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Geometric interpretation of radius of concavity and certain radii results for a class of meromorphic univalent functions

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Abstract. Radii of convexity and starlikeness provide significant insights into the geometric properties for analytic functions f defined in the open unit disk $\mathbb{D}$ of the complex plane. It is well-known that if $f(\mathbb{D})$ is a convex or a starlike domain, then $f(|z| < r)$ is also so for each $0 < r < 1$. But, for meromorphic univalent functions with nonzero pole, if $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is a convex domain, it is not true that $\overline{\mathbb{C}} \setminus f(|z| < r)$ is a convex domain for each $0 < r < 1$. Considering this fact, in Bhowmik, B., Biswas, S.: Distortion, Radius of Concavity and Several Other Radii Results for Certain Classes of Functions, Comput. Methods Funct. Theory, 25 (2025), 393–418, we defined radius of concavity for meromorphic functions with nonzero pole by using the analytic characterization for concave univalent functions. In this article, at first, we provide a geometric interpretation for the definition of radius of concavity. Furthermore, we compute radii of concavity, convexity, and starlikeness for the class $\mathcal{V}_p(\lambda)$, where $\mathcal{V}_p(\lambda)$ consists of functions f that are meromorphic in $\mathbb{D}$, having a simple pole at $z = p$, $p \in (0, 1)$ and satisfying the differential inequality $\left|(z/f(z))^2 f'(z) - 1\right| < \lambda$, $z \in \mathbb{D}$, $\lambda \in (0, 1]$.

1. Introduction and Preliminaries

Throughout this article we will use the following notations. Let $\mathbb{C}$ be the whole complex plane and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. We denote the open unit disk of the complex plane by $\mathbb{D}$, i.e. $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the class of all analytic functions in $\mathbb{D}$ satisfying the normalization $f(0) = 0 = f'(0) - 1$ and $\mathcal{S}$ be the class of all univalent functions in $\mathcal{A}$. For years, researchers have been exploring different subclasses of $\mathcal{S}$ that possess geometric properties. Among these the most prominent classes include the classes of convex functions and starlike functions. We denote the class of all convex functions by C which consists of all functions $f \in \mathcal{S}$ such that f maps $\mathbb{D}$ conformally onto a convex domain. Likewise, the class of starlike functions is denoted by $\mathcal{S}^*$ which consists of all functions $f \in \mathcal{S}$ such that f maps $\mathbb{D}$ conformally onto a starlike domain with respect to the origin. It is well-known that $f \in C$ if and only if

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D},$$

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and $f \in \mathcal{S}^*$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

We refer to the article [9] for some interesting recent results concerning functions in the class of starlike functions. It is well-known that convexity and starlikeness are hereditary property for functions f in C and $\mathcal{S}^*$, that is, if $f(\mathbb{D})$ is a convex or a starlike domain, then $f(|z| < r)$ is also so for each $0 < r < 1$. In this article, we explore the classical radius problem in geometric function theory. Radii of convexity and starlikeness play a crucial role in understanding the geometric properties of analytic functions. We first recall here that, the *radius of convexity* (or *radius of starlikeness*) for a subset $\mathcal{A}_1$ of $\mathcal{A}$, is the largest value of $r \in (0, 1]$ such that every function $f \in \mathcal{A}_1$ is convex (or starlike) within the disk $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$. In 1920, Nevanlinna (c.f. [12]) proved that the radius of convexity of $\mathcal{S}$ is $2 - \sqrt{3}$. Later, in 1934, Grunsky obtained the radius of starlikeness for $\mathcal{S}$ as $\tanh \pi/4$ (see [7, p. 141]). For further results on radii of convexity and starlikeness for analytic functions, we refer the reader to [2, 13, 14]. In order to describe our problem and related results, we need to introduce the following classes of functions. Let $\mathcal{A}(p)$ be the class of meromorphic functions in $\mathbb{D}$ with a simple pole at $z = p$, $p \in (0, 1)$ and normalized by the condition $f(0) = 0 = f'(0) - 1$. Let $S(p)$ denote the class of all univalent functions in $\mathcal{A}(p)$. Let $Co(p)$ be the class of functions $f \in S(p)$ such that $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is a bounded convex domain. In other words, $f(\mathbb{D})$ will be called a concave domain. The functions contained in this class are called as concave univalent functions. In 1971, J. A. Pfaltzgraff and B. Pinchuk (see [15, p. 145]) proved that $f \in Co(p)$ if and only if $f \in S(p)$ such that

$$\mathrm{Re}\, P_f(z) > 0, \quad z \in \mathbb{D}, \quad P_f(p) = \frac{1+p^2}{1-p^2} \quad \text{and} \quad P_f(0) = 1,$$

where

$$P_f(z) := -\left[1 + \frac{zf''(z)}{f'(z)} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz}\right]. \tag{1.1}$$

