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Perturbation bounds and properties for the inverse along a matrix

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Abstract. Let $A \in \mathbb{C}_{m,n}$ and $D \in \mathbb{C}_{n,m}$, we present two different expressions of $A^{\parallel D}$ by using the singular value decomposition and the Schur decomposition and establish several criteria for the existence of $A^{\parallel D}$. If $E \in \mathbb{C}_{m,n}$ is a perturbation matrix, we present the bound of $\|(A+E)^{\parallel D} - A^{\parallel D}\|_2$. Also, the perturbation bounds of $\|A^{\parallel D+T} - A^{\parallel D}\|_2$ and $\|(A+E)^{\parallel D+T} - A^{\parallel D}\|_2$ are given for some $T \in \mathbb{C}_{n,m}$. Finally, we estimate the upper bounds of $\|B^{\parallel C} - A^{\parallel D}\|_F$ provided that $B^{\parallel C}$ and $A^{\parallel D}$ both exist.

1. Introduction

Throughout this paper, $\mathbb{C}_{m,n}$ denotes the set of all $m \times n$ complex matrices. Moreover, I_n stands for the identity matrix of order n. By $\mathrm{rk}(A)$ and A^* we denote the rank of $A \in \mathbb{C}_{m,n}$ and the conjugate transpose of A, respectively. For $A \in \mathbb{C}_{m,n}$, we denote by $\|A\|$, $\|A\|_2$ and $\|A\|_F$ the norm, the spectral norm and the Frobenius norm of A. The symbol $\mathrm{tr}(P)$ stands for trace of a matrix $P \in \mathbb{C}^{n\times n}$. For the matrix A, we know that $\|A\|_2 = max\{|\sigma| : \sigma \text{ is a singular value of } A\}$ and $\|A\|_F = \sqrt{\mathrm{tr}(AA^*)}$.

For a matrix $A \in \mathbb{C}_{m,n}$, the range and the null space of A are respectively defined as $\mathcal{R}(A) = \{Ax : x \in \mathbb{C}^n\} \subseteq \mathbb{C}^m$ and $\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\} \subseteq \mathbb{C}^n$. For any $M \in \mathbb{C}_{m,n}$ and $N \in \mathbb{C}_{m,s}$, recall that $\mathcal{R}(M) \subseteq \mathcal{R}(N)$ if and only if there is some $X \in \mathbb{C}_{s,n}$ such that $M = NX$. Given any $P \in \mathbb{C}_{l,n}$ and $Q \in \mathbb{C}_{m,n}$, $\mathcal{N}(P) \subseteq \mathcal{N}(Q)$ if and only if there exists some $Y \in \mathbb{C}_{m,l}$ such that $Q = YP$.

Let $A \in \mathbb{C}_{m,n}$. The Moore–Penrose inverse of A [21] is the unique matrix $X \in \mathbb{C}_{n,m}$ satisfying $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$. Such a matrix X is usually denoted by $A^\dagger$.

By $\mathrm{ind}(A)$, the index of A, we denote the smallest nonnegative integer k such that $\mathrm{rk}(A^k) = \mathrm{rk}(A^{k+1})$. For any $A \in \mathbb{C}_{n,n}$ with $\mathrm{ind}(A) = k$, the Drazin inverse of A [10], denoted by $A^D \in \mathbb{C}_{n,n}$, is the unique solution to the system of $A^k = A^{k+1}A^D$, $A^DAA^D = A^D$ and $AA^D = A^DA$. If $\mathrm{ind}(A) = 1$, then A is called group invertible, the group inverse of A is denoted by $A^\#$. It is clear that $A^\#$ exists if and only if $\mathrm{rk}(A) = \mathrm{rk}(A^2)$.

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In 2011, the inverse along an element was introduced by Mary [15] in a semigroup. Several writers investigated this new generalized inverses in different settings. For instance, Mary and Patrício [16] in 2013 established the criterion for the inverse along an element in semigroups and rings. In 2017, Benítez et al. [6] redefined and investigated the inverse along an element in the context of rectangular matrices. For any $A \in \mathbb{C}_{m,n}$ and $D \in \mathbb{C}_{n,m}$, the matrix A is called invertible along D if there exists some $X \in \mathbb{C}_{n,m}$ such that $XAD = D = DAX$, $\mathcal{N}(D) \subseteq \mathcal{N}(X)$ and $\mathcal{R}(X) \subseteq \mathcal{R}(D)$. Such an X is called the inverse of A along D. The inverse of A along D is unique if it exists and is denoted by $A^{\|D}$. It was shown in [6] that $A^{\|D}$ exists if and only if $\mathrm{rk}(D) = \mathrm{rk}(DAD)$, moreover, $A^{\|D} = D(DAD)^{\dagger}D$. We refer to [4, 5, 14] for more details on the inverse along an element.

