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Stochastic dynamics and density function of a food chain model with infinite distributed delay

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Abstract. This paper explores a stochastic three-species food chain model with infinite distributed delay. By the linear chain technique, the model is transformed into a new higher-dimensional stochastic differential equation. First, the existence and uniqueness of the global positive solution to the model is proved, as well as the boundedness of the moments of the solution. Then, sufficient conditions for the global attractivity of the system are obtained. Next, we derive an explicit analytical expression for the probability density function of the system around the quasi-positive equilibrium. Finally, we provide some numerical examples to verify the validity of the results.

1. Introduction

In natural ecosystems, the predator-prey relationship is among the most fundamental interspecific relationships. Since the pioneering work of Lotka ([25]) and Volterra ([43]), predator-prey systems, the fundamental building blocks of food webs, have been extensively studied over the past few decades ([5, 16, 37, 44, 46–48, 52]). The well-known Lotka-Volterra predator-prey system describes a single predator species and a single prey species. However, several researchers have pointed out that two-species systems are capable of describing few phenomena in real-world ecosystems ([31, 32]). Certain behaviors critical to the function of ecological communities only emerge in systems that contain three or more species ([14]). For instance, research conducted by Price et al. demonstrates that plant-insect models must be grounded in a three-species framework ([33]). The chain-like structure, formed by a sequence of feeding relationships, is known as a food chain and widely observed in diverse ecosystems. Mathematical advances have further revealed that three-species food chain models exhibit far more complex dynamics than two-species systems do ([14]). Given that the food chains play a vital role in the ecosystem, there have already been numerous studies on food chain models ([4, 20, 22, 42, 49–51]). Many of the food chain models studied in the literature

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are of Lotka-Volterra type and the classical three-species food-chain model can be expressed as follows:

$$\begin{cases} \mathrm{d}N_1(t) = N_1(t)\left[b_1 - C_{11}N_1(t) - C_{12}N_2(t)\right]\mathrm{d}t, \\ \mathrm{d}N_2(t) = N_2(t)\left[-b_2 + C_{21}N_1(t) - C_{22}N_2(t) - C_{23}N_3(t)\right]\mathrm{d}t, \\ \mathrm{d}N_3(t) = N_3(t)\left[-b_3 + C_{32}N_2(t) - C_{33}N_3(t)\right]\mathrm{d}t, \end{cases} \tag{1}$$

where $N_i(t)$ $(i = 1, 2, 3)$ are the population densities of the prey, intermediate predator and top predator at time t, respectively. System (1) has been described in detail in [49], $a_{21} > 0$ and $a_{32} > 0$ denote the efficiency of food conversion. All parameters in system (1) are assumed to be positive constants.

The aforementioned system (1) is limited to describing populations under idealized conditions, which assumes that predator populations can instantaneously convert prey consumption into population growth ([39]). However, in the natural world, "all species should exhibit time delay" ([19]). The neglect of time delays in ecological models may lead to substantial inaccuracies in predicting system behaviors. In theoretical analysis, delay differential equations exhibit far more complex dynamics than ordinary differential equations do, for example, time delays can destabilize an otherwise stable equilibrium and induce population fluctuations ([1, 2, 7, 9, 40]). May theoretically proved that there is no stable equilibrium point in the whole system when the time delay is large ([12, 30]). As Kuang ([19]) demonstrated, animals need time to digest food prior to engaging in further activities or mounting responses. The classical works by Kuang ([19]), Gopalsamy ([11]), and Macdonald ([27]) highlight the importance and utility of time delays in realistic ecological models. Therefore, the introduction of time delays into ecological models is of significant theoretical and practical importance ([8, 17, 34]). Extensive theoretical research indicates that it is more reasonable to incorporate time delays using continuously distributed delays rather than finite discrete delays ([45]). Extensive analyses of population models reveal that growth rates depend not only on current density but also on the lingering effects of past resource consumption and recovery. These effects diminish over time yet lack a definitive endpoint. Consequently, when depicting the long-term cumulative interactions between populations and their environment, infinite distributed delays more accurately reflect the fundamental nature of ecological systems compared to finite distributed delays ([3, 12, 13]). Hence, this paper will consider system (1) with infinite distributed delay:

$$\begin{cases} \mathrm{d}N_1(t) = N_1(t)\left[b_1 - C_{11}N_1(t) - C_{12}N_2(t)\right]\mathrm{d}t, \\ \mathrm{d}N_2(t) = N_2(t)\left[-b_2 + C_{21}\displaystyle\int_{-\infty}^{t} G(t-\phi)N_1(\phi)\,\mathrm{d}\phi - C_{22}N_2(t) - C_{23}N_3(t)\right]\mathrm{d}t, \\ \mathrm{d}N_3(t) = N_3(t)\left[-b_3 + C_{32}N_3(t) - C_{33}N_3(t)\right]\mathrm{d}t, \end{cases} \tag{2}$$

with $N_i(0) > 0$ and $N_1(\phi) \geq 0$ for any $\phi \in (-\infty, 0)$. $G(t)$ is a non-negative normalized delayed-weight function defined on $[0, +\infty)$, namely $\int_0^{\infty} G\left(\phi\right)\mathrm{d}\phi = 1$ ([12]). For the infinite distributed delay, it is reasonable to use the general Gamma distribution as a kernel, that is, $G(t) = \frac{t^n \alpha^{n+1} e^{-\alpha t}}{n!}$, where $\alpha > 0$ is a constant ([54]).

On the other hand, it should be noted that the dynamic evolution of biological systems inherently possesses significant stochastic characteristics ([53, 55, 57]), and the deterministic systems exhibit theoretical limitations in ecosystem modeling, as the parameters involved in such systems cannot capture the influence of environmental noises ([24, 36]). Gaussian white noises can serve as a common and effective approach to characterize environmental disturbances ([6, 23, 29, 56]). Assumed that $r_1 \to r_1 + \sigma_1\dot{B}_1(t)$, $-r_i \to -r_i + \sigma_i\dot{B}_i(t)$, where $B_i(t)$ are mutually-independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, then system (2) with white noise is modified as

$$\begin{cases} \mathrm{d}N_1(t) = N_1(t)\left[b_1 - C_{11}N_1(t) - C_{12}N_2(t)\right]\mathrm{d}t + \sigma_1 N_1(t)\mathrm{d}B_1(t), \\ \mathrm{d}N_2(t) = N_2(t)\left[-b_2 + C_{21}\displaystyle\int_{-\infty}^{t} G(t-\phi)N_1(\phi)\,\mathrm{d}\phi - C_{22}N_2(t) - C_{23}N_3(t)\right]\mathrm{d}t + \sigma_2 N_2(t)\mathrm{d}B_2(t), \\ \mathrm{d}N_3(t) = N_3(t)\left[-b_3 + C_{32}N_3(t) - C_{33}N_3(t)\right]\mathrm{d}t + \sigma_3 N_3(t)\mathrm{d}B_3(t). \end{cases} \tag{3}$$

For convenience, this paper introduces the weak kernel function ($n = 0$) $G(t) = \alpha e^{-\alpha t}$ to describe the historical dependence of species interactions. Hence, system (3) is equal to

$$\begin{cases} dN_1(t) = N_1(t)\left[b_1 - C_{11}N_1(t) - C_{12}N_2(t)\right]dt + \sigma_1 N_1(t)dB_1(t), \\ dN_2(t) = N_2(t)\left[-b_2 + C_{21}\int_{-\infty}^{t}\alpha e^{-\alpha(t-\phi)}N_1(\phi)\,d\phi - C_{22}N_2(t) - C_{23}N_3(t)\right]dt + \sigma_2 N_2(t)dB_2(t), \\ dN_3(t) = N_3(t)\left[-b_3 + C_{32}N_3(t) - C_{33}N_3(t)\right]dt + \sigma_3 N_3(t)dB_3(t). \end{cases} \tag{4}$$

Let $Z_0(t) = \int_{-\infty}^{t}\alpha e^{-\alpha(t-\phi)}N_1(\phi)d\phi$, by the linear chain technique in [18], system (4) is transformed into

$$\begin{cases} dN_1(t) = N_1(t)\left[b_1 - C_{11}N_1(t) - C_{12}N_2(t)\right]dt + \sigma_1 N_1(t)dB_1(t), \\ dN_2(t) = N_2(t)\left[-b_2 + C_{21}Z_0(t) - C_{22}N_2(t) - C_{23}N_3(t)\right]dt + \sigma_2 N_2(t)dB_2(t), \\ dN_3(t) = N_3(t)\left[-b_3 + C_{32}N_2(t) - C_{33}N_3(t)\right]dt + \sigma_3 N_3(t)dB_3(t), \\ dZ_0(t) = \alpha\left[N_1(t) - Z_0(t)\right]dt. \end{cases} \tag{5}$$

The remainder of this paper is organized as follows: In Section 2, the existence and uniqueness of the global positive solution, the moment boundedness of the solution and the asymptotic properties of the solution to system (5) are studied. In Section 3, sufficient conditions for the global attractiveness of system (5) are revealed. In Section 4, we deduce the exact expression of the probability density function of the linearized system corresponding to the stochastic system (5) around the quasi-positive equilibrium. In Section 5, we provide some numerical examples to verify the validity of the results.

2. Existence and Uniqueness of Global Positive Solution

In order to study the long-time dynamical behavior of system (5), we first investigate the existence and uniqueness of its global positive solution.

Theorem 2.1. *For any initial value* $\mathbf{X}(0) = (N_1(0), N_2(0), N_3(0), Z_0(0))^{\mathrm{T}} \in \mathbb{R}_+^4$, *system (5) has a unique solution* $\mathbf{X}(t) = (N_1(t), N_2(t), N_3(t), Z_0(t))^{\mathrm{T}} \in \mathbb{R}_+^4$ *for all* $t \geq 0$ *with probability one.*

Proof. Since the coefficients of system (5) are locally Lipschitz continuous, thanks to Mao ([28]), there exists a unique local solution $(N_1(t), N_2(t), N_3(t), Z_0(t))^{\mathrm{T}} \in \mathbb{R}_+^4$ on $t \in [0, \rho_e)$ for any given initial value $\mathbf{X}(0) \in \mathbb{R}_+^4$, where $\rho_e \in [0, +\infty]$ denotes the explosion time. To prove this solution is global, we only need to show that $\rho_e = +\infty$ a.s.. Let $l_0 > 0$ be sufficiently large such that $N_i(0)$, $Z_0(0)$ all belong to interval $\left[l_0^{-1}, l_0\right]$. For each integer $l \geq l_0$, define the stopping time

$$\tau_l = \inf\left\{t \in (0, \rho_e) : \min\{N_1(t), N_2(t), N_3(t), Z_0(t)\} \leq l^{-1} \text{ or } \max\{N_1(t), N_2(t), N_3(t), Z_0(t)\} \geq l\right\}.$$

Apparently, τ_l is increasing as $l \to +\infty$. Let $\tau_\infty = \lim_{l\to+\infty}\tau_l$, then $\tau_\infty \leq \rho_e$ a.s.. If we can demonstrate that $\tau_\infty = +\infty$ a.s., then $\rho_e = +\infty$ a.s.. Using proof by contradiction, if $\mathbb{P}(\tau_\infty = +\infty) < 1$, then there exist constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}(\tau_\infty \leq T) > \varepsilon$. Consequently, there exists an integer $l_1 \geq l_0$ such that for all $l \geq l_1$, $\mathbb{P}(\tau_l \leq T) \geq \varepsilon$. Define a C^2-function $V : \mathbb{R}_+^4 \to \mathbb{R}_+$ as follows:

$$V(N_1, N_2, N_3, Z_0) = \sum_{i=1}^{3}\left[d_i\left(N_i - 1 - \ln N_i\right)\right] + d_4\left(Z_0^2 + Z_0 - 1 - \ln Z_0\right), \tag{1}$$

where $d_1 = 1$, $d_2 = \frac{C_{11}C_{22}}{2C_{21}^2}$, $d_3 = \frac{C_{11}C_{22}^2C_{33}}{3C_{21}^2C_{32}^2}$, $d_4 = \frac{C_{11}}{2\alpha}$. Applying Itô's formula to V yields

$$dV = \mathcal{L}V dt + \sum_{i=1}^{3} d_i\sigma_i\left(N_i - 1\right)dB_i(t), \tag{2}$$

where

$$\begin{aligned}\mathcal{L}V =& - C_{11}N_1^2 - d_2C_{22}N_2^2 - d_3C_{33}N_3^2 - 2d_4\alpha Z_0^2 - C_{12}N_1N_2 - (d_2C_{23} - d_3C_{32})N_2N_3 \\ &+ 2d_4\alpha N_1Z_0 + d_2C_{21}N_2Z_0 + (b_1 + C_{11})N_1 + (-d_2b_2 + C_{12} + d_2C_{22} - d_3C_{32})N_2 \\ &+ (-d_3b_3 + d_2C_{23} + d_3C_{33})N_3 - d_2C_{21}Z_0 - b_1 + d_2b_2 + d_3b_3 + d_4\alpha N_1 - d_4\alpha Z_0 \\ &- d_4\frac{\alpha N_1}{Z_0} + d_4\alpha + \frac{\sigma_1^2}{2} + \frac{d_2\sigma_2^2}{2} + \frac{d_3\sigma_3^2}{2} \\ \le& (-C_{11} + d_4\alpha)N_1^2 + (b_1 + C_{11} + d_4\alpha)N_1 + \left(-\frac{d_2C_{22}}{2} + \frac{d_3C_{32}^2}{2C_{33}}\right)N_2^2 \\ &+ (C_{12} - d_2b_2 + d_2C_{22} - d_3C_{32})N_2 - \frac{d_3C_{33}}{2}N_3^2 + (-d_3b_3 + d_3C_{33} + d_2C_{23})N_3 \\ &+ \left(-d_4\alpha + \frac{d_2C_{21}^2}{2C_{22}}\right)Z_0^2 - (d_2C_{21} + d_4\alpha)Z_0 - b_1 + \frac{\sigma_1^2}{2} + d_2\left(b_2 + \frac{\sigma_2^2}{2}\right) + d_3\left(b_3 + \frac{\sigma_3^2}{2}\right) + d_4\alpha \\ =& -\frac{C_{11}}{2}N_1^2 + (b_1 + C_{11} + d_4\alpha)N_1 - \frac{C_{11}C_{22}^2}{12C_{21}^2}N_2^2 + (C_{12} - d_2b_2 + d_2C_{22} - d_3C_{32})N_2 \\ &- \frac{C_{11}C_{22}^2C_{33}^2}{6C_{21}^2C_{32}^2}N_3^2 + (-d_3b_3 + d_3C_{33} + d_2C_{23})N_3 - \frac{C_{11}}{4}Z_0^2 - (d_2C_{21} + d_4\alpha)Z_0 \\ &- b_1 + \frac{\sigma_1^2}{2} + d_2\left(b_2 + \frac{\sigma_2^2}{2}\right) + d_3\left(b_3 + \frac{\sigma_3^2}{2}\right) + d_4\alpha \\ \le& k_1,\end{aligned}$$

where $k_1 > 0$ is a constant which is independent of $\mathbf{X}(0)$. Integrating both sides of Eq. (2) from 0 to $\tau_l \wedge T$ and then taking expectations yields

$$\mathbb{E}\left[V\left(N_1(\tau_l \wedge T), N_2(\tau_l \wedge T), N_3(\tau_l \wedge T), Z_0(\tau_l \wedge T)\right)\right] \le V(N_1(0), N_2(0), N_3(0), Z_0(0)) + k_1T.$$

