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On controllability of Hilfer fractional neutral stochastic impulsive differential equations with integral impulses

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Abstract. In this paper, the controllability of Hilfer fractional neutral stochastic impulsive differential equations with integral impulses is investigated. The controllability result is established through the application of semigroup of operators, measure of noncompactness, Mönch's fixed point theorem, and stochastic analysis techniques. An example is presented to validate the theoretical findings.

1. Introduction

Fractional differential equations have emerged as powerful tools for studying complex systems and have attracted considerable attention over recent decades. Their applications span a wide range of disciplines, including physics, chemistry, engineering, aerodynamics, and economics. The flexibility and versatility of fractional differential equations enable researchers to model phenomena characterized by non-locality, memory effects, and complex dynamics with greater precision and accuracy. In recent years, significant progress has been made in both the theoretical understanding and practical applications of fractional calculus [1, 11, 39].

Among the various definitions of fractional derivatives, the Hilfer fractional derivative has received particular attention due to its unifying properties. Specifically, it interpolates between the Riemann–Liouville and Caputo derivatives via a parameter, allowing for more flexible modeling of memory effects [13, 32]. This unique feature renders Hilfer derivatives effective tools for investigating complex real-world phenomena in science and technology. As a result, researchers have increasingly employed Hilfer fractional derivatives to analyze dynamical systems and to gain deeper insights into their underlying behavior [16, 24, 25].

However, many real-world processes cannot be adequately described using classical initial conditions alone. This issue arises particularly in fields such as biology, medicine, chemical technology, physics, and economics, where sudden changes—such as disease outbreaks, harvesting events, or abrupt system shocks play a crucial role in the system dynamics. These situations are better modeled by impulsive differential equations, which combine standard initial conditions with instantaneous perturbations of short duration

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relative to the overall process. Consequently, impulsive differential and integrodifferential equations have proven to be essential in modeling phenomena with sudden transitions [10, 30, 40].

Motivated by this, researchers have studied impulsive differential equations extensively in the integer-order setting due to their broad applications [2, 5, 26]. In contrast, impulsive fractional differential equations remain relatively less explored, despite their potential to capture both memory effects and impulsive behaviors simultaneously. Some recent contributions on impulsive fractional differential equations can be found in [4, 18, 20].

Recent advances further emphasize the growing role of fractional calculus in diverse applied domains, such as controllability of nonlinear fractional systems [34], mixed Volterra–Fredholm integro-differential systems [35], and non-instantaneous impulsive fractional control systems [36]. Moreover, extensions to fractional stochastic and delayed systems continue to emerge [23, 37], demonstrating the relevance of fractional calculus in tackling real-world complexities.

Parallel to this development, the notion of controllability has emerged as a key concept in modern science and engineering. Many researchers have investigated controllability results for various dynamical systems [27, 28, 29, 41]. In particular, fractional-order systems involving the Caputo derivative have been widely studied with respect to approximate controllability [6, 7], while more recent works have examined the existence of solutions for Hilfer fractional differential systems [21, 25, 31].

For instance, Zhou et al. [25] analyzed the existence of mild solutions for Hilfer fractional differential equations of the form

$$ {}^{H}D_{0+}^{\lambda,\nu}\vartheta(t) = A\vartheta(t) + \Lambda(t,\vartheta(t)), \quad t \in (0,T], $$

with corresponding fractional initial conditions. They established controllability results using the Schauder fixed point theorem. Subashini [33] later studied Hilfer integro-differential equations and established existence results via Mönch's fixed point theorem.

Since real-world systems are often influenced by random disturbances, stochastic modeling has become indispensable. Stochastic differential equations provide more accurate descriptions of dynamic systems subject to randomness, leading to a surge in related research. For example, Sivasankar et al. [38] investigated Hilfer fractional neutral stochastic differential equations and established existence results for mild solutions via Mönch's fixed point theorem. Similarly, Bose et al. [8] examined controllability for Hilfer fractional neutral differential equations with and without impulses, employing Mönch's fixed point theorem and the measure of noncompactness. Related works on fractional impulsive and delayed systems include those by Kumar et al. [3] and Karthikeyan et al. [19]. Bose and Udhayakumar [9] further explored the controllability of Hilfer fractional neutral differential equations with integral terms.

To the best of our knowledge, the controllability of neutral stochastic differential systems with integral impulses, particularly those involving the Hilfer fractional derivative, has not yet been addressed in the existing literature. Motivated by this gap, the present work investigates the controllability of a class of Hilfer neutral stochastic impulsive fractional differential equations with integral impulses. The analysis is conducted using semigroup theory, the measure of noncompactness, and fixed point techniques.

In this paper, we consider the following Hilfer fractional neutral stochastic differential equations with integral impulses:

$$ \begin{cases} D_{0^+}^{p,\sigma}[\vartheta(t) - f(t,\vartheta_t)] = A[\vartheta(t) - f(t,\vartheta_t)] + g(t,\vartheta_t) + Bv(t) \\ \qquad\qquad\qquad +h(t,\vartheta_t)\dfrac{dw(t)}{dt}, \quad t \in J := [0,\ell] \setminus \{0\}, \quad t \neq t_j, \\ \Delta\vartheta(t_j) = I_j\left(\displaystyle\int_{t_j-p_j}^{t_j-q_j} \mathcal{G}(\varrho,\vartheta(\varrho))d\varrho\right), \quad j = 1,2,\ldots,r, \\ I_{0^+}^{(1-p)(1-\sigma)}\vartheta(t) = \xi(t) \in L^2(\Omega,\mathfrak{B}_h) \quad \text{for a.e. } t \in J_0 = (-\infty,0], \end{cases} \tag{1} $$

where $D_{0^+}^{p,\sigma}$ denotes the Hilfer derivative with order $0 < p < 1$ and type $0 \le \sigma \le 1$ (see [32]). The state function $\vartheta \in H$, where H is a real separable Hilbert space. Here, A denotes the almost sectorial operator. The histories $\vartheta_t : (-\infty,0] \to \mathfrak{B}_h$ defined by $\vartheta_t(\varrho) = \vartheta(t+\varrho), \varrho \le 0$, in some phase space $\mathfrak{B}_h$. Let K and U be another separable Hilbert spaces. $w(\cdot)$ denotes the K-valued Wiener process. The control function

$\upsilon \in L^2(J,U)$, a set of admissible control functions on U and the operator B is a bounded. f, g and h are given functions to be specified later. Here, $0 = t_0 < t_1 < t_2 < \cdots < t_r < t_{r+1} = \ell$ and $I_j,\ j = 1,2,\ldots,r$ is an impulsive function. $0 \le q_j \le p_j \le t_j - t_{j-1}$ for $j = 1,2,\ldots,r$, $\Delta\vartheta(t_j) = \vartheta(t_j^+) - \vartheta(t_j^-)$, which denotes the jumps at $t = t_j$ in the state ϑ. Also, $\vartheta(t_j^+)$ and $\vartheta(t_j^-)$ denote the right and left limit values of ϑ at $t = t_j$.

The article is structured as follows: Section 2 presents key concepts and definitions relevant to our research, which will be used to prove the main result. Section 3 explores the controllability results concerning Hilfer fractional neutral stochastic impulsive differential equations with integral impulses. Finally, Section 4 offers an example to demonstrate the theoretical findings.

2. Preliminaries and Assumptions

Let $(\Omega, \Upsilon, \mathbb{P})$ be a complete probability space, $\Upsilon_t,\ t \in J$ denotes a normal filtration and Υ_0 contains all $\mathbb{P}$-null sets. Also, $\{w(t)\}_{t\ge 0}$ denotes Q-Wiener process on $(\Omega, \Upsilon, \mathbb{P})$ with a bounded nuclear covariance operator Q, where $Qy_n = \alpha_n y_n,\ n = 1,2,\ldots$, and $Tr(Q) = \sum_{n=1}^{\infty} \alpha_n$, where $\{y_n\}_{n=1}^{\infty}$ be a complete orthonormal basis in K and $\{\alpha_n\}_{n=1}^{\infty}$ denotes the sequence of non-negative real numbers which is bounded. We define $w(t) = \sum_{n=1}^{\infty} \sqrt{\alpha_n} w_n(t) y_n$, where w_n are mutually independent one-dimensional standard Brownian motions over $(\Omega, \Upsilon, \mathbb{P})$. For every $\mu \in L(K,H)$, we define

$$\left\|\mu\right\|_Q^2 = Tr(\mu Q \mu^*) = \sum_{n=1}^{\infty} \left\| \sqrt{\alpha_n} \mu y_n \right\|^2.$$

If $\|\mu\|_Q^2 < \infty$, then μ is called a Q-Hilbert Schmidt operator. We denote the space of all Q-Hilbert-Schmidt operators $\mu : K \to H$ by $L_2^0(K,H)$. The completion $L_2^0(K,H)$ of $L(K,H)$ equipped with the norm $\|\cdot\|_Q$, where $\|\mu\|_Q^2 =< \mu, \mu >$ forms a Hilbert space, where $L(K,H)$ denotes the space of all bounded linear operators from K into H. If $K = H$, we simply denoted it as $L(H)$.