Let $K(p)$ be the class of functions $f \in S(p)$ such that f maps $|z| < r > \rho$ (for some $p < \rho < 1$) onto the complement of a convex set. It is well-known that $f \in K(p)$ if and only if $f \in S(p)$ and there exists ρ, $p < \rho < 1$ such that for each z, $\rho < |z| < 1$,

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0.$$

In 1970, James Miller (see [11]) obtained the radius of convexity for both the classes $K(p)$ and $Co(p)$. More details about these classes can be found in [10, 11, 15, 16]. In 1970, Royster proved that for $0 < p < 2 - \sqrt{3}$, $K(p) = Co(p)$ but for $2 - \sqrt{3} \le p < 1$, $K(p)$ is a proper subset of $Co(p)$ (see [16]). This shows that for $p \in (0, 2 - \sqrt{3})$ if $f \in Co(p)$, then there exists ρ satisfying $p < \rho < 1$ such that $\overline{\mathbb{C}} \setminus f(\mathbb{D}_r)$ is convex for each r satisfying $\rho < r < 1$. However, for $p \in [2 - \sqrt{3}, 1)$ and $f \in Co(p)$, such a conclusion cannot be made. Therefore, for $f \in \mathcal{A}(p)$ if $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is convex, then it does not follow that $\overline{\mathbb{C}} \setminus f(\mathbb{D}_r)$ is convex for each $0 < r < 1$. For example, let us consider the function

$$k_p(z) = \frac{pz}{(p-z)(1-pz)}, \quad z \in \mathbb{D}.$$

Since

$$k_p(\mathbb{D}) = \overline{\mathbb{C}} \setminus \left[\frac{-p}{(1-p)^2}, \frac{-p}{(1+p)^2}\right]$$

(see [1]), $k_p \in Co(p)$ for all $p \in (0, 1)$, i.e. k_p maps $\mathbb{D}$ onto a concave domain; but $k_p \in K(p)$ for $p \in (0, 2 - \sqrt{3})$ (see [16]). For instance, if we take $p = 0.6$ and $r = 0.7$, then it can be easily observed that $\overline{\mathbb{C}} \setminus k_p(\mathbb{D}_r)$ is not a convex domain (see Figure 1).

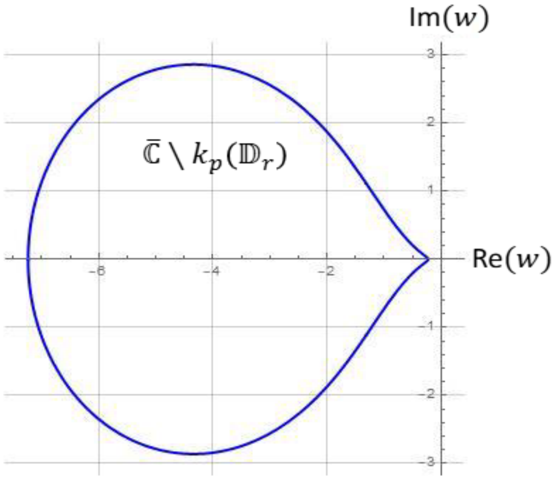

Figure 1: The region $\overline{\mathbb{C}} \setminus k_p(\mathbb{D}_r)$ for $p = 0.6$ and $r = 0.7$

Considering this fact, we defined the *radius of concavity* by using the analytic characterization for a class of meromorphic functions in [3]. We recall this definition from [3, Definition 1.1].

Definition 1.1. *The radius of concavity (with respect to $Co(p)$) of a subset $\mathcal{A}_1(p)$ of $\mathcal{A}(p)$ is the largest number $R_{Co(p)} \in (0, 1]$ such that for each function $f \in \mathcal{A}_1(p)$, $\operatorname{Re} P_f(z) > 0$ for all $|z| < R_{Co(p)}$, where P_f is defined in* (1.1).

