zeta(r) \geqslant 2/5$. That is $r_2(r) \leqslant a_2(r) - 2/5$. Therefore, the disc given in (3.2) is contained in a three leaf shaped region using the inclusion condition in (2.19).

For the function G defined in (3.4), at $z = -\mu_7$ in (3.5), we have

$$\frac{zG'(z)}{G(z)} = \frac{1+2n(-\mu_7)^n - (-\mu_7)^{2n}}{1-(-\mu_7)^{2n}} = \frac{2}{5} = \varphi_{3L}(-1),$$

which proves the sharpness of the radius μ_7 (See Fig.11(a)). □

The class $\mathcal{ST}_{par} = \mathcal{ST}(\varphi_{par})$, where $\varphi_{par}(z) = 1 + (2/\pi^2)(\log((1+\sqrt{z})/(1-\sqrt{z}))^2)$.



(a). μ_7 (b). μ_8

Figure 11: Sharpness of $\mathcal{ST}_{3L}$ and $\mathcal{ST}_{par}$ radius for the class $\mathcal{W}_n$

Theorem 3.8. *The $\mathcal{ST}_{par}$ radius of the class $\mathcal{W}_n$ is given by*

$$\mu_8 = \left(-2n + \sqrt{4n^2+1}\right)^{1/n},$$

where μ_8 is the root of the equation:

$$r^{2n} + 4nr^n - 1 = 0.$$

Proof. Consider the function $\zeta(r)$ which is a decreasing function of r. The number μ_8 be a positive root of the equation $\zeta(r) = 1/2$ in the interval $(0, 1]$. For $0 < r \leqslant \mu_8$, we have $\zeta(r) \geqslant 1/2$. That is $r_2(r) \leqslant a_2(r) - 1/2$. Therefore, the disc given in (3.2) is contained in the region bounded by parabola using the inclusion condition in (2.21).

For the function G defined in (3.4), at $z = -\mu_8$ in (3.5), we have

$$\frac{zG'(z)}{G(z)} = \frac{1 + 2n(-\mu_8)^n - (-\mu_8)^{2n}}{1 - (-\mu_8)^{2n}} = \frac{1}{2} = \varphi_{par}(-1),$$

which proves the sharpness of the radius μ_8(See Fig.11(b)). □

The class $\mathcal{ST}_{car} := \mathcal{ST}(\varphi_{car})$ where $\varphi_{car}(z) = 1 + z + (z^2/2)$.

Theorem 3.9. *The $\mathcal{ST}_{car}$ radius of the class $\mathcal{W}_n$ is given by*

$$\mu_9 = \left(-2n + \sqrt{4n^2+1}\right)^{1/n},$$

where μ_9 is the root of the equation:

$$r^{2n} + 4nr^n - 1 = 0.$$

Proof. Consider the function $\zeta(r)$ which is a decreasing function of r. The number μ_9 be a positive root of the equation $\zeta(r) = 1/2$ in the interval $(0, 1]$. For $0 < r \leqslant \mu_9$, we have $\zeta(r) \geqslant 1/2$. That is $r_2(r) \leqslant a_2(r) - 1/2$. Therefore, the disc given in (3.2) is contained in the region bounded by cardioid using the inclusion condition in (2.20).

For the function G defined in (3.4), at $z = -\mu_9$ in (3.5), we have

$$\frac{zG'(z)}{G(z)} = \frac{1 + 2n(-\mu_9)^n - (-\mu_9)^{2n}}{1 - (-\mu_9)^{2n}} = \frac{1}{2} = \varphi_{car}(-1),$$

which proves the sharpness of the radius μ_9 (See Fig.12(a)). □

(a). μ_9 (b). μ_{10}

Figure 12: Sharpness of $\mathcal{ST}_{car}$ and $\mathcal{ST}_C$ radius for the class $\mathcal{W}_n$

The class $\mathcal{ST}_C := \mathcal{ST}(\varphi_C)$ where $\varphi_C(z) = 1 + (4/3)z + (2/3)z^2$.

Theorem 3.10. *The $\mathcal{ST}_C$ radius of the class $\mathcal{W}_n$ is given by*

$$\mu_{10} = \left(\frac{-3n + \sqrt{9n^2 + 4}}{2} \right)^{1/n},$$

where μ_{10} is the root of the equation:

$$2r^{2n} + 6nr^n - 2 = 0.$$

Proof. Consider the function $\zeta(r)$ which is a decreasing function of r. The number μ_{10} be a positive root of the equation $\zeta(r) = 1/3$ in the interval $(0, 1]$. For $0 < r \leqslant \mu_{10}$, we have $\zeta(r) \geqslant 1/3$. That is $r_2(r) \leqslant a_2(r) - 1/3$. Therefore, the disc given in (3.2) is contained in the region bounded by cardioid using the inclusion condition in (2.7).

For the function G defined in (3.4), at $z = -\mu_{10}$ in (3.5), we have

$$\frac{zG'(z)}{G(z)} = \frac{1 + 2n(-\mu_{10})^n - (-\mu_{10})^{2n}}{1 - (-\mu_{10})^{2n}} = \frac{1}{3} = \varphi_C(-1),$$

which proves the sharpness of the radius μ_{10} (See Fig.12(b)). □

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