Let $L^2(\Omega, H)$ be defined as the set of all H-valued, square integrable and strongly measurable random variables equipped with the norm

$$\left\|\vartheta(\cdot)\right\|_{L^2} = \left(\mathbb{E}\left\|(\cdot,\omega)\right\|^2\right)^{1/2}.$$

We also consider the subspace $L_0^2(\Omega, H)$ given by

$$L_0^2(\Omega,H) = \left\{\vartheta \in L^2(\Omega,H) : \vartheta \text{ is } \Upsilon_0 - \text{measurable}\right\}.$$

Let $J_1 = (-\infty, b]$ and $C(J_1, L^2(\Omega, H))$ denote the Banach space of all Υ_t-adapted, H-valued, strongly measurable stochastic processes that are continuous function from J_1 to $L^2(\Omega, H)$ with the condition $\sup_{t\in J_1} \mathbb{E}\|\vartheta(t)\|^2 < \infty$.

Denoting $J' = (0, b]$ and $\gamma = p + \sigma - p\sigma$, we define

$$C_{1-\gamma}(J, L^2(\Omega,H)) = \left\{\vartheta \in C(J', L^2(\Omega,H)) : t^{1-\gamma}\vartheta(t) \in C(J, L^2(\Omega,H))\right\}$$

with the norm

$$\|\vartheta\|_{C_{1-\gamma}}^2 = \sup_{t\in J} t^{1-\gamma}\mathbb{E}\|\vartheta(t)\|^2.$$

Clearly, $(C_{1-\gamma}(J, L^2(\Omega,H)), \|\cdot\|_{C_{1-\gamma}})$ is a Banach space.

Next, we discuss the abstract phase $\mathfrak{B}_h$ (defined in [42]). Let $h : (-\infty, 0] \to (0,\infty)$ be continuous function with $I_h = \displaystyle\int_{-\infty}^{0} h(t)dt < \infty$. The Banach space $(\mathfrak{B}_h, \|\cdot\|_{\mathfrak{B}_h})$ induced by function h is defined as follows:

$$\mathfrak{B}_h \ = \ \Big\{\xi : (-\infty, 0] \to H : \text{ for any } a > 0, \text{ the function } \xi(\theta) \text{ is bounded and measurable on } [-a, 0]$$

$$\text{and} \int_{-\infty}^{0} h(t) \sup_{-a\leq\theta\leq 0} \mathbb{E}(\|\xi(\theta)\|^2)^{1/2} dt < \infty \Big\}$$

with norm

$$\|\xi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(t) \sup_{-a\leq\theta\leq 0} \mathbb{E}(\|\xi(\theta)\|^2)^{1/2} dt \quad \forall \quad t \in \mathfrak{B}_h.$$

Next, we define the space

$$\begin{aligned} \mathfrak{B}'_h \quad = \quad & \Big\{\vartheta : (-\infty, \ell] \to H : \vartheta_j \in C(J_j, H) \text{ and } \exists\ \vartheta(t_j^-) \text{ and } \vartheta(t_j^+) \text{ with } \vartheta(t_j) = \vartheta(t_j^-), \\ & I_{0^+}^{(1-p)(1-\sigma)} \vartheta(t) = \xi(t),\ t \in (-\infty, 0],\ j = 0, 1, 2, \ldots, r, \Big\}, \end{aligned}$$

where ϑ_j is the restriction of ϑ on the interval $J_j = (t_j, t_{j+1}],\ j = 0, 1, 2, \ldots, r$. Define the semi norm $\|\cdot\|_{\mathfrak{B}'_h}$ be in $\mathfrak{B}'_h$ defined by

$$\|\vartheta\|_{\mathfrak{B}'_h} = \|h\|_{\mathfrak{B}_h} + \sup_{s\in[0,d]} \left(\mathbb{E}\|\vartheta(s)\|^2\right)^{1/2},\ \vartheta \in \mathfrak{B}'_h.$$

Definition 2.1. *[32] The Hilfer fractional derivative of order $0 < p < 1$ and type $0 \leq \sigma \leq 1$ for the function $f : [\ell, \infty) \to \mathbb{R}$, is presented by*

$$D_{b^+}^{p,\sigma} f(t) = \left[I_{b+}^{(1-p)\sigma} D(I_{b+}^{(1-p)(1-\sigma)} f)\right](t).$$

To establish the main result, we need the following assumptions:

(H1) The operator A is the almost sectorial operator of the semigroup $T(t)$ such that $\|T(t)\| \leq \mathcal{K}_1$, where $\mathcal{K}_1 > 1$ is a constant.

(H2) The function $f : J \times \mathfrak{B}_h \to H$ satisfy the following:

- (i) For any $t \in J$, the function f is continuous and for all $\vartheta \in \mathfrak{B}_h$ and satisfies

$$\|f(t, \vartheta)\|^2 \leq M_f \left(1 + \|\vartheta\|^2\right).$$

- (ii) The function f is completely continuous and for any bounded set $\mho \subset C(J, L^2(\Omega, H))$, the set $\{t \to f(t, \vartheta), \vartheta \in \mho\}$ is equicontinuous in H.

(H3) The function $g : J \times \mathfrak{B}_h \to H$ satisfies the following:

- (i) $g(\cdot, \vartheta)$ is strongly measurable for every $\vartheta \in \mathfrak{B}_h$, $g(t, \cdot)$ is continuous for a.e. $t \in J$, and $g(t, \cdot) : J \to H$ is strongly measurable function.
- (ii) $\exists$ an integrable function $m_1 \in L^{1/p_1}(J, \mathbb{R}^+)$, $p_1 \in (0, p)$ and nondecreasing real valued continuous function $\tilde{g} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|g(t, \vartheta)\|^2 \leq m_1(t)\tilde{g}(\|\vartheta\|^2), \quad \vartheta \in \mathfrak{B}_h, \quad t \in J,$$

where $\tilde{g}$ satisfies

$$\liminf_{t\to\infty} \frac{\tilde{g}(t)}{t} = 0.$$

- (iii) $\exists$ a function $m_2 \in L^{1/p_2}(J, \mathbb{R}^+)$, $p_2 \in (0, p)$ such that for each bounded subsets $\mho \subset H$

$$\varsigma(g(t, \mho)) \leq m_2(t)\varsigma(\mho) \quad \text{for a.e. } t \in J.$$

(H4) The function $h : J \times \mathfrak{B}_h \to L_2^0(\Omega, H)$ satisfies the following:

(i) $h(\cdot, \vartheta)$ is strongly measurable for every $\vartheta \in \mathfrak{B}_h$, $h(t, \cdot)$ is continuous for a.e. $t \in J$, and $h(t, \cdot) : J \to L_2^0(\Omega, H)$ is strongly measurable function.

(ii) $\exists$ an integrable function $m_3 \in L^{1/p_3}(J, \mathbb{R}^+)$, $p_3 \in (0, p)$ and nondecreasing real valued continuous function $\tilde{h} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|h(t, \vartheta)\|^2 \le m_3(t)\tilde{h}(\|\vartheta\|^2), \quad \vartheta \in \mathfrak{B}_h, \quad t \in J,$$

where $\tilde{h}$ satisfies

$$\liminf_{t\to\infty} \frac{\tilde{h}(t)}{t} = 0.$$

(iii) $\exists$ a function $m_4 \in L^{1/p_4}(J, \mathbb{R}^+)$, $p_4 \in (0, p)$ such that for each bounded subsets $\mho \subset H$

$$\varsigma(g(t, \mho)) \le m_4(t)\varsigma(\mho) \quad \text{for a.e. } t \in J.$$

(H5) The operator B is bounded, i.e. $\|B\| \le \mathcal{K}_2$, where $\mathcal{K}_2$ is a positive constant.

(H6) The operator $L : L^2(J, U) \to L^2(\Omega, H)$ defined by

$$Lv = \int_0^{\ell} (\ell - \varrho)^{p-1} Q_p(\ell - \varrho) B v(\varrho) d\varrho$$

satisfy the following:

(i) It has an inverse operator L^{-1}, which take the values in $L^2(J, U)/\text{Ker } L$ and there exists a constant $\mathcal{K}_3 > 0$ such that $\left\|L^{-1}\right\| \le \mathcal{K}_3$.

(ii) There exists a function $m_5 \in L^{1/p_5}(J, \mathbb{R}^+)$, $p_5 \in (0, p)$ such that for every bounded sets $\mho \subset H$,

$$\varsigma\left(\left(L^{-1}\mho\right)(t)\right) \le m_5(t)\varsigma(\mho).$$

(H7) The function $\mathcal{G}$ satisfies the following conditions:

(i) There are constants $L_{\mathcal{G}}$ and $\tilde{L}_{\mathcal{G}}$ such that

$$\|\mathcal{G}(t, \vartheta_1) - \mathcal{G}(t, \vartheta_2)\|^2 \le L_{\mathcal{G}}\|\vartheta_1 - \vartheta_2\|^2, \quad \tilde{L}_{\mathcal{G}} = \sup_{t\in J} \|\mathcal{G}(t, 0)\|^2,$$

for every $\vartheta_1, \vartheta_2 \in H$, and $t \in J$.