It is well known that perturbation theory is an important content in generalized inverse theory. Many authors considered numerous theoretical results on the perturbation analysis of generalized inverses. For instance, Ben-Israel [2] established an upper bound for $\frac{\|(A+E)^{\dagger}-A^{\dagger}\|}{\|A^{\dagger}\|}$. Afterwards, much effort has been made for estimating the perturbation bounds of the Moore–Penrose inverse, see [17–20, 24, 25] for more details. Rong in [22] presented an explicit upper bound of $\frac{\|(A+E)^{D}-A^{D}\|}{\|A^{D}\|}$ under certain circumstances. Later, Wei et al. in [26–29] established perturbation bounds for Drazin inverses. The general perturbation and acute perturbation for the group inverse were studied in [11, 12, 23, 26]. In 2018, Ma in [13] considered perturbation analysis for the core inverse, introduced by Baksalary and Trenkler in [1].

The inverse along an element is a type of uniquely-defined outer generalized inverse, which encompass Moore–Penrose inverses, Drazin inverses and core inverses. It is natural to consider perturbation bounds for the inverse along an element.

The main purpose of this article is to investigate perturbation bounds and properties for the inverse along a matrix and to improve perturbation theory of some generalized inverses. The article is organized as follows. For $A \in \mathbb{C}_{m,n}$ and $D \in \mathbb{C}_{n,m}$, in Section 2, we present the formula of $A^{\|D}$ by using the singular value decomposition and the Schur decomposition and give the characterizations for the existence of $A^{\|D}$. Let $E \in \mathbb{C}_{m,n}$ be a perturbation matrix. We investigate the problem of bounding of $\|(A+E)^{\|D} - A^{\|D}\|_2$. In addition, the explicit bounds of $\|(A+E)^{\|D}(A+E) - A^{\|D}A\|_2$ and $\|(A+E)(A+E)^{\|D} - AA^{\|D}\|_2$ are given by Theorem 2.4. Let $T \in \mathbb{C}_{n,m}$ be a perturbation matrix. We study the upper bound of $\|A^{\|D+T} - A^{\|D}\|_2$. In general, the perturbation bound of $\|(A+E)^{\|D+T} - A^{\|D}\|_2$ is investigated in Theorem 2.14, generalizing the previous result on Moore–Penrose inverses and Drazin inverses. As a corollary, we give the sufficient condition for the continuity of the inverse along a matrix. Finally, when $B^{\|C}$ and $A^{\|D}$ both exist, the bound of $\|B^{\|C} - A^{\|D}\|_F$ is given.

2. Properties of the inverse along a matrix

In what follows, we assume $A \in \mathbb{C}_{m,n}$ and $D \in \mathbb{C}_{n,m}$. First, some auxiliary lemmas are given.

Lemma 2.1. [3] *Let $A \in \mathbb{C}_{n,m}$ and $\mathrm{rk(A)} = r > 0$. The singular value decomposition of A is given by*

$$A = U\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}V^*, \tag{1}$$

where $U \in \mathbb{C}_{n,n}$ and $V \in \mathbb{C}_{m,m}$ are unitary matrices i.e. $U^*U = I_n$ and $V^*V = I_m$. $\Sigma = diag(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \sigma_3 I_{r_3}, \cdots, \sigma_t I_{r_t})$ is a diagonal matrix, the σ_i is singular values of A and $\sigma_1 > \sigma_2 > \sigma_3 > \cdots > \sigma_t$, $r_1 + r_2 + r_3 + \cdots + r_t = r$. It is well known that

$$A^{\dagger} = V\begin{bmatrix}\Sigma^{-1} & 0\\ 0 & 0\end{bmatrix}U^*.$$

We give following result that is helpful to consider the inverse of $A+E$ along D, where $A, E \in \mathbb{C}_{m,n}$.

