



## Study of Riemannian maps with total manifolds admitting a Ricci-Bourguignon soliton

Rajendra Prasad<sup>a</sup>, Abhinav Verma<sup>a,\*</sup>, Vindhyachal Singh Yadav<sup>a</sup>

<sup>a</sup>Department of Mathematics and Astronomy, University of Lucknow, 226007-Lucknow, Uttar Pradesh, India

**Abstract.** In this article, we look at Riemannian maps with total manifolds that accept a Ricci-Bourguignon soliton and provide an illustration. We have obtained the requirements for any fiber of such a Riemannian map to be Ricci-Bourguignon soliton, almost Ricci-Bourguignon soliton, and Einstein. We also find that the range space of such a Riemannian map must be Ricci-Bourguignon soliton and Einstein. Moreover, we have studied  $\eta$ -Ricci-Bourguignon soliton on a totally geodesic Riemannian map. Furthermore, we investigate the harmonicity and biharmonicity of a Riemannian map derived from the Ricci-Bourguignon soliton and determine the necessary and sufficient conditions for such a Riemannian map to be harmonic and biharmonic.

### 1. Introduction

In the mid 1980s of the 20<sup>th</sup> century, R. Hamilton developed the idea of Ricci soliton in relation to the self-similar solution of Ricci flow [12, 13]. G. Canteno and L. Mazziere studied the gradient Einstein solitons [2, 3].

The idea of generalization of Ricci flow as Ricci-Bourguignon flow was introduced by Jean-Pierre Bourguignon in 1981 [1]. Special solutions to the Ricci-Bourguignon flows, such as the Schouten flow, the Einstein flow, the traceless Ricci flow, and the Ricci flow, are known as Ricci-Bourguignon soliton (in short,  $\mathcal{RBS}$ )

Let  $(\mathcal{B}, g_{\mathcal{B}})$  be an  $n$ -dimensional Riemannian manifold. The Ricci-Bourguignon flow on  $(\mathcal{B}, g_{\mathcal{B}})$  is defined by

$$\frac{\partial g_{\mathcal{B}}(t)}{\partial t} + 2\mathcal{S} - 2\rho\kappa g_{\mathcal{B}} = 0, \quad (1)$$

where,  $\mathcal{S}$  is the Ricci tensor of the metric,  $\kappa$  is the scalar curvature of the Riemannian metric  $g$  and  $\rho \in \mathbb{R}$ .

Let  $(\mathcal{B}, g_{\mathcal{B}})$  be an  $n$ -dimensional Riemannian manifold. The  $\mathcal{RBS}$  that are self-similar solutions to Ricci-Bourguignon flow and is defined by

$$\mathcal{S} + \frac{1}{2}\mathcal{L}_{\lambda}g_{\mathcal{B}} = -(\lambda - \rho\kappa)g_{\mathcal{B}}, \quad (2)$$

---

2020 Mathematics Subject Classification. Primary 53C21; Secondary 53C25, 53C43.

Keywords. Riemannian maps, Ricci-Bourguignon soliton, Harmonicity, Biharmonicity.

Received: 04 April 2025; Accepted: 04 December 2025

Communicated by Ljubica Velimirović

\* Corresponding author: Abhinav Verma

Email addresses: [rp.lucknow@rediffmail.com](mailto:rp.lucknow@rediffmail.com) (Rajendra Prasad), [vabhinav831@gmail.com](mailto:vabhinav831@gmail.com) (Abhinav Verma),

[vs.yadav4@gmail.com](mailto:vs.yadav4@gmail.com) (Vindhyachal Singh Yadav)

ORCID iDs: <https://orcid.org/0000-0002-7502-0239> (Rajendra Prasad), <https://orcid.org/0009-0004-7998-5224> (Abhinav Verma), <https://orcid.org/0009-0009-2810-2723> (Vindhyachal Singh Yadav)

where,  $\mathcal{L}_\Lambda g_{\mathcal{B}}$  represents the Lie derivative of the metric  $g_{\mathcal{B}}$  w.r.t. a vector field  $\Lambda$ , and  $\lambda$  denotes soliton constant. Let  $\eta$  be a 1-form, then the Riemannian manifold  $(\mathcal{B}, g_{\mathcal{B}})$  is said to be  $\eta$ - $\mathcal{RBS}$  if there exists a vector field  $\Lambda$ , and  $\lambda \in \mathbb{R}$ , such that

$$\mathcal{S} + \frac{1}{2}\mathcal{L}_\Lambda g_{\mathcal{B}} = -(\lambda - \rho\kappa)g_{\mathcal{B}} - \mu\eta \otimes \eta, \tag{3}$$

where,  $\mu \in \mathbb{R}$ . The  $\eta$ - $\mathcal{RBS}$  is shrinking, steady, and expanding, if  $\lambda < 0$ ,  $\lambda = 0$ , and  $\lambda > 0$ , respectively. For specific values of  $\rho$ , the following results hold:

- If,  $\rho = \frac{1}{2}$ , then  $\eta$ - $\mathcal{RBS}$  changes to  $\eta$ -Einstein soliton.
- If,  $\rho = \frac{1}{2(n-1)}$ , then  $\eta$ - $\mathcal{RBS}$  changes to  $\eta$ -Schouten soliton.
- If,  $\mu = 0$ , then  $\eta$ - $\mathcal{RBS}$  changes to  $\mathcal{RBS}$ .

If vector field  $\Lambda$  is Killing vector field, i.e.,  $\mathcal{L}_\Lambda g_{\mathcal{B}} = 0$ , then for  $n \geq 3$ ,  $\eta$ - $\mathcal{RBS}$  becomes trivial soliton. So, we obtain an  $\eta$ -Einstein manifold. For further study, see [4–8, 19, 20]

However, in 1992, Fischer expanded the concept of an isometric immersion and Riemannian submersion by introducing the Riemannian map (in short,  $\mathcal{RM}$ ) between Riemannian manifolds in [10, 14]. The geometry of Riemannian submersions has been discussed in [9]. For further study, see [15–17, 21, 27]. It is noteworthy that  $\mathcal{RM}$  has the extraordinary characteristics satisfying the generalized eikonal equation  $\|\mathcal{F}_*\|^2 = \text{Rank } \mathcal{F}$ , which serves as a link between geometric optics and physical optics [10]. Cauchy’s method of characteristics was used to solve eikonal equation of the geometrical optics. Fischer also suggested a method for creating a quantum model in [11]. He noted that the success of such a programme would offer an intriguing connection between Maxwell’s equation, Schödinger’s equation, and their suggested generalization on the physical side, whereas,  $\mathcal{RM}$ , harmonic maps, and Lagrangian field theory on the mathematical side.

This article has been organised in the following manner: Section 1 covers introduction, where related concepts and their brief histories are described. The next section contains the fundamental formulae, that are essential for the progress of this article. In section 3, the exploration of  $\mathcal{RM}$  has been done for which total manifolds accept an  $\mathcal{RBS}$  as well as  $\eta$ - $\mathcal{RBS}$ . Further, we study an  $\mathcal{RM}$  from  $\mathcal{RBS}$  to a Riemannian manifold and explore the essential requirements for which any fiber of  $\mathcal{RM}$  is an  $\mathcal{RBS}$ , almost- $\mathcal{RBS}$ , and Einstein manifold. In section 4, we explore the harmonicity and biharmonicity of an  $\mathcal{RM}$  derived from the  $\mathcal{RBS}$  and determine the necessary and sufficient conditions for such an  $\mathcal{RM}$  to be harmonic and biharmonic. In the last section, we construct an illustration for  $\mathcal{RM}$  for which total manifold accepts an  $\mathcal{RBS}$ .

## 2. Preliminaries

Let  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$  be a smooth map between Riemannian manifolds such that  $0 < \text{Rank } \mathcal{F} \leq \min\{m, n\}$ , where  $\dim(\mathcal{B}) = m$  and  $\dim(\mathcal{N}) = n$ . Then we denote the kernel space of  $\mathcal{F}_*$  by  $\mathcal{V}_p = \text{Ker } \mathcal{F}_*|_p$  at  $p \in \mathcal{B}$  and consider the orthogonal complementary space  $\mathcal{H}_p = (\text{Ker } \mathcal{F}_*|_p)^\perp$  to  $\text{Ker } \mathcal{F}_*|_p$  in  $\mathcal{T}_p\mathcal{B}$ . Thus, the tangent space  $\mathcal{T}_p\mathcal{B}$  of  $\mathcal{B}$  at  $p$  has the decomposition

$$\mathcal{T}_p\mathcal{B} = (\text{Ker } \mathcal{F}_*|_p) \oplus (\text{Ker } \mathcal{F}_*|_p)^\perp = \mathcal{V}_p \oplus \mathcal{H}_p.$$

We denote the range of  $\mathcal{F}_*$  by  $\text{Range } \mathcal{F}_*$  at  $p \in \mathcal{B}$  and consider the orthogonal complementary space  $(\text{Range } \mathcal{F}_*|_p)^\perp$  to  $\text{Range } \mathcal{F}_*|_p$  in the tangent space  $\mathcal{T}_{\mathcal{F}(p)}\mathcal{N}$  of  $\mathcal{N}$  at  $\mathcal{F}(p) \in \mathcal{N}$ . Since  $\text{Rank } \mathcal{F} \leq \min\{m, n\}$ , we have  $(\text{Range } \mathcal{F}_*)^\perp = \{0\}$ . Thus, the tangent space  $\mathcal{T}_{\mathcal{F}(p)}\mathcal{N}$  of  $\mathcal{N}$  at  $\mathcal{F}(p) \in \mathcal{N}$  has the decomposition

$$\mathcal{T}_{\mathcal{F}(p)}\mathcal{N} = (\text{Range } \mathcal{F}_*|_p) \oplus (\text{Range } \mathcal{F}_*|_p)^\perp.$$

Now, a smooth map  $\mathcal{F} : (\mathcal{B}^m, g_{\mathcal{B}}) \rightarrow (\mathcal{N}^n, g_{\mathcal{N}})$  is called an  $\mathcal{RM}$  at  $p \in \mathcal{B}$  if the horizontal restriction

$$\mathcal{F}_{*h}|_p : (\text{Ker } \mathcal{F}_*|_p)^\perp \rightarrow \text{Range } \mathcal{F}_*|_p$$

is a linear isometry between the inner product spaces

$$((\text{Ker } \mathcal{F}_*|_p)^\perp, g_{\mathcal{B}}(p)|_{(\text{Ker } \mathcal{F}_*|_p)^\perp}) \quad \text{and} \quad (\text{Range } \mathcal{F}_*|_p, g_{\mathcal{N}}(\mathcal{F}(p))|_{\text{Range } \mathcal{F}_*|_p}),$$

where  $\mathcal{F}(p) = p_1$ . In other words,  $\mathcal{F}_*$  satisfies the equation

$$g_{\mathcal{N}}(\mathcal{F}_*X, \mathcal{F}_*Y) = g_{\mathcal{B}}(X, Y), \tag{4}$$

$\forall X, Y \in \Gamma((\text{Ker } \mathcal{F}_*|_p)^\perp)$ .