(ii) There exists a constant $\lambda > 0$ such that for each bounded subsets $\mho \subset H$

$$\varsigma(\mathcal{G}(t, \mho)) \le \lambda\varsigma(\mho) \quad \text{for a.e. } t \in J.$$

(H8) The function $I_j : H \to H$, $j = 1, 2, \ldots, r$ satisfies the following assumptions:

(i) I_j is continuous and there exists a constant L_j such that for all $\vartheta_1, \vartheta_2 \in H$, we have

$$\|I_j(\vartheta_1) - I_j(\vartheta_2)\|^2 \le L_j\|\vartheta_1 - \vartheta_2\|^2.$$

(ii) There exists constant $L_j^* > 0$ such that for all $\vartheta \in H$

$$\|I_j(\vartheta)\|^2 \le L_j^*.$$

Definition 2.2. *[44] A stochastic process $\vartheta : (-\infty, \ell] \to H$ is said to be a mild solution of the system (1) if $I_{0^+}^{(1-p)(1-\sigma)}\vartheta(t) = \xi(t) \in L^2(\Omega, \mathfrak{B}_h)$, for a.e. $t \in J_0$, and for $t \in J$, ϑ satisfies the following integral equation*

$$\vartheta(t) = S_{p,\sigma}(t)[\xi(0) - f(0, \xi(0))] + f(t, \vartheta_t) + \int_0^t (t - \varrho)^{p-1} Q_p(t - \varrho) g(\varrho, \vartheta_\varrho) d\varrho + \int_0^t (t - \varrho)^{p-1} Q_p(t - \varrho) B v(\varrho) d\varrho$$

$$+\int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)h(\varrho,\vartheta_\varrho)dw(\varrho)+\sum_{j=1}^{r}S_{p,\sigma}(t-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j}\mathcal{G}(\varrho,\vartheta(\varrho))d\varrho\right), \tag{2}$$

where

$$S_{p,\sigma}(t)=I_{0^+}^{\sigma(1-p)}\mathcal{K}_p(t),\quad \mathcal{K}_p(t)=t^{p-1}Q_p(t)\quad and\quad Q_p(t)=\int_0^\infty p\varrho W_p(\varrho)T(t^p\varrho)d\varrho,$$

where $W_p(\cdot)$ is the Wright function

$$W_p(\tau)=\sum_{k\in\mathbb{N}}\frac{(-\tau)^{k-1}}{(k-1)!\Gamma(1-pk)},\quad \tau\in\mathbb{C},\quad 0<p<1,$$

which satisfies the following property

$$\int_0^\infty \varrho^\alpha W_p(\varrho)d\varrho=\frac{\Gamma(1+\alpha)}{\Gamma(1+p\alpha)},\quad \tau\geq 0.$$

Lemma 2.3. *[25] Under the assumption (H1), $Q_p(t)$ and $S_{p,\sigma}(t)$ are strongly continuous for $t>0$*

Lemma 2.4. *[25] Under the assumption (H1), for any fixed $t>0$, the operators $\mathcal{K}_p(t)$ and $S_{p,\sigma}(t)$ are linear and for all $\vartheta\in H$,*

$$\|\mathcal{K}_p(t)\vartheta\|\leq L't^{p-1}\|\vartheta\|,\quad \|S_{p,\sigma}(t)\vartheta\|\leq L''t^{(p-1)(\sigma-1)}\|\vartheta\|,$$

where $L'=\frac{\mathcal{K}_1}{\Gamma(p)}$ and $L''=\frac{\mathcal{K}_1}{\Gamma(p(1-\sigma)+\sigma)}$.

Definition 2.5. *The system (1) is said to be controllable on interval J if for each $\xi\in L^2(\Omega,\mathfrak{B}_h)$, $\vartheta^1\in H$, there exists $v\in L^2(J,U)$ such that the mild solution ϑ of (1) satisfies $\vartheta(\ell)=\vartheta^1$.*

Now, let us discuss some of the definitions and properties of the measure of noncompactness.

Definition 2.6. *Suppose $\mho$ is the bounded set in a Hilbert space H, then the Hausdorff measure of noncompactness ς is defined by*

$$\varsigma(\mho)=\inf\{\epsilon>0:\mho \text{ can be covered by a finite number of balls with radii } \epsilon\}.$$

Lemma 2.7. *[15] Let H be a Hilbert space and $\mho_1,\mho_2$ are bounded subsets of H. Then the following statements hold:*

1. *$\mho_1$ is relatively compact $\Leftrightarrow \varsigma(G_1)=0$;*
2. *$\varsigma(\mho_1)\leq\varsigma(\mho_2)$ if $\mho_1\subseteq\mho_2$;*
3. *$\varsigma(\mho_1\cup\mho_2)\leq\max\{\varsigma(\mho_1),\varsigma(\mho_2)\}$;*
4. *for every $\lambda\in\mathbb{R}$, $\varsigma(\lambda\mho_1)=|\lambda|\varsigma(\mho_1)$;*
5. *$\varsigma(\mho_1+\mho_2)\leq\varsigma(\mho_1)+\varsigma(\mho_2)$, where $\mho_1+\mho_2=\{\vartheta_1+\vartheta_2:\vartheta_1\in\mho_1,\vartheta_2\in\mho_2\}$;*
6. *$\varsigma(\mho_1)=\varsigma\left(\overline{\mho_1}\right)=\varsigma(conv(\mho_1))$, where $conv(\mho_1)$ and $\overline{\mho_1}$ denote the convex hull and closure of $\mho_1$, respectively;*
7. *If the operator $\Psi:D(\Psi)\subseteq H\to H_1$ is Lipschitz continuous and Λ_1 is the constant, then we know that $\rho(\Psi(\mho_1))\leq\Lambda_1\varsigma(\mho_1)$ for any bounded subset $\mho_1\subset D(\Psi)$, where ρ represents the Hausdorff measure of non-compactness in a Hilbert space H_1.*

Theorem 2.8. *[17] If $\{\vartheta_m\}_{m=1}^\infty$ is a sequence of Bochner-integrable functions from J to H such that $\|\vartheta_m(t)\|\leq\xi(t)$ for almost every $t\in J$ and every $m\geq 1$, where $\xi\in L^1(J,\mathbb{R})$, then the function $\varphi(t)=\mu(\{\vartheta_m(t),m\geq 1\})\in L^1(J,\mathbb{R})$ and satisfies*

$$\mu\left(\left\{\int_0^t\vartheta_m(\varrho)d\varrho,m\geq 1\right\}\right)\leq 2\int_0^t\varphi(\varrho)d\varrho.$$

Lemma 2.9. *[15] If $\mho \subset C([a,b],H)$ is bounded and equicontinuous, then $\varsigma(\mho(t))$ is continuous for $t \in [a,b]$, and*

$$\varsigma(\mho) = \sup\{\varsigma(\mho(t)), t \in [a,b]\},$$

where $\mho(t) = \{\vartheta(t), \vartheta \in \mho\} \subseteq H$.

Lemma 2.10. *[22] If $\mho \subset L^p([0,T], L_0^2)$ and $w(t)$ is a Q-Wiener process. For every $p \geq 2$, the Hausdorff measure of noncompactness ς satisfies*

$$\varsigma\left(\int_0^t \mho(\varrho)dw(\varrho)\right) \leq \sqrt{T\frac{p}{2}(p-1)}\varsigma(\mho(t)),$$

where

$$\int_0^t \mho(\varrho)dw(\varrho) = \left\{\int_0^t \vartheta(\varrho)dw(\varrho) \text{ for all } \vartheta \in \mho, \quad t \in [0,T]\right\}.$$

Remark 2.11. *From Lemma 2.10, we take $p = 2$, then*

$$\varsigma\left(\int_0^t \mho(\varrho)dw(\varrho)\right) \leq \sqrt{TTr(Q)}\varsigma(\mho(t)).$$

3. Controllability Result

For our convenience, we take the following notations:

$$K_{m_2} = \|m_2\|_{L^{1/p_2}(J,\mathbb{R}^+)}, \quad K_{m_4} = \|m_4\|_{L^{1/p_4}(J,\mathbb{R}^+)},$$

$$K_{p_2} = \left[\left(\frac{2-p-p_2}{1-p_2}\right)\ell^{(2-p-p_2)/(1-p_2)}\right]^{1-p_2}, \quad K_{p_4} = \left[\left(\frac{2-p-p_4}{1-p_4}\right)\ell^{(2-p-p_4)/(1-p_4)}\right]^{1-p_4}.$$