We mention here that, in [3, Theorem 2] we obtained lower bound of the radius of concavity for $S(p)$. Furthermore, we determined the same for the linear combination of functions belonging to $S(p)$ with complex coefficients. In this article, our first concern is to provide a geometric interpretation of the *radius of concavity* provided in the Definition 1.1. This is done in Section 2 of this article. After that, we obtained radii of concavity, convexity and starlikeness for the the subclass $\mathcal{V}_p(\lambda)$ of $\mathcal{A}(p)$, $\lambda \in (0, 1]$. For a detailed discussion for functions in this class, we refer the reader to the articles [4–6, 8]. Here, at first, we present a brief overview about this class of functions. For $\lambda \in (0, 1]$, let

$$\mathcal{V}_p(\lambda) = \{f \in \mathcal{A}(p) : |U_f(z)| < \lambda,\ z \in \mathbb{D}\},$$

where $U_f(z) := (z/f(z))^2 f'(z) - 1$. It is well-known that $\mathcal{V}_p(\lambda) \subsetneq S(p)$, for $0 < \lambda \leq 1$ (see [6]). In [4], it has been proved that $\mathcal{V}_p(\lambda)$ does not contain $Co(p)$ for $\lambda \in (0, 1]$ and $Co(p)$ does not contain $\mathcal{V}_p(\lambda)$ for $\lambda \in \left((1 - p^2)/(1 + p^2), 1\right]$. In this article, we compute the radii of concavity, convexity, and starlikeness for the class $\mathcal{V}_p(\lambda)$.

The article is organized as follows. In Section 2, we provide geometric interpretation for radius of concavity. In Section 3, we obtain the radii of concavity, convexity, and starlikeness for the class $\mathcal{V}_p(\lambda)$ which are presented in Theorems 2, 3, and 4, respectively.

2. Geometric interpretation of the radius of concavity

We mentioned in Section 1 that concavity is not a hereditary property for meromorphic functions with non-zero pole. Consequently, if $R_{Co(p)}$ represents the radius of concavity for a subset $\mathcal{A}_1(p)$ of $\mathcal{A}(p)$, it does not follow that $f \in \mathcal{A}_1(p)$ maps $\mathbb{D}_{R_{Co(p)}}$ onto a concave domain. In view of this, we provide geometric interpretation of the radius of concavity in Theorem 1. In order to prove this theorem, we need to introduce the class MC which consists of functions of the form

$$f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots, \quad z \in \mathbb{D}, \tag{2.1}$$

that are univalent and analytic in $\mathbb{D}$ except for the simple pole at $z = 0$ and map $\mathbb{D}$ onto a concave domain. It is well-known that a necessary and sufficient condition for f having an expansion of the form (2.1) to belong to MC is

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0, \quad z \in \mathbb{D};$$

(see [15]). It is important to note that concavity is a hereditary property for functions in this class. For more details about this class, we encourage the readers to go through the article [15].

Theorem 1. *Let* $p \in (0,1)$. *If* $R_{Co(p)}$ $(p < R_{Co(p)} < 1)$ *is the radius of concavity for a subclass* $\mathcal{A}_1(p)$ *of* $\mathcal{A}(p)$, *then* $\mathcal{A}_1(p) \ni f$ *maps the disk* $|z - p| < R$ *onto a concave domain, where* R *is the smallest positive root of the equation* $\phi(r) = 0$, *with*

$$\begin{aligned}\phi(r) =& p(1+p^2)r^3 + \left(p(1+p^2)R_{Co(p)} + 1 - 2p^2 - p^4\right) r^2 \\ &+ (1-p^2)\left(2p + 3p^3 - (2p^2+3)R_{Co(p)}\right) r + p(1-p^2)^2(R_{Co(p)} - p).\end{aligned}$$

Proof. For $f \in \mathcal{A}(p)$, let us consider the function

$$F(z) = \frac{1}{R_{Co(p)} - p} f\left(p + z(R_{Co(p)} - p)\right), \quad z \in \mathbb{D}.$$

It is easy to check that F is meromorphic in $\mathbb{D}$ with a simple pole at $z = 0$, and by simple calculations it can be seen that

$$1 + \frac{zF''(z)}{F'(z)} = 1 + z\left(R_{Co(p)} - p\right) \frac{f''\left(p + z(R_{Co(p)} - p)\right)}{f'\left(p + z(R_{Co(p)} - p)\right)}, \quad z \in \mathbb{D}. \tag{2.2}$$

