Triple product L-functions related to $\mathrm{GL}(6)$ automorphic forms

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Abstract. In this paper, we study the distribution of Fourier coefficients of triple product L-functions associated with certain GL(6) automorphic forms. More precisely, we derive asymptotic formulas for the first moment of Fourier coefficients of the automorphic L-functions attached to these GL(6) forms. As an application, we derive subconvexity bounds for the associated L-functions on the critical line.

1. Introduction and Main Results

The analytic properties of automorphic L-functions and the behavior of their Fourier coefficients are central topics in modern number theory. Let $g \in \mathcal{H}_k^*$ be a primitive holomorphic cusp form of even weight k for the full modular group $\mathrm{SL}(2, \mathbb{Z})$. The associated Hecke L-function is defined by

$$L(s, g) := \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^s}, \quad (\Re(s) > 1).$$

Extensive research has been dedicated to the summatory function $\sum_{n \leq x} \lambda_g(n)$ (see [3, 7, 12, 19–22]).

For the triple product L-function $L(s, g \otimes g \otimes g)$, Lü and Sankaranarayanan [15] established that for any $\epsilon > 0$, $\sum_{n \leq x} \lambda_{g \otimes g \otimes g}(n) \ll x^{\frac{7}{10}+\epsilon}$. Subsequently, results on the second moment were refined by Venkatasubbareddy and Sankaranarayanan [18].

In this paper, we extend these investigations to GL(6) automorphic forms constructed from Maass forms. Let f be a Maass cusp form for $\mathrm{SL}(2, \mathbb{Z})$ and F be the Gelbart-Jacquet lift of f to $\mathrm{SL}(3, \mathbb{Z})$ (see Gelbart-Jacquet [4]). We consider the following three GL(6) automorphic L-functions:

$$L(s, G_1) := \zeta(s)L(s, f)L(s, F), \tag{1.1}$$

$$L(s, G_2) := \zeta(s)^3 L(s, F), \tag{1.2}$$

$$L(s, G_3) := \zeta(s)^4 L(s, f). \tag{1.3}$$

We investigate the triple product L-functions $L(s, G_i \otimes G_i \otimes G_i)$ for $i = 1, 2, 3$, defined for $\Re(s) > 1$ by the Dirichlet series $\sum \lambda_{G_i \otimes G_i \otimes G_i}(n)n^{-s}$. Note that each of these triple product L-functions is of degree 216. Our main results are the following asymptotic formulas.

Theorem 1.1. *For sufficiently large x and any $\epsilon > 0$, we have*

$$\sum_{n \leq x} \lambda_{G_1 \otimes G_1 \otimes G_1}(n) = xP_1(\log x) + O\left(x^{\frac{50881}{51441} + \epsilon}\right),$$

where $P_1(t)$ is a polynomial in t of degree 10.

Theorem 1.2. *For sufficiently large x and any $\epsilon > 0$, we have*

$$\sum_{n \leq x} \lambda_{G_2 \otimes G_2 \otimes G_2}(n) = xP_2(\log x) + O\left(x^{\frac{37021}{37441} + \epsilon}\right),$$

where $P_2(t)$ is a polynomial in t of degree 36.

Theorem 1.3. *For sufficiently large x and any $\epsilon > 0$, we have*

$$\sum_{n \leq x} \lambda_{G_3 \otimes G_3 \otimes G_3}(n) = xP_3(\log x) + O\left(x^{\frac{2529}{2564} + \epsilon}\right),$$

where $P_3(t)$ is a polynomial in t of degree 75.

Remark 1.4. The exponents in Theorems 1.1–1.3 improve upon the general bound derived from the result of Lau and Lü [13]. This improvement stems from the specific decompositions of the *L*-functions established in Section 2 and recent advances in subconvexity bounds.

2. Preliminaries and Key Lemmas

2.1. L-functions and Symmetric Powers

Let f be a Hecke-Maass cusp form for $SL(2, \mathbb{Z})$ with normalized coefficients $\lambda_f(n)$. The associated *L*-function is given by

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s} = \prod_p (1 - \alpha(p)p^{-s})^{-1}(1 - \beta(p)p^{-s})^{-1}, \quad (\Re(s) > 1). \tag{2.1}$$

Let F be the self-dual Maass form for $SL(3, \mathbb{Z})$ obtained via the Gelbart-Jacquet lift of f. Its *L*-function is defined as

$$L(s, F) = \prod_p (1 - \alpha(p)^2 p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - \beta(p)^2 p^{-s})^{-1}, \quad (\Re(s) > 2). \tag{2.2}$$