Theorem 3.1. *If (H1)–(H8) holds, then the system (1) is controllable on J if $\tilde{\mathcal{K}} < 1$, where*

$$\tilde{\mathcal{K}} = 2L'\ell^{(1-p)(1-\sigma)}\left\{K_{p_2}K_{m_2}\left[1 + L'\mathcal{K}_2 m_5 K_{p_2}\right] + Tr(Q)K_{p_4}K_{m_4}\left[1 + L'\mathcal{K}_4 m_5 K_{p_4}\right]\right\} + 2r\lambda L_j \ell^{2(1-p)(1-\sigma)+1}L''.$$

Proof. From (H6), we define the control function as

$$\begin{aligned} \upsilon_\vartheta(t) &= L^{-1}\Bigg[\vartheta^1 - S_{p,\sigma}(\ell)[\xi(0) - f(0,\xi(0))] - f(\ell,\vartheta_\ell) - \int_0^\ell (\ell-\varrho)^{p-1}Q_p(\ell-\varrho)g(\varrho,\vartheta_\varrho)d\varrho \\ &\quad - \int_0^\ell (\ell-\varrho)^{p-1}Q_p(\ell-\varrho)h(\varrho,\vartheta_\varrho)dw(\varrho) - \sum_{j=1}^r S_{p,\sigma}(\ell-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j} \mathcal{G}(\varrho,\vartheta(\varrho))d\varrho\right)\Bigg](t). \end{aligned} \tag{3}$$

Let $\mathfrak{B}_\ell$ be the space of all functions $\vartheta : (-\infty,\ell] \to H$ such that $\vartheta_0 \in \mathfrak{B}_h$ and the restriction $\vartheta : J \to H$ is continuous. Let $\|\cdot\|_\ell$ be the seminorm in $\mathfrak{B}_\ell$ defined by

$$\|\vartheta\|_\ell = \|\vartheta_0\|_{\mathfrak{B}_h} + \sup\{\|\vartheta(\varrho)\|, \ 0 \leq \varrho \leq \ell\}, \quad \vartheta \in \mathfrak{B}_\ell.$$

Let $Z_\ell = C(J_1, L^2(\Omega;\mathfrak{B}_\ell))$. Consider the operator $\Psi : Z_\ell \to Z_\ell$ defined by

$$\Psi(\vartheta(t)) = \begin{cases} \Psi_1(t), \quad t \in J_0, \\ S_{p,\sigma}(t)\left[\xi(0) - f(0,\xi(0))\right] + f(t,\vartheta_t) + \int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)g(\varrho,\vartheta_\varrho)d\varrho \\ + \int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)B\upsilon_\vartheta(\varrho)d\varrho + \int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)h(\varrho,\vartheta_\varrho)dw(\varrho) \\ + \sum_{j=1}^r S_{p,\sigma}(t-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j} \mathcal{G}(\varrho,\vartheta(\varrho))d\varrho\right), \quad t \in J. \end{cases} \tag{4}$$

We show that the operator Ψ possesses a fixed point, which is a mild solution of system (1). It is easy to see that $\Psi(\vartheta(\ell)) = \vartheta^1$. Therefore, (1) is controllable on J_0.

For $\Psi_1 \in Z$, let $\widehat{\Psi}(\cdot) : (-\infty, \ell) \to Z_\ell$ be the function defined by

$$\widehat{\Psi}(t) = \begin{cases} \Psi_1(t), & t \in J_0, \\ S_{p,\sigma}(t)\xi(0), & t \in J. \end{cases} \tag{5}$$

Set $\vartheta(t) = \kappa(t) + \widehat{\Psi}(t)$, $-\infty < t \le \ell$. It is easy to see that ϑ satisfies equation (2) if and only if κ satisfies $\kappa_0 = 0$ and

$$\begin{aligned} \kappa(t) &= -S_{p,\sigma}(t)f(0,\xi(0)) + f(t,\kappa_t + \widehat{\Psi}_t) + \int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)g(\varrho,\kappa_\varrho + \widehat{\Psi}_\varrho)d\varrho \\ &\quad + \int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)Bv_{\kappa+\widehat{\Psi}}(\varrho)d\varrho + \int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)h(\varrho,\kappa_\varrho + \widehat{\Psi}_\varrho)dw(\varrho) \\ &\quad + \sum_{j=1}^r S_{p,\sigma}(t-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j} \mathcal{G}(\varrho,\vartheta(\varrho))d\varrho\right), \end{aligned} \tag{6}$$

where $v_{\kappa+\widehat{\Psi}}$ is obtained from equation (3) by replacing $\vartheta = \kappa + \widehat{\Psi}$.

Let $\mathfrak{B}_\ell^0 = \{\kappa \in \mathfrak{B}_\ell : \kappa_0 = 0 \in \mathfrak{B}_h\}$. For any $\kappa \in \mathfrak{B}_\ell^0$, we have

$$\|\kappa\|_\ell = \|\kappa_0\|_{\mathfrak{B}_h} + \sup\{\|\kappa(\varrho)\|,\quad 0 \le \varrho \le \ell\} = \sup\{\|\kappa(\varrho)\|,\quad 0 \le \varrho \le \ell\}.$$

Thus, if $Z_\ell^0 = C(J_1, L^2(\Omega; \mathfrak{B}_\ell^0))$, then $(Z_\ell^0, \|\cdot\|_\ell)$ is a Banach space. Define $B_q = \left\{\kappa : \|\kappa\|_\ell^2 \le q, \kappa \in \mathfrak{B}_\ell^0\right\}$ for some $q > 0$. Clearly, B_q is a bounded and closed convex set in $\mathfrak{B}_\ell^0$ for each q, then $B_q \subseteq Z_\ell^0$ is uniformly bounded. For every $\kappa \in B_q$, we see that

$$\begin{aligned} \left\|\kappa_t + \widehat{\Psi}_t\right\|_{\mathfrak{B}_h} &\le \|\kappa_t\|_{\mathfrak{B}_h} + \|\widehat{\Psi}_t\|_{\mathfrak{B}_h} \\ &\le \bar{K}\left(q + \frac{\Gamma(\beta)}{\Gamma(\sigma(1-p)+p\sigma)}M_1\|\Psi_1\|\right) + 2\bar{N}\|\Psi_1\|_{\mathfrak{B}_h} := q'. \end{aligned}$$

Consider the operator $\Phi : Z_\ell^0 \to Z_\ell^0$ defined by

$$\Phi(\kappa(t)) = \begin{cases} 0, \quad t \in J_0, \\ -S_{p,\sigma}(t)f(0,\xi(0)) + f(t,\kappa_t + \widehat{\Psi}_t) + \int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)g(\varrho,\kappa_\varrho + \widehat{\Psi}_\varrho)d\varrho \\ + \int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)Bv_{\kappa+\widehat{\Psi}}(\varrho)d\varrho + \int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)h(\varrho,\kappa_\varrho + \widehat{\Psi}_\varrho)dw(\varrho) \\ + \sum_{j=1}^r S_{p,\sigma}(t-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j} \mathcal{G}(\varrho,\vartheta(\varrho))d\varrho\right), \quad t \in J. \end{cases} \tag{7}$$

Clearly, the existence of a fixed point of the operator Ψ is equivalent to existence of a fixed point of the operator Φ. We show that Φ has a fixed point. We divide the proof into several steps.

Step 1. Show that $\Phi\left(B_q\right) \subseteq B_q$ for some $q > 0$. Suppose that the condition is not true, then for all $q > 0$, there exists $\kappa^q \in B_q$ such that $\Phi(\kappa^q) \notin B_q$. From (7), we have

$$\begin{aligned} q &< \mathbb{E}\|t^{(1-p)(1-\sigma)}\Phi(\kappa^q(t))\|^2 \\ &\le \sup t^{(1-p)(1-\sigma)}\mathbb{E}\left\|S_{p,\sigma}(t)f(0,\xi(0)) + f(t,\kappa_t^q + \widehat{\Psi}_t) + \int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)g(\varrho,\kappa_\varrho^q + \widehat{\Psi}_\varrho)d\varrho\right. \\ &\quad + \int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)BL^{-1}\left\{\vartheta^1 - S_{p,\sigma}(\ell)[\xi(0) - f(0,\xi(0))] - f(\ell,\kappa_\ell^q + \widehat{\Psi}_\ell)\right. \end{aligned}$$