It follows that isometric immersions and Riemannian submersions are particular  $\mathcal{R}\mathcal{M}$  with  $\text{Ker } \mathcal{F}_* = \{0\}$  and  $(\text{Range } \mathcal{F}_*)^\perp = \{0\}$ .

The O'Neill tensors  $A$  and  $T$ , defined in [18], are given as

$$A_E F = \mathcal{H}(\nabla_{\mathcal{H}E} \mathcal{V}F) + \mathcal{V}(\nabla_{\mathcal{H}E} \mathcal{H}F), \tag{5}$$

$$T_E F = \mathcal{H}(\nabla_{\mathcal{V}E} \mathcal{V}F) + \mathcal{V}(\nabla_{\mathcal{V}E} \mathcal{H}F), \tag{6}$$

for all vector fields  $E, F$  on  $\mathcal{B}$ , where  $\nabla$  is the Levi-Civita connection of  $g_{\mathcal{B}}$ , and  $\mathcal{V}, \mathcal{H}$  denote the projections to vertical and horizontal subbundles, respectively. For any  $E \in \Gamma(\mathcal{T}\mathcal{B})$ ,  $T(E, \cdot)$  and  $A(E, \cdot)$  are skew-symmetric operators on  $(\Gamma(\mathcal{T}\mathcal{B}), g_{\mathcal{B}})$  reversing the horizontal and the vertical distributions. It is also easy to see that  $T$  is vertical ( $T(E) = T(\mathcal{V}E)$ ) and  $A$  is horizontal ( $A(E) = A(\mathcal{H}E)$ ). Moreover, the tensor field  $T$  satisfies  $T(U, W) = T(W, U)$  for all  $U, W \in \Gamma(\text{Ker } \mathcal{F}_*)$ . Relations (5) and (6) together give

$$\nabla_{\mathcal{V}}W = T_{\mathcal{V}}W + \hat{\nabla}_{\mathcal{V}}W, \tag{7}$$

$$\nabla_{\mathcal{X}}V = A_{\mathcal{X}}V + \mathcal{V}\nabla_{\mathcal{X}}V, \tag{8}$$

$$\nabla_{\mathcal{X}}Y = A_{\mathcal{X}}Y + \mathcal{H}\nabla_{\mathcal{X}}Y, \tag{9}$$

$$\nabla_{\mathcal{V}}X = T_{\mathcal{V}}X + \mathcal{H}\nabla_{\mathcal{V}}X, \tag{10}$$

for all  $X, Y \in (\text{Ker } \mathcal{F}_*)^\perp$  and  $V, W \in \text{Ker } \mathcal{F}_*$ , where  $\hat{\nabla}_{\mathcal{V}}W = \mathcal{V}\nabla_{\mathcal{V}}W$ .

Also, an  $\mathcal{R}\mathcal{M}$  is an  $\mathcal{R}\mathcal{M}$  with totally umbilical fibers if [23]

$$T_U V = g_{\mathcal{B}}(U, V)H, \tag{11}$$

for all  $U, V \in \text{Ker } \mathcal{F}_*$ , where  $H$  is the mean curvature vector field of fibers. Let  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$  be a smooth map between Riemannian manifolds  $(\mathcal{B}, g_{\mathcal{B}})$  and  $(\mathcal{N}, g_{\mathcal{N}})$ . Then the differential  $\mathcal{F}_*$  of  $\mathcal{F}$  can be viewed as a section of the bundle  $\text{Hom}(\mathcal{T}\mathcal{B}, \mathcal{F}^{-1}\mathcal{T}\mathcal{N}) \rightarrow \mathcal{B}$ , where  $\mathcal{F}^{-1}\mathcal{T}\mathcal{N}$  is the pullback bundle whose fibers at  $p \in \mathcal{B}$  are  $(\mathcal{F}^{-1}\mathcal{T}\mathcal{N})_p = \mathcal{T}_{\mathcal{F}(p)}\mathcal{N}$ ,  $p \in \mathcal{B}$ . The bundle  $\text{Hom}(\mathcal{T}\mathcal{B}, \mathcal{F}^{-1}\mathcal{T}\mathcal{N})$  has a connection  $\nabla$  induced from the Levi-Civita connection  $\overset{\mathcal{B}}{\nabla}$  and the pullback connection  $\overset{\mathcal{F}}{\nabla}$ . Then the second fundamental form of  $\mathcal{F}$  is given by

$$(\nabla \mathcal{F}_*)(X, Y) = \overset{\mathcal{F}}{\nabla}_X \mathcal{F}_*Y - \mathcal{F}_*(\overset{\mathcal{B}}{\nabla}_X Y), \tag{12}$$

for all  $X, Y \in \mathcal{T}\mathcal{B}$ . It is known that the second fundamental form is symmetric. In [22], B. Sahin has proved that

$$(\nabla \mathcal{F}_*)(X, Y) \in \Gamma(\text{Range } \mathcal{F}_*)^\perp, \tag{13}$$

for all  $X, Y \in (\text{Ker } \mathcal{F}_*)^\perp$ .

Also in [26], an  $\mathcal{R}\mathcal{M}$   $\mathcal{F}$  is an umbilical  $\mathcal{R}\mathcal{M}$  iff

$$(\nabla \mathcal{F}_*)(X, Y) = g_{\mathcal{B}}(X, Y)H', \tag{14}$$

for  $X, Y \in (\text{Ker } \mathcal{F}_*)^\perp$  and  $H'$  is a nowhere zero vector field on  $(\text{Range } \mathcal{F}_*)^\perp$ .

The tension field  $\tau(\mathcal{F})$  is defined to be the trace of the second fundamental form of  $\mathcal{F}$ , i.e.

$$\tau(\mathcal{F}) = \text{trace}(\nabla \mathcal{F}_*) = \sum_{i=1}^m (\nabla \mathcal{F}_*)(\mathcal{E}_i, \mathcal{E}_i), \tag{15}$$

here  $m = \dim(\mathcal{B})$  and  $\{\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_m\}$  is the orthonormal base frame on  $\mathcal{B}$ .

We denote the Riemannian curvature tensor of  $(\mathcal{B}, g_{\mathcal{B}})$ ,  $\text{Range } \mathcal{F}_*$ , and any fiber of  $\mathcal{F}$  by  $K, K'$ , and  $\hat{K}$ , respectively. Then [25], we have

$$g_{\mathcal{B}}(K(U, V)\bar{W}, F) = g_{\mathcal{B}}(\hat{K}(U, V)\bar{W}, F) - g_{\mathcal{B}}(T_U F, T_V \bar{W}) + g_{\mathcal{B}}(T_V F, T_U \bar{W}),$$

$$g_{\mathcal{B}}(K(X, Y)Z, H) = g_{\mathcal{N}}(K'(\mathcal{F}_* X, \mathcal{F}_* Y)\mathcal{F}_* Z, \mathcal{F}_* H) - g_{\mathcal{N}}((\nabla \mathcal{F}_*)(X, Z), (\nabla \mathcal{F}_*)(Y, H)) + g_{\mathcal{N}}((\nabla \mathcal{F}_*)(Y, Z), (\nabla \mathcal{F}_*)(X, H)),$$

for any  $U, V, \bar{W}, F \in \text{Ker } \mathcal{F}_*$  and  $X, Y, Z, H \in (\text{Ker } \mathcal{F}_*)^\perp$ . Now we denote the Ricci tensor and scalar curvature by  $\mathcal{S}$  and  $\kappa$ , respectively, and define them as

$$\mathcal{S}(X, Y) = \text{trace}(Z \rightarrow K(Z, X)Y),$$

and

$$\kappa = \text{trace } \mathcal{S}(X, Y),$$

for all  $X, Y \in \mathcal{F}\mathcal{B}$ .

The Ricci tensor  $\mathcal{S}$  on  $(\mathcal{B}, g_{\mathcal{B}})$  is given by

$$\mathcal{S}(U, \bar{W}) = \hat{\mathcal{S}}(U, \bar{W}) - g_{\mathcal{B}}(N, T_U \bar{W}) + \sum_{j=r+1}^m [g_{\mathcal{B}}((\nabla_{X_j} T)_U \bar{W}, X_j) + g_{\mathcal{B}}(A_{X_j} U, A_{X_j} \bar{W})], \tag{16}$$

$$\begin{aligned} \mathcal{S}(X, Y) = & \mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_* X, \mathcal{F}_* Y) + \sum_{i=1}^r [g_{\mathcal{B}}((\nabla_X T)_{U_i} U_i, Y) + g_{\mathcal{B}}((\nabla_{U_i} A)_X Y, U_i) - g_{\mathcal{B}}(T_{U_i} X, T_{U_i} Y) + g_{\mathcal{B}}(A_X U_i, A_Y U_i)] \\ & + g_{\mathcal{N}}((\nabla \mathcal{F}_*)(X, Y), \tau^{(\text{Ker } \mathcal{F}_*)^\perp}) - \sum_{j=r+1}^m g_{\mathcal{N}}((\nabla \mathcal{F}_*)(X_j, Y), (\nabla \mathcal{F}_*)(X, X_j)), \end{aligned} \tag{17}$$

$$\mathcal{S}(X, U) = \sum_{i=1}^r [g_{\mathcal{B}}((\nabla_U T)_{U_i} U_i, X) - g_{\mathcal{B}}((\nabla_{U_i} T)_U U_i, X)] + \sum_{j=r+1}^m [g_{\mathcal{B}}((\nabla_{X_j} A)_{X_j} X, U) + 2g_{\mathcal{B}}(T_U X_j, A_{X_j} X)], \tag{18}$$

for all  $U, \bar{W} \in \Gamma(\text{Ker } \mathcal{F}_*)$  and  $X, Y \in \Gamma((\text{Ker } \mathcal{F}_*)^\perp)$ , where  $\hat{\mathcal{S}}(U, \bar{W})$ ,  $\mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_* X, \mathcal{F}_* Y)$  and  $\tau^{(\text{Ker } \mathcal{F}_*)^\perp}$  are the Ricci tensors of the fibers,  $\text{Range } \mathcal{F}_*$ , and the component of the tension field  $\tau$ , respectively.