For $j \geq 0$, the j-th symmetric power *L*-function is defined as

$$L(s, \text{sym}^j f) = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^j f}(n)}{n^s} = \prod_p \prod_{i=0}^{j} (1 - \alpha^{j-i}(p)\beta^i(p)p^{-s})^{-1}.$$

Remark 2.1. For a Maass cusp form f for $SL_2(\mathbb{Z})$, the analytic continuation and functional equation of the symmetric-power *L*-functions $L(s, \text{sym}^j f)$ are known unconditionally only for $j \leq 6$. These cases follow from the work of Gelbart-Jacquet ($j = 2$) [4], Kim-Shahidi ($j = 3$) [11], and Kim-Sarnak ($j = 4, 5, 6$) [10]. Since in our factorisations (Lemmas 2.7-2.9) only symmetric powers with $j \leq 6$ appear, the standard convexity bound for $L(s, \text{sym}^j f)$ is fully justified in our setting.

2.2. Auxiliary Bounds

We require the following bounds for the Riemann zeta function and automorphic *L*-functions.

Lemma 2.2. *For any $\epsilon > 0$, we have uniformly for $T \geq 1$:*

$$\int_0^T \left|\zeta(\tfrac{5}{7} + it)\right|^{12} dt \ll_\epsilon T^{1+\epsilon}.$$

Also, uniformly for $1/2 \leq \sigma \leq 1 + \epsilon$:

$$\zeta(\sigma + it) \ll_\epsilon (1 + |t|)^{\frac{13}{42}(1-\sigma)+\epsilon}.$$

Proof. See [8, Theorem 8.4] and [1, Theorem 5], respectively. □

Lemma 2.3. *For any $\epsilon > 0$, we have uniformly for $1/2 \leq \sigma \leq 1 + \epsilon$:*

$$L(\sigma + it, F) \ll_{f,\epsilon} (1 + |t|)^{\max\{\frac{6}{5}(1-\sigma),0\}+\epsilon}.$$

Proof. This follows from the Phragmén-Lindelöf principle and the result of Lin, Nunes, and Qi [14, Corollary 1.2]. □

Lemma 2.4. *For any $\epsilon > 0$, we have uniformly for $T \geq 1$:*

$$\int_0^T \left|L(\tfrac{5}{8} + it, f)\right|^4 dt \ll_\epsilon T^{1+\epsilon}.$$

Furthermore, for $1/2 \leq \sigma \leq 2$:

$$L(\sigma + it, f) \ll_{f,\epsilon} (1 + |t|)^{\max\{\frac{2}{3}(1-\sigma),0\}+\epsilon}.$$

Proof. See Ivić [9, Theorem 2] and Good [5]. □

Lemma 2.5. *For $T \geq T_0$, where T_0 is sufficiently large, there exists a $T^* \in (T, 2T)$ such that*

$$\max_{\sigma \geq 1/2} |\zeta(\sigma + iT^*)| \leq \exp(C(\log\log T)^2).$$

Proof. See Ramachandra and Sankaranarayanan [17, Lemma 2]. □

Lemma 2.6. *Suppose $\mathcal{L}(s)$ is a general L-function of degree d. Then for any $\epsilon > 0$:*

$$\int_T^{2T} |\mathcal{L}(\sigma + it)|^2 dt \ll T^{\max\{d(1-\sigma),1\}+\epsilon}.$$

Proof. This is a standard result for general *L*-functions (see Perelli [16]). □

2.3. Decomposition of Triple Product L-functions

The improvement in our results relies on explicit decompositions of the triple product *L*-functions. We present these as separate lemmas for clarity.

Lemma 2.7. *The triple product L-function $L(s, G_1 \otimes G_1 \otimes G_1)$ satisfies the decomposition:*

$$L(s, G_1 \otimes G_1 \otimes G_1) = \zeta(s)^{11} L(s, f)^{17} L(s, F)^{18} L(s, \mathrm{sym}^3 f)^{13} L(s, \mathrm{sym}^4 f)^8 L(s, \mathrm{sym}^5 f)^3 L(s, \mathrm{sym}^6 f).$$