$$
\begin{aligned}
& -\int_0^{\ell}(\ell-\varrho)^{p-1}Q_p(\ell-\varrho)g(\varrho,\kappa_\varrho^q+\widehat{\Psi}_\varrho)d\varrho-\int_0^{\ell}(\ell-\varrho)^{p-1}Q_p(\ell-\varrho)h(\varrho,\kappa_\varrho^q+\widehat{\Psi}_\varrho)dw(\varrho) \\
& -\sum_{j=1}^{r}S_{p,\sigma}(\ell-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j}\mathcal{G}(\varrho,\kappa^q(\varrho)+\widehat{\Psi}(\varrho))d\varrho\right)\Bigg\}(\varrho)d\varrho+\int_0^t(t-\varrho)^{p-1}Q_p(t-\varrho)h(\varrho,\kappa_\varrho^q+\widehat{\Psi}_\varrho)dw(\varrho) \\
& +\sum_{j=1}^{r}S_{p,\sigma}(t-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j}\mathcal{G}(\varrho,\kappa^q(\varrho)+\widehat{\Psi}(\varrho))d\varrho\right)\Bigg\|^2 \\
\leq\ & 11\sup t^{(1-p)(1-\sigma)}\Bigg[\mathbb{E}\|S_{p,\sigma}(t)f(0,\xi(0))\|^2+\mathbb{E}\|f(t,\kappa_t^q+\widehat{\Psi}_t)\|^2+\mathbb{E}\left\|\int_0^t(t-\varrho)^{p-1}Q_p(t-\varrho)g(\varrho,\kappa_\varrho^q+\widehat{\Psi}_\varrho)d\varrho\right\|^2 \\
& +\mathbb{E}\Bigg\|\int_0^t(t-\varrho)^{p-1}Q_p(t-\varrho)BL^{-1}\Bigg\{\vartheta^1-S_{p,\sigma}(\ell)[\xi(0)-f(0,\xi(0))]-f(\ell,\kappa_\ell^q+\widehat{\Psi}_\ell) \\
& -\int_0^{\ell}(\ell-\varrho)^{p-1}Q_p(\ell-\varrho)g(\varrho,\kappa_\varrho^q+\widehat{\Psi}_\varrho)d\varrho-\int_0^{\ell}(\ell-\varrho)^{p-1}Q_p(\ell-\varrho)h(\varrho,\kappa_\varrho^q+\widehat{\Psi}_\varrho)dw(\varrho) \\
& -\sum_{j=1}^{r}S_{p,\sigma}(t-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j}\mathcal{G}(\varrho,\kappa^q(\varrho)+\widehat{\Psi}(\varrho))d\varrho\right)\Bigg\}(\varrho)d\varrho\Bigg\|^2+\mathbb{E}\left\|\int_0^t(t-\varrho)^{p-1}Q_p(t-\varrho)h(\varrho,\kappa_\varrho^q+\widehat{\Psi}_\varrho)dw(\varrho)\right\|^2 \\
& +\mathbb{E}\left\|\sum_{j=1}^{r}S_{p,\sigma}(t-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j}\mathcal{G}(\varrho,\kappa^q(\varrho)+\widehat{\Psi}(\varrho))d\varrho\right)\right\|^2\Bigg] \\
\leq\ & 11\ell^{2(1-p)(1-\sigma)}\Bigg[(L'')^2\ell^{2(1-p)(1-\sigma)}M_f(1+\mathbb{E}\|\xi(0)\|^2)+M_f(1+q')+(L')^2m_1(\ell)\tilde{g}(q')\times\frac{\ell^{2p-1}}{2p-1} \\
& +(L')^2\mathcal{K}_2^2\mathcal{K}_3^2\times\frac{\ell^{2p-1}}{2p-1}\Big\{\|\vartheta^1\|+(L'')^2\ell^{2(1-p)(1-\sigma)}(\mathbb{E}\|\xi(0)\|^2+M_f(1+\mathbb{E}\|\xi(0)\|^2)) \\
& +M_f(1+q')+(L')^2m_1(\ell)\tilde{g}(q')\times\frac{\ell^{2p-1}}{2p-1}+Tr(Q)(L')^2m_3(\ell)\tilde{h}(q')\times\frac{\ell^{2p-1}}{2p-1}+\sum_{j=1}^{r}(L'')^2\ell^{2(1-p)(1-\sigma)}L_j^*\Big\} \\
& +Tr(Q)(L')^2m_3(\ell)\tilde{h}(q')\times\frac{\ell^{2p-1}}{2p-1}+\sum_{j=1}^{r}(L'')^2\ell^{2(1-p)(1-\sigma)}L_j^*\Bigg] \\
\leq\ & 11\ell^{(1-p)(1-\sigma)}\Big[M^*+(L')^2\mathcal{K}_2^2\mathcal{K}_3^2\times\frac{\ell^{2p-1}}{2p-1}\Big\{\|\vartheta^1\|+(L'')^2\ell^{2(1-p)(1-\sigma)}\mathbb{E}\|\xi(0)\|^2+M^*\Big\}\Big],
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
M^* =\ & (L'')^2\ell^{2(1-p)(1-\sigma)}M_f(1+\mathbb{E}\|\xi(0)\|^2)+M_f(1+q')+\frac{(L')^2\ell^{2p-1}}{2p-1}\Big[m_1(\ell)\tilde{g}(q')+Tr(Q)m_3(\ell)\tilde{h}(q')\Big] \\
& +\sum_{j=1}^{r}(L'')^2\ell^{2(1-p)(1-\sigma)}L_j^*.
\end{aligned}
$$

Now, dividing both sides of equation (8) by q and taking the limit $q\to\infty$, we get a contradiction. Therefore, $\Phi\left(B_q\right)\subseteq B_q$.

Step 2. Φ is continuous on B_q. For any $\vartheta^m,\vartheta\in B_q$, $m=0,1,2,\ldots$, such that $\lim_{m\to\infty}\vartheta^m=\vartheta$, then we get

$$\lim_{m\to\infty}\vartheta^m(t)=\vartheta(t)\quad\text{and}\quad\lim_{m\to\infty}t^{(1-p)(1-\sigma)}\vartheta^m(t)=t^{(1-p)(1-\sigma)}\vartheta(t).$$

Let $\vartheta(t) = \kappa(t) + \widehat{\Psi}(t)$. Then $\{\kappa^m(t) + \widehat{\Psi}(t)\} \subset B_q$ with $\kappa^m(t) + \widehat{\Psi}(t) \to \kappa(t) + \widehat{\Psi}(t)$ in B_q as $m \to \infty$, and we have from (H2)

$$f(t, \vartheta_t^m) = f(t, \kappa_t^m + \widehat{\Psi}_t) \to f(t, \kappa_t + \widehat{\Psi}_t) = f(t, \vartheta_t) \quad \text{as} \quad m \to \infty. \tag{9}$$

Also, from (H3) and (H4), we have

$$g(t, \vartheta_t^m) = g(t, \kappa_t^m + \widehat{\Psi}_t) \to g(t, \kappa_t + \widehat{\Psi}_t) = g(t, \vartheta_t), \tag{10}$$

$$h(t, \vartheta_t^m) = h(t, \kappa_t^m + \widehat{\Psi}_t) \to h(t, \kappa_t + \widehat{\Psi}_t) = h(t, \vartheta_t) \quad \text{as} \quad m \to \infty. \tag{11}$$

Also, from the dominated convergence theorem, we have

$$\int_0^t (t-\varrho)^{2(p-1)} \mathbb{E}\|g(\varrho, \kappa_\varrho^m + \widehat{\Psi}_\varrho) - g(\varrho, \kappa_\varrho + \widehat{\Psi}_\varrho)\|^2 d\varrho \quad \to \quad 0, \tag{12}$$

$$\int_0^t (t-\varrho)^{2(p-1)} \mathbb{E}\|h(\varrho, \kappa_\varrho^m + \widehat{\Psi}_\varrho) - h(\varrho, \kappa_\varrho + \widehat{\Psi}_\varrho)\|^2 d\varrho \quad \to \quad 0 \quad \text{as} \quad m \to \infty. \tag{13}$$

Next, from equation (3)

$$\begin{aligned} v_\vartheta^m(t) \quad = \quad & L^{-1}\Big\{\vartheta^1 - S_{p,\sigma}(\ell)[\xi(0) - f(0,\xi(0))] - f(\ell, \kappa_\ell^m + \widehat{\Psi}_\ell) - \int_0^\ell (\ell-\varrho)^{p-1} Q_p(\ell-\varrho) g(\varrho, \kappa_\varrho^m + \widehat{\Psi}_\varrho) d\varrho \\ & - \int_0^\ell (\ell-\varrho)^{p-1} Q_p(\ell-\varrho) h(\varrho, \kappa_\varrho^m + \widehat{\Psi}_\varrho) dw(\varrho) - \sum_{j=1}^r S_{p,\sigma}(\ell - t_j) I_j \left(\int_{t_j-p_j}^{t_j-q_j} \mathcal{G}(\varrho, \kappa^m(\varrho) + \widehat{\Psi}(\varrho)) d\varrho \right) \Big\}(t). \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{E}\|v_\vartheta^m(t) - v_\vartheta(t)\|^2 \quad \leq \quad & 4\left\|L^{-1}\right\| \Big[\mathbb{E}\|f(\ell, \kappa_\ell^m + \widehat{\Psi}_\ell) - f(\ell, \kappa_\ell + \widehat{\Psi}_\ell)\|^2 \\ & + \mathbb{E}\left\| \int_0^\ell (\ell-\varrho)^{p-1} Q_p(\ell-\varrho) \Big[g(\varrho, \kappa_\varrho^m + \widehat{\Psi}_\varrho) - g(\varrho, \kappa_\varrho + \widehat{\Psi}_\varrho) \Big] d\varrho \right\|^2 \\ & + \mathbb{E}\left\| \int_0^\ell (\ell-\varrho)^{p-1} Q_p(\ell-\varrho) \Big[h(\varrho, \kappa_\varrho^m + \widehat{\Psi}_\varrho) - h(\varrho, \kappa_\varrho + \widehat{\Psi}_\varrho) \Big] dw(\varrho) \right\|^2 \\ & + \mathbb{E}\left\| \sum_{j=1}^r S_{p,\sigma}(\ell - t_j) I_j \left(\int_{t_j-p_j}^{t_j-q_j} \Big[\mathcal{G}\left(\varrho, \kappa^m(\varrho) + \widehat{\Psi}(\varrho)\right) - \mathcal{G}\left(\varrho, \kappa(\varrho) + \widehat{\Psi}(\varrho)\right) \Big] d\varrho \right) \right\|^2 \Big] \\ & \to 0 \quad \text{as} \quad m \to \infty. \end{aligned} \tag{14}$$