The scalar curvature  $\kappa$  on  $(\mathcal{B}, g_{\mathcal{B}})$  is defined as

$$\kappa = \sum_{i=1}^r \mathcal{S}(U_i, U_i) + \sum_{j=r+1}^m \mathcal{S}(X_j, X_j),$$

or,

$$\kappa = \mathcal{V}(\kappa) + \mathcal{H}(\kappa), \tag{19}$$

where,  $\{U_1, U_2, \dots, U_r\}$  and  $\{X_{r+1}, X_{r+2}, \dots, X_m\}$  are orthonormal bases of  $\text{Ker } \mathcal{F}_*$  and  $(\text{Ker } \mathcal{F}_*)^\perp$ , respectively. On the other hand, the mean curvature vector field  $H$  on any fiber of an  $\mathcal{R}\mathcal{M}$  is given by

$$N = rH \quad \text{such that} \quad N = \sum_{i=1}^r T_{U_i} U_i, \tag{20}$$

where  $r$  denotes the dimension of any fiber of  $\mathcal{F}$ . We know that the horizontal vector field  $\mathbf{N}$  vanishes iff any fiber of the  $\mathcal{R}\mathcal{M}$  is minimal.

We denote the divergence of the horizontal vector field  $\mathbf{X}$  on  $(\text{Ker}\mathcal{F}_*)^\perp$  by  $\hat{\delta}(\mathbf{X})$ , and is given by [9]

$$\hat{\delta}(\mathbf{X}) = \sum_{j=r+1}^m g_{\mathcal{B}}(\nabla_{\mathbf{X}_j}\mathbf{X}, \mathbf{X}_j). \tag{21}$$

Hence, relations (20) and (21) together give

$$\hat{\delta}(\mathbf{N}) = \sum_{i=1}^r \sum_{j=r+1}^m g_{\mathcal{B}}(\nabla_{\mathbf{X}_j}\mathbf{T})_{\mathbf{U}_i}\mathbf{U}_i, \mathbf{X}_j). \tag{22}$$

where  $\{\mathbf{X}_j\}$  and  $\{\mathbf{U}_i\}$  are orthonormal bases of  $(\text{Ker}\mathcal{F}_*)^\perp$  and  $\text{Ker}\mathcal{F}_*$ , respectively.

**Proposition 2.1.** [23] Let  $\mathcal{F}$  be an  $\mathcal{R}\mathcal{M}$  from a Riemannian manifold  $(\mathcal{B}, g_{\mathcal{B}})$  to a Riemannian manifold  $(\mathcal{N}, g_{\mathcal{N}})$ . Then  $\mathcal{F}$  is totally geodesic iff:

- (i)  $\mathbf{A}_X\mathbf{Y} = 0$  for all  $\mathbf{X}, \mathbf{Y} \in (\text{Ker}\mathcal{F}_*)^\perp$ ,
- (ii) The fibers are totally geodesic, i.e.,  $\mathbf{T}_U\mathbf{V} = 0$  for all  $\mathbf{U}, \mathbf{V} \in \text{Ker}\mathcal{F}_*$ ,
- (iii)  $\mathbf{S}_{V\mathcal{F}_*}\mathbf{X} = 0$ , for  $\mathbf{V} \in (\text{Range}\mathcal{F}_*)^\perp$  and  $\mathbf{X} \in (\text{Ker}\mathcal{F}_*)^\perp$ .

**Definition 2.2.** Let us consider a smooth map between Riemannian manifolds,  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$ . Then tension field  $\tau(\mathcal{F})$  of map  $\mathcal{F}$  vanishes at every point  $p \in \mathcal{B}$  iff  $\mathcal{F}$  is harmonic [24], i.e.

$$\tau(\mathcal{F}) = \text{trace}(\nabla\mathcal{F}_*) = \sum_{i=1}^m (\nabla\mathcal{F}_*)(\mathcal{E}_i, \mathcal{E}_i) = 0,$$

where,  $\{\mathcal{E}_i\}_{i=1}^m$  is local orthonormal base frame at each point  $p \in \mathcal{B}$  and  $\nabla\mathcal{F}_*$  is the second fundamental form of  $\mathcal{F}$ .

**Lemma 2.3.** [23] Let us consider an  $\mathcal{R}\mathcal{M}$  between Riemannian manifolds,  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$ . Then,  $\tau(\mathcal{F}) = -r\mathcal{F}_*(\mathbf{H}) + (m - r)\mathbf{H}'$  yields the tension field  $\tau$  of  $\mathcal{F}$ , which suggests that  $\tau(\mathcal{F}) = -\mathcal{F}_*(r\mathbf{H}) + (m - r)\mathbf{H}'$ . Using  $\mathbf{N} = r\mathbf{H}$  in this Lemma, we obtain

$$\tau(\mathcal{F}) = -\mathcal{F}_*(\mathbf{N}) + (m - r)\mathbf{H}', \tag{23}$$

where  $r = \dim(\text{Ker}\mathcal{F}_*)$ ,  $(m - r) = \text{Rank}\mathcal{F}$ ,  $\mathbf{H}$ , and  $\mathbf{H}'$  are the mean curvature vector fields of the distribution  $\text{Ker}\mathcal{F}_*$  and  $\text{Range}\mathcal{F}_*$ , respectively.

In the past ten years, biharmonic maps have been the subject of much research, and numerous authors have produced categorization results. B. Sahin examined the biharmonicity of  $\mathcal{R}\mathcal{M}$  in [25]. We have the subsequent result on the biharmonicity of  $\mathcal{R}\mathcal{M}$ :

**Theorem 2.4.** [25] Let  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}(c), g_{\mathcal{N}})$  be an  $\mathcal{R}\mathcal{M}$  from a Riemannian manifold to a space form. Then,  $\mathcal{F}$  is biharmonic iff

$$r \text{ trace}_{(\nabla\mathcal{F}_*)(\cdot, \mathbf{H})} \mathcal{F}_*(\cdot) - r \text{ trace}_{\mathcal{F}_*(\nabla(\cdot)\nabla(\cdot)\mathbf{H})} \mathcal{F}_*(\cdot) - (m - r) \text{ trace}_{\mathcal{F}_*(\nabla(\cdot)\mathcal{F}_*(\mathbf{S}_{\mathbf{H}'}\mathcal{F}_*(\cdot)))} \mathcal{F}_*(\cdot) - (m - r) \text{ trace}_{\nabla_0^{\mathcal{F}_*}\mathbf{H}} \mathcal{F}_*(\cdot) - rc(m - r - 1)\mathcal{F}_*(\mathbf{H}) = 0, \tag{24}$$

and

$$r \text{ trace}_{\nabla(\cdot)^\perp} (\nabla\mathcal{F}_*)(\cdot, \mathbf{H}) + r \text{ trace}_{(\nabla\mathcal{F}_*)(\cdot, \nabla(\cdot)\mathbf{H})} \mathcal{F}_*(\cdot) + (m - r) \text{ trace}_{(\nabla\mathcal{F}_*)(\cdot, \mathcal{F}_*(\mathbf{S}_{\mathbf{H}'}\mathcal{F}_*(\cdot)))} \mathcal{F}_*(\cdot) - (m - r)\Delta^{R^1}\mathbf{H}' - (m - r)^2c\mathbf{H}' = 0, \tag{25}$$

where,  $\dim(\text{Ker}\mathcal{F}_*) = r$  and  $\dim(\text{Ker}\mathcal{F}_*)^\perp = m - r$ .

### 3. Characteristics of Riemannian map with total manifold admitting a Ricci-Bourguignon soliton

In this section, we examine  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$ , an  $\mathcal{R.M}$  from  $\mathcal{R.B.S}$  to Riemannian manifold. We provide some characterizations for every fiber of such  $\mathcal{R.M}$  and Range  $\mathcal{F}_*$ .

**Lemma 3.1.** *Let  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$  be totally geodesic  $\mathcal{R.M}$  between Riemannian manifolds. Then vertical and horizontal component of  $\kappa$  is given*

$$\mathcal{V}(\kappa) = \hat{\kappa}$$

and

$$\mathcal{H}(\kappa) = \kappa^{\text{Range } \mathcal{F}_*},$$

where,  $\hat{\kappa}$  and  $\kappa^{\text{Range } \mathcal{F}_*}$  are scalar curvatures of fibers of  $\mathcal{F}$ .

**Theorem 3.2.** *Let  $(\mathcal{B}, \Lambda, \rho, \lambda, g_{\mathcal{B}})$  be an  $\mathcal{R.B.S}$  with the potential vector field  $\Lambda$  and  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$  be a totally geodesic  $\mathcal{R.M}$  between Riemannian manifolds. In this case, if the vector field  $\Lambda$  is vertical, then any fiber of  $\mathcal{F}$  is an  $\mathcal{R.B.S}$ , and if the vector field  $\Lambda$  is horizontal, then any fiber of  $\mathcal{F}$  is an Einstein.*

*Proof.* As  $(\mathcal{B}, \rho, \Lambda, \lambda, g_{\mathcal{B}})$  be an  $\mathcal{R.B.S}$ , so for any  $U, W \in \text{Ker } \mathcal{F}_*$ , using (16) in (2), we obtain

$$\begin{aligned} \frac{1}{2} \{g_{\mathcal{B}}(\nabla_U \Lambda, W) + g_{\mathcal{B}}(\nabla_W \Lambda, U)\} + \hat{S}(U, W) - g_{\mathcal{B}}(N, T_U W) \\ + \sum_{j=r+1}^m g_{\mathcal{B}}((\nabla_{X_j} T)_U W, X_j) + g_{\mathcal{B}}(A_{X_j} U, A_{X_j} W) + (\lambda - \mathcal{V}(\kappa)\rho)g_{\mathcal{B}}(U, W) = 0. \end{aligned} \quad (26)$$

From the preceding relation, if the vector field  $\Lambda = \zeta$  is vertical and  $\mathcal{F}$  is a totally geodesic  $\mathcal{R.M}$ , then we obtain

$$\frac{1}{2} \{g_{\mathcal{B}}(\hat{\nabla}_U \zeta, W) + g_{\mathcal{B}}(\hat{\nabla}_W \zeta, U)\} + \hat{S}(U, W) + (\lambda - \hat{\kappa}\rho)g_{\mathcal{B}}(U, W) = 0. \quad (27)$$

Thus, the relation (27) implies that any fiber of  $\mathcal{F}$  is an  $\mathcal{R.B.S}$ .