Let $\Omega_l = \{\tau_l \le T\}$ for $l \ge l_1$ and I_{Ω_l} be the indicator function of Ω_l. Note that for every $\omega \in \Omega_l$, $N_1(\tau_l, \omega)$, $N_2(\tau_l, \omega)$, $N_3(\tau_l, \omega)$, $Z_0(\tau_l, \omega)$ equals either l or $\frac{1}{l}$, then

$$\begin{aligned}& V(N_1(0), N_2(0), N_3(0), Z_0(0)) + k_1T \\ \ge& \mathbb{E}\left[I_{\Omega_l}V(N_1(\tau_l, \omega), N_2(\tau_l, \omega), N_3(\tau_l, \omega), Z_0(\tau_l, \omega))\right] \\ \ge& \varepsilon\left[(l - 1 - \ln l) \wedge \left(\frac{1}{l} - 1 - \ln\frac{1}{l}\right) \wedge d_2(l - 1 - \ln l) \wedge d_2\left(\frac{1}{l} - 1 - \ln\frac{1}{l}\right)\right. \\ &\left. \wedge d_3(l - 1 - \ln l) \wedge d_3\left(\frac{1}{l} - 1 - \ln\frac{1}{l}\right) \wedge d_4\left(l^2 + l - 1 - \ln l\right) \wedge d_4\left(\frac{1}{l^2} + \frac{1}{l} - 1 - \ln\frac{1}{l}\right)\right] \\ \ge& \varepsilon\min\{1, d_2, d_3, d_4\}\min\left\{l - 1 - \ln l, \frac{1}{l} - 1 - \ln\frac{1}{l}\right\}.\end{aligned}$$

Taking $l \to +\infty$ yields $V(N_1(0), N_2(0), N_3(0), Z_0(0)) + k_1T = +\infty$, which is a contradiction. This completes the proof of Theorem 2.1. □

Consider the following stochastic differential equation:

$$d\Phi(t) = \Phi(t)\left[r - a\Phi(t)\right]dt + \sigma\Phi(t)dB(t). \tag{3}$$

Lemma 2.2. *([18]) For any $p > 1$, system (3) satisfies*

$$\limsup_{t\to+\infty}\mathbb{E}\left[\Phi^p(t)\right] \le \left(\frac{r + \frac{p-1}{2}\sigma^2}{a}\right)^p.$$

Theorem 2.3. *Let $\mathbf{X}(t)$ be the solution to system (5) with initial value $\mathbf{X}(0) \in \mathbb{R}_+^4$, then for any $p > 1$, there exists a constant $K(p) > 0$ such that*

$$\mathbb{E}\left[Z_0^p(t)\right] \le K(p),\ \mathbb{E}\left[N_i^p(t)\right] \le K(p)\ (i = 1,2,3).$$

Proof. Thanks to the comparison theorem for stochastic differential equations ([17]) and Lemma 2.2, we have

$$\limsup_{t\to+\infty} \mathbb{E}\left[N_1^p(t)\right] \le \left(\frac{b_1 + \frac{p-1}{2}\sigma_1^2}{C_{11}}\right)^p,$$

which implies that there exists a constant $K_1(p) > 0$ such that $\mathbb{E}\left[N_1^p(t)\right] \le K_1(p)$ for all $t \ge 0$. Compute

$$\begin{aligned}
\mathrm{d}Z_0^p(t) &= \alpha p Z_0^{p-1}(t)\left[N_1(t) - Z_0(t)\right]\mathrm{d}t \le \alpha\left[N_1^p(t) - Z_0^p(t)\right]\mathrm{d}t,\\
\frac{\mathrm{d}\mathbb{E}\left[Z_0^p(t)\right]}{\mathrm{d}t} &\le \alpha\left(\mathbb{E}\left[N_1^p(t)\right] - \mathbb{E}\left[Z_0^p(t)\right]\right) \le \alpha\left(K_1(p) - \mathbb{E}\left[Z_0^p(t)\right]\right),
\end{aligned}$$

which implies that $\limsup_{t\to+\infty} \mathbb{E}\left[Z_0^p(t)\right] \le K_1(p)$. Hence, there exists a constant $K_4(p) > 0$ such that $\mathbb{E}\left[Z_0^p(t)\right] \le K_4(p)$ for all $t \ge 0$. By Itô's formula, we have

$$\begin{aligned}
\mathcal{L}\left[N_2^p\right] &= pN_2^p\left[-b_2 + C_{21}Z_0 - C_{22}N_2 - C_{23}N_3\right] + \frac{p(p-1)}{2}\sigma_2^2 N_2^p\\
&\le -pb_2N_2^p + pC_{21}N_2^pZ_0 - pC_{22}N_2^{p+1} + \frac{p(p-1)}{2}\sigma_2^2N_2^p\\
&\le -pb_2N_2^p + \frac{pC_{22}}{2}N_2^{p+1} + \left(\frac{p}{p+1}\right)^{p+1} C_{21}^{p+1}\left(\frac{2}{C_{22}}\right)^p Z_0^{p+1} - pC_{22}N_2^{p+1} + \frac{p(p-1)}{2}\sigma_2^2N_2^p\\
&= -pb_2N_2^p - \frac{pC_{22}}{2}N_2^{p+1} + \frac{p(p-1)}{2}\sigma_2^2N_2^p + \left(\frac{p}{p+1}\right)^{p+1} C_{21}^{p+1}\left(\frac{2}{C_{22}}\right)^p Z_0^{p+1},\\
\mathrm{d}\left[e^tN_2^p\right] &= e^t\left(N_2^p + \mathcal{L}\left[N_2^p\right]\right)\mathrm{d}t + p\sigma_2 e^t N_2^p \mathrm{d}B_2(t)\\
&\le e^t\left[H_2 + \left(\frac{p}{p+1}\right)^{p+1} C_{21}^{p+1}\left(\frac{2}{C_{22}}\right)^p Z_0^{p+1}\right]\mathrm{d}t + p\sigma_2 e^t N_2^p \mathrm{d}B_2(t),
\end{aligned}$$

where $H_2 = \sup_{N_2\in\mathbb{R}_+}\left[N_2^p - pb_2N_2^p - \frac{pC_{22}}{2}N_2^{p+1} + \frac{p(p-1)}{2}\sigma_2^2N_2^p\right]$. Hence,

$$\begin{aligned}
\mathbb{E}\left[e^tN_2^p(t)\right] - N_2^p(0) &\le \int_0^t e^s\left\{H_2 + \left(\frac{p}{p+1}\right)^{p+1} C_{21}^{p+1}\left(\frac{2}{C_{22}}\right)^p \mathbb{E}\left[Z_0^{p+1}(s)\right]\right\}\mathrm{d}s\\
&\le \int_0^t e^s\left\{H_2 + \left(\frac{p}{p+1}\right)^{p+1} C_{21}^{p+1}\left(\frac{2}{C_{22}}\right)^p K_4(p+1)\right\}\mathrm{d}s\\
&= \left\{H_2 + \left(\frac{p}{p+1}\right)^{p+1} C_{21}^{p+1}\left(\frac{2}{C_{22}}\right)^p K_4(p+1)\right\}\left(e^t - 1\right),
\end{aligned}$$

which implies

$$\limsup_{t\to+\infty} \mathbb{E}\left[N_2^p(t)\right] \le H_2 + \left(\frac{p}{p+1}\right)^{p+1} C_{21}^{p+1}\left(\frac{2}{C_{22}}\right)^p K_4(p+1).$$

Thus, there exists a constant $K_2(p) > 0$ such that $\mathbb{E}\left[N_2^p(t)\right] \le K_2(p)$ for all $t \ge 0$. Compute

$$\begin{aligned}\mathcal{L}\left[N_3^p\right] &= pN_3^p\left[-b_3 + C_{32}N_2 - C_{33}N_3\right] + \frac{p(p-1)}{2}\sigma_3^2 N_3^p \\ &\le -pb_3N_3^p - \frac{pC_{33}}{2}N_3^{p+1} + \frac{p(p-1)}{2}\sigma_3^2N_3^p + \left(\frac{p}{p+1}\right)^{p+1} C_{32}^{p+1}\left(\frac{2}{C_{33}}\right)^p N_2^{p+1}, \\ \mathrm{d}\left[\mathrm{e}^t N_3^p\right] &= \mathrm{e}^t\left(N_3^p + \mathcal{L}\left[N_3^p\right]\right)\mathrm{d}t + p\sigma_3\mathrm{e}^t N_3^p \mathrm{d}B_3(t) \\ &\le \mathrm{e}^t\left[H_3 + \left(\frac{p}{p+1}\right)^{p+1} C_{32}^{p+1}\left(\frac{2}{C_{33}}\right)^p N_2^{p+1}\right]\mathrm{d}t + p\sigma_3\mathrm{e}^t N_3^p\mathrm{d}B_3(t),\end{aligned}$$

where $H_3 = \sup_{N_3\in\mathbb{R}_+}\left[N_3^p - pb_3N_3^p - \frac{pC_{33}}{2}N_3^{p+1} + \frac{p(p-1)}{2}\sigma_3^2N_3^p\right]$. Hence,

$$\mathbb{E}\left[\mathrm{e}^t N_3^p(t)\right] - N_3^p(0) \le \left\{H_3 + \left(\frac{p}{p+1}\right)^{p+1} C_{32}^{p+1}\left(\frac{2}{C_{33}}\right)^p K_2(p+1)\right\}\left(\mathrm{e}^t - 1\right),$$

which implies

$$\limsup_{t\to+\infty}\mathbb{E}\left[N_3^p(t)\right] \le H_3 + \left(\frac{p}{p+1}\right)^{p+1} C_{32}^{p+1}\left(\frac{2}{C_{33}}\right)^p K_2(p+1).$$

Therefore, there exists a constant $K_3(p) > 0$ such that $\mathbb{E}\left[N_3^p(t)\right] \le K_3(p)$ for all $t \ge 0$. □

Theorem 2.4. *Let $\mathbf{X}(t)$ be the solution to system (5) with initial value $\mathbf{X}(0) \in \mathbb{R}_+^4$. Then*

$$\lim_{t\to+\infty} t^{-1}Z_0(t) = 0, \ \limsup_{t\to+\infty} t^{-1}\ln N_i(t) \le 0 \ a.s. \ (i = 1, 2, 3).$$

Proof. Let $Q(t) = \sum_{i=1}^3 d_iN_i(t) + d_4Z_0^2(t)$. Choose $\beta = 1 + \frac{\alpha}{2(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)}$, we compute

$$\mathrm{d}\,(1+Q)^\beta = \mathcal{L}\,(1+Q)^\beta\,\mathrm{d}t + \beta\,(1+Q)^{\beta-1}\sum_{i=1}^3 \sigma_i d_i N_i(t)\mathrm{d}B_i(t),$$

where

$$\begin{aligned}\mathcal{L}\,(1+Q)^\beta = {} & \beta\,(1+Q)^{\beta-1}\left[N_1\left(b_1 - C_{11}N_1 - C_{12}N_2\right) + d_2N_2\left(-b_2 + C_{21}Z_0 - C_{22}N_2 - C_{23}N_3\right)\right. \\ & \left. + d_3N_3\left(-b_3 + C_{32}N_2 - C_{33}N_3\right) + 2d_4\alpha Z_0\left(N_1 - Z_0\right)\right] \\ & + \frac{1}{2}\beta(\beta-1)\,(1+Q)^{\beta-2}\left[\sigma_1^2N_1^2 + \sigma_2^2d_2^2N_2^2 + \sigma_3^2d_3^2N_3^2\right].\end{aligned}$$

Compute $I_1 = \sigma_1^2N_1^2 + \sigma_2^2d_2^2N_2^2 + \sigma_3^2d_3^2N_3^2 \le \left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)Q^2$ and

$$\begin{aligned}I_2 = {} & (1+Q)\left[N_1\left(b_1 - C_{11}N_1 - C_{12}N_2\right) + d_2N_2\left(-b_2 + C_{21}Z_0 - C_{22}N_2 - C_{23}N_3\right)\right. \\ & \left. + d_3N_3\left(-b_3 + C_{32}N_2 - C_{33}N_3\right) + 2d_4\alpha Z_0\left(N_1 - Z_0\right)\right] \\ \le {} & (1+Q)\left[-\left(C_{11} - \alpha d_4\right)N_1^2 + b_1N_1 - \left(\frac{d_2C_{22}}{2} - \frac{d_3C_{32}^2}{2C_{33}}\right)N_2^2 - d_2b_2N_2 - \frac{d_3C_{33}}{2}N_3^2 - d_3b_3N_3\right. \\ & \left. - \left(\alpha - \frac{d_2C_{21}^2}{2d_4C_{22}}\right)d_4Z_0^2\right] \\ = {} & (1+Q)\left[-\frac{\alpha}{2}Q - \frac{C_{11}}{2}N_1^2 + \left(b_1 + \frac{\alpha}{2}\right)N_1 - \frac{C_{11}C_{22}^2}{12C_{21}^2}N_2^2 + d_2\left(\frac{\alpha}{2} - b_2\right)N_2 - \frac{C_{11}C_{22}^2C_{33}^2}{6C_{21}^2C_{32}^2}N_3^2 + d_3\left(\frac{\alpha}{2} - b_3\right)N_3\right] \\ \le {} & -\frac{\alpha}{2}Q^2 + h(1+Q),\end{aligned}$$

where

$$h=\sup_{(N_1,N_2,N_3)\in\mathbb{R}_+^3}\left\{-\frac{C_{11}}{2}N_1^2+\left(b_1+\frac{\alpha}{2}\right)N_1-\frac{C_{11}C_{22}^2}{12C_{21}^2}N_2^2+d_2\left(\frac{\alpha}{2}-b_2\right)N_2-\frac{C_{11}C_{22}^2C_{33}^2}{6C_{21}^2C_{32}^2}N_3^2+d_3\left(\frac{\alpha}{2}-b_3\right)N_3\right\}.$$