Proposition 2.2. *Let $A \in \mathbb{C}_{m,n}$ and $D \in \mathbb{C}_{n,m}$ such that $\mathrm{rk(DAD)} = r > 0$. Then $A^{\|D}$ exists if and only if there exist two unitary matrices $U \in \mathbb{C}_{n,n}$ and $V \in \mathbb{C}_{m,m}$ such that $D = U\begin{bmatrix}D_1 & 0\\ 0 & 0\end{bmatrix}V^*$ and $A = V\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}U^*$, where $D_1, A_1 \in \mathbb{C}_{r,r}$ are invertible. Moreover, $A^{\|D} = U\begin{bmatrix}A_1^{-1} & 0\\ 0 & 0\end{bmatrix}V^*$.*

Proof. Suppose first $A^{\|D}$ exists and DAD can be partitioned as (2.1). Then we have

$$(DAD)^{\dagger} = V\begin{bmatrix}\Sigma^{-1} & 0\\ 0 & 0\end{bmatrix}U^*,$$

thus

$$(DAD)^{\dagger}DAD = V\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}V^*$$

and

$$DAD(DAD)^{\dagger} = U\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}U^*.$$

Since $A^{\|D} = D(DAD)^{\dagger}D$, one has

$$D = A^{\|D}AD = D(DAD)^{\dagger}DAD = U\begin{bmatrix}D_1 & 0\\ D_3 & 0\end{bmatrix}V^*.$$

Note also that

$$D = DAA^{\|D} = DAD(DAD)^{\dagger}D = U\begin{bmatrix}D_1 & D_2\\ 0 & 0\end{bmatrix}V^*.$$

Then $D = U\begin{bmatrix}D_1 & 0\\ 0 & 0\end{bmatrix}V^*$. By $A = V\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}U^*$, one has

$$DAD = U\begin{bmatrix}D_1A_1D_1 & 0\\ 0 & 0\end{bmatrix}V^*,$$

so $D_1A_1D_1 = \Sigma$. As rk(D) = rk(DAD) = r, we have rk(D_1) = r, and hence D_1 and A_1 are invertible. So

$$(DAD)^{\dagger} = V\begin{bmatrix}\Sigma^{-1} & 0\\ 0 & 0\end{bmatrix}U^* = V\begin{bmatrix}D_1^{-1}A_1^{-1}D_1^{-1} & 0\\ 0 & 0\end{bmatrix}U^*.$$

Whence

$$A^{\|D} = D(DAD)^{\dagger}D = U\begin{bmatrix}A_1^{-1} & 0\\ 0 & 0\end{bmatrix}V^*.$$

Conversely, given A and D as the form, it follows that

$$(DAD)^{\dagger} = V\begin{bmatrix}D_1^{-1}A_1^{-1}D_1^{-1} & 0\\ 0 & 0\end{bmatrix}U^*,$$

so that

$$(DAD)^{\dagger}DAD = V\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}V^*$$

and

$$DAD(DAD)^{\dagger} = U\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}U^*.$$

If $D = U\begin{bmatrix}D_1 & 0\\ 0 & 0\end{bmatrix}V^*$, then we have

$$D(DAD)^{\dagger}DAD = U\begin{bmatrix}D_1 & 0\\ 0 & 0\end{bmatrix}V^* = D$$

and

$$DAD(DAD)^{\dagger}D = U\begin{bmatrix}D_1 & 0\\ 0 & 0\end{bmatrix}V^* = D.$$

Let $X = D(DAD)^{\dagger}D$. We can obtain $X = A^{\|D}$ by $\mathcal{R}(X) \subseteq \mathcal{R}(D)$ and $\mathcal{N}(D) \subseteq \mathcal{N}(X)$. Consequently, $A^{\|D}$ exists. $\square$

Proposition 2.3. *Let $A, E \in \mathbb{C}_{m,n}$ and $D \in \mathbb{C}_{n,m}$ such that $A^{\|D}$ exists. If $B = A + E$, then the following statements are equivalent*:

(i) *$B^{\|D}$ exists.*

(ii) *$I_n + A^{\|D}E$ is invertible.*

(iii) *$I_m + EA^{\|D}$ is invertible.*

Moreover, $B^{\|D} = (I_n + A^{\|D}E)^{-1}A^{\|D} = A^{\|D}(I_m + EA^{\|D})^{-1}$.