Proof. Recall $L(s, G_1) = \zeta(s)L(s, f)L(s, F)$. Using the local parameters defined in (2.1) and (2.2), the Euler factor at p for $L(s, G_1)$ is:

$$(1 - \alpha(p)^2 p^{-s})^{-1}(1 - \alpha(p)p^{-s})^{-1}(1 - p^{-s})^{-2}(1 - \beta(p)p^{-s})^{-1}(1 - \beta(p)^2 p^{-s})^{-1}.$$

The triple product corresponds to the tensor cube of the representation. By comparing the resulting Euler factors with the definition of symmetric power *L*-functions, we obtain the decomposition stated above. We omit the routine algebraic expansion. □

Lemma 2.8. *The triple product L-function $L(s, G_2 \otimes G_2 \otimes G_2)$ satisfies:*

$$L(s, G_2 \otimes G_2 \otimes G_2) = \zeta(s)^{37} L(s, F)^{39} L(s, \text{sym}^4 f)^{11} L(s, \text{sym}^6 f).$$

Proof. From (1.2), $L(s, G_2) = \zeta(s)^3 L(s, F)$. The decomposition follows similarly to Lemma 2.7 by expanding the triple product of the Euler factors of $\zeta(s)^3 L(s, F)$. $\square$

Lemma 2.9. *The triple product L-function $L(s, G_3 \otimes G_3 \otimes G_3)$ satisfies:*

$$L(s, G_3 \otimes G_3 \otimes G_3) = \zeta(s)^{76} L(s, f)^{50} L(s, F)^{12} L(s, \text{sym}^3 f).$$

Proof. From (1.3), $L(s, G_3) = \zeta(s)^4 L(s, f)$. The decomposition follows by direct comparison of Euler factors. $\square$

3. Proofs of Main Theorems

We employ Perron's formula and the decompositions established in Section 2.

3.1. Proof of Theorem 1.1 (Case G_1)

Applying Perron's formula (see [6]) to $L(s, G_1 \otimes G_1 \otimes G_1)$ with $\eta = 1 + \epsilon$ and $10 \leq T \leq x$:

$$\sum_{n \leq x} \lambda_{G_1 \otimes G_1 \otimes G_1}(n) = \frac{1}{2\pi i} \int_{\eta - iT}^{\eta + iT} L(s, G_1 \otimes G_1 \otimes G_1) \frac{x^s}{s} ds + O\left(\frac{x^{1+\epsilon}}{T}\right).$$

We shift the line of integration to $\Re(s) = 5/8$. By Lemma 2.7, the function has a pole of order 11 at $s = 1$ due to the factor $\zeta(s)^{11}$. This pole contributes the main term $x P_1(\log x)$, where P_1 is a polynomial of degree 10.

Let C be the rectangular contour with vertices $\eta \pm iT$ and $5/8 \pm iT$, where T is replaced by T^* from Lemma 2.5. The integral splits into:

$$\frac{1}{2\pi i} \left(\int_{5/8 - iT}^{5/8 + iT} + \int_{\eta - iT}^{5/8 - iT} + \int_{5/8 + iT}^{\eta + iT} \right) L(s, G_1 \otimes G_1 \otimes G_1) \frac{x^s}{s} ds =: J_1^{(1)} + J_2^{(1)} + J_3^{(1)}.$$

Horizontal Integrals ($J_2^{(1)}, J_3^{(1)}$): Using the convexity bound (Lemma 2.6) and Lemma 2.3, we estimate the integrand on the horizontal segments. The contribution is dominated by the bounds near $\sigma = 5/8$. We have:

$$J_2^{(1)} + J_3^{(1)} \ll x^{\frac{5}{8} + \epsilon} T^{\frac{2663}{80}} + \frac{x^{1+\epsilon}}{T}.$$

Vertical Integral ($J_1^{(1)}$): On the line $\sigma = 5/8$, we use the decomposition from Lemma 2.7:

$$\left| L\left(\tfrac{5}{8} + it, G_1 \otimes G_1 \otimes G_1\right) \right| = |\zeta|^{11} |L(f)|^{17} |L(F)|^{18} |L(\text{sym}^3 f)|^{13} |L(\text{sym}^4 f)|^8 |L(\text{sym}^5 f)|^3 |L(\text{sym}^6 f)|.$$