Also, if the vector field  $\Lambda = Z$  is horizontal and  $\mathcal{F}$  is a totally geodesic  $\mathcal{R.M}$ , then from the relation (26), we get

$$-\frac{1}{2} \{g_{\mathcal{B}}(\nabla_U W, Z) + g_{\mathcal{B}}(Z, \nabla_W U)\} + \hat{S}(U, W) + (\lambda - \hat{\kappa}\rho)g_{\mathcal{B}}(U, W) = 0.$$

In view of relation (7), the above relation takes the form

$$-g_{\mathcal{B}}(T_U W, Z) + \hat{S}(U, W) + (\lambda - \hat{\kappa}\rho)g_{\mathcal{B}}(U, W) = 0.$$

As the totally geodesic foliation on  $\mathcal{B}$  is defined by fibers of  $\mathcal{F}$ , so the above relation gives

$$\hat{S}(U, W) = -(\lambda - \hat{\kappa}\rho)g_{\mathcal{B}}(U, W). \quad (28)$$

Thus, fibers of  $\mathcal{F}$  are Einstein.

This completes the proof.  $\square$

**Theorem 3.3.** *Let  $(\mathcal{B}, \Lambda, \rho, \lambda, \mu, g_{\mathcal{B}})$  be an  $\eta$ - $\mathcal{R.B.S}$  with the potential vector field  $\Lambda$  and  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$  be a totally geodesic  $\mathcal{R.M}$  between Riemannian manifolds. In this case, If the vector field  $\Lambda$  is vertical, then any fiber of  $\mathcal{F}$  is an  $\eta$ - $\mathcal{R.B.S}$ , and if the vector field  $\Lambda$  is horizontal, then any fiber of  $\mathcal{F}$  is an  $\eta$ -Einstein.*

*Proof.* Assuming  $(\mathcal{B}, \rho, \Lambda, \lambda, \mu, g_{\mathcal{B}})$  be an  $\eta$ - $\mathcal{RBS}$ , then for any  $U, W \in \text{Ker } \mathcal{F}_*$ , using (16) in (3), we obtain

$$\begin{aligned} & \frac{1}{2} \{g_{\mathcal{B}}(\nabla_U \Lambda, W) + g_{\mathcal{B}}(\nabla_W \Lambda, U)\} + \hat{S}(U, W) - g_{\mathcal{B}}(N, T_U W) \\ & + \sum_{j=r+1}^m \{g_{\mathcal{B}}((\nabla_{X_j} T)_U W, X_j) + g_{\mathcal{B}}(A_{X_j} U, A_{X_j} W)\} + (\lambda - \mathcal{V}(\kappa)\rho)g_{\mathcal{B}}(U, W) + \mu\eta(U)\eta(W) = 0. \end{aligned} \quad (29)$$

From the preceding relation, if the vector field  $\Lambda = \zeta$  is vertical and  $\mathcal{F}$  is a totally geodesic  $\mathcal{RM}$ , then we obtain

$$\frac{1}{2} \{g_{\mathcal{B}}(\hat{\nabla}_U \zeta, W) + g_{\mathcal{B}}(\hat{\nabla}_W \zeta, U)\} + \hat{S}(U, W) + (\lambda - \hat{\kappa}\rho)g_{\mathcal{B}}(U, W) + \mu\eta(U)\eta(W) = 0. \quad (30)$$

Thus, the relation (30) shows that any fiber of  $\mathcal{F}$  is an  $\mathcal{RBS}$ .

Also, if the vector field  $\Lambda = Z$  is horizontal and  $\mathcal{F}$  is a totally geodesic  $\mathcal{RM}$ , then from the relation (29), we get

$$-\frac{1}{2} \{g_{\mathcal{B}}(\nabla_U W, Z) + g_{\mathcal{B}}(Z, \nabla_U W)\} + \hat{S}(U, W) + (\lambda - \hat{\kappa}\rho)g_{\mathcal{B}}(U, W) + \mu\eta(U)\eta(W) = 0. \quad (31)$$

In the light of (7), relation (31) gives

$$-g_{\mathcal{B}}(T_U W, Z) + \hat{S}(U, W) + (\lambda - \hat{\kappa}\rho)g_{\mathcal{B}}(U, W) + \mu\eta(U)\eta(W) = 0.$$

Because, fibers of  $\mathcal{F}$  define totally geodesic foliation on  $\mathcal{B}$ , therefore from the above equation, we obtain

$$\hat{S}(U, W) = -(\lambda - \hat{\kappa}\rho)g_{\mathcal{B}}(U, W) - \mu\eta(U)\eta(W). \quad (32)$$

Thus, fibers of  $\mathcal{F}$  are  $\eta$ -Einstein.

This completes the proof.  $\square$

**Theorem 3.4.** Let  $(\mathcal{B}, \Lambda, \lambda, \rho, g_{\mathcal{B}})$  be an  $\mathcal{RBS}$  with  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$  and the vertical potential vector field  $\Lambda$ . Let  $\mathcal{F}$  be an  $\mathcal{RM}$  between Riemannian manifolds in such a way that fibers of  $\mathcal{F}$  are totally umbilical and  $(\text{Ker } \mathcal{F}_*)^\perp$  defines a totally geodesic foliation on  $\mathcal{B}$ . Then, any fiber of  $\mathcal{F}$  is a soliton that is almost a Ricci-Bourguignon.

*Proof.* It is given that  $(\mathcal{B}, \Lambda, \lambda, \rho, g_{\mathcal{B}})$  an  $\mathcal{RBS}$ , is admitted by the total space  $(\mathcal{B}, g_{\mathcal{B}})$  of the Riemannian map  $\mathcal{F}$ , we have

$$\frac{1}{2} (\mathcal{L}_\Lambda g_{\mathcal{B}})(U, W) + \mathcal{S}(U, W) + (\lambda - \mathcal{V}(\kappa)\rho)g_{\mathcal{B}}(U, W) = 0, \quad (33)$$

for any  $U, W \in \Gamma(\text{Ker } \mathcal{F}_*)$ . Using (16) with  $\Lambda = \zeta$ , a vertical vector, into relation (33), we get

$$\begin{aligned} & \frac{1}{2} \{g_{\mathcal{B}}(\hat{\nabla}_U \zeta, W) + g_{\mathcal{B}}(\hat{\nabla}_W \zeta, U)\} + \hat{S}(U, W) - g_{\mathcal{B}}(N, T_U W) + \sum_{j=r+1}^m [g_{\mathcal{B}}((\nabla_{X_j} T)_U W, X_j) + g_{\mathcal{B}}(A_{X_j} U, A_{X_j} W)] \\ & + (\lambda - \mathcal{V}(\kappa)\rho)g_{\mathcal{B}}(U, W) = 0. \end{aligned} \quad (34)$$

If the  $\mathcal{RBS} (\mathcal{B}, \zeta, \lambda, \rho, g_{\mathcal{B}})$  possesses totally umbilical fibers then plugging relations (7) & (11) in (34), it yields

$$\begin{aligned} & \frac{1}{2} \{g_{\mathcal{B}}(\hat{\nabla}_U \zeta, W) + g_{\mathcal{B}}(\hat{\nabla}_W \zeta, U)\} + \hat{S}(U, W) - g_{\mathcal{B}}(N, H)g_{\mathcal{B}}(U, W) \\ & + \sum_{j=r+1}^m [( \nabla_{X_j} g_{\mathcal{B}})(U, W)g_{\mathcal{B}}(H, X_j) - g_{\mathcal{B}}(\nabla_{X_j} H, X_j)g_{\mathcal{B}}(U, W) + g_{\mathcal{B}}(A_{X_j} U, A_{X_j} W)] + (\lambda - \mathcal{V}(\kappa)\rho)g_{\mathcal{B}}(U, W) = 0. \end{aligned} \quad (35)$$

As the totally geodesic foliation on  $\mathcal{B}$  is defined by horizontal distribution  $(\text{Ker } \mathcal{F}_*)^\perp$ , then in view of (21) &  $\mathbf{N} = r\mathbf{H}$  in (35), it provides

$$\frac{1}{2}(\mathcal{L}_\zeta g_{\mathcal{B}})(\mathbf{U}, \mathbf{W}) + \hat{\mathbf{S}}(\mathbf{U}, \mathbf{W}) - r g_{\mathcal{B}}(\mathbf{H}, \mathbf{H})g_{\mathcal{B}}(\mathbf{U}, \mathbf{W}) - \hat{\delta}(\mathbf{H})g_{\mathcal{B}}(\mathbf{U}, \mathbf{W}) + \left(\lambda - (\hat{\kappa} - r^2\|\mathbf{H}\|^2 - r\hat{\delta}(\mathbf{H}))\rho\right)g_{\mathcal{B}}(\mathbf{U}, \mathbf{W}) = 0. \quad (36)$$

From here, we have

$$\frac{1}{2}(\mathcal{L}_\zeta g_{\mathcal{B}})(\mathbf{U}, \mathbf{W}) + \hat{\mathbf{S}}(\mathbf{U}, \mathbf{W}) + (h - \hat{\kappa}\rho)g_{\mathcal{B}}(\mathbf{U}, \mathbf{W}) = 0, \quad (37)$$

where,  $h = \lambda + (r\|\mathbf{H}\|^2 + \hat{\delta}(\mathbf{H}))(r\rho - 1)$ . The above relation implies that any fiber of  $\mathcal{F}$  is an almost  $\mathcal{RBS}$ . This completes the proof.  $\square$

**Theorem 3.5.** Let  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$  be a totally geodesic  $\mathcal{RM}$  between Riemannian manifolds and  $(\mathcal{B}, \Lambda, \lambda, \rho, g_{\mathcal{B}})$  be an  $\mathcal{RBS}$  with the potential vector field  $\Lambda$ . In this case, if vector field  $\Lambda$  is vertical then  $\text{Range } \mathcal{F}_*$  is an Einstein, and if the vector field  $\Lambda$  is horizontal then the  $\text{Range } \mathcal{F}_*$  is  $\mathcal{RBS}$ .