Hence,

$$\mathrm{d}\,(1+Q)^\beta\le\beta\,(1+Q)^{\beta-2}\left[-\frac{\alpha}{4}Q^2+h(1+Q)\right]\mathrm{d}t+\beta\,(1+Q)^{\beta-1}\sum_{i=1}^3\sigma_id_iN_i(t)\mathrm{d}B_i(t). \tag{4}$$

Compute

$$\begin{aligned}\mathrm{d}\left[\mathrm{e}^{\frac{\alpha\beta}{8}t}\,(1+Q)^\beta\right]&=\left\{\frac{\alpha\beta}{8}\mathrm{e}^{\frac{\alpha\beta}{8}t}\,(1+Q)^\beta+\mathrm{e}^{\frac{\alpha\beta}{8}t}\mathcal{L}\left[(1+Q)^\beta\right]\right\}\mathrm{d}t+\mathrm{e}^{\frac{\alpha\beta}{8}t}\beta\,(1+Q)^{\beta-1}\sum_{i=1}^3\sigma_id_iN_i(t)\mathrm{d}B_i(t)\\&\le\mathrm{e}^{\frac{\alpha\beta}{8}t}\left\{\frac{\alpha\beta}{8}\,(1+Q)^\beta+\beta\,(1+Q)^{\beta-2}\left[-\frac{\alpha}{4}Q^2+h(1+Q)\right]\right\}\mathrm{d}t\\&\quad+\mathrm{e}^{\frac{\alpha\beta}{8}t}\beta\,(1+Q)^{\beta-1}\sum_{i=1}^3\sigma_id_iN_i(t)\mathrm{d}B_i(t)\\&\le H\mathrm{e}^{\frac{\alpha\beta}{8}t}\mathrm{d}t+\mathrm{e}^{\frac{\alpha\beta}{8}t}\beta\,(1+Q)^{\beta-1}\sum_{i=1}^3\sigma_id_iN_i(t)\mathrm{d}B_i(t),\end{aligned}$$

where $H=\sup_{Q\in\mathbb{R}_+}\left\{\frac{\alpha\beta}{8}\,(1+Q)^\beta+\beta\,(1+Q)^{\beta-2}\left[-\frac{\alpha}{4}Q^2+h(1+Q)\right]\right\}<+\infty$. Hence,

$$\mathrm{e}^{\frac{\alpha\beta}{8}t}\mathbb{E}\left[(1+Q)^\beta\right]-(1+Q(0))^\beta\le\int_0^tH\mathrm{e}^{\frac{\alpha\beta}{8}s}\mathrm{d}s=\frac{8H}{\alpha\beta}\left(\mathrm{e}^{\frac{\alpha\beta}{8}t}-1\right).$$

Therefore,

$$\limsup_{t\to+\infty}\mathbb{E}\left[(1+Q)^\beta\right]\le\frac{8H}{\alpha\beta},$$

which together with the continuity of $\mathbb{E}\left[(1+Q)^\beta\right]$ tells us that there exist an $H_0>0$ such that

$$\mathbb{E}\left[(1+Q)^\beta\right]\le H_0\ \text{ for all } t\ge0.$$

Choose $\delta>0$ satisfying $\beta\left(\frac{\alpha}{4}+|h|\right)\delta+\sqrt{32}\beta\sqrt{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2}\sqrt{\delta}=\frac{1}{2}$. Thanks to (4), for $k=1,2,3,...,$

$$\begin{aligned}&\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}(1+Q(t))^\beta\right]\\&\le\mathbb{E}\left[(1+Q\,(k\delta))^\beta\right]+\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}\left|\int_{k\delta}^t\beta\,(1+Q(s))^{\beta-2}\left[-\frac{\alpha}{4}Q^2(s)+h\,(1+Q(s))\right]\mathrm{d}s\right|\right]\\&\quad+\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}\left|\int_{k\delta}^t\beta\,(1+Q(s))^{\beta-1}\sum_{i=1}^3\sigma_id_iN_i(s)\mathrm{d}B_i(s)\right|\right]\\&\le H_0+I_1^*+I_2^*,\end{aligned}$$

where

$$\begin{aligned}
I_1^* &= \mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}\left|\int_{k\delta}^{t}\beta\,(1+Q(s))^{\beta-2}\left[-\frac{\alpha}{4}Q^2(s)+h\,(1+Q(s))\right]\mathrm{d}s\right|\right]\\
&\le \mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}\int_{k\delta}^{t}\beta\left(\frac{\alpha}{4}+|h|\right)(1+Q(s))^{\beta}\,\mathrm{d}s\right]\\
&\le \beta\left(\frac{\alpha}{4}+|h|\right)\delta\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}(1+Q(t))^{\beta}\right],
\end{aligned}$$

and thanks to the Burkholder-Davis-Gundy inequality ([28]),

$$\begin{aligned}
I_2^* &= \mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}\left|\int_{k\delta}^{t}\beta\,(1+Q(s))^{\beta-1}\sum_{i=1}^{3}\sigma_i d_i N_i(s)\mathrm{d}B_i(s)\right|\right]\\
&\le \sqrt{32}\mathbb{E}\left[\int_{k\delta}^{(k+1)\delta}\beta^2\,(1+Q(s))^{2\beta-2}\sum_{i=1}^{3}\sigma_i^2 d_i^2 N_i^2(s)\mathrm{d}s\right]^{\frac{1}{2}}\\
&\le \sqrt{32}\mathbb{E}\left[\int_{k\delta}^{(k+1)\delta}\beta^2\,(1+Q(s))^{2\beta-2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)(1+Q(s))^2\,\mathrm{d}s\right]^{\frac{1}{2}}\\
&\le \sqrt{32}\mathbb{E}\left[\beta^2\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)\int_{k\delta}^{(k+1)\delta}\sup_{k\delta\le s\le(k+1)\delta}(1+Q(s))^{2\beta}\,\mathrm{d}s\right]^{\frac{1}{2}}\\
&= \sqrt{32}\mathbb{E}\left[\beta^2\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)\delta\sup_{k\delta\le t\le(k+1)\delta}(1+Q(t))^{2\beta}\right]^{\frac{1}{2}}\\
&\le \sqrt{32}\beta\sqrt{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2}\,\sqrt{\delta}\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}(1+Q(t))^{\beta}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}(1+Q(t))^{\beta}\right] &\le H_0+\left[\beta\left(\frac{\alpha}{4}+|h|\right)\delta+\sqrt{32}\beta\sqrt{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2}\,\sqrt{\delta}\right]\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}(1+Q(t))^{\beta}\right]\\
&= H_0+\frac{1}{2}\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}(1+Q(t))^{\beta}\right],
\end{aligned}$$

which implies

$$\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}(1+Q(t))^{\beta}\right]\le 2H_0.$$

Thanks to Chebyshev's inequality, for any $\gamma\in(0,1)$ and $k=1,2,3,\cdots$, it follows

$$\mathbb{P}\left\{\sup_{k\delta\le t\le(k+1)\delta}(1+Q(t))^{\beta}\ge(k\delta)^{1+\gamma}\right\}\le\frac{2H_0}{(k\delta)^{1+\gamma}},$$

and

$$\sum_{k=1}^{\infty}\mathbb{P}\left\{\sup_{k\delta\le t\le(k+1)\delta}(1+Q(t))^{\beta}\ge(k\delta)^{1+\gamma}\right\}\le\frac{2H_0}{\delta^{1+\gamma}}\sum_{k=1}^{\infty}\frac{1}{k^{1+\gamma}}<+\infty,$$

Thanks to Borel-Cantelli's lemma, we have

$$\mathbb{P}\left[\limsup_{k\to+\infty}\left(\sup_{k\delta\le t\le(k+1)\delta}(1+Q(t))^{\beta}\ge(k\delta)^{1+\gamma}\right)\right]=0.$$

Thus, there exists an integer-valued random variable $k_0(\omega)$ such that for almost all $\omega\in\Omega$,

$$\sup_{k\delta\le t\le(k+1)\delta}(1+Q(t))^{\beta}<(k\delta)^{1+\gamma}$$

holds whenever $k\ge k_0(\omega)$. Hence, for almost all $\omega\in\Omega$, if $k\ge k_0$ and $k\delta\le t\le(k+1)\delta$, it is true that

$$\frac{\ln(1+Q(t))^{\beta}}{\ln t}\le\frac{(1+\gamma)\ln(k\delta)}{\ln(k\delta)}=1+\gamma,\quad \limsup_{t\to+\infty}\frac{\ln(1+Q(t))^{\beta}}{\ln t}\le 1+\gamma.$$

Letting $\gamma\to0^{+}$ yields

$$\limsup_{t\to+\infty}\frac{\ln(1+Q(t))^{\beta}}{\ln t}\le 1,\quad \limsup_{t\to+\infty}\frac{\ln Q(t)}{\ln t}\le\frac{1}{\beta}<1.$$

Hence, we deduce

$$\limsup_{t\to+\infty}t^{-1}\ln N_i(t)\le 0\quad(i=1,2,3),$$

and there exists a constant $T\gg1$ such that for any $t>T$,

$$\ln Q(t)\le\frac{\beta+1}{2\beta}\ln t,$$

which implies

$$\lim_{t\to+\infty}t^{-1}Q(t)=0,\ \lim_{t\to+\infty}t^{-1}Z_0^2(t)=0,\ \lim_{t\to+\infty}t^{-1}Z_0(t)=0.$$

□

3. Global Attractivity

Definition 3.1. *([18]) Let $\widetilde{\mathbf{X}}(t)=\left(\widetilde{N}_1(t),\widetilde{N}_2(t),\widetilde{N}_3(t),\widetilde{Z}_0(t)\right)$ be a positive solution of system (5) and $\mathbf{X}(t)$ be the solution of system (5) with initial value $\mathbf{X}(0)\in\mathbb{R}_+^4$. If*

$$\lim_{t\to+\infty}\left|Z_0(t)-\widetilde{Z}_0(t)\right|=0,\ \lim_{t\to+\infty}\left|N_i(t)-\widetilde{N}_i(t)\right|=0,\ (i=1,2,3),$$

then $\widetilde{\mathbf{X}}(t)$ is globally attractive.

Lemma 3.2. *Let $f(t)$ be a nonnegative function defined on $[0,\infty)$ such that $f(t)$ is integrable on $[0,\infty)$ and is uniformly continuous on $[0,\infty)$. Then $\lim_{t\to+\infty}f(t)=0$.*

Lemma 3.3. *([18]) Let $X(\cdot)$ be a stochastic process with continuous sample paths a.s., such that*

$$\mathbb{E}\left[|X(t)-X(s)|^{\varpi}\right]\le C\,|t-s|^{1+\varrho}$$

for constants $\varpi,\varrho>0$, $C\ge0$ and for all $0\le t,s$. Then for each $0<\zeta<\frac{\varrho}{\varpi}$, $T>0$, and almost every ω, there exists a constant $K=K(\omega,\zeta,T)$ such that

$$|X(t,\omega)-X(s,\omega)|\le K\,|t-s|^{\zeta}\ \text{ for all } 0\le t,s\le T.$$

Hence the sample path $t\mapsto X(t,\omega)$ is uniformly Hölder continuous with exponent ζ on $[0,T]$.