Proof. Since $A^{\|D}$ exists, there exist two unitary matrices $U \in \mathbb{C}_{n,n}$ and $V \in \mathbb{C}_{m,m}$ such that

$$D = U\begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}V^*,\ A = V\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}U^* \text{ and } A^{\|D} = U\begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*,$$

where $D_1, A_1 \in \mathbb{C}_{r,r}$ are invertible. We assume that $E = V\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}U^*$ with $E_1 \in \mathbb{C}_{r,r}$. Then $B = A + E = V\begin{bmatrix} A_1 + E_1 & A_2 + E_2 \\ A_3 + E_3 & A_4 + E_4 \end{bmatrix}U^*$.

(i) ⇒ (ii) As $B^{\|D}$ exists, it follows by Proposition 2.2 that $A_1 + E_1$ is invertible. Since A_1 is invertible and $A_1 + E_1 = A_1(I_r + A_1^{-1}E_1)$, we deduce that $I_r + A_1^{-1}E_1$ is invertible. Note that

$$I_n + A^{\|D}E = U\begin{bmatrix} I_r + A_1^{-1}E_1 & A_1^{-1}E_2 \\ 0 & I_{n-r} \end{bmatrix}U^*. \tag{2}$$

Then $I_n + A^{\|D}E$ is invertible.

(ii) ⇒ (iii) Given (ii), by (2), we can obtain that $I_r + A_1^{-1}E_1$ is invertible. Note that $A_1(I_r + A_1^{-1}E_1) = A_1 + E_1 = (I_r + E_1A_1^{-1})A_1$. Then $I_r + E_1A_1^{-1}$ is invertible. Also, from

$$I_m + EA^{\|D} = V\begin{bmatrix} I_r + E_1A_1^{-1} & 0 \\ E_3A_1^{-1} & I_{m-r} \end{bmatrix}V^*, \tag{3}$$

we deduce that $I_m + EA^{\|D}$ is invertible.

(iii) ⇒ (i) As $I_r + E_1A_1^{-1}$ is invertible, then $A_1 + E_1$ is invertible, so that $B^{\|D}$ exists by Proposition 2.2.

From Proposition 2.2, we have

$$B^{\|D} = U\begin{bmatrix} (A_1 + E_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*.$$

So

$$B^{\|D} = U\begin{bmatrix} (I_r + A_1^{-1}E_1)^{-1}A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^* = U\begin{bmatrix} A_1^{-1}(I_r + E_1A_1^{-1})^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*.$$

From the formula (2.2), we have

$$(I_n + A^{\|D}E)^{-1} = U\begin{bmatrix} (I_r + A_1^{-1}E_1)^{-1} & -(A_1 + E_1)^{-1}E_2 \\ 0 & I_{n-r} \end{bmatrix}U^*.$$

Then

$$(I_n + A^{\|D}E)^{-1}A^{\|D} = U\begin{bmatrix} (I_r + A_1^{-1}E_1)^{-1}A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^* = B^{\|D}.$$

Similarly, we have $B^{\|D} = A^{\|D}(I_m + EA^{\|D})^{-1}$. □

Recall that [9] if $F \in \mathbb{C}_{n,n}$ and $\|F\|_F < 1$ then $I_n + F$ is invertible. We claim the implication that $\|A^{\|D}E\|_F < 1$ leads to $I_n + A^{\|D}E$ is invertible. Jacobson's Lemma guarantees that $I_m + EA^{\|D}$ is invertible.

Next, we present the bound for $\|B^{\|D}\|_F$, $\|B^{\|D} - A^{\|D}\|_F$, $\|BB^{\|D} - AA^{\|D}\|_2$ and $\|B^{\|D}B - A^{\|D}A\|_F$ under the condition $\|A^{\|D}E\|_F < 1$.

Theorem 2.4. *Let $A, E \in \mathbb{C}_{m,n}$ and $D \in \mathbb{C}_{n,m}$ such that $A^{\|D}$ exists. If $B = A + E$ and $\|A^{\|D}E\|_F < 1$, then*

(i) $\frac{\|A^{\|D}\|_F}{1+\|A^{\|D}E\|_F} \leq \|B^{\|D}\|_F \leq \frac{\|A^{\|D}\|_F}{1-\|A^{\|D}E\|_F}$.

(ii) $\|B^{\|D} - A^{\|D}\|_F \leq \frac{\|A^{\|D}\|_F\|A^{\|D}E\|_F}{1-\|A^{\|D}E\|_F}$.

(iii) $\|B^{\|D}B - A^{\|D}A\|_F \leq \frac{\|I_n-A^{\|D}A\|_F\|A^{\|D}E\|_F}{1-\|A^{\|D}E\|_F}$.