Now,

$$\begin{aligned} \mathbb{E}\|\Phi\vartheta^m(t) - \Phi\vartheta(t)\|_\ell^2 \quad \leq \quad & 5\mathbb{E}\|f(t, \kappa_t^m + \widehat{\Psi}_t) - f(t, \kappa_t + \widehat{\Psi}_t)\|^2 \\ & + 5\mathbb{E}\left\| \int_0^t (t-\varrho)^{p-1} Q_p(t-\varrho) \Big[g(\varrho, \kappa_\varrho^m + \widehat{\Psi}_\varrho) - g(\varrho, \kappa_\varrho + \widehat{\Psi}_\varrho) \Big] d\varrho \right\|^2 \\ & + 5\mathbb{E}\left\| \int_0^t (t-\varrho)^{p-1} Q_p(t-\varrho) B \Big[v_\vartheta^m(t) - v_\vartheta(t) \Big] d\varrho \right\|^2 \\ & + 5\mathbb{E}\left\| \int_0^t (t-\varrho)^{p-1} Q_p(t-\varrho) \Big[h(\varrho, \kappa_\varrho^m + \widehat{\Psi}_\varrho) - h(\varrho, \kappa_\varrho + \widehat{\Psi}_\varrho) \Big] dw(\varrho) \right\|^2 \\ & + 5\mathbb{E}\left\| \sum_{j=1}^r S_{p,\sigma}(t - t_j) I_j \left(\int_{t_j-p_j}^{t_j-q_j} \Big[\mathcal{G}\left(\varrho, \kappa^m(\varrho) + \widehat{\Psi}(\varrho)\right) - \mathcal{G}\left(\varrho, \kappa(\varrho) + \widehat{\Psi}(\varrho)\right) \Big] d\varrho \right) \right\|^2. \end{aligned}$$

Using the estimates (9)-(14) in the above inequality, we obtain that $\mathbb{E}\|\Phi\vartheta^m(t) - \Phi\vartheta(t)\|_\ell^2 \to 0$ as $m \to \infty$. Therefore, Φ is continuous.

Step 3. Φ is equicontinuous. For $\vartheta \in B_q$ and $0 \leq t_1 \leq t_2 \leq \ell$, we have

$$
\begin{aligned}
&\mathbb{E}\|\Phi\vartheta(t_2) - \Phi\vartheta(t_1)\|^2 \\
&\leq 6\mathbb{E}\Big\|t_2^{(1-p)(1-\sigma)}S_{p,\sigma}(t_2)f(0,\xi(0)) - t_1^{(1-p)(1-\sigma)}S_{p,\sigma}(t_1)f(0,\xi(0))\Big\|^2 \\
&\quad +6\mathbb{E}\Big\|t_2^{(1-p)(1-\sigma)}f(t_2,\kappa_{t_2}+\widehat{\Psi}_{t_2}) - t_1^{(1-p)(1-\sigma)}f(t_1,\kappa_{t_1}+\widehat{\Psi}_{t_1})\Big\| \\
&\quad +6\mathbb{E}\Bigg\|t_2^{(1-p)(1-\sigma)}\int_0^{t_2}(t_2-\varrho)^{p-1}Q_p(t_2-\varrho)g(\varrho,\kappa_\varrho+\widehat{\Psi}_\varrho)d\varrho \\
&\qquad -t_1^{(1-p)(1-\sigma)}\int_0^{t_1}(t_1-\varrho)^{p-1}Q_p(t_1-\varrho)g(\varrho,\kappa_\varrho+\widehat{\Psi}_\varrho)d\varrho\Bigg\|^2 \\
&\quad +6\mathbb{E}\Bigg\|t_2^{(1-p)(1-\sigma)}\int_0^{t_2}(t_2-\varrho)^{p-1}Q_p(t_2-\varrho)B\upsilon_{\kappa+\widehat{\Psi}}(\varrho)d\varrho \\
&\qquad -t_1^{(1-p)(1-\sigma)}\int_0^{t_1}(t_1-\varrho)^{p-1}Q_p(t_1-\varrho)B\upsilon_{\kappa+\widehat{\Psi}}(\varrho)d\varrho\Bigg\|^2 \\
&\quad +6\mathbb{E}\Bigg\|t_2^{(1-p)(1-\sigma)}\int_0^{t_2}(t_2-\varrho)^{p-1}Q_p(t_2-\varrho)h(\varrho,\kappa_\varrho+\widehat{\Psi}_\varrho)dw(\varrho) \\
&\qquad -t_1^{(1-p)(1-\sigma)}\int_0^{t_1}(t_1-\varrho)^{p-1}Q_p(t_1-\varrho)h(\varrho,\kappa_\varrho+\widehat{\Psi}_\varrho)dw(\varrho)\Bigg\|^2 \\
&\quad +6\mathbb{E}\Bigg\|t_2^{(1-p)(1-\sigma)}\sum_{j=1}^{r}S_{p,\sigma}(t_2-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j}\mathcal{G}\left(\varrho,\kappa(\varrho)+\widehat{\Psi}(\varrho)\right)d\varrho\right) \\
&\qquad -t_1^{(1-p)(1-\sigma)}\sum_{j=1}^{r}S_{p,\sigma}(t_1-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j}\mathcal{G}\left(\varrho,\kappa(\varrho)+\widehat{\Psi}(\varrho)\right)d\varrho\right)\Bigg\|^2 \\
\leq\ & 12\mathbb{E}\Big\|t_2^{(1-p)(1-\sigma)}\Big[S_{p,\sigma}(t_2)-S_{p,\sigma}(t_1)\Big]f(0,\xi(0))\Big\|^2 + 12\mathbb{E}\Big\|\Big(t_2^{(1-p)(1-\sigma)}-t_1^{(1-p)(1-\sigma)}\Big)S_{p,\sigma}(t_1)f(0,\xi(0))\Big\|^2 \\
&+12\mathbb{E}\Big\|t_2^{(1-p)(1-\sigma)}\Big[f(t_2,\kappa_{t_2}+\widehat{\Psi}_{t_2})-f(t_1,\kappa_{t_1}+\widehat{\Psi}_{t_1})\Big]\Big\|^2 + 12\mathbb{E}\Big\|\Big(t_2^{(1-p)(1-\sigma)}-t_1^{(1-p)(1-\sigma)}\Big)f(t_1,\kappa_{t_1}+\widehat{\Psi}_{t_1})\Big\|^2 \\
&+12\mathbb{E}\left\|t_2^{(1-p)(1-\sigma)}\int_0^{t_1}\Big[(t_2-\varrho)^{p-1}Q_p(t_2-\varrho)-(t_1-\varrho)^{p-1}Q_p(t_1-\varrho)\Big]g(\varrho,\kappa_\varrho+\widehat{\Psi}_\varrho)d\varrho\right\|^2 \\
&+12\mathbb{E}\left\|t_2^{(1-p)(1-\sigma)}\int_{t_1}^{t_2}(t_2-\varrho)^{p-1}Q_p(t_2-\varrho)g(\varrho,\kappa_\varrho+\widehat{\Psi}_\varrho)d\varrho\right\|^2 \\
&+12\mathbb{E}\left\|\Big(t_2^{(1-p)(1-\sigma)}-t_1^{(1-p)(1-\sigma)}\Big)\int_0^{t_1}(t_1-\varrho)^{p-1}Q_p(t_1-\varrho)g(\varrho,\kappa_\varrho+\widehat{\Psi}_\varrho)d\varrho\right\|^2 \\
&+12\mathbb{E}\left\|t_2^{(1-p)(1-\sigma)}\int_0^{t_1}\Big[(t_2-\varrho)^{p-1}Q_p(t_2-\varrho)-(t_1-\varrho)^{p-1}Q_p(t_1-\varrho)\Big]B\upsilon_{\kappa+\widehat{\Psi}}(\varrho)d\varrho\right\|^2 \\
&+12\mathbb{E}\left\|t_2^{(1-p)(1-\sigma)}\int_{t_1}^{t_2}(t_2-\varrho)^{p-1}Q_p(t_2-\varrho)B\upsilon_{\kappa+\widehat{\Psi}}(\varrho)d\varrho\right\|^2 \\
&+12\mathbb{E}\left\|\Big(t_2^{(1-p)(1-\sigma)}-t_1^{(1-p)(1-\sigma)}\Big)\int_0^{t_1}(t_1-\varrho)^{p-1}Q_p(t_1-\varrho)B\upsilon_{\kappa+\widehat{\Psi}}(\varrho)d\varrho\right\|^2 \\
&+12\mathbb{E}\left\|t_2^{(1-p)(1-\sigma)}\int_0^{t_1}\Big[(t_2-\varrho)^{p-1}Q_p(t_2-\varrho)-(t_1-\varrho)^{p-1}Q_p(t_1-\varrho)\Big]h(\varrho,\kappa_\varrho+\widehat{\Psi}_\varrho)dw(\varrho)\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+12\mathbb{E}\left\|t_2^{(1-p)(1-\sigma)}\int_{t_1}^{t_2}(t_2-\varrho)^{p-1}Q_p(t_2-\varrho)h(\varrho,\kappa_\varrho+\widehat{\Psi}_\varrho)dw(\varrho)\right\|^2\\
&+12\mathbb{E}\left\|\left(t_2^{(1-p)(1-\sigma)}-t_1^{(1-p)(1-\sigma)}\right)\int_0^{t_1}(t_1-\varrho)^{p-1}Q_p(t_1-\varrho)h(\varrho,\kappa_\varrho+\widehat{\Psi}_\varrho)dw(\varrho)\right\|^2\\
&+6\mathbb{E}\left\|\sum_{j=1}^{r}\left[t_2^{(1-p)(1-\sigma)}S_{p,\sigma}(t_2-t_j)-t_1^{(1-p)(1-\sigma)}S_{p,\sigma}(t_1-t_j)\right]I_j\left(\int_{t_j-p_j}^{t_j-q_j}\mathcal{G}\left(\varrho,\kappa(\varrho)+\widehat{\Psi}(\varrho)\right)d\varrho\right)\right\|^2\\
\to\ & 0 \quad \text{as} \quad t_2\to t_1.
\end{aligned}
$$