We apply Hölder's inequality. We extract the max norms of ζ, $L(F)$, $L(\text{sym}^3 f)$, $L(\text{sym}^4 f)$, and $L(\text{sym}^5 f)$ using pointwise bounds (Lemmas 2.2–2.4). We integrate the remaining factors using the fourth moment of $L(s, f)$ and the second moment of $L(s, \text{sym}^6 f)$. Specifically:

$$\begin{aligned}
J_1^{(1)} &\ll x^{5/8} \int_{-T}^{T} \frac{|L(s, G_1 \otimes G_1 \otimes G_1)|}{|5/8 + it|} dt \\
&\ll x^{5/8+\epsilon} + \frac{x^{5/8}}{T} \max_{|t| \leq T} \left(|\zeta|^{11} |L(f)|^{15} |L(F)|^{18} |L(\text{sym}^3 f)|^{13} |L(\text{sym}^4 f)|^8 |L(\text{sym}^5 f)|^3 \right) \\
&\quad \times \left(\int_{-T}^{T} |L(s, f)|^4 dt \right)^{1/2} \left(\int_{-T}^{T} |L(s, \text{sym}^6 f)|^2 dt \right)^{1/2} \\
&\ll x^{5/8+\epsilon} T^{\frac{149843}{4480} + \epsilon}.
\end{aligned}$$

Balancing the error terms $x^{1+\epsilon}/T$ and the vertical integral estimate yields $T \asymp x^{\frac{560}{51441}}$. This results in the final error term:

$$O\left(x^{\frac{50881}{51441} + \epsilon} \right).$$

3.2. Proof of Theorem 1.2 (Case G_2)

For $L(s, G_2 \otimes G_2 \otimes G_2)$, we follow the same contour integration strategy, but we shift the line of integration to $\Re(s) = 5/7$ along $T = \pm T^*$. By Lemma 2.8, $L(s, G_2 \otimes G_2 \otimes G_2)$ has a factor $\zeta(s)^{37}$, giving a pole of order 37 at $s = 1$. The residue is $xP_2(\log x)$, where $\deg(P_2) = 36$.

Horizontal Integrals: Using Lemma 2.3 and Lemma 2.6, the horizontal integrals contribute:

$$J_2^{(2)} + J_3^{(2)} \ll x^{\frac{5}{7}+\epsilon}T^{\frac{743}{35}} + \frac{x^{1+\epsilon}}{T}.$$

Vertical Integral: On the line $\sigma = 5/7$, we use the decomposition:

$$|L(\tfrac{5}{7} + it, G_2 \otimes G_2 \otimes G_2)| = |\zeta|^{37}|L(F)|^{39}|L(\mathrm{sym}^4 f)|^{11}|L(\mathrm{sym}^6 f)|.$$

We extract the maximum of $|\zeta|^{31}|L(F)|^{39}|L(\mathrm{sym}^4 f)|^{11}$ and apply Hölder's inequality with exponents $(2, 2)$ to the remaining factors (pairing $|\zeta|^6$ and $|L(\mathrm{sym}^6 f)|$). This yields:

$$J_1^{(2)} \ll x^{\frac{5}{7}+\epsilon} + \frac{x^{\frac{5}{7}}}{T} \max_t \left(|\zeta|^{31}|L(F)|^{39}|L(\mathrm{sym}^4 f)|^{11}\right) \times \left(\int |\zeta|^{12}\right)^{1/2} \left(\int |L(\mathrm{sym}^6 f)|^2\right)^{1/2}$$

$$\ll x^{\frac{5}{7}+\epsilon}T^{\frac{35971}{1470}+\epsilon}.$$

Optimizing T yields $T \asymp x^{\frac{420}{37441}}$, resulting in the error term:

$$O\left(x^{\frac{37021}{37441}+\epsilon}\right).$$

3.3. Proof of Theorem 1.3 (Case G_3)

For $L(s, G_3 \otimes G_3 \otimes G_3)$, we shift the contour to $\Re(s) = 5/8$. By Lemma 2.9, $L(s, G_3 \otimes G_3 \otimes G_3)$ has a factor $\zeta(s)^{76}$, giving a pole of order 76 at $s = 1$. The residue is $xP_3(\log x)$, where $\deg(P_3) = 75$.