(iv) $\|BB^{\|D} - AA^{\|D}\|_F \leq \|I_m - AA^{\|D}\|_F\|EA^{\|D}\|_F\|(I_m + EA^{\|D})^{-1}\|_F$.

Proof. (i) It follows by Proposition 2.3 that $B^{\|D}$ exists and $B^{\|D} = (I_n + A^{\|D}E)^{-1}A^{\|D}$, then $\|A^{\|D}\|_F = \|(I_n + A^{\|D}E)B^{\|D}\|_F \leq (1 + \|A^{\|D}E\|_F)\|B^{\|D}\|_F$, so that

$$\frac{\|A^{\|D}\|_F}{1 + \|A^{\|D}E\|_F} \leq \|B^{\|D}\|_F.$$

Note that

$$(I_n + A^{\|D}E)^{-1} = I_n - (I_n + A^{\|D}E)^{-1}A^{\|D}E. \tag{4}$$

Then we have $\|(I_n + A^{\|D}E)^{-1}\|_F \leq 1 + \|(I_n + A^{\|D}E)^{-1}\|_F\|A^{\|D}E\|_F$, hence $\|(I_n + A^{\|D}E)^{-1}\|_F \leq \frac{1}{1-\|A^{\|D}E\|_F}$. Therefore, one has

$$\|B^{\|D}\|_F = \|(I_n + A^{\|D}E)^{-1}A^{\|D}\|_F \leq \frac{\|A^{\|D}\|_F}{1 - \|A^{\|D}E\|_F}.$$

(ii) By the equality (4), we have

$$B^{\|D} = (I_n + A^{\|D}E)^{-1}A^{\|D} = A^{\|D} - (I_n + A^{\|D}E)^{-1}A^{\|D}EA^{\|D}.$$

Hence,

$$\|B^{\|D} - A^{\|D}\|_F = \| - (I_n + A^{\|D}E)^{-1}A^{\|D}EA^{\|D}\|_F \leq \frac{\|A^{\|D}\|_F\|A^{\|D}E\|_F}{1 - \|A^{\|D}E\|_F}.$$

(iii) As $B = A + E$, then $B^{\|D}B = (I_n + A^{\|D}E)^{-1}A^{\|D}A + (I_n + A^{\|D}E)^{-1}A^{\|D}E$. Again by (4), one has

$$B^{\|D}B = A^{\|D}A - (I_n + A^{\|D}E)^{-1}A^{\|D}EA^{\|D}A + (I_n + A^{\|D}E)^{-1}A^{\|D}E. \tag{5}$$

Consequently, we obtain

$$\|B^{\|D}B - A^{\|D}A\|_F \leq \frac{\|I_n-A^{\|D}A\|_F\|A^{\|D}E\|_F}{1-\|A^{\|D}E\|_F}.$$

(iv) It is shown in Proposition 2.3 that

$$B^{\|D} = A^{\|D}(I_m + EA^{\|D})^{-1}.$$

These equalities $B = A + E$ and $(I_m + EA^{\|D})^{-1} = I_m - EA^{\|D}(I_m + EA^{\|D})^{-1}$ lead to

$$BB^{\|D} = AA^{\|D} - AA^{\|D}EA^{\|D}(I_m + EA^{\|D})^{-1} + EA^{\|D}(I_m + EA^{\|D})^{-1}, \tag{6}$$

so that $\|BB^{\|D} - AA^{\|D}\|_F \leq \|I_m - AA^{\|D}\|_F\|EA^{\|D}\|_F\|(I_m + EA^{\|D})^{-1}\|_F$. □

Remark 2.5. (i) Adding the condition $EA^{\|D}A = E$ in Theorem 2.4, then by (5), it follows that $B^{\|D}B = A^{\|D}A$. Similarly, if $E = AA^{\|D}E$, then $BB^{\|D} = AA^{\|D}$ by (6).

(ii) Let $A \in \mathbb{C}_{n,n}$. We know that A^{-1} exists if and only if $A^{\|I_n}$ exists [6]. In Theorem 2.4, let $m = n$ and $D = I_n$. Then $\|B^{-1} - A^{-1}\|_F \leq \frac{\|A^{-1}\|_F\|A^{-1}E\|_F}{1-\|A^{-1}E\|_F}$ [9].