*Proof.* Let us consider a Riemannian map  $\mathcal{F}$  with total space admits  $\mathcal{RBS}$ . So relation (2) & (17) together give

$$\begin{aligned} &\frac{1}{2}\{g_{\mathcal{B}}(\nabla_{\mathbf{X}}\Lambda, \mathbf{Y}) + g_{\mathcal{B}}(\nabla_{\mathbf{Y}}\Lambda, \mathbf{X})\} + \mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_*\mathbf{X}, \mathcal{F}_*\mathbf{Y}) \\ &+ \sum_{i=1}^r \{g_{\mathcal{B}}((\nabla_{\mathbf{X}}\mathbf{T})_{\mathbf{U}_i}\mathbf{U}_i, \mathbf{Y}) + g_{\mathcal{B}}((\nabla_{\mathbf{U}_i}\mathbf{A})_{\mathbf{X}}\mathbf{Y}, \mathbf{U}_i) - g_{\mathcal{B}}(\mathbf{T}_{\mathbf{U}_i}\mathbf{X}, \mathbf{T}_{\mathbf{U}_i}\mathbf{Y}) + g_{\mathcal{B}}(\mathbf{A}_{\mathbf{X}}\mathbf{U}_i, \mathbf{A}_{\mathbf{Y}}\mathbf{U}_i)\} + g_{\mathcal{N}}((\nabla_{\mathcal{F}_*})_{\mathbf{X}}\mathbf{Y}, \tau^{(\text{Ker } \mathcal{F}_*)^\perp}) \\ &- \sum_{j=r+1}^m g_{\mathcal{N}}((\nabla_{\mathcal{F}_*})_{\mathbf{X}_j}\mathbf{Y}, (\nabla_{\mathcal{F}_*})_{\mathbf{X}}\mathbf{X}_j) + (\lambda - \mathcal{H}(\kappa)\rho)g_{\mathcal{B}}(\mathbf{X}, \mathbf{Y}) = 0. \end{aligned} \quad (38)$$

Now, if the vector field  $\Lambda = \zeta$  is vertical and  $\mathcal{F}$  is totally geodesic, then relation (38) provides

$$\frac{1}{2}\{g_{\mathcal{B}}(\mathcal{H}(\nabla_{\mathbf{X}}\zeta), \mathbf{Y}) + g_{\mathcal{B}}(\mathcal{H}(\nabla_{\mathbf{Y}}\zeta), \mathbf{X})\} + \mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_*\mathbf{X}, \mathcal{F}_*\mathbf{Y}) + (\lambda - \kappa^{\text{Range } \mathcal{F}_*}\rho)g_{\mathcal{B}}(\mathbf{X}, \mathbf{Y}) = 0,$$

$\forall \mathbf{X}, \mathbf{Y} \in \Gamma((\text{Ker } \mathcal{F}_*)^\perp)$ .

As,  $\mathcal{F}$  is  $\mathcal{RM}$ , so relation (4) & above relation together give:

$$\frac{1}{2}\{g_{\mathcal{N}}(\mathcal{F}_*(\nabla_{\mathbf{X}}\zeta), \mathcal{F}_*\mathbf{Y}) + g_{\mathcal{N}}(\mathcal{F}_*(\nabla_{\mathbf{Y}}\zeta), \mathcal{F}_*\mathbf{X})\} + \mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_*\mathbf{X}, \mathcal{F}_*\mathbf{Y}) + (\lambda - \kappa^{\text{Range } \mathcal{F}_*}\rho)g_{\mathcal{B}}(\mathbf{X}, \mathbf{Y}) = 0. \quad (39)$$

In view of (12), relation (39) gives

$$\begin{aligned} &\frac{1}{2}\{g_{\mathcal{N}}(-(\nabla_{\mathcal{F}_*})_{\mathbf{X}}\zeta + \nabla_{\mathbf{X}}^{\mathcal{F}}\zeta, \mathcal{F}_*\mathbf{Y}) + g_{\mathcal{N}}(-(\nabla_{\mathcal{F}_*})_{\mathbf{Y}}\zeta + \nabla_{\mathbf{Y}}^{\mathcal{F}}\zeta, \mathcal{F}_*\mathbf{X})\} \\ &+ \mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_*\mathbf{X}, \mathcal{F}_*\mathbf{Y}) + (\lambda - \kappa^{\text{Range } \mathcal{F}_*}\rho)g_{\mathcal{N}}(\mathcal{F}_*\mathbf{X}, \mathcal{F}_*\mathbf{Y}) = 0. \end{aligned} \quad (40)$$

Because  $\mathcal{F}$  is totally geodesic and  $\zeta$  is vertical, therefore  $\mathcal{F}_*\zeta = 0$ . In view of this relation, the relation (40) yields

$$\mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_*\mathbf{X}, \mathcal{F}_*\mathbf{Y}) = -(\lambda - \kappa^{\text{Range } \mathcal{F}_*}\rho)g_{\mathcal{N}}(\mathcal{F}_*\mathbf{X}, \mathcal{F}_*\mathbf{Y}). \quad (41)$$

The relation (41) implies that  $\text{Range } \mathcal{F}_*$  is an Einstein manifold.

Moreover, if the vector field  $\Lambda = \mathbf{Z}$  is horizontal and  $\mathcal{F}$  is totally geodesic, then the relation (38) gives

$$\frac{1}{2}(\mathcal{L}_{\mathbf{Z}}g_{\mathcal{B}})(\mathbf{X}, \mathbf{Y}) + \mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_*\mathbf{X}, \mathcal{F}_*\mathbf{Y}) + (\lambda - \kappa^{\text{Range } \mathcal{F}_*}\rho)g_{\mathcal{B}}(\mathbf{X}, \mathbf{Y}) = 0,$$

Using (4) in the above equation, it yields

$$\frac{1}{2}\{g_N(\mathcal{F}_*(\nabla_X Z), \mathcal{F}_*Y) + g_N(\mathcal{F}_*(\nabla_Y Z), \mathcal{F}_*X)\} + \mathcal{S}^{\text{Range}\mathcal{F}_*}(\mathcal{F}_*X, \mathcal{F}_*Y) + (\lambda - \kappa^{\text{Range}\mathcal{F}_*}\rho)g_N(\mathcal{F}_*X, \mathcal{F}_*Y) = 0. \tag{42}$$

Setting (12) in the relation (42), it gives

$$\frac{1}{2}\{g_N(-(\nabla_{\mathcal{F}_*})X, Z) + \nabla_X \mathcal{F}_*Z, \mathcal{F}_*Y\} + g_N(-(\nabla_{\mathcal{F}_*})Y, Z) + \nabla_Y \mathcal{F}_*Z, \mathcal{F}_*X\} + \mathcal{S}^{\text{Range}\mathcal{F}_*}(\mathcal{F}_*X, \mathcal{F}_*Y) + (\lambda - \kappa^{\text{Range}\mathcal{F}_*}\rho)g_N(\mathcal{F}_*X, \mathcal{F}_*Y) = 0. \tag{43}$$

Relations (13) & (43) together give

$$\frac{1}{2}\{g_N(\nabla_X \mathcal{F}_*Z, \mathcal{F}_*Y) + g_N(\nabla_Y \mathcal{F}_*Z, \mathcal{F}_*X)\} + \mathcal{S}^{\text{Range}\mathcal{F}_*}(\mathcal{F}_*X, \mathcal{F}_*Y) + (\lambda - \kappa^{\text{Range}\mathcal{F}_*}\rho)g_N(\mathcal{F}_*X, \mathcal{F}_*Y) = 0. \tag{44}$$

This implies that  $\text{Range}\mathcal{F}_*$  is an  $\mathcal{RBS}$ .

This completes the proof.  $\square$

**Theorem 3.6.** Let  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$  be a totally geodesic  $\mathcal{RM}$  from a Riemannian manifold to an Einstein manifold, and  $(\mathcal{B}, \Lambda, \lambda, \rho, g_{\mathcal{B}})$  be an  $\mathcal{RBS}$  with the horizontal potential vector field  $\Lambda$ . The vector field  $\Lambda$  is then conformal on the horizontal distribution  $(\text{Ker}\mathcal{F}_*)^\perp$ .

*Proof.* Being  $(\mathcal{B}, \Lambda, \lambda, \rho, g_{\mathcal{B}})$  an  $\mathcal{RBS}$ , we have the following relation

$$\frac{1}{2}(\mathcal{L}_\Lambda g_{\mathcal{B}})(X, Y) + \mathcal{S}(X, Y) + (\lambda - \mathcal{H}(\kappa)\rho)g_{\mathcal{B}}(X, Y) = 0,$$

$\forall X, Y \in \Gamma((\text{Ker}\mathcal{F}_*)^\perp)$ .

Using (17) in the above relation, it provides

$$\begin{aligned} &\frac{1}{2}(\mathcal{L}_\Lambda g_{\mathcal{B}})(X, Y) + \mathcal{S}^{\text{Range}\mathcal{F}_*}(\mathcal{F}_*X, \mathcal{F}_*Y) \\ &+ \sum_{i=1}^r \{g_{\mathcal{B}}((\nabla_X T)_{U_i}, U_i, Y) + g_{\mathcal{B}}((\nabla_{U_i} A)_X, Y, U_i) - g_{\mathcal{B}}(T_{U_i} X, T_{U_i} Y) + g_{\mathcal{B}}(A_X U_i, A_Y U_i)\} \\ &- \sum_{j=r+1}^m g_N((\nabla_{\mathcal{F}_*})X_j, Y, (\nabla_{\mathcal{F}_*})X, X_j) + g_N((\nabla_{\mathcal{F}_*})X, Y, \tau^{(\text{Ker}\mathcal{F}_*)^\perp}) + (\lambda - \kappa^{\text{Range}\mathcal{F}_*}\rho)g_{\mathcal{B}}(X, Y) = 0, \end{aligned} \tag{45}$$

here,  $\{X_j\}$  denotes an orthonormal basis of  $(\text{Ker}\mathcal{F}_*)$ . Being  $\mathcal{F}$ , a totally geodesic  $\mathcal{RM}$ , then from relation (45) we have

$$\frac{1}{2}(\mathcal{L}_\Lambda g_{\mathcal{B}})(X, Y) + \mathcal{S}^{\text{Range}\mathcal{F}_*}(\mathcal{F}_*X, \mathcal{F}_*Y) + (\lambda - \kappa^{\text{Range}\mathcal{F}_*}\rho)g_{\mathcal{B}}(X, Y) = 0. \tag{46}$$

As  $\mathcal{F}$ , an  $\mathcal{RM}$  therefore relation (4) together with relation (46), we have

$$\frac{1}{2}(\mathcal{L}_\Lambda g_{\mathcal{B}})(X, Y) + \mathcal{S}^{\text{Range}\mathcal{F}_*}(\mathcal{F}_*X, \mathcal{F}_*Y) + (\lambda - \kappa^{\text{Range}\mathcal{F}_*}\rho)g_N(\mathcal{F}_*X, \mathcal{F}_*Y) = 0 \tag{47}$$

Since,  $(\mathcal{N}, g_{\mathcal{N}})$  is an Einstein manifold, therefore  $\text{Range}\mathcal{F}_*$  is also an Einstein. So,  $\mathcal{S}^{\text{Range}\mathcal{F}_*}(\mathcal{F}_*X, \mathcal{F}_*Y) = \nu g_N(\mathcal{F}_*X, \mathcal{F}_*Y)$ . Plugging this relation into (47), we find

$$(\mathcal{L}_\Lambda g_{\mathcal{B}})(X, Y) = -2(\nu + \lambda - \kappa^{\text{Range}\mathcal{F}_*}\rho)g_N(\mathcal{F}_*X, \mathcal{F}_*Y) \tag{48}$$

This shows that vector field  $\Lambda$  is conformal.