Let us consider

$$w = p + z\left(R_{Co(p)} - p\right), \quad z \in \mathbb{D}. \tag{2.3}$$

Then we get $|w - p| < R_{Co(p)} - p$, which implies $w \in \mathbb{D}_{R_{Co(p)}}$. Since $\operatorname{Re} P_f(z) > 0$ for all $z \in \mathbb{D}_{R_{Co(p)}}$, we have $\operatorname{Re} P_f(w) > 0$, where P_f is defined in (1.1). Using (2.3), the equation (2.2) takes the following form

$$1 + \frac{zF''(z)}{F'(z)} = 1 + (w - p)\frac{f''(w)}{f'(w)}. \tag{2.4}$$

By a little computation, we see that

$$1 + (w-p)\frac{f''(w)}{f'(w)} = -\left[1 + \left(1 - \frac{p}{w}\right) P_f(w) - \left(1 - \frac{p}{w}\right)\frac{1 + pw}{1 - pw}\right], \tag{2.5}$$

for $|w - p| < R_{Co(p)} - p$. Let us consider the function

$$Q(z) = \left(\frac{1-p^2}{1+p^2}\right) P_f\left(p + z(R_{Co(p)} - p\right), \quad z \in \mathbb{D}.$$

Using (2.3), the function Q can be written as

$$Q(z) = \left(\frac{1-p^2}{1+p^2}\right) P_f(w).$$

Since $\operatorname{Re} P_f(w) > 0$, we get $\operatorname{Re} Q(z) > 0$, $z \in \mathbb{D}$. Also, $Q(0) = 1$. Thus, we get $Q \prec (1+z)/(1-z)$, $z \in \mathbb{D}$, which implies

$$Q(z) = \frac{1+u(z)}{1-u(z)}, \quad z \in \mathbb{D},$$

for some analytic function u in $\mathbb{D}$ with $|u(z)| \le |z|$, $z \in \mathbb{D}$. By simple computations, we get

$$|Q(z)| \le \frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D}.$$

The above inequality yields

$$|P_f(w)| \le \left(\frac{1+p^2}{1-p^2}\right)\frac{R_{Co(p)} - p + |w-p|}{R_{Co(p)} - p - |w-p|}, \quad |w-p| < R_{Co(p)} - p. \tag{2.6}$$

It is easy to see that

$$\operatorname{Re}\left(\left(1-\frac{p}{w}\right)P_f(w)\right) \ge -\left|1-\frac{p}{w}\right| |P_f(w)| \ge \frac{-|w-p|}{\big||w-p|-p\big|}|P_f(w)|,$$

for $|w-p| < R_{Co(p)} - p$. Using (2.6), the above inequality implies

$$\operatorname{Re}\left(\left(1-\frac{p}{w}\right)P_f(w)\right) \ge \frac{-|w-p|}{\big||w-p|-p\big|}\left(\frac{1+p^2}{1-p^2}\right)\left(\frac{R_{Co(p)} - p + |w-p|}{R_{Co(p)} - p - |w-p|}\right). \tag{2.7}$$

By simple calculations, we have

$$\operatorname{Re}\left(\left(1-\frac{p}{w}\right)\frac{1+pw}{1-pw}\right) \le \frac{|w-p|}{\big||w-p|-p\big|}\left(\frac{1+p^2+p|w-p|}{1-p^2-p|w-p|}\right). \tag{2.8}$$

If we let $|w-p| = r$, then from (2.5), (2.7) and (2.8) we get

$$\begin{aligned}\operatorname{Re}\left(1+(w-p)\frac{f''(w)}{f'(w)}\right) \le -1 &+ \frac{r}{\big|r-p\big|}\left(\frac{1+p^2}{1-p^2}\right)\frac{R_{Co(p)} - p + r}{R_{Co(p)} - p - r} \\ &+ \frac{r}{|r-p|}\left(\frac{1+p^2+pr}{1-p^2-pr}\right).\end{aligned} \tag{2.9}$$

Now we consider two cases.
Case I: Let $R_{Co(p)} \le 2p$. Then $r = |w-p| < R_{Co(p)} - p \le p$. Thus from (2.9), we get

$$\operatorname{Re}\left(1+(w-p)\frac{f''(w)}{f'(w)}\right) \le -1 + \frac{r}{p-r}\left(\frac{(1+p^2)(R_{Co(p)} - p + r)}{(1-p^2)(R_{Co(p)} - p - r)} + \frac{1+p^2+pr}{1-p^2-pr}\right).$$

The right hand side of the above inequality is strictly negative if $|w-p| < R$, where R is the smallest positive root of the equation $\phi(r) = 0$, where ϕ is defined in the statement of the theorem. We now investigate the existence of the number R for each $p \in (0,1)$. The function ϕ is continuous on $[0, R_{Co(p)} - p]$ with

$$\phi(0) = p(1-p^2)^2(R_{Co(p)} - p) > 0$$

and

$$\phi(R_{Co(p)} - p) = -2(1+p^2)(1-pR_{Co(p)})(R_{Co(p)} - p)^2 < 0.$$

Therefore, by the intermediate value theorem we conclude that ϕ has at least one root in $(0, R_{Co(p)} - p)$. Hence, the number R exists for each $p \in (0,1)$.