Hence, Φ is equicontinuous on B_q.

Step 4. The Mönch's condition holds (see [14]). Let $\mathcal{P}\subseteq B_q$ be countable and $\mathcal{P}\subseteq \operatorname{conv}(\{0\}\cup\Phi(\mathcal{P}))$. We show that $\varsigma(\mathcal{P})=0$. We assume that $\mathcal{P}=\{\kappa^m\}_{m=1}^{\infty}$. We need to show that $\Phi(\mathcal{P})(t)$ is relatively compact for all $t\in J$ in H.

By Lemma 2.7, we obtain that

$$
\begin{aligned}
\varsigma(\mathcal{P}(t)) &= \varsigma\left(\{\Phi(\kappa^m)(t)\}_{m=1}^{\infty}\right)\\
&\le \varsigma\left(\left\{t^{(1-p)(1-\sigma)}\int_0^t(t-\varrho)^{p-1}Q_p(t-\varrho)g(\varrho,\kappa_\varrho^m+\widehat{\Psi}_\varrho)d\varrho\right\}_{m=1}^{\infty}\right)\\
&\quad+\varsigma\left(\left\{t^{(1-p)(1-\sigma)}\int_0^t(t-\varrho)^{p-1}Q_p(t-\varrho)Bv_{\kappa^m}(\varrho)d\varrho\right\}_{m=1}^{\infty}\right)\\
&\quad+\varsigma\left(\left\{t^{(1-p)(1-\sigma)}\int_0^t(t-\varrho)^{p-1}Q_p(t-\varrho)h(\varrho,\kappa_\varrho^m+\widehat{\Psi}_\varrho)dw(\varrho)\right\}_{m=1}^{\infty}\right)\\
&\quad+\varsigma\left(\left\{t^{(1-p)(1-\sigma)}\sum_{j=1}^{r}S_{p,\sigma}(t-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j}\mathcal{G}\left(\varrho,\kappa^m(\varrho)+\widehat{\Psi}(\varrho)\right)d\varrho\right)\right\}_{m=1}^{\infty}\right)\\
&= \varsigma(\mho_1)+\varsigma(\mho_2)+\varsigma(\mho_3)+\varsigma(\mho_4).
\end{aligned}
\tag{15}
$$

By Lemma 2.7 and (H3)(iii), we have

$$
\begin{aligned}
\varsigma(\mho_1) &= \varsigma\left(\left\{t^{(1-p)(1-\sigma)}\int_0^t(t-\varrho)^{p-1}Q_p(t-\varrho)g(\varrho,\kappa_\varrho^m+\widehat{\Psi}_\varrho)d\varrho\right\}_{m=1}^{\infty}\right)\\
&\le 2L'\ell^{(1-p)(1-\sigma)}\int_0^t(t-\varrho)^{1-p}\varsigma\left(g\left(\varrho,\{\kappa_\varrho^m+\widehat{\Psi}_\varrho\}_{m=1}^{\infty}\right)\right)d\varrho\\
&\le 2L'\ell^{(1-p)(1-\sigma)}\int_0^t(t-\varrho)^{1-p}m_2(\varrho)\varsigma(\mathcal{P})\,d\varrho\\
&\le 2L'\ell^{(1-p)(1-\sigma)}\left(\int_0^t(t-\varrho)^{\frac{1-p}{1-p_2}}d\varrho\right)^{p_2-1}\left(\int_0^t\|m_2(\varrho)\|^{p_2}d\varrho\right)^{1/p_2}\varsigma(\mathcal{P})\\
&\le 2L'\ell^{(1-p)(1-\sigma)}K_{p_2}\|m_2\|_{L^{1/p_2}(J,\mathbb{R}^+)}\varsigma(\mathcal{P}).
\end{aligned}
\tag{16}
$$

By Lemma 2.7 and (H4)(iii), we have

$$
\begin{aligned}
\varsigma(\mho_2) &= \varsigma\left(\left\{t^{(1-p)(1-\sigma)}\int_0^t(t-\varrho)^{p-1}Q_p(t-\varrho)h(\varrho,\kappa_\varrho^m+\widehat{\Psi}_\varrho)dw(\varrho)\right\}_{m=1}^{\infty}\right)\\
&\le 2Tr(Q)\ell^{(1-p)(1-\sigma)}\int_0^t(t-\varrho)^{1-p}\varsigma\left(h\left(\varrho,\{\kappa_\varrho^m+\widehat{\Psi}_\varrho\}_{m=1}^{\infty}\right)\right)d\varrho\\
&\le 2Tr(Q)L'\ell^{(1-p)(1-\sigma)}\int_0^t(t-\varrho)^{1-p}m_4(\varrho)\mu(\mathcal{P})\,d\varrho
\end{aligned}
$$

$$\leq 2Tr(Q)L'\ell^{(1-p)(1-\sigma)}\left(\int_0^t (t-\varrho)^{\frac{p-1}{1-p_4}}d\varrho\right)^{p_4-1}\left(\int_0^t \|m_4(\varrho)\|^{p_4}d\varrho\right)^{1/p_4}\varsigma(\mathcal{P})$$
$$\leq 2Tr(Q)L'\ell^{(1-p)(1-\sigma)}K_{p_4}\|m_4\|_{L^{1/p_4}(J,\mathbb{R}^+)}\varsigma(\mathcal{P}). \tag{17}$$