This completes the proof.  $\square$

#### 4. Harmonicity and Biharmonicity of Riemannian maps by Ricci-Bourguignon solitons

The harmonicity and biharmonicity of an  $\mathcal{R.M}$  from an  $\mathcal{R.B.S}$  to a Riemannian manifold are covered in this part. Now, we begin with the subsequent lemma:

**Lemma 4.1.** *Let  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$  be an umbilical  $\mathcal{R.M}$  between Riemannian manifolds such that distribution  $(\mathcal{F})^{\perp}$  defines a totally geodesic foliation on  $\mathcal{B}$ . Then,*

$$\mathcal{V}(\kappa) = \hat{\kappa} - \|\mathbf{N}\|^2 + \hat{\delta}(\mathbf{N}),$$

and

$$\mathcal{H}(\kappa) = \kappa^{\text{Range } \mathcal{F}_*} + (m - r)\|\mathbf{H}'\|^2 + (m - r)^2\|\mathbf{H}\|^2.$$

**Theorem 4.2.** *Let  $(\mathcal{B}, \Lambda, \lambda, \rho, g_{\mathcal{B}})$  be an  $\mathcal{R.B.S}$  with  $\Lambda$ , a horizontal potential vector field, and  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$  be an umbilical  $\mathcal{R.M}$  across Riemannian manifolds such that distribution  $(\text{Ker } \mathcal{F}_*)^{\perp}$  defines a totally geodesic foliation on  $\mathcal{B}$ . Then,  $\mathcal{F}$  is harmonic iff the subsequent requirements are met:*

- (i) The scalar curvature  $\frac{\lambda r}{r\rho - 1}$ , (where,  $r\rho \neq 1$ ) of fibers of  $\mathcal{F}$  is constant.
- (ii)  $\text{Range } \mathcal{F}_*$  possesses a constant scalar curvature  $\frac{(m - r)\lambda\rho}{(m - r)\rho - 1}$ , (where,  $(m - r)\rho \neq 1$ ).

*Proof.* Let us consider  $(\mathcal{B}, \Lambda, \lambda, \rho, g_{\mathcal{B}})$  be an  $\mathcal{R.B.S}$ , then from relation (2), we have

$$\frac{1}{2}\{g_{\mathcal{B}}(\nabla_{\mathbf{U}}\Lambda, \mathbf{W}) + g_{\mathcal{B}}(\mathbf{U}, \nabla_{\mathbf{W}}\Lambda)\} + \mathcal{S}(\mathbf{U}, \mathbf{W}) + (\lambda - \mathcal{V}(\kappa)\rho)g_{\mathcal{B}}(\mathbf{U}, \mathbf{W}) = 0, \tag{49}$$

for all  $\mathbf{U}, \mathbf{W} \in \Gamma(\text{Ker } \mathcal{F}_*)$ .

Trace of the relation (49) gives

$$\sum_{i=1}^r \left( \frac{1}{2} (g_{\mathcal{B}}(\nabla_{\mathbf{U}_i}\Lambda, \mathbf{U}_i) + g_{\mathcal{B}}(\mathbf{U}_i, \nabla_{\mathbf{U}_i}\Lambda)) + \mathcal{S}(\mathbf{U}_i, \mathbf{U}_i) + (\lambda - \mathcal{V}(\kappa)\rho)g_{\mathcal{B}}(\mathbf{U}_i, \mathbf{U}_i) \right) = 0, \tag{50}$$

where,  $\{\mathbf{U}_i\}_{i=1}^r$  is an orthonormal basis of  $\mathcal{F}_*$ . The relation (50) implies

$$\sum_{i=1}^r [g_{\mathcal{B}}(\nabla_{\mathbf{U}_i}\Lambda, \mathbf{U}_i) + \mathcal{S}(\mathbf{U}_i, \mathbf{U}_i) + (\lambda - \mathcal{V}(\kappa)\rho)g_{\mathcal{B}}(\mathbf{U}_i, \mathbf{U}_i)] = 0. \tag{51}$$

Plugging (16) into the relation (51), it yields

$$\sum_{i=1}^r [g_{\mathcal{B}}(\nabla_{\mathbf{U}_i}\Lambda, \mathbf{U}_i) + \hat{\mathcal{S}}(\mathbf{U}_i, \mathbf{U}_i) - g_{\mathcal{B}}(\mathbf{N}, \mathbf{T}_{\mathbf{U}_i}\mathbf{U}_i) + \sum_{j=r+1}^m \{g_{\mathcal{B}}((\nabla_{\mathbf{X}_j}\mathbf{T})_{\mathbf{U}_i}\mathbf{U}_i, \mathbf{X}_j) + g_{\mathcal{B}}(\mathbf{A}_{\mathbf{X}_j}\mathbf{U}_i, \mathbf{A}_{\mathbf{X}_j}\mathbf{U}_i)\} + (\lambda - \mathcal{V}(\kappa)\rho)g_{\mathcal{B}}(\mathbf{U}_i, \mathbf{U}_i)] = 0, \tag{52}$$

here,  $\{\mathbf{X}_j\}_{j=r+1}^m$  is an orthonormal base frame of  $(\text{Ker } \mathcal{F}_*)^{\perp}$ . Because,  $(\text{Ker } \mathcal{F}_*)^{\perp}$  defines totally geodesic foliation on  $\mathcal{B}$  therefore, the relation (52) with lemma (4.1), gives

$$\sum_{i=1}^r (\hat{\mathcal{S}}(\mathbf{U}_i, \mathbf{U}_i) + \sum_{j=r+1}^m g_{\mathcal{B}}((\nabla_{\mathbf{X}_j}\mathbf{T})_{\mathbf{U}_i}\mathbf{U}_i, \mathbf{X}_j) + (\lambda - \{\hat{\kappa} - \|\mathbf{N}\|^2 + \hat{\delta}(\mathbf{N})\}\rho)g_{\mathcal{B}}(\mathbf{U}_i, \mathbf{U}_i)) - g_{\mathcal{B}}(\mathbf{N}, \mathbf{N}) - g_{\mathcal{B}}(\mathbf{N}, \Lambda) = 0. \tag{53}$$

Plugging  $\hat{\kappa} = \sum_{i=1}^r \hat{S}(U_i, U_i) = \left(\frac{\lambda r}{r\rho - 1}\right)$  into relation (53), we obtain

$$\left(\frac{\lambda r}{r\rho - 1}\right) + \sum_{i=1}^r \sum_{j=r+1}^m g_{\mathcal{B}}((\nabla_{X_j} T)_{U_i} U_i, X_j) + \left(\lambda - \left\{\left(\frac{\lambda r}{r\rho - 1}\right) - \|N\|^2 + \delta(N)\right\}\rho\right)r - g_{\mathcal{B}}(N, N) - g_{\mathcal{B}}(N, \Lambda) = 0. \tag{54}$$

Relations (21) & (54) together give

$$-(r\rho - 1)\hat{\delta}(N) + g_{\mathcal{B}}((r\rho - 1)N - \Lambda, N) = 0 \quad \text{iff } N = 0. \tag{55}$$

This establishes that  $\text{Ker } \mathcal{F}_*$  is minimal. Therefore,  $\text{Ker } \mathcal{F}_*$  is totally geodesic. Now,  $(\mathcal{B}, \Lambda, \lambda, \rho, g_{\mathcal{B}})$  is an  $\mathcal{HBS}$ , so from relation (2), we have

$$\frac{1}{2}(\mathcal{L}_{\Lambda} g_{\mathcal{B}})(X, Y) + \mathcal{S}(X, Y) + (\lambda + \kappa\rho)g_{\mathcal{B}}(X, Y) = 0,$$

$\forall X, Y \in \Gamma((\text{Ker } \mathcal{F}_*)^\perp)$ . Plugging relation (17) into the foregoing relation, it gives

$$\begin{aligned} &\frac{1}{2} \left( g_N(\overset{\mathcal{F}}{\nabla}_X \mathcal{F}_* \Lambda, \mathcal{F}_* Y) + g_N(\overset{\mathcal{F}}{\nabla}_Y \mathcal{F}_* \Lambda, \mathcal{F}_* X) \right) + \mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_* X, \mathcal{F}_* Y) \\ &+ \sum_{i=1}^r (g_{\mathcal{B}}((\nabla_X T)_{U_i} U_i, Y) + g_{\mathcal{B}}((\nabla_{U_i} A)_X Y, U_i) - g_{\mathcal{B}}(T_{U_i} X, T_{U_i} Y) + g_{\mathcal{B}}(A_X U_i, A_Y U_i)) \\ &- \sum_{j=r+1}^m g_N((\nabla \mathcal{F}_*)(X_j, Y), (\nabla \mathcal{F}_*)(X, X_j)) + g_N((\nabla \mathcal{F}_*)(X, Y), \tau^{(\text{Ker } \mathcal{F}_*)^\perp}) + (\lambda - \mathcal{H}(\kappa)\rho)g_{\mathcal{B}}(X, Y) = 0, \end{aligned} \tag{56}$$

here,  $\{X_j\}_{j=r+1}^m$  is an orthonormal base frame of  $(\text{Ker } \mathcal{F}_*)^\perp$ . As,  $(\text{Ker } \mathcal{F}_*)^\perp$  &  $\text{Ker } \mathcal{F}_*$  both are totally geodesic, so relation (56) with lemma (4.1) gives the subsequent result

$$\begin{aligned} &\frac{1}{2} \left\{ g_N(\overset{\mathcal{F}}{\nabla}_X \mathcal{F}_* \Lambda, \mathcal{F}_* Y) + g_N(\overset{\mathcal{F}}{\nabla}_Y \mathcal{F}_* \Lambda, \mathcal{F}_* X) \right\} + \mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_* X, \mathcal{F}_* Y) \\ &- \sum_{j=r+1}^m g_N((\nabla \mathcal{F}_*)(X_j, Y), (\nabla \mathcal{F}_*)(X, X_j)) + g_N((\nabla \mathcal{F}_*)(X, Y), \tau^{(\text{Ker } \mathcal{F}_*)^\perp}) + (\lambda - \mathcal{H}(\kappa)\rho)g_{\mathcal{B}}(X, Y) = 0. \end{aligned} \tag{57}$$