Case II: Let $R_{Co(p)} > 2p$. Then $R_{Co(p)} - p > p$ and $r = |w - p| < R_{Co(p)} - p$. Since $2p < R_{Co(p)} < 1$, it follows that $p < 1/2$. Now we first consider $r < p$. Thus from (2.9), we get

$$\operatorname{Re}\left(1 + (w-p)\frac{f''(w)}{f'(w)}\right) \leq -1 + \frac{r}{p-r}\left(\frac{(1+p^2)(R_{Co(p)} - p + r)}{(1-p^2)(R_{Co(p)} - p - r)} + \frac{1+p^2+pr}{1-p^2-pr}\right).$$

The right hand side of the above inequality is strictly negative if $|w - p| < R$. We now investigate the existence of the number R for each $p \in (0, 1)$. The function ϕ which is defined in the statement of the theorem is continuous on $[0, p]$ with

$$\phi(0) = p(1-p^2)^2(R_{Co(p)} - p) > 0$$

and

$$\phi(p) = 2p\left((R_{Co(p)} - 2p)(2p^4 - 1) + p(1+p^2)(2p^2 - 1)\right) < 0 \quad \text{since } p < 1/2.$$

Therefore, by the intermediate value theorem we conclude that ϕ has at least one root in $(0, p)$. Hence, the number R exists for each $p \in (0, 1)$. Therefore, we don't need to consider the case when $p < r < R_{Co(p)} - p$ and proceed further. Thus, combining the above two cases we get

$$\operatorname{Re}\left(1 + (w-p)\frac{f''(w)}{f'(w)}\right) < 0, \quad |w - p| < R.$$

Hence, from (2.4) we conclude

$$\operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) < 0, \quad |z| < \frac{R}{R_{Co(p)} - p}. \tag{2.10}$$

Let

$$F_1(z) = F\left(\frac{Rz}{R_{Co(p)} - p}\right), \quad z \in \mathbb{D}.$$

It is easy to check that F_1 is meromorphic in $\mathbb{D}$ with a simple pole at $z = 0$. If we let

$$\xi = \frac{Rz}{R_{Co(p)} - p}, \quad z \in \mathbb{D},$$

then we get $|\xi| < R/(R_{Co(p)} - p)$. Thus, from (2.10) we get

$$\operatorname{Re}\left(1 + \frac{\xi F''(\xi)}{F'(\xi)}\right) < 0.$$

The above inequality yields

$$\operatorname{Re}\left(1 + \frac{zF_1''(z)}{F_1'(z)}\right) < 0, \quad z \in \mathbb{D}.$$

This shows that $(1/m)F_1 \in MC$, where m is the residue of F_1 at the pole $z = 0$. Thus, F_1 maps $\mathbb{D}$ onto a concave domain which implies that F maps the disk $|z| < R/(R_{Co(p)} - p)$ onto a concave domain. Therefore, f maps the disk $|z - p| < R$ onto a concave domain. □

3. Radii of concavity, convexity, and starlikeness of the class $\mathcal{V}_p(\lambda)$

In this section, we consider radius problems for the class $\mathcal{V}_p(\lambda)$. In the following theorem we determine a lower bound of the radius of concavity for the class $\mathcal{V}_p(\lambda)$.

Theorem 2. *The radius of concavity of $\mathcal{V}_p(\lambda)$ is at least R, where R is the smallest positive root of the equation $\phi(r) = 0$ with*

$$\phi(r) = (\lambda^2 p^2)r^5 + \lambda p(3\lambda p^2 - \lambda + 1)r^4 + \lambda(5\lambda p^2 - 4p^2 + 3)r^3 + p(1 - 4\lambda - \lambda p^2)r^2 - (1 + p^2 + 3\lambda p^2)r + p.$$

Proof. From [6, p. 1004], we know that if $f \in \mathcal{V}_p(\lambda)$, then there exists a holomorphic function w_1 such that $w_1(\mathbb{D}) \subseteq \overline{\mathbb{D}}$ and

$$\frac{z}{f(z)} = 1 - \left(\frac{f''(0)}{2}\right)z + \lambda z \int_0^z w_1(t)dt, \quad z \in \mathbb{D}. \tag{3.1}$$