Horizontal Integrals: Similar estimation yields:

$$J_2^{(3)} + J_3^{(3)} \ll x^{\frac{5}{8}+\epsilon}T^{\frac{353}{20}} + \frac{x^{1+\epsilon}}{T}.$$

Vertical Integral: On $\sigma = 5/8$, we decompose:

$$|L(\tfrac{5}{8} + it, G_3 \otimes G_3 \otimes G_3)| = |\zeta|^{76}|L(f)|^{50}|L(F)|^{12}|L(\mathrm{sym}^3 f)|.$$

We apply Hölder's inequality, utilizing the 4th moment of $L(f)$ (Lemma 2.4) and the 2nd moment of $L(\mathrm{sym}^3 f)$.

$$J_1^{(3)} \ll \frac{x^{5/8}}{T} \max_t \left(|\zeta|^{76}|L(f)|^{48}|L(F)|^{12}\right) \left(\int |L(f)|^4\right)^{1/2} \left(\int |L(\mathrm{sym}^3 f)|^2\right)^{1/2}$$

$$\ll x^{\frac{5}{8}+\epsilon}T^{\frac{1853}{70}+\epsilon}.$$

Optimizing T yields $T \asymp x^{\frac{35}{2564}}$, leading to the error term:

$$O\left(x^{\frac{2529}{2564}+\epsilon}\right).$$

4. Application to Subconvexity Bounds and Concluding Remarks

The study of discrete power moments of the Riemann zeta-function on the critical line, and more generally, investigating discrete power moments of generalized Dirichlet series, holds significant importance. As an application of the decompositions in Lemmas 2.7–2.9, we can derive subconvex bounds for the corresponding L-functions on the critical line $\Re(s) = 1/2$.

Consider the mean square of $L(s, G_2 \otimes G_2 \otimes G_2)$ on the critical line:

$$\int_{-T}^{T} |L(\tfrac{1}{2} + it, G_2 \otimes G_2 \otimes G_2)|^2 dt = \int_{-T}^{T} \left| \zeta(\tfrac{1}{2} + it)^{37} L(\tfrac{1}{2} + it, F)^{39} L(\tfrac{1}{2} + it, \mathrm{sym}^4 f)^{11} L(\tfrac{1}{2} + it, \mathrm{sym}^6 f) \right|^2 dt.$$

Using the convexity bounds for the terms in the max norm and the mean square for the remaining terms, we obtain:

$$\ll \max_{|t| \leq T} \left(\left| \zeta(\tfrac{1}{2} + it)^{37} L(\tfrac{1}{2} + it, F)^{39} \right|^2 \right) \times \int_{-T}^{T} \left| L(\tfrac{1}{2} + it, \mathrm{sym}^4 f)^{11} L(\tfrac{1}{2} + it, \mathrm{sym}^6 f) \right|^2 dt$$

$$\ll T^{74 \times \frac{13}{84} + 78 \times \frac{3}{5} + \frac{62}{2} + \epsilon} \ll T^{\frac{18743}{210} + \epsilon}.$$

This implies the individual bound:

$$|L(\tfrac{1}{2} + it, G_2 \otimes G_2 \otimes G_2)| \ll (1 + |t|)^{\frac{18743}{420} + \epsilon}.$$

Performing similar calculations for G_1 and G_3, we arrive at the following corollary.

Corollary 4.1. *For $\frac{1}{2} \leq \sigma < 1$, we have the following bounds on the critical line $\sigma = 1/2$:*

$$|L(\tfrac{1}{2} + it, G_1 \otimes G_1 \otimes G_1)| \ll (1 + |t|)^{\frac{4979}{105} + \epsilon},$$

$$|L(\tfrac{1}{2} + it, G_2 \otimes G_2 \otimes G_2)| \ll (1 + |t|)^{\frac{18743}{420} + \epsilon},$$

$$|L(\tfrac{1}{2} + it, G_3 \otimes G_3 \otimes G_3)| \ll (1 + |t|)^{\frac{3846}{105} + \epsilon}.$$

Remark 4.2. *We compare these with the general convexity bound for an L-function of degree 216. According to Perelli [16], the convexity bound is $O(T^{d/4+\epsilon}) = O(T^{54+\epsilon})$. Note that:*

$$54 > \frac{18743}{420} \approx 44.62, \quad 54 > \frac{4979}{105} \approx 47.41, \quad and \quad 54 > \frac{3846}{105} \approx 36.62.$$

Thus, our results provide explicit subconvexity bounds for these degree 216 L-functions.

Remark 4.3. *We note that the error terms in our main theorems and the subconvexity exponents in Corollary 4.1 depend heavily on the bounds for the component L-functions. Recent strong subconvexity results, such as those by Dasgupta, Leung, and Young [2] for* GL(3) *L-functions, indicate that further progress is possible. Any future improvements in the t-aspect subconvexity bounds for* GL(3) *or symmetric power L-functions would directly yield sharper estimates for these triple product moments.*

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