Theorem 3.4. *Let $\mathbf{X}(t)$ be the solution to system (5) with initial value $\mathbf{X}(0) \in \mathbb{R}^4_+$. Then almost every sample path of $\mathbf{X}(t)$ is uniformly continuous on $t \geq 0$.*

Proof. Let $p > 2$ and $t - s \in [0, 1]$. Thanks to system (5), we compute

$$\mathbb{E}\left[|N_1(t) - N_1(s)|^p\right] \leq 2^{p-1}\left(J_{11} + J_{12}\right),$$

where

$$\begin{aligned} J_{11} &= \mathbb{E}\left|\int_s^t N_1(\theta)\left[b_1 - C_{11}N_1(\theta) - C_{12}N_2(\theta)\right]\mathrm{d}\theta\right|^p \\ &\leq 3^{p-1}\left[b_1^p K_1(p) + \left(C_{11} + \frac{C_{12}}{2}\right)^p K_1(2p) + \left(\frac{C_{12}}{2}\right)^p K_2(2p)\right]|t-s|^p, \\ J_{12} &= \mathbb{E}\left[\left|\int_s^t \sigma_1 N_1(\theta)\mathrm{d}B_1(\theta)\right|^p\right] \leq \left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}} \sigma_1^p K_1(p)\,|t-s|^{\frac{p}{2}}, \end{aligned}$$

where Theorem 7.1 in [28] has been used. Therefore, $\mathbb{E}\left[|N_1(t) - N_1(s)|^p\right] \leq L_1\,|t-s|^{\frac{p}{2}}$, where

$$L_1 = 6^{p-1}\left[b_1^p K_1(p) + \left(C_{11} + \frac{C_{12}}{2}\right)^p K_1(2p) + \left(\frac{C_{12}}{2}\right)^p K_2(2p)\right] + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}} \sigma_1^p K_1(p).$$

Similarly, we can derive

$$\mathbb{E}\left[|N_2(t) - N_2(s)|^p\right] \leq L_2\,|t-s|^{\frac{p}{2}}, \quad \mathbb{E}\left[|N_3(t) - N_3(s)|^p\right] \leq L_3\,|t-s|^{\frac{p}{2}}, \quad \mathbb{E}\left[|Z_0(t) - Z_0(s)|^p\right] \leq L_4\,|t-s|^{\frac{p}{2}},$$

where

$$\begin{aligned} L_2 &= 8^{p-1}\left[b_2^p K_2(p) + \left(\frac{C_{21}}{2} + C_{22} + \frac{C_{23}}{2}\right)^p K_2(2p) + \left(\frac{C_{23}}{2}\right)^p K_3(2p) + \left(\frac{C_{21}}{2}\right)^p K_4(2p)\right] + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}} \sigma_2^p K_2(p), \\ L_3 &= 6^{p-1}\left[b_3^p K_3(p) + \left(\frac{C_{32}}{2} + C_{33}\right)^p K_3(2p) + \left(\frac{C_{32}}{2}\right)^p K_2(2p)\right] + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}} \sigma_3^p K_3(p), \\ L_4 &= 2^{p-1}\alpha^p\left[K_1(p) + K_4(p)\right]. \end{aligned}$$

Define $C = 4^{\frac{p-2}{2}}\sum_{i=1}^4 L_i > 0$, $\varrho = \frac{p-2}{2} > 0$. Then,

$$\begin{aligned} \mathbb{E}\left[|\mathbf{X}(t) - \mathbf{X}(s)|^p\right] &= \mathbb{E}\left[\sum_{i=1}^3 |N_i(t) - N_i(s)|^2 + |Z_0(t) - Z_0(s)|^2\right]^{\frac{p}{2}} \\ &\leq 4^{\frac{p-2}{2}}\mathbb{E}\left[\sum_{i=1}^3 |N_i(t) - N_i(s)|^p + |Z_0(t) - Z_0(t)|^p\right] \\ &\leq 4^{\frac{p-2}{2}}\sum_{i=1}^4 L_i\,|t-s|^{\frac{p}{2}} = C\,|t-s|^{1+\varrho}. \end{aligned}$$

Therefore, almost every sample path $\mathbf{X}(t)$ is uniformly continuous on $t \geq 0$ according to Lemma 3.3. □

Theorem 3.5. *Assume that $C_{11}C_{22}C_{33} - C_{11}C_{23}C_{32} - C_{12}C_{21}C_{33} > 0$. Then $\widetilde{\mathbf{X}}(t)$ is globally attractive.*

Proof. From system (5), we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Z_0(t)-\widetilde{Z}_0(t)\right)=\alpha\left[\left(N_1(t)-\widetilde{N}_1(t)\right)-\left(Z_0(t)-\widetilde{Z}_0(t)\right)\right],$$

$$Z_0(t)-\widetilde{Z}_0(t)=\left[Z_0(0)-\widetilde{Z}_0(0)\right]\mathrm{e}^{-\alpha t}+\alpha\int_0^t \mathrm{e}^{-\alpha(t-s)}\left(N_1(s)-\widetilde{N}_1(s)\right)\mathrm{d}s.$$

Therefore,

$$\left|Z_0(t)-\widetilde{Z}_0(t)\right|\le\left|Z_0(0)-\widetilde{Z}_0(0)\right|\mathrm{e}^{-\alpha t}+\alpha\int_0^t \mathrm{e}^{-\alpha(t-s)}\left|N_1(s)-\widetilde{N}_1(s)\right|\mathrm{d}s. \tag{1}$$

Integrating both sides of (1) from 0 to t leads to

$$\begin{aligned}\int_0^t\left|Z_0(\theta)-\widetilde{Z}_0(\theta)\right|\mathrm{d}\theta &\le \left|Z_0(0)-\widetilde{Z}_0(0)\right|\int_0^t \mathrm{e}^{-\alpha\theta}\mathrm{d}\theta+\alpha\int_0^t\mathrm{d}\theta\int_0^\theta \mathrm{e}^{-\alpha(\theta-s)}\left|N_1(s)-\widetilde{N}_1(s)\right|\mathrm{d}s\\
&=\left|Z_0(0)-\widetilde{Z}_0(0)\right|\frac{1-\mathrm{e}^{-\alpha t}}{\alpha}+\alpha\int_0^t\mathrm{d}s\int_s^t \mathrm{e}^{-\alpha(\theta-s)}\left|N_1(s)-\widetilde{N}_1(s)\right|\mathrm{d}\theta\\
&=\left|Z_0(0)-\widetilde{Z}_0(0)\right|\frac{1-\mathrm{e}^{-\alpha t}}{\alpha}+\alpha\int_0^t \mathrm{e}^{\alpha s}\left|N_1(s)-\widetilde{N}_1(s)\right|\int_s^t \mathrm{e}^{-\alpha\theta}\mathrm{d}\theta\mathrm{d}s\\
&=\left|Z_0(0)-\widetilde{Z}_0(0)\right|\frac{1-\mathrm{e}^{-\alpha t}}{\alpha}+\int_0^t\left|N_1(s)-\widetilde{N}_1(s)\right|\left(1-\mathrm{e}^{-\alpha(t-s)}\right)\mathrm{d}s\\
&\le\frac{\left|Z_0(0)-\widetilde{Z}_0(0)\right|}{\alpha}+\int_0^t\left|N_1(s)-\widetilde{N}_1(s)\right|\mathrm{d}s\\
&=\frac{\left|Z_0(0)-\widetilde{Z}_0(0)\right|}{\alpha}+\int_0^t\left|N_1(\theta)-\widetilde{N}_1(\theta)\right|\mathrm{d}\theta.\end{aligned} \tag{2}$$

Define $\overline{V}=\sum_{i=1}^3 H_i\left|\ln N_i(t)-\ln\widetilde{N}_i(t)\right|$, where

$$H_1=\frac{C_{33}\left(C_{11}C_{22}+C_{12}C_{21}\right)}{C_{11}\left(C_{11}C_{22}C_{33}-C_{11}C_{23}C_{32}-C_{12}C_{21}C_{33}\right)},\quad H_2=\frac{2C_{12}C_{21}C_{33}+C_{11}C_{23}C_{32}}{C_{21}\left(C_{11}C_{22}C_{33}-C_{11}C_{23}C_{32}-C_{12}C_{21}C_{33}\right)},$$
$$H_3=\frac{C_{23}\left(C_{11}C_{22}+C_{12}C_{21}\right)}{C_{21}\left(C_{11}C_{22}C_{33}-C_{11}C_{23}C_{32}-C_{12}C_{21}C_{33}\right)}.$$

By Itô's formula, we have

$$\begin{aligned}\mathcal{L}\left[\overline{V}(t)\right]\le\ & H_1\left[-C_{11}\left|N_1(t)-\widetilde{N}_1(t)\right|+C_{12}\left|N_2(t)-\widetilde{N}_2(t)\right|\right]\\
&+H_2\left[C_{21}\left|Z_0(t)-\widetilde{Z}_0(t)\right|-C_{22}\left|N_2(t)-\widetilde{N}_2(t)\right|+C_{23}\left|N_3(t)-\widetilde{N}_3(t)\right|\right]\\
&+H_3\left[C_{32}\left|N_2(t)-\widetilde{N}_2(t)\right|-C_{33}\left|N_3(t)-\widetilde{N}_3(t)\right|\right].\end{aligned} \tag{3}$$

Integrating both sides of (3) from 0 to t and together with (2) yields

$$\begin{aligned}\overline{V}(t)-\overline{V}(0)\le\ & H_2\frac{C_{21}}{\alpha}\left|Z_0(0)-\widetilde{Z}_0(0)\right|-\left[C_{11}H_1-C_{21}H_2\right]\int_0^t\left|N_1(\theta)-\widetilde{N}_1(\theta)\right|\mathrm{d}\theta\\
&-\left[-C_{12}H_1+C_{22}H_2-C_{32}H_3\right]\int_0^t\left|N_2(\theta)-\widetilde{N}_2(\theta)\right|\mathrm{d}\theta-\left[-C_{23}H_2+C_{33}H_3\right]\int_0^t\left|N_3(\theta)-\widetilde{N}_3(\theta)\right|\mathrm{d}\theta\\
=\ & H_2\frac{C_{21}}{\alpha}\left|Z_0(0)-\widetilde{Z}_0(0)\right|-\int_0^t\left|N_1(\theta)-\widetilde{N}_1(\theta)\right|\mathrm{d}\theta\\
&-\frac{C_{12}}{C_{11}}\int_0^t\left|N_2(\theta)-\widetilde{N}_2(\theta)\right|\mathrm{d}\theta-\frac{C_{23}}{C_{21}}\int_0^t\left|N_3(\theta)-\widetilde{N}_3(\theta)\right|\mathrm{d}\theta,\end{aligned}$$

which implies that $\left|N_i(t) - \widetilde{N}_i(t)\right| \in L^1[0,+\infty)$ $(i = 1,2,3)$.

Making use of (2) yields $\left|Z_0(t) - \widetilde{Z}_0(t)\right| \in L^1[0,+\infty)$. Consequently, we have

$$\lim_{t\to+\infty} \left|Z_0(t) - \widetilde{Z}_0(t)\right| = 0, \quad \lim_{t\to+\infty} \left|N_i(t) - \widetilde{N}_i(t)\right| = 0 \quad (i = 1,2,3).$$

□

4. Probability Density Function

For convenience, define

$$\mathrm{Q} = \begin{vmatrix} C_{11} & C_{12} \\ -C_{21} & C_{22} \end{vmatrix}, \ \mathrm{Q}_1 = \begin{vmatrix} b_1 - \frac{\sigma_1^2}{2} & C_{12} \\ -\left(b_2 + \frac{\sigma_2^2}{2}\right) & C_{22} \end{vmatrix}, \ \mathrm{Q}_2 = \begin{vmatrix} C_{11} & b_1 - \frac{\sigma_1^2}{2} \\ -C_{21} & -\left(b_2 + \frac{\sigma_2^2}{2}\right) \end{vmatrix},$$

$$\mathrm{T} = \begin{vmatrix} C_{11} & C_{12} & 0 \\ -C_{21} & C_{22} & C_{23} \\ 0 & -C_{32} & C_{33} \end{vmatrix}, \ \mathrm{T}_1 = \begin{vmatrix} b_1 - \frac{\sigma_1^2}{2} & C_{12} & 0 \\ -\left(b_2 + \frac{\sigma_2^2}{2}\right) & C_{22} & C_{23} \\ -\left(b_3 + \frac{\sigma_3^2}{2}\right) & -C_{32} & C_{33} \end{vmatrix},$$

$$\mathrm{T}_2 = \begin{vmatrix} C_{11} & b_1 - \frac{\sigma_1^2}{2} & 0 \\ -C_{21} & -\left(b_2 + \frac{\sigma_2^2}{2}\right) & C_{23} \\ 0 & -\left(b_3 + \frac{\sigma_3^2}{2}\right) & C_{33} \end{vmatrix}, \ \mathrm{T}_3 = \begin{vmatrix} C_{11} & C_{12} & b_1 - \frac{\sigma_1^2}{2} \\ -C_{21} & C_{22} & -\left(b_2 + \frac{\sigma_2^2}{2}\right) \\ 0 & -C_{32} & -\left(b_3 + \frac{\sigma_3^2}{2}\right) \end{vmatrix}.$$

Based on the fourth equation of system (5), we obtain

$$\frac{Z_0(t)}{t} - \frac{Z_0(0)}{t} = \alpha\left(t^{-1}\int_0^t N_1(s)\mathrm{d}s - t^{-1}\int_0^t Z_0(s)\mathrm{d}s\right). \tag{1}$$

Combining (1) with Theorem 2.4 yields

$$\lim_{t\to+\infty}\left(t^{-1}\int_0^t N_1(s)\mathrm{d}s - t^{-1}\int_0^t Z_0(s)\mathrm{d}s\right) = 0. \tag{2}$$

Theorem 4.1. *Assume that $C_{22}C_{33}\left(C_{11}C_{22} + C_{12}C_{21}\right) > C_{12}C_{21}C_{23}C_{32}$, then system (5) possesses the following properties:*
(i) *If* $\mathrm{T}_3 > 0$, *then*

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_i(s)\mathrm{d}s = \frac{\mathrm{T}_i}{\mathrm{T}} \quad a.s. \quad (i = 1,2,3).$$

(ii) *If* $\mathrm{Q}_2 > 0 > \mathrm{T}_3$, *then*

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_i(s)\mathrm{d}s = \frac{\mathrm{Q}_i}{\mathrm{Q}}, \quad \lim_{t\to+\infty} N_3(t) = 0 \quad a.s. \quad (i = 1,2).$$

(iii) *If* $b_1 - \frac{\sigma_1^2}{2} > 0 > \mathrm{Q}_2$, *then*

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_1(s)\mathrm{d}s = \frac{b_1 - \frac{\sigma_1^2}{2}}{C_{11}} \quad \lim_{t\to+\infty} N_i(t) = 0 \quad a.s. \quad (i = 2,3).$$

(iv) *If* $0 > b_1 - \frac{\sigma_1^2}{2}$, *then* $\lim_{t\to+\infty} N_i(t) = 0$ *a.s.* $(i = 1,2,3)$.