Also, from Lemma 2.7 and (H6) (iii), we have

$$\begin{aligned}
\varsigma(\mho_3) &= \varsigma\left(\left\{t^{(1-p)(1-\sigma)}\int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)B\upsilon_{\kappa^m}(\varrho)d\varrho\right\}_{m=1}^{\infty}\right)\\
&= \varsigma\Bigg(t^{(1-p)(1-\sigma)}\int_0^t (t-\varrho)^{p-1}Q_p(t-\varrho)BL^{-1}\Big\{x^1 - S_{p,\sigma}(\ell)[\xi(0)-f(0,\xi(0))] - f(\ell,\{\kappa_\ell^m+\widehat{\Psi}_\ell\}_{m=1}^{\infty})\\
&\quad -\int_0^\ell (\ell-\varrho)^{p-1}Q_p(\ell-\varrho)g(\varrho,\{\kappa_\varrho^m+\widehat{\Psi}_\varrho\}_{m=1}^{\infty})d\varrho - \int_0^\ell (\ell-\varrho)^{p-1}Q_p(\ell-\varrho)h(\varrho,\{\kappa_\varrho^m+\widehat{\Psi}_\varrho\}_{m=1}^{\infty})dw(\varrho)\Big\}d\varrho\\
&\quad -\sum_{j=1}^{r} S_{p,\sigma}(t-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j}\mathcal{G}\left(\varrho,\{\kappa^m(\varrho)+\widehat{\Psi}(\varrho)\}_{m=1}^{\infty}\right)d\varrho\right)\Bigg)\\
&\leq 2L'^2\mathcal{K}_2 m_5\ell^{(1-p)(1-\sigma)}\Bigg[\left(\left(\int_0^t (t-\varrho)^{\frac{1-p}{1-p_2}}d\varrho\right)^{2(p_2-1)}\left(\int_0^t \|m_2(\varrho)\|^{p_2}d\varrho\right)^{1/p_2}\varsigma(\mathcal{P})\right.\\
&\quad +Tr(Q)\left(\int_0^t (t-\varrho)^{\frac{1-p}{1-p_4}}d\varrho\right)^{2(p_4-1)}\left(\int_0^t \|m_4(\varrho)\|^{p_4}d\varrho\right)^{1/p_4}\varsigma(\mathcal{P}) + r\lambda L_j L''\ell^{(1-p)(1-\sigma)}(2\ell)\varsigma(\mathcal{P})\Bigg]\\
&\leq 2L'^2\mathcal{K}_2 m_5\ell^{(1-p)(1-\sigma)}\Big[K_{p_2}^2\|m_2\|_{L^{1/p_2}(J,\mathbb{R}^+)} + Tr(Q)K_{p_4}^2\|m_4\|_{L^{1/p_4}(J,\mathbb{R}^+)} + r\lambda L_j L''\ell^{(1-p)(1-\sigma)}(2\ell)\Big]\varsigma(\mathcal{P}).
\end{aligned} \tag{18}$$

From (H8), we have

$$\begin{aligned}
\varsigma(\mho_4) &= \varsigma\left(\left\{t^{(1-p)(1-\sigma)}\sum_{j=1}^{r}S_{p,\sigma}(t-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j}\mathcal{G}\left(\varrho,\kappa^m(\varrho)+\widehat{\Psi}(\varrho)\right)d\varrho\right)\right\}_{m=1}^{\infty}\right)\\
&\leq \ell^{(1-p)(1-\sigma)}\sum_{j=1}^{r}S_{p,\sigma}(t-t_j)I_j\left(\int_{t_j-p_j}^{t_j-q_j}\mathcal{G}\left(\varrho,\left\{\kappa^m(\varrho)+\widehat{\Psi}(\varrho)\right\}_{m=1}^{\infty}\right)d\varrho\right)\\
&\leq r\lambda L_j\ell^{(1-p)(1-\sigma)}L''\ell^{(1-p)(1-\sigma)}(2\ell)\varsigma(\mathcal{P})
\end{aligned} \tag{19}$$

Using the above estimates, we find that

$$\varsigma(\Phi(\mathcal{P})) \leq \tilde{\mathcal{K}}\varsigma(\mathcal{P}),$$

where

$$\tilde{\mathcal{K}} = 2L'\ell^{(1-p)(1-\sigma)}\left\{K_{p_2}K_{m_2}\Big[1+L'\mathcal{K}_2 m_5 K_{p_2}\Big] + Tr(Q)K_{p_4}K_{m_4}\Big[1+L'\mathcal{K}_4 m_5 K_{p_4}\Big]\right\} + r\lambda L_j\ell^{(1-p)(1-\sigma)}L''\ell^{(1-p)(1-\sigma)}(2\ell).$$

Therefore, from Mönch's condition, we find that

$$\varsigma(\Phi(\mathcal{P})) \leq \varsigma(\text{conv}(\{0\}\cup\Phi(\mathcal{P})) = \varsigma(\Phi(\mathcal{P})) \leq \tilde{\mathcal{K}}\varsigma(\mathcal{P}).$$

This implies that $\varsigma(\mathcal{P}) = 0$. Then, by using [14, Lemma], Φ has a fixed point in B_q. So, $\vartheta = \kappa + \widehat{\Psi}$ is the mild solution of equation (1) such that $\vartheta(\ell) = \vartheta^1$. Therefore, the system (1) is controllable in H. □

4. Application

Consider the following Hilfer fractional neutral stochastic impulsive differential equation given by

$$
\begin{aligned}
&D_{0^+}^{3/4,\sigma}\left[\vartheta(t,z) - \int_{-\infty}^{t} \zeta_1(\varrho,z)\vartheta(\varrho,z)d\varrho\right] = \frac{\partial^2}{\partial z^2}\vartheta(t,z) + \gamma_1\left(t, \int_{-\infty}^{t} \delta_1(\varrho - t)\vartheta(\varrho,z)d\varrho\right)\\
&\quad +Bv(t,z) + \gamma_2\left(t, \int_{-\infty}^{t} \delta_2(\varrho - t)\vartheta(\varrho,z)dw(\varrho)\right), t \in (0,1], \quad t \neq t_1, \quad z \in [0,\pi],\\
&I_{0^+}^{(1-3/4)(1-\sigma)}\vartheta(0,z) = \xi(z), \quad z \in [0,\pi],\\
&\Delta\vartheta(t_1,z) = I_1\left(\int_{t_1-p_1}^{t_1-q_1} \mathcal{G}(\varrho,z,\vartheta(\varrho,z))d\varrho\right),\\
&\vartheta(t,0) = \vartheta(t,\pi) = 0, \quad t \in (0,1],\\
&\vartheta(t,z) = \phi(t,z), \quad 0 \le z \le \pi, \quad t \in (-\infty,0),
\end{aligned}
\tag{20}
$$

where $D_{0^+}^{3/4,\sigma}$ denotes the Hilfer fractional derivative with order $0 < 3/4 < 1$ and type $0 \le \sigma \le 1$. γ_1,γ_2 and ϕ are continuous functions, ζ_1,δ_1 and δ_2 are some appropriate functions. $w(\cdot)$ is the one-dimensional Brownian motion in K on the filtered probability space $(\Omega,\Upsilon,\mathbb{P})$. v is the control function in Hilbert space U.

To change the system into an abstract form, let $H = K = L^2[0,\pi]$ be endowed with the norm $\|\cdot\|_{L^2}$ and $A : D(A) \subset H \to H$ is defined as $A\vartheta = \vartheta''$ with

$$D(A) = \{\vartheta : H : \ \vartheta, \vartheta' \text{ are absolutely continuous}, \ \vartheta'' \in H, \ \vartheta(0) = \vartheta(\pi) = 0\},$$

and

$$A\vartheta = \sum_{m\in\mathbb{N}} m^2\langle\vartheta, e_m\rangle e_m, \ \vartheta \in D(A),$$

where $e_m(z) = \frac{1}{\sqrt{2\pi}}e^{imz}$, $m \in \mathbb{N}$ is the orthogonal set of eigen vectors of A.

Here, A generates an analytic semigroup $\{T(t) : t \ge 0\}$ on H, $T(t)$ is not a compact semigroup on H with $\varsigma(T(t)D) \le \varsigma(D)$ and there exists $\mathcal{K}_1 > 1$ such that $\sup_{t\in(0,1]} \|T(t)\| \le \mathcal{K}_1$. Also, $t \to \varpi(t^{3/4}\beta + r)\vartheta$ is equicontinuous, $t > 0$, and $\beta \in (0,1)$.

Define $\vartheta(t)(z) = \vartheta(t,z)$ and $v(t)(z) = v(t,z)$, then

$$
\begin{aligned}
f(t,\vartheta_t)(z) &= \int_{-\infty}^{t} \zeta_1(\varrho,z)\vartheta(\varrho,z)d\varrho,\\
g(t,\vartheta_t)(z) &= \gamma_1\left(t, \int_{-\infty}^{t} \delta_1(\varrho - t)\vartheta(\varrho,z)d\varrho\right),\\
h(t,\vartheta_t)(z) &= \gamma_2\left(t, \int_{-\infty}^{t} \delta_2(\varrho - t)\vartheta(\varrho,z)dw(\varrho)\right),\\
\mathcal{G}(t,\vartheta(t))(z) &= \mathcal{G}(t,z,\vartheta(t,z)).
\end{aligned}
$$

Then, the system (20) can be written as the abstract form of (1). We also assume that the functions satisfy the assumptions of the theorem. Therefore, using Theorem 3.1, we can say the the system (20) is controllable.

5. Conclusion

In this article, we studied the controllability of a class of Hilfer fractional neutral stochastic impulsive differential equations involving integral impulses. Using appropriate assumptions, the analysis was conducted via semigroup theory, fractional calculus, and the measure of noncompactness combined with Mönch's fixed point theorem, which together provided a robust framework for establishing controllability results. The obtained results also generalize to other types of fractional derivatives, including Caputo and Riemann–Liouville derivatives, illustrating the broad applicability of the approach.

As future work, we plan to extend these results to Ψ-Hilfer and (k, Ψ)-Hilfer impulsive fractional differential equations with infinite delay using the measure of noncompactness. This extension will further enhance the modeling capabilities and applicability of fractional impulsive systems in complex dynamical scenarios.

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