Let

$$w(z) := \left(\int_p^z w_1(t)dt\right)/(z - p), \quad z \in \mathbb{D}. \tag{3.2}$$

Then from [5, Theorem 1], we get

$$\frac{z}{f(z)} = \frac{-(z-p)(1 - \lambda p z w(z))}{p}, \quad z \in \mathbb{D}, \tag{3.3}$$

where w is analytic in $\mathbb{D}$ and $|w(z)| \leq 1$, $z \in \mathbb{D}$. By differentiating (3.1) with respect to z and after a little computation, we get

$$-z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} = 1 - \lambda z^2 w_1(z), \quad z \in \mathbb{D}.$$

This implies that

$$\left(\frac{z}{f(z)}\right)^2 f'(z) = 1 - \lambda z^2 w_1(z), \quad z \in \mathbb{D}.$$

Then from (3.3), we get

$$f'(z) = \frac{p^2(1 - \lambda z^2 w_1(z))}{(z-p)^2(1 - \lambda p z w(z))^2}, \quad z \in \mathbb{D}. \tag{3.4}$$

Thus, we have

$$\frac{zf''(z)}{f'(z)} = \frac{-2z}{z-p} - \frac{\lambda z^2(2w_1(z) + zw_1'(z))}{1 - \lambda z^2 w_1(z)} + \frac{2\lambda p z(w(z) + zw'(z))}{1 - \lambda p z w(z)}, \quad z \in \mathbb{D}. \tag{3.5}$$

From (3.2), we get

$$(z-p)w'(z) + w(z) = w_1(z).$$

Then by a simple calculation we see that

$$zw'(z) + w(z) = \frac{1}{z-p}(zw_1(z) - pw(z)).$$

By using this, we deduce from (3.5) that for $z \in \mathbb{D}$,

$$1 + \frac{zf''(z)}{f'(z)} + \frac{z+p}{z-p} = \frac{-\lambda z^2(2w_1(z) + zw_1'(z))}{1 - \lambda z^2 w_1(z)} + \frac{2\lambda p z(zw_1(z) - pw(z))}{(z-p)(1 - \lambda p z w(z))}. \tag{3.6}$$

Let us define

$$u(z) = z^2 w_1(z), \quad z \in \mathbb{D}. \tag{3.7}$$

Then we get $u(0) = 0 = u'(0)$ and $|u(z)| \le |z|^2$, $z \in \mathbb{D}$. Using (3.7), the equation (3.6) takes the following form

$$1 + \frac{zf''(z)}{f'(z)} + \frac{z+p}{z-p} = \frac{-\lambda z u'(z)}{1-\lambda u(z)} + \frac{2\lambda p(u(z) - pzw(z))}{(z-p)(1-\lambda pzw(z))}. \tag{3.8}$$

Since $w_2(z) = (u(z))/z$ is an analytic function from $\mathbb{D}$ to $\mathbb{D}$, we get

$$|w_2'(z)| \le \frac{1 - |w_2(z)|^2}{1-|z|^2}, \quad z \in \mathbb{D},$$

which is equivalent to

$$|zu'(z) - u(z)| \le \frac{|z|^2 - |u(z)|^2}{1-|z|^2}, \quad z \in \mathbb{D}.$$

From (3.8) and the above inequality, we get

$$\left|1 + \frac{zf''(z)}{f'(z)} + \frac{z+p}{z-p}\right| \le \frac{\lambda(|z|^2 - |u(z)|^2)}{(1-|z|^2)(1-\lambda|u(z)|)} + \frac{\lambda|u(z)|}{(1-\lambda|u(z)|)} + \frac{2\lambda p(|u(z)| + p|z|)}{||z|-p|(1-\lambda p|z|)},$$

for $z \in \mathbb{D}$. If we let $|u(z)| = x$ and $|z| = r < p$ (note that $0 \le x \le r^2$), then the above inequality becomes

$$\left|1 + \frac{zf''(z)}{f'(z)} + \frac{z+p}{z-p}\right| \le \frac{\lambda(r^2 - x^2)}{(1-r^2)(1-\lambda x)} + \frac{\lambda x}{1-\lambda x} + \frac{2\lambda p(x+pr)}{(p-r)(1-\lambda pr)}. \tag{3.9}$$

Let us define

$$\eta(x) = \frac{\lambda(r^2 - x^2)}{(1-r^2)(1-\lambda x)} + \frac{\lambda x}{1-\lambda x} + \frac{2\lambda p(x+pr)}{(p-r)(1-\lambda pr)}, \quad x \in [0, r^2].$$