Proof. Let $U_4(t) = \ln Z_0(t)$, $U_i(t) = \ln N_i(t)$ $(i = 1, 2, 3)$. By Itô's formula, we have

$$\begin{cases} dU_1 = \left[\left(b_1 - \frac{\sigma_1^2}{2}\right) - C_{11}e^{U_1} - C_{12}e^{U_2}\right]dt + \sigma_1 dB_1(t), \\ dU_2 = \left[-\left(b_2 + \frac{\sigma_2^2}{2}\right) - C_{22}e^{U_2} - C_{23}e^{U_3} + C_{21}e^{U_4}\right]dt + \sigma_2 dB_2(t), \\ dU_3 = \left[-\left(b_3 + \frac{\sigma_3^2}{2}\right) + C_{32}e^{U_2} - C_{33}e^{U_3}\right]dt + \sigma_3 dB_3(t), \\ dU_4 = \alpha\left[e^{U_1 - U_4} - 1\right]dt. \end{cases} \tag{3}$$

Based on system (3), Eq.(2) and the strong law of large numbers for local martingales ([21]), we have

$$\begin{pmatrix} \ln N_1(t) \\ \ln N_2(t) \\ \ln N_3(t) \end{pmatrix} = \begin{pmatrix} \left(b_1 - \frac{\sigma_1^2}{2}\right) \\ -\left(b_2 + \frac{\sigma_2^2}{2}\right) \\ -\left(b_3 + \frac{\sigma_3^2}{2}\right) \end{pmatrix} t - \begin{pmatrix} C_{11} & C_{12} & 0 \\ -C_{21} & C_{22} & C_{23} \\ 0 & -C_{32} & C_{33} \end{pmatrix} \begin{pmatrix} \int_0^t N_1(s)ds \\ \int_0^t N_2(s)ds \\ \int_0^t N_3(s)ds \end{pmatrix} + o(t) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where $\lim_{t\to+\infty} \frac{o(t)}{t} = 0$. The subsequent part of the proof is similar to [45], and here is omitted. □

Now, define a quasi-positive equilibrium $E^* = \left(N_1^*, N_2^*, N_3^*, Z_0^*\right) = \left(e^{U_1^*}, e^{U_2^*}, e^{U_3^*}, e^{U_4^*}\right)$. Next, we will deduce the exact expression of the probability density function of the linearized system corresponding to system (5) around E^*. Let $S_i(t) = U_i(t) - U_i^*$ $(i = 1, 2, 3, 4)$, then the linearized equation of system (5) around E^* is

$$\begin{cases} dS_1(t) = \left[-C_{11}N_1^*S_1(t) - C_{12}N_2^*S_2(t)\right]dt + \sigma_1 dB_1(t), \\ dS_2(t) = \left[-C_{22}N_2^*S_2(t) - C_{23}N_3^*S_3(t) + C_{21}N_1^*S_4(t)\right]dt + \sigma_2 dB_2(t), \\ dS_3(t) = \left[C_{32}N_2^*S_2(t) - C_{33}N_3^*S_3(t)\right]dt + \sigma_3 dB_3(t), \\ dS_4(t) = \alpha\left[S_1(t) - S_4(t)\right]dt. \end{cases} \tag{4}$$

For a square matrix A, denote $A > \mathbf{0}$ be a positive definite matrix; denote $A \geq \mathbf{0}$ be a positive semi-definite matrix.

Definition 4.2. *([26]) The characteristic polynomial of a square matrix $A_{n\times n}$ is defined as $\phi_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$, then A is called a Hurwitz matrix if and only if A has all negative real-part eigenvalues, that is to say,*

$$D_k = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2k-1} \\ 1 & a_2 & a_4 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & \cdots & a_{2k-3} \\ 0 & 1 & a_2 & \cdots & a_{2k-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_k \end{vmatrix} > 0, \quad k = 1, 2, \ldots, n$$

with the complementary definition $a_i = 0$ if $i > n$.

Lemma 4.3. *([38]) For the four-dimensional real algebraic equation $\Theta_0^2 + B\Omega + \Omega B^T = \mathbf{0}$, with $\Theta_0 = diag(1, 0, 0, 0)$ and Ω is a real symmetric matrix.*

(*i*) *If*

$$B=\begin{pmatrix}-\rho_1 & -\rho_2 & -\rho_3 & -\rho_4\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix},$$

with $\rho_1>0$, $\rho_3>0$, $\rho_4>0$ *and* $\rho_1\rho_2\rho_3-\rho_3^2-\rho_1^2\rho_4>0$, *then*

$$\Omega=\begin{pmatrix}\frac{\rho_2\rho_3-\rho_1\rho_4}{2(\rho_1\rho_2\rho_3-\rho_3^2-\rho_1^2\rho_4)} & 0 & -\frac{\rho_3}{2(\rho_1\rho_2\rho_3-\rho_3^2-\rho_1^2\rho_4)} & 0\\ 0 & \frac{\rho_3}{2(\rho_1\rho_2\rho_3-\rho_3^2-\rho_1^2\rho_4)} & 0 & -\frac{\rho_1}{2(\rho_1\rho_2\rho_3-\rho_3^2-\rho_1^2\rho_4)}\\ -\frac{\rho_3}{2(\rho_1\rho_2\rho_3-\rho_3^2-\rho_1^2\rho_4)} & 0 & \frac{\rho_1}{2(\rho_1\rho_2\rho_3-\rho_3^2-\rho_1^2\rho_4)} & 0\\ 0 & -\frac{\rho_1}{2(\rho_1\rho_2\rho_3-\rho_3^2-\rho_1^2\rho_4)} & 0 & \frac{\rho_1\rho_2-\rho_3}{2\rho_4(\rho_1\rho_2\rho_3-\rho_3^2-\rho_1^2\rho_4)}\end{pmatrix}\succ\mathbf{0}.$$

(*ii*) *If*

$$B=\begin{pmatrix}-\rho_1 & -\rho_2 & -\rho_3 & \rho_4\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & \rho_5\end{pmatrix},$$

with $\rho_1>0$, $\rho_3>0$, $\rho_1\rho_2-\rho_3>0$, *then*

$$\Omega=\begin{pmatrix}\frac{\rho_2}{2(\rho_1\rho_2-\rho_3)} & 0 & -\frac{1}{2(\rho_1\rho_2-\rho_3)} & 0\\ 0 & \frac{1}{2(\rho_1\rho_2-\rho_3)} & 0 & 0\\ -\frac{1}{2(\rho_1\rho_2-\rho_3)} & 0 & \frac{\rho_1}{2\rho_3(\rho_1\rho_2-\rho_3)} & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\succeq\mathbf{0}.$$

Theorem 4.4. *For any initial value* $\mathbf{X}(0)\in\mathbb{R}_+^4$, *if* $C_{11}C_{22}>C_{12}C_{21}$, $2C_{22}C_{33}>C_{23}C_{32}$ *and* $\mathrm{T}_3>0$, *then the solution of system (5) obeys a normal probability density function around* E^*, *which satisfies*

$$\Phi(N_1,N_2,N_3,Z_0)=(2\pi)^{-2}|\Sigma|^{-\frac{1}{2}}(N_1N_2N_3Z_0)^{-1}\,\mathrm{e}^{-\frac{1}{2}\left(\ln\frac{N_1}{N_1^*},\ln\frac{N_2}{N_2^*},\ln\frac{N_3}{N_3^*},\ln\frac{Z_0}{N_1^*}\right)\Sigma^{-1}\left(\ln\frac{N_1}{N_1^*},\ln\frac{N_2}{N_2^*},\ln\frac{N_3}{N_3^*},\ln\frac{Z_0}{N_1^*}\right)^{\mathrm{T}}},$$

where the covariance matrix Σ *is as follows:*

(1) *If* $C_{11}N_1^*\neq C_{33}N_3^*$, $\alpha\neq C_{33}N_3^*$, *then*

$$\Sigma=\theta_1^2(M_1I_{12}I_{11})^{-1}\Sigma_{01}\left[(M_1I_{12}I_{11})^{-1}\right]^{\mathrm{T}}+\theta_{21}^2(M_{21}I_{24}I_{23}I_{22}I_{21})^{-1}\Sigma_{021}\left[(M_{21}I_{24}I_{23}I_{22}I_{21})^{-1}\right]^{\mathrm{T}}$$
$$+\theta_3^2(M_3I_{31})^{-1}\Sigma_{03}\left[(M_3I_{31})^{-1}\right]^{\mathrm{T}};$$

(2) *If* $C_{11}N_1^*\neq C_{33}N_3^*$, $\alpha=C_{33}N_3^*$, *then*

$$\Sigma=\theta_1^2(M_1I_{12}I_{11})^{-1}\Sigma_{01}\left[(M_1I_{12}I_{11})^{-1}\right]^{\mathrm{T}}+\theta_{22}^2(M_{22}I_{24}I_{23}I_{22}I_{21})^{-1}\Sigma_{022}\left[(M_{22}I_{24}I_{23}I_{22}I_{21})^{-1}\right]^{\mathrm{T}}$$
$$+\theta_3^2(M_3I_{31})^{-1}\Sigma_{03}\left[(M_3I_{31})^{-1}\right]^{\mathrm{T}};$$

(3) *If* $C_{11}N_1^*=C_{33}N_3^*$, *then*

$$\Sigma=\theta_1^2(M_1I_{12}I_{11})^{-1}\Sigma_{01}\left[(M_1I_{12}I_{11})^{-1}\right]^{\mathrm{T}}+\theta_{23}^2(M_{23}I_{25}I_{23}I_{22}I_{21})^{-1}\Sigma_{023}\left[(M_{23}I_{25}I_{23}I_{22}I_{21})^{-1}\right]^{\mathrm{T}}$$
$$+\theta_3^2(M_3I_{31})^{-1}\Sigma_{03}\left[(M_3I_{31})^{-1}\right]^{\mathrm{T}},$$

where the above matrixes will be defined in the following proof process and

$$
\begin{aligned}
\theta_1 &= \sigma_1\alpha C_{21}C_{32}N_1^*N_2^*,\ \theta_{21} = \sigma_2\alpha C_{12}\left(\alpha - C_{33}N_3^*\right)N_2^*,\ \theta_{22} = \sigma_2 C_{12}N_2^*\left(C_{11}N_1^* - C_{33}N_3^*\right),\\
\theta_{23} &= \sigma_2\alpha C_{12}N_2^*,\ \theta_3 = \sigma_3\alpha C_{12}C_{23}N_2^*N_3^*,\ r_{11} = \alpha + C_{11}N_1^* + C_{22}N_2^* + C_{33}N_3^*,\\
r_{12} &= \alpha C_{11}N_1^* + \alpha C_{22}N_2^* + \alpha C_{33}N_3^* + (C_{22}C_{33} + C_{23}C_{32})N_2^*N_3^* + C_{11}C_{33}N_1^*N_3^* + C_{11}C_{22}N_1^*N_2^*,\\
r_{13} &= (\alpha C_{12}C_{21} + \alpha C_{11}C_{22})N_1^*N_2^* + (\alpha C_{22}C_{33} + \alpha C_{23}C_{32})N_2^*N_3^* + \alpha C_{11}C_{33}N_1^*N_3^*\\
&\quad + (C_{11}C_{22}C_{33} + C_{11}C_{23}C_{32})N_1^*N_2^*N_3^*,\\
r_{14} &= (\alpha C_{12}C_{21}C_{33} + \alpha C_{11}C_{22}C_{33} + \alpha C_{11}C_{23}C_{32})N_1^*N_2^*N_3^*,\ r_{21} = C_{11}N_1^* + C_{22}N_2^* + C_{33}N_3^*,\\
r_{22} &= C_{11}C_{22}N_1^*N_2^* + C_{11}C_{33}N_1^*N_3^* + (C_{22}C_{33} + C_{23}C_{32})N_2^*N_2^*,\\
r_{23} &= (C_{11}C_{22}C_{33} + C_{11}C_{23}C_{32})N_1^*N_2^*N_3^* + \alpha C_{12}C_{21}N_1^*N_2^*,\\
r_{24} &= C_{11}C_{12}C_{21}\left(N_1^*\right)^2N_2^* - C_{12}C_{21}C_{33}N_1^*N_2^*N_3^*,\ r_{31} = \alpha + C_{22}N_2^* + C_{33}N_3^*,\\
r_{32} &= \alpha C_{22}N_2^* + \alpha C_{33}N_3^* + (C_{22}C_{33} + C_{23}C_{32})N_2^*N_3^*,\\
r_{33} &= \alpha C_{12}C_{21}N_1^*N_2^* + (\alpha C_{22}C_{33} + \alpha C_{23}C_{32})N_2^*N_3^*,\\
r_{34} &= \alpha\left[C_{11}N_1^*\left(C_{11}N_1^* - C_{22}N_2^* - C_{33}N_3^*\right) + (C_{22}C_{33} + C_{23}C_{32})N_2^*N_3^*\right],\\
\eta &= r_{11}r_{12}r_{13} - r_{13}^2 - r_{11}^2r_{14},\ \iota = r_{12}r_{13} - r_{11}r_{14},\ \zeta = r_{11}r_{12} - r_{13}.
\end{aligned}
$$

Proof. Let $S(t) = (S_1(t), S_2(t), S_3(t), S_4(t))^{\mathrm{T}}$, $B(t) = (B_1(t), B_2(t), B_3(t), 0)^{\mathrm{T}}$ and

$$
A = \begin{pmatrix} -C_{11}N_1^* & -C_{12}N_2^* & 0 & 0\\ 0 & -C_{22}N_2^* & -C_{23}N_3^* & C_{21}N_1^*\\ 0 & C_{32}N_2^* & -C_{33}N_3^* & 0\\ \alpha & 0 & 0 & -\alpha \end{pmatrix},\ H = \begin{pmatrix} \sigma_1 & 0 & 0 & 0\\ 0 & \sigma_2 & 0 & 0\\ 0 & 0 & \sigma_3 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

then system (4) can be expressed as $\mathrm{d}S(t) = AS(t)\mathrm{d}t + H\mathrm{d}B(t)$. Thanks to Gardiner ([10]), system (4) has a unique probability density function $\Phi(S)$. Since H is a constant matrix, thanks to Roozen ([35]), $\Phi(S)$ can be described as a Gaussian distribution, i.e., $\Phi(S) = c\mathrm{e}^{-\frac{1}{2}SQS^T}$ with $\int_{\mathbb{R}^4}\Phi(S)\mathrm{d}S = 1$, where $\mathrm{Q} > \mathbf{0}$ is a symmetric matrix determined by

$$\mathrm{Q}H^2\mathrm{Q} + \mathrm{Q}A + A^T\mathrm{Q} = \mathbf{0}. \tag{5}$$

Let $\mathrm{Q}^{-1} = \Sigma$, then (5) is equivalent to

$$H^2 + A\Sigma + \Sigma A^T = \mathbf{0}. \tag{6}$$

Thanks to the finite independent superposition principle ([41]), equation (6) can be equivalently converted to the sum of the following three sub-equations:

$$H_i^2 + A\Sigma_i + \Sigma_i A^{\mathrm{T}} = \mathbf{0}, \tag{7}$$

where $H_1^2 = diag\left(\sigma_1^2, 0, 0, 0\right)$, $H_2^2 = diag\left(0, \sigma_2^2, 0, 0\right)$, $H_3^2 = diag\left(0, 0, \sigma_3^2, 0\right)$ and Σ_i $(i = 1, 2, 3)$ are their corresponding solutions, respectively. Obviously, $\Sigma = \sum_{i=1}^3 \Sigma_i$. The characteristic polynomial of matrix A is $\phi_A(\lambda) = \lambda^4 + r_{11}\lambda^3 + r_{12}\lambda^2 + r_{13}\lambda + r_{14}$. Denote

$$
D_2 = \begin{vmatrix} r_{11} & r_{13}\\ 1 & r_{12} \end{vmatrix},\ D_3 = \begin{vmatrix} r_{11} & r_{13} & 0\\ 1 & r_{12} & r_{14}\\ 0 & r_{11} & r_{13} \end{vmatrix},\ D_4 = \begin{vmatrix} r_{11} & r_{13} & 0 & 0\\ 1 & r_{12} & r_{14} & 0\\ 0 & r_{11} & r_{13} & 0\\ 0 & 1 & r_{12} & r_{14} \end{vmatrix}.
$$