Being  $\mathcal{F}$  is an umbilical  $\mathcal{RM}$ , so relations (14) & (57) together give

$$\begin{aligned} &\frac{1}{2} \left\{ g_N(\overset{\mathcal{F}}{\nabla}_X \mathcal{F}_* \Lambda, \mathcal{F}_* Y) + g_N(\overset{\mathcal{F}}{\nabla}_Y \mathcal{F}_* \Lambda, \mathcal{F}_* X) \right\} - \sum_{j=r+1}^m g_{\mathcal{B}}(X_j, Y)g_{\mathcal{B}}(X_j, X)g_N(H', H') \\ &+ \mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_* X, \mathcal{F}_* Y) + g_{\mathcal{B}}(X, Y)g_N(H', \tau^{(\text{Ker } \mathcal{F}_*)^\perp}) + (\lambda - \mathcal{H}(\kappa)\rho)g_N(\mathcal{F}_* X, \mathcal{F}_* Y) = 0. \end{aligned} \tag{58}$$

Plugging  $(m - r)H'$  in place of  $\tau^{(\text{Ker } \mathcal{F}_*)^\perp}$  [23] in relation (58), it provides

$$\begin{aligned} &\frac{1}{2} \left\{ g_N(\overset{\mathcal{F}}{\nabla}_X \mathcal{F}_* \Lambda, \mathcal{F}_* Y) + g_N(\overset{\mathcal{F}}{\nabla}_Y \mathcal{F}_* \Lambda, \mathcal{F}_* X) \right\} - g_{\mathcal{B}}(X, Y)g_N(H', H') + g_{\mathcal{B}}(X, Y)g_N(H', (m - r)H') \\ &+ \mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_* X, \mathcal{F}_* Y) + \left( \lambda - \left\{ \kappa^{\text{Range } \mathcal{F}_*} + (m - r)\|H'\|^2 + (m - r)^2\|H'\|^2 \right\} \rho \right) g_N(\mathcal{F}_* X, \mathcal{F}_* Y) = 0. \end{aligned} \tag{59}$$

Using  $\kappa^{\text{Range } \mathcal{F}_*} = \sum_{j=r+1}^m \mathcal{S}^{\text{Range } \mathcal{F}_*}(\mathcal{F}_* X_j, \mathcal{F}_* Y_j) = \left(\frac{(m - r)\lambda\rho}{(m - r)\rho - 1}\right)$  into (59), then taking trace of this, we have

$$(m - r)(m - r - 1 - (m - r)\rho - (m - r)^2\rho)\|H'\|^2 = 0. \tag{60}$$

Relation (60) implies that  $H' = 0$ . Then, relations (23), (55) & (60) together provide  $\tau(\mathcal{F}) = 0$ . Hence,  $\mathcal{F}$  is a harmonic map. This completes the proof.  $\square$

**Theorem 4.3.** Let  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}(c), g_{\mathcal{N}})$  be an  $\mathcal{R.M}$  from a Riemannian manifold to a space form such that the distribution  $(\text{Ker}\mathcal{F}_*)^\perp$  defines a totally geodesic foliation on  $\mathcal{B}$  and  $(\mathcal{B}, \Lambda, \lambda, \rho, g_{\mathcal{B}})$  be an  $\mathcal{R.B.S}$  with the horizontal potential vector field  $\Lambda$ . Then,  $\mathcal{F}$  is biharmonic iff fibers of  $\mathcal{F}$  have constant scalar curvature  $\frac{\lambda r}{r\rho - 1}$  and  $\text{Range}\mathcal{F}_*$  has constant scalar curvature  $\frac{(m-r)\lambda\rho}{(m-r)\rho - 1}$  hold.

*Proof.* Using the theorem (4.2), we can observe that  $\mathcal{F}$  has a minimal  $\text{Ker}\mathcal{F}_*$  if the fiber of  $\mathcal{F}$  has a constant scalar curvature  $\frac{\lambda r}{r\rho - 1}$ . The constant scalar curvature  $\frac{(m-r)\lambda\rho}{(m-r)\rho - 1}$  of  $\text{Range}\mathcal{F}_*$  implies that  $H' = 0$ . The proof is therefore finished since  $\mathcal{F}$  is biharmonic based on (24) & (25).  $\square$

**5. Illustration**

Let us consider two dimensional Riemannian manifold  $\mathcal{B} = \{(u_1, u_2) : 0 \neq u_1 \in \mathbb{R}, 0 \neq u_2 \in \mathbb{R}\}$  with metric  $g_{\mathcal{B}} = e^{2u_2} du_1^2 + du_2^2$ . We assume  $\mathcal{N} = \{(v_1, v_2, v_3) : v_1, v_2, v_3 \in \mathbb{R}\}$  with metric  $g_{\mathcal{N}} = e^{2v_3} dv_1^2 + e^{2v_3} dv_2^2 + dv_3^2$ . A map  $\mathcal{F} : (\mathcal{B}, g_{\mathcal{B}}) \rightarrow (\mathcal{N}, g_{\mathcal{N}})$  is defined by

$$\mathcal{F}(u_1, u_2) = \left( \frac{u_1 + u_2}{\sqrt{2}}, 0, 0 \right).$$

After evaluation of properties of  $\mathcal{F}$ , it yields

$$\text{Ker}\mathcal{F}_* = \text{Span}\{U = \mathcal{E}_1 - \mathcal{E}_2\},$$

and

$$(\text{Ker}\mathcal{F}_*)^\perp = \text{Span}\{X = \mathcal{E}_1 + \mathcal{E}_2\},$$

here,  $\{\mathcal{E}_1 = e^{-u_2} \frac{\partial}{\partial u_1}, \mathcal{E}_2 = \frac{\partial}{\partial u_2}\}$  is basis on  $\mathcal{T}_p\mathcal{B}$  and  $\{\mathcal{E}_1^* = e^{-v_3} \frac{\partial}{\partial v_1}, \mathcal{E}_2^* = e^{-v_3} \frac{\partial}{\partial v_2}, \mathcal{E}_3^* = \frac{\partial}{\partial v_3}\}$  is basis on  $\mathcal{T}_{\mathcal{F}(p)}\mathcal{N}$ , for all  $p \in \mathcal{B}$ .

On evaluation of  $\mathcal{F}_*X$ , it yields the following relations

$$\begin{aligned} \mathcal{F}_*(X) &= \sqrt{2}\mathcal{E}_1^*, \\ g_{\mathcal{B}}(X, X) &= g_{\mathcal{N}}(\mathcal{F}_*X, \mathcal{F}_*X), \end{aligned}$$

for  $X \in (\text{Ker}\mathcal{F}_*)^\perp$ . Therefore,  $\mathcal{F}$  is  $\mathcal{R.M}$  with range  $(\text{Range}\mathcal{F}_*)^\perp = \text{Span}\{\mathcal{E}_2^*, \mathcal{E}_3^*\}$ .

We represent Lie-brackets of E, F by  $[E, F]$ , defined as  $[E, F] = EF - FE$ . The non-vanishing constituents of Lie-brackets are evaluated as below:

$$[\mathcal{E}_1, \mathcal{E}_2] = \mathcal{E}_1, \quad [\mathcal{E}_2, \mathcal{E}_1] = -\mathcal{E}_1.$$

Let Riemannian connection with respect to  $g_{\mathcal{B}}$  be denoted by  $\nabla$ . So, by Koszul’s formula, we have the subsequent results:

$$\left. \begin{aligned} \nabla_{\mathcal{E}_1}\mathcal{E}_1 &= -\mathcal{E}_2, & \nabla_{\mathcal{E}_1}\mathcal{E}_2 &= \mathcal{E}_1, \\ \nabla_{\mathcal{E}_2}\mathcal{E}_1 &= 0, & \nabla_{\mathcal{E}_2}\mathcal{E}_2 &= 0. \end{aligned} \right\} \tag{61}$$

In view of relations(61), we have the following relations

$$\left. \begin{aligned} \nabla_U U &= \nabla_{\mathcal{E}_1 - \mathcal{E}_2}(\mathcal{E}_1 - \mathcal{E}_2) = -(\mathcal{E}_1 + \mathcal{E}_2) = -X, \\ \nabla_U X &= \nabla_{\mathcal{E}_1 - \mathcal{E}_2}(\mathcal{E}_1 + \mathcal{E}_2) = (\mathcal{E}_1 - \mathcal{E}_2) = U, \\ \nabla_X U &= \nabla_{\mathcal{E}_1 + \mathcal{E}_2}(\mathcal{E}_1 - \mathcal{E}_2) = -(\mathcal{E}_1 + \mathcal{E}_2) = -X, \\ \nabla_X X &= \nabla_{\mathcal{E}_1 + \mathcal{E}_2}(\mathcal{E}_1 + \mathcal{E}_2) = (\mathcal{E}_1 - \mathcal{E}_2) = U, \end{aligned} \right\} \tag{62}$$

Now, from relations (62), decomposing  $\nabla_U U, \nabla_U X, \nabla_X U$  &  $\nabla_X X$  into horizontal and vertical components together with relations (6), (7), (8) & (9), we find

$$\left. \begin{aligned} \mathcal{H}(\nabla_U U) &= -X, & \mathcal{V}(\nabla_U U) &= 0, & T_U U &= -X \\ \mathcal{H}(\nabla_U X) &= 0, & \mathcal{V}(\nabla_U X) &= U, & T_U X &= U \\ \mathcal{H}(\nabla_X U) &= -X, & \mathcal{V}(\nabla_X U) &= 0, & A_X U &= -X \\ \mathcal{H}(\nabla_X X) &= 0, & \mathcal{V}(\nabla_X X) &= U, & A_X X &= U. \end{aligned} \right\} \tag{63}$$

Now, we know that the  $\mathcal{RB}\mathcal{S}$  is defined by the following relation:

$$\frac{1}{2}(\mathcal{L}_\Lambda g_{\mathcal{B}})(E, F) + \mathcal{S}(E, F) + (\lambda + \kappa\rho)g_{\mathcal{B}}(E, F) = 0, \tag{64}$$

for any  $E, F, \Lambda \in \Gamma(\mathcal{TB})$ . Since, dimensions of  $\text{Ker } \mathcal{F}_*$  and  $(\text{Ker } \mathcal{F}_*)^\perp$  are one, so we can decompose  $E, F$ , and  $\Lambda$  such that:

$$\left. \begin{aligned} E &= \alpha U + aX, \\ F &= \beta U + bX, \\ \Lambda &= \gamma U + cX, \end{aligned} \right\} \tag{65}$$

where  $U$  and  $X$  denote components of  $\text{Ker } \mathcal{F}_*$  and  $(\text{Ker } \mathcal{F}_*)^\perp$ , respectively, and  $\alpha, \beta, \gamma, a, b, c \in \mathbb{R}$  are some scalars.