Then by a little computation, we see that if $r < p$ then

$$\eta'(x) \ge 0, \quad x \in [0, r^2].$$

Since the function η is increasing on $[0, r^2]$ for $r < p$, we have

$$\eta(x) \le \eta(r^2) = \frac{2\lambda r^2}{1-\lambda r^2} + \frac{2\lambda pr(p+r)}{(p-r)(1-\lambda pr)}.$$

Thus, from (3.9) we get

$$\left|1 + \frac{zf''(z)}{f'(z)} + \frac{z+p}{z-p}\right| \le \frac{2\lambda r^2}{1-\lambda r^2} + \frac{2\lambda pr(p+r)}{(p-r)(1-\lambda pr)}, \quad |z| = r < p. \tag{3.10}$$

This implies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)} + \frac{z+p}{z-p}\right) \le \frac{2\lambda r^2}{1-\lambda r^2} + \frac{2\lambda pr(p+r)}{(p-r)(1-\lambda pr)}, \quad |z| = r < p. \tag{3.11}$$

It is easy to see that

$$\mathrm{Re}\left(\frac{1+pz}{1-pz}\right) \ge \frac{1-pr}{1+pr}, \quad |z| = r < 1.$$

Applying (3.11) and the above inequality, we get

$$\mathrm{Re}\, P_f(z) \ge \frac{1-pr}{1+pr} - \frac{2\lambda r^2}{1-\lambda r^2} - \frac{2\lambda pr(p+r)}{(p-r)(1-\lambda pr)}, \quad |z| = r < p.$$

The right hand side of the above inequality is strictly positive if $|z| = r < R$ where R is the smallest positive root of the equation $\phi(r) = 0$. Here ϕ is defined in the statement of the theorem. Thus, $\operatorname{Re} P_f(z) > 0$ if $|z| < R$. We now investigate the existence of the number R for each $\lambda \in (0, 1]$ and $p \in (0, 1)$. The function ϕ is continuous on $[0, p]$ with

$$\phi(0) = p > 0 \quad \text{and} \quad \phi(p) = -4\lambda p^3(1 + p^2)(1 - \lambda p^2) < 0.$$

Therefore, by the intermediate value theorem ϕ has at least one root in $(0, p)$. Hence, the number R exists for each $\lambda \in (0, 1]$ and $p \in (0, 1)$. □

In the following theorem we find the radius of convexity for the class $\mathcal{V}_p(\lambda)$ by applying the estimate derived in (3.10).

Theorem 3. *If $f \in \mathcal{V}_p(\lambda)$, then f maps $|z| < R$ onto a convex set where R is the smallest positive root of the equation $\psi(r) = 0$ with*

$$\psi(r) = (\lambda^2 p)r^5 + \lambda(1 + 2\lambda p^2)r^4 - \lambda p(1 - 5\lambda p^2)r^3 + (1 - 5\lambda p^2)r^2 - p(2 + 3\lambda p^2)r + p^2.$$

Proof. From (3.10), we have

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \operatorname{Re}\left(\frac{p+z}{p-z}\right) - \frac{2\lambda r^2}{1 - \lambda r^2} - \frac{2\lambda pr(p+r)}{(p-r)(1-\lambda pr)}, \quad |z| = r < p.$$

Since

$$\operatorname{Re}\left(\frac{p+z}{p-z}\right) \geq \frac{p-r}{p+r}, \quad |z| = r < 1,$$

we get

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \frac{p-r}{p+r} - \frac{2\lambda r^2}{1 - \lambda r^2} - \frac{2\lambda pr(p+r)}{(p-r)(1-\lambda pr)}, \quad |z| = r < p.$$

The right hand side of the above inequality is strictly positive if $|z| < R$ where R is the smallest positive root of the equation $\psi(r) = 0$. Here ψ is defined in the statement of the theorem. Thus, $\operatorname{Re}\,(1 + zf''(z)/f'(z)) > 0$ if $|z| < R$. We now investigate the existence of the number R for each $\lambda \in (0, 1]$ and $p \in (0, 1)$. The function ψ is continuous on $[0, p]$ with

$$\psi(0) = p^2 > 0 \quad \text{and} \quad \psi(p) = -8\lambda p^4(1 - \lambda p^2) < 0.$$

Therefore, by the intermediate value theorem ψ has at least one root in $(0, p)$. Thus, the existence of the number R is confirmed for every $\lambda \in (0, 1]$ and $p \in (0, 1)$. □

In the following theorem we find the radius of starlikeness for the class $\mathcal{V}_p(\lambda)$.