Clearly, $D_4 = r_{14} D_3$. Compute

$$
\begin{aligned}
D_2 =\ & C_{23}C_{32}C_{33}N_2^*(N_3^*)^2 + C_{22}C_{33}^2N_2^*(N_3^*)^2 + C_{22}C_{23}C_{32}(N_2^*)^2N_3^* + C_{22}^2C_{33}(N_2^*)^2N_3^* + C_{11}C_{33}^2N_1^*(N_3^*)^2 \\
& + 2C_{11}C_{22}C_{33}N_1^*N_2^*N_3^* + C_{11}C_{22}^2N_1^*(N_2^*)^2 + C_{11}^2C_{33}(N_1^*)^2N_3^* + C_{11}^2C_{22}(N_1^*)^2N_2^* + C_{33}^2\alpha(N_3^*)^2 \\
& + 2C_{22}C_{33}\alpha N_2^*N_3^* + C_{22}^2\alpha(N_2^*)^2 + [2C_{11}C_{22}\alpha - C_{12}C_{21}\alpha]\,N_1^*N_2^* + 2C_{11}C_{33}\alpha N_1^*N_3^* + C_{11}^2\alpha(N_1^*)^2 \\
& + C_{33}\alpha^2N_3^* + C_{22}\alpha^2N_2^* + C_{11}\alpha^2N_1^*,
\end{aligned}
$$

$$
\begin{aligned}
D_3 =\ & \left[C_{11}^3C_{22}^2C_{33} + C_{11}^3C_{22}C_{23}C_{32}\right]\left(N_1^*\right)^3\left(N_2^*\right)^2N_3^* + \left[C_{11}^3C_{22}^2\alpha + C_{11}^2C_{12}C_{21}C_{22}\alpha\right]\left(N_1^*\right)^3\left(N_2^*\right)^2 \\
& + \left[C_{11}^3C_{22}C_{33}^2 + C_{11}^3C_{23}C_{32}C_{33}\right]\left(N_1^*\right)^3N_2^*\left(N_3^*\right)^2 + \left[2C_{11}^3C_{22}C_{33}\alpha\right]\left(N_1^*\right)^3N_2^*N_3^* \\
& + \left[C_{11}^3C_{22}\alpha^2 + C_{11}^2C_{12}C_{21}\alpha^2\right]\left(N_1^*\right)^3N_2^* + C_{11}^3C_{33}^2\alpha\left(N_1^*\right)^3\left(N_3^*\right)^2 + C_{11}^3C_{33}\alpha^2\left(N_1^*\right)^3N_3^* \\
& + \left[C_{11}^2C_{22}^3C_{33} + C_{11}^2C_{22}^2C_{23}C_{32}\right]\left(N_1^*\right)^2\left(N_2^*\right)^3N_3^* + \left[C_{11}^2C_{22}^3\alpha + C_{11}C_{12}C_{21}C_{22}^2\alpha\right]\left(N_1^*\right)^2\left(N_2^*\right)^3 \\
& + \left[2C_{11}^2C_{22}^2C_{33}^2 + 2C_{11}^2C_{22}C_{23}C_{32}C_{33}\right]\left(N_1^*\right)^2\left(N_2^*\right)^2\left(N_3^*\right)^2 \\
& + \left[\left(4C_{11}^2C_{22}^2C_{33}\alpha - C_{11}C_{12}C_{21}C_{22}C_{33}\alpha\right) + \left(C_{11}^2C_{22}C_{23}C_{32}\alpha - C_{11}C_{12}C_{21}C_{23}C_{32}\alpha\right)\right]\left(N_1^*\right)^2\left(N_2^*\right)^2N_3^* \\
& + \left[2C_{11}^2C_{22}^2\alpha^2 - C_{12}^2C_{21}^2\alpha^2 + C_{11}C_{12}C_{21}C_{22}\alpha^2\right]\left(N_1^*\right)^2\left(N_2^*\right)^2 \\
& + \left[C_{11}^2C_{22}C_{33}^3 + C_{11}^2C_{23}C_{32}C_{33}^2\right]\left(N_1^*\right)^2N_2^*\left(N_3^*\right)^3 \\
& + \left[4C_{11}^2C_{22}C_{33}^2\alpha - C_{11}C_{12}C_{21}C_{33}^2\alpha + C_{11}^2C_{23}C_{32}C_{33}\alpha\right]\left(N_1^*\right)^2N_2^*\left(N_3^*\right)^2 \\
& + \left[4C_{11}^2C_{22}C_{33}\alpha^2 - C_{11}C_{12}C_{21}C_{33}\alpha^2\right]\left(N_1^*\right)^2N_2^*N_3^* + \left[C_{11}^2C_{22}\alpha^3 + C_{11}C_{12}C_{21}\alpha^3\right]\left(N_1^*\right)^2N_2^* \\
& + C_{11}^2C_{33}^3\alpha\left(N_1^*\right)^2\left(N_3^*\right)^3 + 2C_{11}^2C_{33}^2\alpha^2\left(N_1^*\right)^2\left(N_3^*\right)^2 + C_{11}^2C_{33}\alpha^3\left(N_1^*\right)^2N_3^* \\
& + \left[C_{11}C_{22}^3C_{33}^2 + C_{11}C_{22}C_{23}^2C_{32}^2 + 2C_{11}C_{22}^2C_{23}C_{32}C_{33}\right]N_1^*\left(N_2^*\right)^3\left(N_3^*\right)^2 \\
& + \left[2C_{11}C_{22}^3C_{33}\alpha + 2C_{11}C_{22}^2C_{23}C_{32}\alpha + C_{12}C_{21}C_{22}C_{23}C_{32}\alpha\right]N_1^*\left(N_2^*\right)^3N_3^* \\
& + \left[C_{11}C_{22}^3\alpha^2 + C_{12}C_{21}C_{22}^2\alpha^2\right]N_1^*\left(N_2^*\right)^3 + \left[C_{11}C_{22}^2\alpha^3 + C_{12}C_{21}C_{22}\alpha^3\right]N_1^*\left(N_2^*\right)^2 \\
& + \left[C_{11}C_{22}^2C_{33}^3 + C_{11}C_{23}^2C_{32}^2C_{33} + 2C_{11}C_{22}C_{23}C_{32}C_{33}^2\right]N_1^*\left(N_2^*\right)^2\left(N_3^*\right)^3 \\
& + \left[4C_{11}C_{22}^2C_{33}^2\alpha - C_{12}C_{21}C_{22}C_{33}^2\alpha + 4C_{11}C_{22}C_{23}C_{32}C_{33}\alpha + C_{12}C_{21}C_{23}C_{32}C_{33}\alpha\right]N_1^*\left(N_2^*\right)^2\left(N_3^*\right)^2 \\
& + \left[\left(4C_{11}C_{22}^2C_{33}\alpha^2 - C_{12}C_{21}C_{22}C_{33}\alpha^2\right) + \left(C_{11}C_{22}C_{23}C_{32}\alpha^2 - C_{12}C_{21}C_{23}C_{32}\alpha^2\right)\right]N_1^*\left(N_2^*\right)^2N_3^* \\
& + \left[2C_{11}C_{22}C_{33}^3\alpha - C_{12}C_{21}C_{33}^3\alpha + 2C_{11}C_{23}C_{32}C_{33}^2\alpha\right]N_1^*N_2^*\left(N_3^*\right)^3 \\
& + \left[4C_{11}C_{22}C_{33}^2\alpha^2 - C_{12}C_{21}C_{33}^2\alpha^2 + C_{11}C_{23}C_{32}C_{33}\alpha^2\right]N_1^*N_2^*\left(N_3^*\right)^2 \\
& + 2C_{11}C_{22}C_{33}\alpha^3N_1^*N_2^*N_3^* + C_{11}C_{33}^3\alpha^2N_1^*\left(N_3^*\right)^3 + C_{11}C_{33}^2\alpha^3N_1^*\left(N_3^*\right)^2 \\
& + \left[C_{22}^3C_{33}^2\alpha + C_{22}C_{23}^2C_{32}^2\alpha + 2C_{22}^2C_{23}C_{32}C_{33}\alpha\right]\left(N_2^*\right)^3\left(N_3^*\right)^2 + \left[C_{22}^3C_{33}\alpha^2 + C_{22}^2C_{23}C_{32}\alpha^2\right]\left(N_2^*\right)^3N_3^* \\
& + \left[C_{22}^2C_{33}^3\alpha + C_{23}^2C_{32}^2C_{33}\alpha + 2C_{22}C_{23}C_{32}C_{33}^2\alpha\right]\left(N_2^*\right)^2\left(N_3^*\right)^3 \\
& + \left[2C_{22}^2C_{33}^2\alpha^2 + 2C_{22}C_{23}C_{32}C_{33}\alpha^2\right]\left(N_2^*\right)^2\left(N_3^*\right)^2 + \left[C_{22}^2C_{33}\alpha^3 + C_{22}C_{23}C_{32}\alpha^3\right]\left(N_2^*\right)^2N_3^* \\
& + \left[C_{22}C_{33}^3\alpha^2 + C_{23}C_{32}C_{33}^2\alpha^2\right]N_2^*\left(N_3^*\right)^3 + \left[C_{22}C_{33}^2\alpha^3 + C_{23}C_{32}C_{33}\alpha^3\right]N_2^*\left(N_3^*\right)^2.
\end{aligned}
$$

By means of the Routh-Hurwitz criterion in [26], if $C_{11}C_{22} > C_{12}C_{21}$, then the matrix A has all negative real-part eigenvalues. Now, we will prove Theorem 4.4 based on three steps. Denote $G = diag(1, 0, 0, 0)$.

Step 1. Consider the algebraic equation:

$$H_1^2 + A\Sigma_1 + \Sigma_1 A^{\mathrm{T}} = \mathbf{0}. \tag{8}$$

Let $A_{11} = I_{11} A I_{11}^{-1}$, where

$$I_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

then we have

$$A_{11} = \begin{pmatrix} -C_{11}N_1^* & 0 & 0 & -C_{12}N_2^* \\ \alpha & -\alpha & 0 & 0 \\ 0 & 0 & -C_{33}N_3^* & C_{32}N_2^* \\ 0 & C_{21}N_1^* & -C_{23}N_3^* & -C_{22}N_2^* \end{pmatrix}.$$

Let $A_{12} = I_{12} A_{11} I_{12}^{-1}$, where

$$I_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

then we have

$$A_{12} = \begin{pmatrix} -C_{11}N_1^* & 0 & -C_{12}N_2^* & 0 \\ \alpha & -\alpha & 0 & 0 \\ 0 & C_{21}N_1^* & -C_{22}N_2^* & -C_{23}N_3^* \\ 0 & 0 & C_{32}N_2^* & -C_{33}N_3^* \end{pmatrix}.$$

Let $A_{13} = M_1 A_{12} M_1^{-1}$, where

$$M_1 = \begin{pmatrix} \alpha C_{21}C_{32}N_1^*N_2^* & m_1 & m_2 & m_3 \\ 0 & C_{21}C_{32}N_1^*N_2^* & -C_{32}N_2^*\left(C_{22}N_2^* + C_{33}N_3^*\right) & -C_{23}C_{32}N_2^*N_3^* + \left(C_{33}N_3^*\right)^2 \\ 0 & 0 & C_{32}N_2^* & -C_{33}N_3^* \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$m_1 = -\alpha C_{21}C_{32}N_1^*N_2^* - C_{21}C_{22}C_{32}N_1^*\left(N_2^*\right)^2 - C_{21}C_{32}C_{33}N_1^*N_2^*N_3^*,$$
$$m_2 = C_{22}^2C_{32}\left(N_2^*\right)^3 + C_{22}C_{32}C_{33}\left(N_2^*\right)^2N_3^* - C_{23}C_{32}^2\left(N_2^*\right)^2N_3^* + C_{32}C_{33}^2N_2^*\left(N_3^*\right)^2,$$
$$m_3 = C_{22}C_{23}C_{32}\left(N_2^*\right)^2N_3^* + 2C_{23}C_{32}C_{33}N_2^*\left(N_3^*\right)^2 - \left(C_{33}N_3^*\right)^3.$$

Then, we have

$$A_{13} = \begin{pmatrix} -r_{11} & -r_{12} & -r_{13} & -r_{14} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence, equation (8) is equivalently transformed into

$$\left(M_1I_{12}I_{11}\right)H_1^2\left(M_1I_{12}I_{11}\right)^{\mathrm{T}} + A_{13}\left(M_1I_{12}I_{11}\right)\Sigma_1\left(M_1I_{12}I_{11}\right)^{\mathrm{T}} + \left(M_1I_{12}I_{11}\right)\Sigma_1\left(M_1I_{12}I_{11}\right)^{\mathrm{T}}A_{13}^{\mathrm{T}} = \mathbf{0}.$$

Consequently,

$$G + A_{13}\Sigma_{01} + \Sigma_{01}A_{13}^{\mathrm{T}} = \mathbf{0},$$

where $\Sigma_{01} = \frac{1}{\theta_1^2}(M_1 I_{12} I_{11})\,\Sigma_1\,(M_1 I_{12} I_{11})^{\mathrm{T}}$ taking the form

$$\Sigma_{01} = \begin{pmatrix} \frac{\iota}{2\eta} & 0 & -\frac{r_{13}}{2\eta} & 0 \\ 0 & \frac{r_{13}}{2\eta} & 0 & -\frac{r_{11}}{2\eta} \\ -\frac{r_{13}}{2\eta} & 0 & \frac{r_{11}}{2\eta} & 0 \\ 0 & -\frac{r_{11}}{2\eta} & 0 & \frac{\zeta}{2r_{14}\eta} \end{pmatrix} \succ \mathbf{0}.$$

Hence, $\Sigma_1 = \theta_1^2 (M_1 I_{12} I_{11})^{-1}\,\Sigma_{01}\left[(M_1 I_{12} I_{11})^{-1}\right]^{\mathrm{T}} \succ \mathbf{0}$.