Now, we have the relation

$$\frac{1}{2}(\mathcal{L}_\Lambda g_{\mathcal{B}})(E, F) = \frac{1}{2}[g_{\mathcal{B}}(\nabla_E \Lambda, F) + g_{\mathcal{B}}(\nabla_E \Lambda, E)]. \tag{66}$$

Putting values of  $E, F$  &  $\Lambda$  from the relations (65) into (66), we have

$$\frac{1}{2}(\mathcal{L}_\Lambda g_{\mathcal{B}})(E, F) = \frac{1}{2}[g_{\mathcal{B}}(\nabla_{\alpha U + aX}(\gamma U + cX), \beta U + bX) + g_{\mathcal{B}}(\nabla_{\beta U + bX}(\gamma U + cX), \alpha U + aX)]. \tag{67}$$

On simplification, relations (67) give

$$\begin{aligned} \frac{1}{2}(\mathcal{L}_\Lambda g_{\mathcal{B}})(E, F) &= \frac{1}{2}[2\alpha\beta\gamma g_{\mathcal{B}}(\nabla_U U, U) + 2\alpha\beta c g_{\mathcal{B}}(\nabla_U X, U) + a\gamma\beta g_{\mathcal{B}}(\nabla_X U, U) + a\beta c g_{\mathcal{B}}(\nabla_X X, U) + \alpha b\gamma g_{\mathcal{B}}(\nabla_U U, X) \\ &\quad + \alpha b c g_{\mathcal{B}}(\nabla_U X, X) + 2ab\gamma g_{\mathcal{B}}(\nabla_X U, X) + abc g_{\mathcal{B}}(\nabla_X X, X) + ab\gamma g_{\mathcal{B}}(\nabla_X U, U) + abc g_{\mathcal{B}}(\nabla_X X, U) \\ &\quad + a\beta\gamma g_{\mathcal{B}}(\nabla_U U, X) + a\beta c g_{\mathcal{B}}(\nabla_U X, X) + abc g_{\mathcal{B}}(\nabla_X X, X)]. \end{aligned} \tag{68}$$

Plugging the values of relations (62) into relation (68), we have

$$\frac{1}{2}(\mathcal{L}_\Lambda g_{\mathcal{B}})(E, F) = -a\beta\gamma + 2\alpha\beta c + a\beta c + abc - 2ab\gamma - ab\gamma. \tag{69}$$

We evaluate the values of  $g_{\mathcal{B}}$  &  $\mathcal{S}$  from the relations (65), we have

$$g_{\mathcal{B}}(E, F) = g_{\mathcal{B}}(\alpha U + aX, \beta U + bX) = 2(\alpha\beta + ab), \tag{70}$$

and

$$\mathcal{S}(E, F) = \alpha\beta\mathcal{S}(U, U) + ab\mathcal{S}(X, X) + (ab + a\beta)\mathcal{S}(X, U). \tag{71}$$

As,  $\dim(\text{Ker } \mathcal{F}_*) = 1$ , so  $\hat{\mathcal{S}}(U, U) = 0$ . Using this relation together with (62) & (63) into (16), we get:

$$\mathcal{S}(U, U) = 0. \tag{72}$$

Because, dimension of  $\text{Range } \mathcal{F}_* = 1$ , therefore  $\mathcal{S}^{\text{Range } \mathcal{F}_*} = 0$ . Now, using this relation together with (62) & (63) into (17), we get:

$$S(\mathbf{x}, \mathbf{x}) = 0. \quad (73)$$

Further, relations (18), (62) & (63) together give

$$S(\mathbf{x}, \mathbf{U}) = 0. \quad (74)$$

Plugging (72), (73) & (74) into (71), it gives

$$S(\mathbf{E}, \mathbf{F}) = 0. \quad (75)$$

Contracting relation (75), it provides

$$\kappa = 0. \quad (76)$$

Thus, plugging (69),(70),(75) & (76) into (64), it yields

$$\lambda = \frac{1}{2(\alpha\beta + ab)} \left[ -a\beta\gamma + 2\alpha\beta c + a\beta c + abc - 2ab\gamma - ab\gamma \right], \quad (77)$$

where  $\alpha\beta \neq -ab$ . Since  $\alpha, \beta, \gamma, a, b, c \in \mathbb{R}$ , then  $\mathcal{RBS}$   $(\mathcal{B}, \lambda, \Lambda, \rho, g_{\mathcal{B}})$  is shrinking, expanding, or steady according to  $\lambda < 0$ ,  $\lambda > 0$ , or  $\lambda = 0$ .

From the foregoing discussions, we arrive to the conclusion that the total manifold admits  $\mathcal{RBS}$ .

## References

- [1] J. P. Bourguignon, *Ricci Curvature and Einstein Metrics*, Global Differential Geometry and Global Analysis, Springer Berlin Heidelberg (1981), 42–63.
- [2] G. Catino, L. Mazzieri, *Gradient Einstein solitons*, Nonlinear Anal. **132** (2016), 66–94.
- [3] G. Catino, L. Mazzieri, S. Mongodi, *Rigidity of gradient Einstein shrinkers*, Commun. Contemp. Math. **17** (2015), 1550046, 18.
- [4] S. K. Chaubey, M. D. Siddiqi, D. G. Prakasha, *Invariant submanifolds of hyperbolic Sasakian manifolds and  $\eta$ -Ricci-Bourguignon solitons*, Filomat **36** (2022), 409–421.
- [5] B.-Y. Chen, M. D. Siddiqi, A. N. Siddiqui, *On Ricci-Bourguignon solitons for statistical submersions*, Bull. Korean Math. Soc. **62** (2025), 91–110.
- [6] A. W. Cunha, R. Lemos, F. Roing, *On Ricci-Bourguignon solitons: triviality, uniqueness and scalar curvature estimates*, J. Math. Anal. Appl. **526** (2023), Paper No. 127333, 16.
- [7] U. C. De, C. A. Mantica, S. Shenawy, B. Ünal, *Ricci solitons on singly warped product manifolds and applications*, J. Geom. Phys. **166** (2021), Paper No. 104257, 10.
- [8] U. C. De, K. De, *K-Ricci-Bourguignon almost solitons*, Int. Electron. J. Geom. **17** (2024), 63–71.
- [9] M. Falcitelli, S. Ianus, A. M. Pastore, *Riemannian submersions and related topics*, World Scientific Publishing Co., Inc., River Edge, NJ, 2004, xiv+277.
- [10] A. E. Fischer, *Riemannian maps between Riemannian manifolds*, Mathematical aspects of classical field theory (Seattle, WA, 1991), Contemp. Math. **132** (1992), 331–366.
- [11] A. E. Fischer, *An introduction to conformal Ricci flow*, Classical and Quantum Gravity **21** (2004), S171–S218.
- [12] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982), 255–306.
- [13] R. S. Hamilton, *The Ricci Flow on Surfaces*, Mathematics and General Relativity (1988), 237–262.
- [14] S. Kumar, and R. Prasad *Semi-slant Riemannian maps from cosymplectic manifolds into Riemannian manifolds*, Gulf J. Math. **9** (2020), 62–80.
- [15] S. Kumar, R. Prasad, P. K. Singh, *CSI- $\xi^\perp$ -Riemannian maps from Kenmotsu manifolds to Riemannian manifolds*, Gulf J. Math. **15** (2023), 96–108.
- [16] S. Kumar, R. Prasad, S. K. Verma, *Hemi-slant Riemannian Submersions from Cosymplectic Manifolds*, Advanced Studies: Euro-Tbilisi Mathematical Journal **15** (2022).
- [17] S. E. Meriç, E. Kiliç, *Riemannian submersions whose total manifolds admit a Ricci soliton*, Int. J. Geom. Methods Mod. Phys. **16** (2019), 1950196, 12.
- [18] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.
- [19] S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti, *Ricci almost solitons*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **10** (2011), 757–799.
- [20] R. Prasad, A. Verma, and V S Yadav, *Study of  $\eta$ -Ricci-Bourguignon solitons on  $\xi$ -Conformally flat Lorentzian para-Kenmotsu manifolds*, Commun. Korean Math. Soc., **40**, no. 4 (2025), 869–885.
- [21] R. Prasad, S. Pandey, *Slant Riemannian maps from an almost contact manifold*, Filomat **31** (2017), 3999–4007.

- [22] B. Şahin, *Invariant and anti-invariant Riemannian maps to Kähler manifolds*, Int. J. Geom. Methods Mod. Phys. **7** (2010), 337–355.
- [23] B. Şahin, *Conformal Riemannian maps between Riemannian manifolds, their harmonicity and decomposition theorems*, Acta Appl. Math. **109** (2010), 829–847.
- [24] B. Şahin, *Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications*, Elsevier/Academic Press, London, 2017, xvii+342.
- [25] B. Şahin, *Biharmonic Riemannian maps*, Ann. Polon. Math. **102** (2011), 39–49.
- [26] B. Şahin, *Semi-invariant Riemannian maps to Kähler manifolds*, Int. J. Geom. Methods Mod. Phys. **8** (2011), 1439–1454.
- [27] A. Yadav, K. Meena, *Riemannian maps whose total manifolds admit a Ricci soliton*, J. Geom. Phys. **168** (2021), Paper No. 104317, 13.