Theorem 4. *If $f \in \mathcal{V}_p(\lambda)$, then f maps $|z| < R$ onto a starlike domain where R is the smallest positive root of the equation $\xi(r) = 0$ with*

$$\xi(r) = -\lambda^2 pr^4 - \lambda(1 + \lambda p^2)r^3 - 2\lambda pr^2 - (1 + \lambda p^2)r + p.$$

Proof. If $f \in \mathcal{V}_p(\lambda)$, then from (3.3) and (3.4) we get

$$\frac{zf'(z)}{f(z)} = \frac{p(1 - \lambda z^2 w_1(z))}{(p-z)(1 - \lambda pzw(z))}, \quad z \in \mathbb{D},$$

where w and w_1 are analytic in $\mathbb{D}$, satisfying $|w(z)| \leq 1$ and $|w_1(z)| \leq 1$, $z \in \mathbb{D}$. Therefore, we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \frac{\operatorname{Re}\left(p\left(1 - \lambda z^2 w_1(z)\right)(p - \overline{z})\left(1 - \lambda p\overline{zw(z)}\right)\right)}{|p-z|^2|1 - \lambda pzw(z)|^2}, \quad z \in \mathbb{D}. \tag{3.12}$$

If we let

$$Q(z) = p\left(1 - \lambda z^2 w_1(z)\right)(p - \overline{z})\left(1 - \lambda p \overline{z w(z)}\right), \quad z \in \mathbb{D},$$

then the equation (3.12) takes the form

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \frac{\operatorname{Re} Q(z)}{|p - z|^2 |1 - \lambda p z w(z)|^2}, \quad z \in \mathbb{D}. \tag{3.13}$$

The function Q can be written as

$$Q(z) = p\left(p - \overline{z} + \lambda p \overline{z w(z)}(\overline{z} - p) - \lambda p z^2 w_1(z) + \lambda |z|^2 z w_1(z) + \lambda^2 p |z|^2 w_1(z) \overline{w(z)}(pz - |z|^2)\right), \quad z \in \mathbb{D}.$$

As $-|u(z)| \leq \operatorname{Re} u(z) \leq |u(z)|$, $z \in \mathbb{C}$ for any function $u : \mathbb{C} \to \mathbb{C}$, from the above equation we get

$$\operatorname{Re} Q(z) \geq p\left(p - |\overline{z}| - \lambda p |\overline{z}| \left|\overline{w(z)}\right| (|\overline{z}| + p) - \lambda p |z|^2 |w_1(z)| - \lambda |z|^3 |w_1(z)| - \lambda^2 p |z|^2 |w_1(z)| \left|\overline{w(z)}\right| (p|z| + |z|^2)\right), \quad z \in \mathbb{D}.$$

Since $|w(z)| \leq 1$ and $|w_1(z)| \leq 1$ for $z \in \mathbb{D}$, the above inequality yields

$$\operatorname{Re} Q(z) \geq p\left(p - (1 + \lambda p^2) r - 2\lambda p r^2 - \lambda (1 + \lambda p^2) r^3 - \lambda^2 p r^4\right), \quad |z| = r < 1.$$

By applying the triangle inequality and using the aforementioned inequality, we derive from (3.13) that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{p\left(p - (1 + \lambda p^2) r - 2\lambda p r^2 - \lambda (1 + \lambda p^2) r^3 - \lambda^2 p r^4\right)}{(p + r)^2 (1 + \lambda p r)^2}, \quad |z| = r < 1.$$

The right hand side of the above inequality is strictly positive if $|z| = r < R$ where R is the smallest positive root of the equation $\xi(r) = 0$, where ξ is defined in the statement of the theorem. Thus, we obtain $\operatorname{Re}(zf'(z)/f(z)) > 0$ if $|z| < R$. We now investigate the existence of the number R for each $\lambda \in (0, 1]$ and $p \in (0, 1)$. The function ξ is continuous on $[0, p]$ with

$$\xi(0) = p > 0 \quad \text{and} \quad \xi(p) = -2\lambda p^3 (2 + \lambda p^2) < 0.$$

Therefore, by the intermediate value theorem ξ has at least one root in $(0, p)$. Hence, the number R exists for each $\lambda \in (0, 1]$ and $p \in (0, 1)$. This proves the theorem. □

Statements and Declarations

Competing interests: The authors declare that they have no conflict of interest.
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