Step 2. Consider the algebraic equation:

$$H_2^2 + A\Sigma_2 + \Sigma_2 A^{\mathrm{T}} = \mathbf{0}. \tag{9}$$

Let $A_{21} = I_{21} A I_{21}^{-1}$, where

$$I_{21} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then we have

$$A_{21} = \begin{pmatrix} -C_{22}N_2^* & 0 & -C_{23}N_3^* & C_{21}N_1^* \\ -C_{12}N_2^* & -C_{11}N_1^* & 0 & 0 \\ C_{32}N_2^* & 0 & -C_{33}N_3^* & 0 \\ 0 & \alpha & 0 & -\alpha \end{pmatrix}.$$

Let $A_{22} = I_{22} A_{21} I_{22}^{-1}$, where

$$I_{22} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then we have

$$A_{22} = \begin{pmatrix} -C_{22}N_2^* & -C_{23}N_3^* & 0 & C_{21}N_1^* \\ C_{32}N_2^* & -C_{33}N_3^* & 0 & 0 \\ -C_{12}N_2^* & 0 & -C_{11}N_1^* & 0 \\ 0 & 0 & \alpha & -\alpha \end{pmatrix}.$$

Let $A_{23} = I_{23} A_{22} I_{23}^{-1}$, where

$$I_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{C_{12}}{C_{32}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then we have

$$A_{23} = \begin{pmatrix} -C_{22}N_2^* & -C_{23}N_3^* & 0 & C_{21}N_1^* \\ C_{32}N_2^* & -C_{33}N_3^* & 0 & 0 \\ 0 & l_1 & -C_{11}N_1^* & 0 \\ 0 & -\frac{\alpha C_{12}}{C_{32}} & \alpha & -\alpha \end{pmatrix}.$$

where $l_1 = \frac{C_{12}\left(C_{11}N_1^*-C_{33}N_3^*\right)}{C_{32}}$. The following analysis will be divided into two parts depending on the value of l_1.

Case 1. $l_1 \neq 0$. Let $A_{24} = I_{24}A_{23}I_{24}^{-1}$, where

$$I_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\alpha}{C_{11}N_1^*-C_{33}N_3^*} & 1 \end{pmatrix},$$

then we have

$$A_{24} = \begin{pmatrix} -C_{22}N_2^* & -C_{23}N_3^* & -\frac{\alpha C_{21}N_1^*}{C_{11}N_1^*-C_{33}N_3^*} & C_{21}N_1^* \\ C_{32}N_2^* & -C_{33}N_3^* & 0 & 0 \\ 0 & l_1 & -C_{11}N_1^* & 0 \\ 0 & 0 & \frac{\alpha\left(\alpha-C_{33}N_3^*\right)}{C_{11}N_1^*-C_{33}N_3^*} & -\alpha \end{pmatrix}.$$

Subcase 1. $\alpha \neq C_{33}N_3^*$. Let $A_{25} = M_{21}A_{24}M_{21}^{-1}$, where

$$M_{21} = \begin{pmatrix} \alpha C_{12}N_2^*\left(\alpha - C_{33}N_3^*\right) & m_4 & m_5 & -\alpha^3 \\ 0 & \frac{\alpha C_{12}\left(\alpha-C_{33}N_3^*\right)}{C_{32}} & m_6 & \alpha^2 \\ 0 & 0 & \frac{\alpha\left(\alpha-C_{33}N_3^*\right)}{C_{11}N_1^*-C_{33}N_3^*} & -\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$m_4 = -\frac{\alpha C_{12}\left[\alpha^2 - C_{33}^2\left(N_3^*\right)^2 + C_{11}N_1^*\left(\alpha - C_{33}N_3^*\right)\right]}{C_{32}},$$
$$m_5 = \frac{\alpha\left(\alpha - C_{33}N_3^*\right)\left[\alpha^2 + C_{11}^2\left(N_1^*\right)^2 + \alpha C_{11}N_1^*\right]}{C_{11}N_1^* - C_{33}N_3^*}, \quad m_6 = -\frac{\alpha\left(\alpha - C_{33}N_3^*\right)\left(\alpha + C_{11}N_1^*\right)}{C_{11}N_1^* - C_{33}N_3^*},$$

and we have

$$A_{25} = \begin{pmatrix} -r_{11} & -r_{12} & -r_{13} & -r_{14} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then equation (9) is equivalently transformed into

$$\left(M_{21}I_{24}I_{23}I_{22}I_{21}\right)H_2^2\left(M_{21}I_{24}I_{23}I_{22}I_{21}\right)^{\mathrm{T}} + A_{25}\left(M_{21}I_{24}I_{23}I_{22}I_{21}\right)\Sigma_{21}\left(M_{21}I_{24}I_{23}I_{22}I_{21}\right)^{\mathrm{T}}$$
$$+ \left(M_{21}I_{24}I_{23}I_{22}I_{21}\right)\Sigma_{21}\left(M_{21}I_{24}I_{23}I_{22}I_{21}\right)^{\mathrm{T}}A_{25}^{\mathrm{T}} = \mathbf{0},$$

i.e.,

$$G + A_{25}\Sigma_{021} + \Sigma_{021}A_{25}^{\mathrm{T}} = \mathbf{0},$$

where $\Sigma_{021} = \frac{1}{\theta_{21}^2}\left(M_{21}I_{24}I_{23}I_{22}I_{21}\right)\Sigma_{21}\left(M_{21}I_{24}I_{23}I_{22}I_{21}\right)^{\mathrm{T}} = \Sigma_{01} \succ \mathbf{0}$. Therefore,

$$\Sigma_{21} = \theta_{21}^2\left(M_{21}I_{24}I_{23}I_{22}I_{21}\right)^{-1}\Sigma_{021}\left[\left(M_{21}I_{24}I_{23}I_{22}I_{21}\right)^{-1}\right]^{\mathrm{T}} \succ \mathbf{0}.$$

Subcase 2. $\alpha = C_{33}N_3^*$. Denote

$$B_{24} = \begin{pmatrix} -C_{22}N_2^* & -C_{23}N_3^* & -\frac{\alpha C_{21}N_1^*}{C_{11}N_1^* - C_{33}N_3^*} & C_{21}N_1^* \\ C_{32}N_2^* & -C_{33}N_3^* & 0 & 0 \\ 0 & l_1 & -C_{11}N_1^* & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix}.$$

Let $B_{25} = M_{22}B_{24}M_{22}^{-1}$, where

$$M_{22} = \begin{pmatrix} C_{12}N_2^*\left(C_{11}N_1^* - C_{33}N_3^*\right) & -\frac{C_{12}\left[\left(C_{11}N_1^*\right)^2 - \left(C_{33}N_3^*\right)^2\right]}{C_{32}} & \left(C_{11}N_1^*\right)^2 & 0 \\ 0 & \frac{C_{12}\left(C_{11}N_1^* - C_{33}N_3^*\right)}{C_{32}} & -C_{11}N_1^* & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and we have

$$B_{25} = \begin{pmatrix} -r_{21} & -r_{22} & -r_{23} & r_{24} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix}.$$

Then equation (9) is equivalently transformed into

$$\begin{aligned} &(M_{22}I_{24}I_{23}I_{22}I_{21})\, H_2^2\, (M_{22}I_{24}I_{23}I_{22}I_{21})^{\mathrm{T}} + B_{25}\, (M_{22}I_{24}I_{23}I_{22}I_{21})\, \Sigma_{22}\, (M_{22}I_{24}I_{23}I_{22}I_{21})^{\mathrm{T}} \\ &+ (M_{22}I_{24}I_{23}I_{22}I_{21})\, \Sigma_{22}\, (M_{22}I_{24}I_{23}I_{22}I_{21})^{\mathrm{T}}\, B_{25}^{\mathrm{T}} = \mathbf{0}, \end{aligned}$$

i.e.,

$$G + B_{25}\Sigma_{022} + \Sigma_{022}B_{25}^{\mathrm{T}} = \mathbf{0},$$

where $\Sigma_{022} = \frac{1}{\theta_{22}^2}\, (M_{22}I_{24}I_{23}I_{22}I_{21})\, \Sigma_{22}\, (M_{22}I_{24}I_{23}I_{22}I_{21})^{\mathrm{T}}$ taking the form

$$\Sigma_{022} = \begin{pmatrix} \frac{r_{22}}{2(r_{21}r_{22}-r_{23})} & 0 & -\frac{1}{2(r_{21}r_{22}-r_{23})} & 0 \\ 0 & \frac{1}{2(r_{21}r_{22}-r_{23})} & 0 & 0 \\ -\frac{1}{2(r_{21}r_{22}-r_{23})} & 0 & \frac{r_{21}}{2r_{23}(r_{21}r_{22}-r_{23})} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \geq \mathbf{0}.$$

Hence, $\Sigma_{22} = \theta_{22}^2\, (M_{22}I_{24}I_{23}I_{22}I_{21})^{-1}\, \Sigma_{022} \left[(M_{22}I_{24}I_{23}I_{22}I_{21})^{-1}\right]^{\mathrm{T}} \geq \mathbf{0}$.

Case 2. $l_1 = 0$, which is tantamount to $C_{11}N_1^* = C_{33}N_3^*$. Denote

$$C_{23} = \begin{pmatrix} -C_{22}N_2^* & -C_{23}N_3^* & 0 & C_{21}N_1^* \\ C_{32}N_2^* & -C_{33}N_3^* & 0 & 0 \\ 0 & 0 & -C_{11}N_1^* & 0 \\ 0 & -\frac{\alpha C_{12}}{C_{32}} & \alpha & -\alpha \end{pmatrix}.$$

Let $C_{24} = I_{25}C_{23}I_{25}^{-1}$, where

$$I_{25} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

then we have

$$C_{24} = \begin{pmatrix} -C_{22}N_2^* & -C_{23}N_3^* & C_{21}N_1^* & 0 \\ C_{32}N_2^* & -C_{33}N_3^* & 0 & 0 \\ 0 & -\frac{\alpha C_{12}}{C_{32}} & -\alpha & \alpha \\ 0 & 0 & 0 & -C_{11}N_1^* \end{pmatrix}.$$

Let $C_{25} = M_{23}C_{24}M_{23}^{-1}$, where

$$M_{23} = \begin{pmatrix} -\alpha C_{12}N_2^* & \frac{\alpha C_{12}\left(\alpha + C_{33}N_3^*\right)}{C_{32}} & \alpha^2 & -\alpha^2 - \alpha C_{11}N_1^* \\ 0 & -\frac{\alpha C_{12}}{C_{32}} & -\alpha & \alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and we have

$$C_{25} = \begin{pmatrix} -r_{31} & -r_{32} & -r_{33} & r_{34} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -C_{11}N_1^* \end{pmatrix}.$$

Then equation (9) is equivalently transformed into

$$(M_{23}I_{25}I_{23}I_{22}I_{21})\,H_2^2\,(M_{23}I_{25}I_{23}I_{22}I_{21})^{\mathrm{T}} + C_{25}\,(M_{23}I_{24}I_{23}I_{22}I_{21})\,\Sigma_{23}\,(M_{23}I_{25}I_{23}I_{22}I_{21})^{\mathrm{T}}$$
$$+ (M_{23}I_{25}I_{23}I_{22}I_{21})\,\Sigma_{23}\,(M_{23}I_{25}I_{23}I_{22}I_{21})^{\mathrm{T}}\,C_{25}^{\mathrm{T}} = \mathbf{0},$$

i.e.,

$$G + C_{25}\Sigma_{023} + \Sigma_{023}C_{25}^{\mathrm{T}} = \mathbf{0},$$

where $\Sigma_{023} = \frac{1}{\theta_{23}^2}\,(M_{23}I_{25}I_{23}I_{22}I_{21})\,\Sigma_{23}\,(M_{23}I_{25}I_{23}I_{22}I_{21})^{\mathrm{T}}$ taking the form

$$\Sigma_{023} = \begin{pmatrix} \frac{r_{32}}{2(r_{31}r_{32}-r_{33})} & 0 & -\frac{1}{2(r_{31}r_{32}-r_{33})} & 0 \\ 0 & \frac{1}{2(r_{31}r_{32}-r_{33})} & 0 & 0 \\ -\frac{1}{2(r_{31}r_{32}-r_{33})} & 0 & \frac{r_{31}}{2r_{33}(r_{31}r_{32}-r_{33})} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \geq \mathbf{0}.$$

Hence, $\Sigma_{23} = \theta_{23}^2\,(M_{23}I_{25}I_{23}I_{22}I_{21})^{-1}\,\Sigma_{023}\left[(M_{23}I_{25}I_{23}I_{22}I_{21})^{-1}\right]^{\mathrm{T}} \geq \mathbf{0}$.

Step 3. Consider the algebraic equation:

$$H_3^2 + A\Sigma_3 + \Sigma_3A^{\mathrm{T}} = \mathbf{0}. \tag{10}$$

Let $A_{31} = I_{31}AI_{31}^{-1}$, where

$$I_{31} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then we have

$$A_{31} = \begin{pmatrix} -C_{33}N_3^* & C_{32}N_2^* & 0 & 0 \\ -C_{23}N_3^* & -C_{22}N_2^* & 0 & C_{21}N_1^* \\ 0 & -C_{12}N_2^* & -C_{11}N_1^* & 0 \\ 0 & 0 & \alpha & -\alpha \end{pmatrix}.$$

Let $A_{32} = M_3 A_{31} M_3^{-1}$, where

$$M_3 = \begin{pmatrix} \alpha C_{12}C_{23}N_2^*N_3^* & m_7 & \alpha\left(C_{11}N_1^*\right)^2 + \alpha^2 C_{11}N_1^* + \alpha^3 & -\alpha C_{12}C_{21}N_1^*N_2^* - \alpha^3 \\ 0 & -\alpha C_{12}N_2^* & -\alpha C_{11}N_1^* - \alpha^2 & \alpha^2 \\ 0 & 0 & \alpha & -\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $m_7 = \alpha^2 C_{12}N_2^* + \alpha C_{11}C_{22}N_1^*N_2^* + \alpha C_{12}C_{22}\left(N_2^*\right)^2$ and we obtain

$$A_{32} = \begin{pmatrix} -r_{11} & -r_{12} & -r_{13} & -r_{14} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then equation (10) is equivalently transformed into

$$(M_3 I_{31})\, H_3^2\, (M_3 I_{31})^{\mathrm{T}} + A_{32}\,(M_3 I_{31})\,\Sigma_3\,(M_3 I_{31})^{\mathrm{T}} + (M_3 I_{31})\,\Sigma_3\,(M_3 I_{31})^{\mathrm{T}} A_{32}^{\mathrm{T}} = \mathbf{0},$$

i.e.,

$$G + A_{32}\Sigma_{03} + \Sigma_{03}A_{32}^{\mathrm{T}} = \mathbf{0},$$

where $\Sigma_{03} = \frac{1}{\theta_3^2}(M_3 I_{31})\,\Sigma_3\,(M_3 I_{31})^{\mathrm{T}} = \Sigma_{01} \succ \mathbf{0}$. Therefore, $\Sigma_3 = \theta_3^2 (M_3 I_{31})^{-1}\,\Sigma_{03}\left[(M_3 I_{31})^{-1}\right]^{\mathrm{T}} \succ \mathbf{0}$.

To sum up, $\Sigma = \sum_{i=1}^{3}\Sigma_i \succ \mathbf{0}$. Based on the relationship of system (5) and system (4), the desired conclusion is proved. □

Remark 4.5. *In the proof of Theorem 4.4, we need to verify the following conditions in order to use Lemma 4.3:*

$$\Xi_1 = r_{11}r_{12}r_{13} - r_{13}^2 - r_{11}^2 r_{14} > 0,\ \Xi_2 = r_{21}r_{22} - r_{23} > 0,\ \Xi_3 = r_{31}r_{32} - r_{33} > 0.$$

Direct calculation implies that $\Xi_1 = D_3$*. Hence,* $\Xi_1 > 0$ *holds if* $C_{11}C_{22} > C_{12}C_{21}$*. Compute*

$$\begin{aligned} \Xi_2 = {} & C_{23}C_{32}C_{33}N_2^*\left(N_3^*\right)^2 + C_{22}C_{33}^2N_2^*\left(N_3^*\right)^2 + C_{22}C_{23}C_{32}\left(N_2^*\right)^2N_3^* + C_{22}^2C_{33}\left(N_2^*\right)^2N_3^* \\ & + C_{11}C_{33}^2N_1^*\left(N_3^*\right)^2 + C_{11}^2C_{22}\left(N_1^*\right)^2N_2^* + C_{11}C_{22}^2N_1^*\left(N_2^*\right)^2 + C_{11}^2C_{33}\left(N_1^*\right)^2N_3^* \\ & + \left[2C_{11}C_{22}C_{33}N_1^*N_2^*N_3^* - \alpha C_{12}C_{21}N_1^*N_2^*\right], \\ \Xi_3 = {} & C_{23}C_{32}C_{33}N_2^*\left(N_3^*\right)^2 + C_{22}C_{33}^2N_2^*\left(N_3^*\right)^2 + C_{22}C_{23}C_{32}\left(N_2^*\right)^2N_3^* + C_{22}^2C_{33}\left(N_2^*\right)^2N_3^* \\ & + \alpha C_{33}^2\left(N_3^*\right)^2 + \alpha C_{22}^2\left(N_2^*\right)^2 + \alpha^2C_{33}N_3^* + \alpha^2C_{22}N_2^* + \left[2\alpha C_{22}C_{33}N_2^*N_3^* - \alpha C_{12}C_{21}N_1^*N_2^*\right]. \end{aligned}$$

Since Ξ_2 *appears in* **Subcase 2** *with* $\alpha = C_{33}N_3^*$*, substituting* $\alpha = C_{33}N_3^*$ *into the expression for* Ξ_2 *verifies that* $\Xi_2 > 0$ *whenever* $C_{11}C_{22} > C_{12}C_{21}$*. Similarly, since* Ξ_3 *arises in* **Case 2** *under the condition* $C_{11}N_1^* = C_{33}N_3^*$*, substituting* $C_{11}N_1^* = C_{33}N_3^*$ *into the expression for* Ξ_3 *verifies that* $\Xi_3 > 0$ *when* $C_{11}C_{22} > C_{12}C_{21}$*.*

5. Examples and Numerical Simulations

Thanks to the classical Milstein's higher-order method ([15]), the corresponding discretization equation of system (5) is as follows:

$$\begin{cases} N_1^{k+1} = N_1^k + N_1^k\left[b_1 - C_{11}N_1^k - C_{12}N_2^k\right]\Delta t + \sigma_1 N_1^k \xi_{1,k}\sqrt{\Delta t} + \dfrac{\sigma_1^2}{2}N_1^k\left(\xi_{1,k}^2 - 1\right)\Delta t, \\ N_2^{k+1} = N_2^k + N_2^k\left[-b_2 + C_{21}Z_0^k - C_{22}N_2^k - C_{23}N_3^k\right]\Delta t + \sigma_2 N_2^k \xi_{2,k}\sqrt{\Delta t} + \dfrac{\sigma_2^2}{2}N_2^k\left(\xi_{2,k}^2 - 1\right)\Delta t, \\ N_3^{k+1} = N_3^k + N_3^k\left[-b_3 + C_{32}N_2^k - C_{33}N_3^k\right]\Delta t + \sigma_3 N_3^k \xi_{3,k}\sqrt{\Delta t} + \dfrac{\sigma_3^2}{2}N_3^k\left(\xi_{3,k}^2 - 1\right)\Delta t, \\ Z_0^{k+1} = Z_0^k + \alpha\left[N_1^k - Z_0^k\right]\Delta t, \end{cases} \tag{1}$$

where the time increment $\Delta t > 0$. $\xi_{i,k}$ $(i = 1, 2, 3)$ are three independent Gaussian random variables which follow the Gaussian distribution $N(0, 1)$ for $k = 1, 2, 3 \cdots$.

Let $b_1 = 0.8505$, $b_2 = 0.2492$, $b_3 = 0.1478$, $C_{11} = 0.1625$, $C_{12} = 0.1067$, $C_{21} = 0.5217$, $C_{22} = 0.9219$, $C_{23} = 0.4675$, $C_{32} = 0.6483$, $C_{33} = 0.8608$, $\alpha = 0.6486$, $\sigma_1 = 0.05$, $\sigma_2 = 0.05$, $\sigma_3 = 0.05$. Compute

$$\begin{aligned} &E^* = \left(N_1^*, N_2^*, N_3^*, Z_0^*\right) = (4.1875, 1.5818, 1.0181, 4.1875), \\ &C_{11}C_{22} - C_{12}C_{21} = 0.0941 > 0,\ 2C_{22}C_{33} - C_{23}C_{32} = 1.2841 > 0, \mathrm{T}_3 = 0.2302 > 0, \\ &C_{11}N_1^* - C_{33}N_3^* = -0.1959,\ \alpha - C_{33}N_3^* = -0.2278. \end{aligned}$$

Thanks to Theorem 4.1, all three species in system (5) can coexist in the long term (see the left-hand column of Fig 1). Thanks to Theorem 4.4, the variance matrix Σ takes the following form

$$\Sigma = \begin{pmatrix} 0.0019 & 0.0005 & 0.0002 & 0.0007 \\ 0.0005 & 0.0019 & 0.0008 & 0.0008 \\ 0.0002 & 0.0008 & 0.0026 & 0.0006 \\ 0.0007 & 0.0008 & 0.0006 & 0.0007 \end{pmatrix},$$

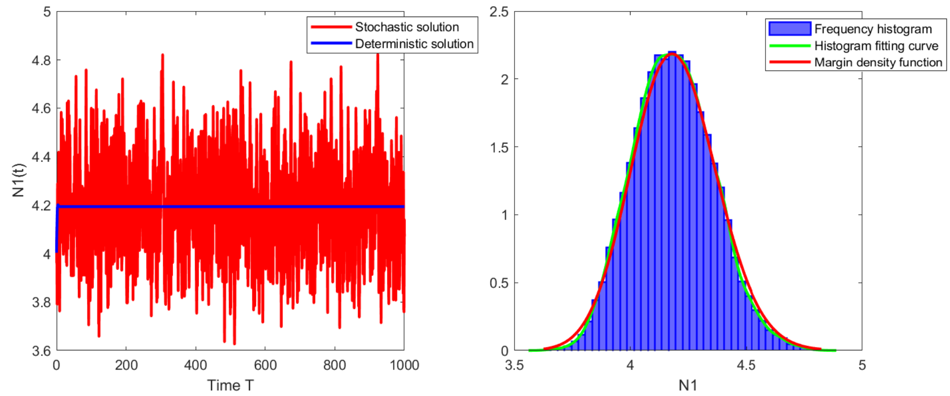
and the four-dimensional density function $\Phi(N_1, N_2, N_3, Z_0)$ is given by

$$\Phi(N_1, N_2, N_3, Z_0) = \frac{19628}{N_1N_2N_3Z_0}\mathrm{e}^{-\frac{1}{2}\left(\ln\frac{N_1}{4.1875}, \ln\frac{N_2}{1.5818}, \ln\frac{N_3}{1.0181}, \ln\frac{Z_0}{4.1875}\right)\Sigma^{-1}\left(\ln\frac{N_1}{4.1875}, \ln\frac{N_2}{1.5818}, \ln\frac{N_3}{1.0181}, \ln\frac{Z_0}{4.1875}\right)^{\mathrm{T}}}.$$

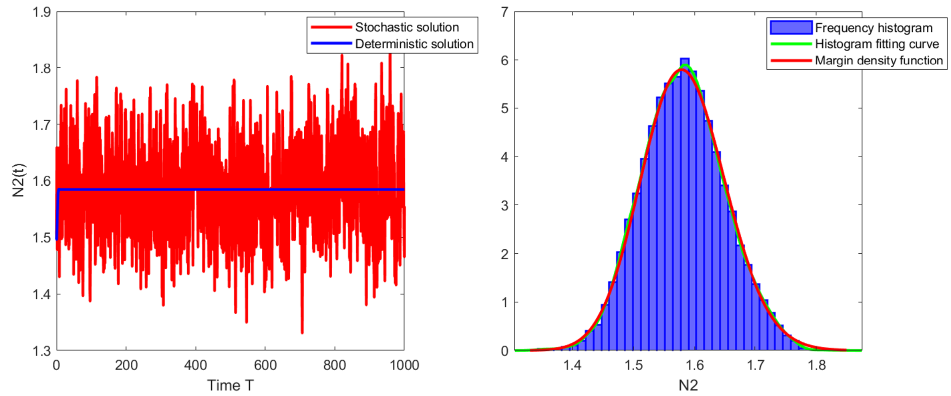
Moreover, it has the corresponding marginal density functions:

$$\begin{aligned} &\Phi^{(N_1)}(N_1) = \frac{9.1524}{N_1}\mathrm{e}^{-263.1579(\ln N_1 - 1.4321)^2},\quad \Phi^{(N_2)}(N_2) = \frac{9.1524}{N_2}\mathrm{e}^{-263.1579(\ln N_2 - 0.4586)^2}, \\ &\Phi^{(N_3)}(N_3) = \frac{7.8239}{N_3}\mathrm{e}^{-192.3077(\ln N_3 - 0.0179)^2}, \end{aligned}$$

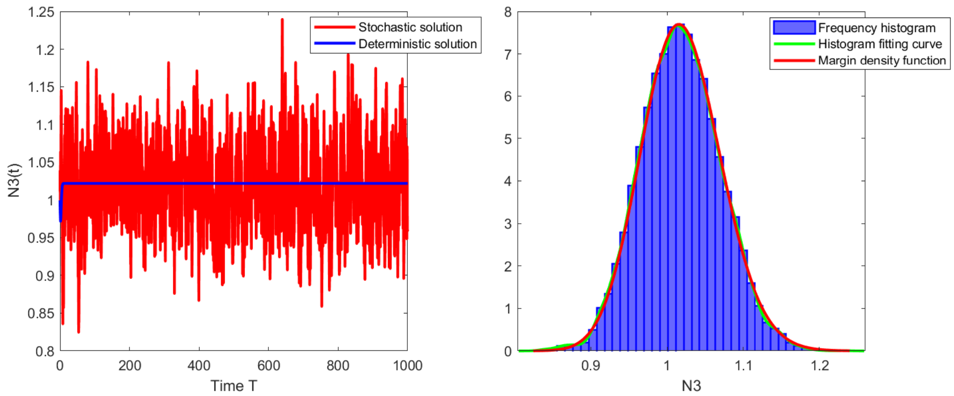
which are shown in the right-hand column of Fig 1.



(a)



(b)



(c)

Figure 1: The left-hand column shows the numbers of prey, intermediate predator and top predator in system (5) and its deterministic system, respectively. The right-hand column presents the frequency histogram, the frequency histogram fitting curve and the marginal density function of prey, intermediate predator and top predator in system (5), respectively. The size of the increment Δt is $\Delta t = \frac{1}{50}$.

Declarations

Author's Contribution

Sheng Wang: Methodology, Writing, Editing. Jiarui Meng: Methodology, Writing, Editing. Both authors read and approved the final manuscript.

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The authors declare that there are no conflicts of interest.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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