Recall from [6, Theorem 3.7] that $A^{\|D}$ exists if and only if AD is group invertible and $\mathrm{rk}(AD) = \mathrm{rk}(D)$ if and only if DA is group invertible and $\mathrm{rk}(DA) = \mathrm{rk}(D)$, in this case, $A^{\|D} = D(AD)^{\#} = (DA)^{\#}D$.

Let $A, D \in \mathbb{C}_{n,n}$ and $A = I_n$. We obtain that $D^{\#}$ exists if and only if $I_n^{\|D}$ exists and $I_n^{\|D} = DD^{\#}$. If T is a perturbation matrix, many authors presented the bound for $\|(D+T)(D+T)^{\#} - DD^{\#}\|_2$ under different conditions, for instance [11, 12, 23, 26, 28]. Note that $\|(D+T)(D+T)^{\#} - DD^{\#}\|_2 = \|I_n^{\|D+T} - I_n^{\|D}\|_2$, it is natural to establish the bound for $\|A^{\|D+T} - A^{\|D}\|_2$ in rectangular matrices as a generalization for previous works.

To solve this problem, we give the existence of $A^{\|D}$ by using the Schur decomposition. According to [7], an arbitrary square matrix $G \in \mathbb{C}_{n,n}$ of rank r with index 1 can be partitioned as

$$G = U\begin{bmatrix} M & N \\ 0 & 0 \end{bmatrix}U^*, \tag{7}$$

where $U \in \mathbb{C}_{n,n}$ is a unitary matrix and $M \in \mathbb{C}_{r,r}$ is invertible, the group inverse of G can be expressed by

$$G^{\#} = U\begin{bmatrix} M^{-1} & M^{-2}N \\ 0 & 0 \end{bmatrix}U^*.$$

The symbol P_G stands for the orthogonal projector onto $\mathcal{R}(G)$. Obviously, $P_G = GG^{\dagger} = U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^*$ and $G^{\dagger}G = P_{G^*}$.

Theorem 2.6. *Let $A \in \mathbb{C}_{m,n}$ and $D \in \mathbb{C}_{n,m}$ with $\mathrm{rk}(D) = r$. Then the following statements are equivalent*:

(i) *$A^{\|D}$ exists.*

(ii) *There exist two unitary matrices $U \in \mathbb{C}_{n,n}$ and $V \in \mathbb{C}_{m,m}$ such that $A = V\begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}U^*$ and $D = U\begin{bmatrix} D_1 & D_2 \\ 0 & 0 \end{bmatrix}V^*$ are invertible, where $A_1, D_1 \in \mathbb{C}_{r,r}$.*

(iii) *There exist two unitary matrices $U \in \mathbb{C}_{n,n}$ and $V \in \mathbb{C}_{m,m}$ such that $A = V\begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}U^*$ and $D = U\begin{bmatrix} D_1 & D_2 \\ 0 & 0 \end{bmatrix}V^*$ and D_1A_1 is invertible, where $A_1, D_1 \in \mathbb{C}_{r,r}$.*

In this case, $A^{\|D} = D(AD)^{\#} = U\begin{bmatrix} A_1^{-1} & A_1^{-1}D_1^{-1}D_2 \\ 0 & 0 \end{bmatrix}V^*$.

Proof. (i)⇒(ii) Given (i), then DA is group invertible and $\mathrm{rk}(DA) = \mathrm{rk}(D)$. So we assume that DA has the form (7), then

$$(DA)^{\#} = U\begin{bmatrix} M^{-1} & M^{-2}N \\ 0 & 0 \end{bmatrix}U^*,$$

so

$$(DA)^{\#}DA = U\begin{bmatrix} I_r & M^{-1}N \\ 0 & 0 \end{bmatrix}U^*.$$

Similarly, for corresponding matrices V, P, Q, we have

$$AD(AD)^{\#} = V\begin{bmatrix} I_r & P^{-1}Q \\ 0 & 0 \end{bmatrix}V^*.$$

Set $A = V\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}U^*$ and $D = U\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}V^*$. From $A^{\|D} = (DA)^{\#}D$, it follows that $D = A^{\|D}AD = (DA)^{\#}DAD$. By a direct calculation, one has $D_3 = D_4 = 0$. So

$$AD = V\begin{bmatrix} A_1D_1 & A_1D_2 \\ A_3D_1 & A_3D_2 \end{bmatrix}V^*.$$

We have $A_1D_1 = P$ and $A_3D_1 = 0$. Since P is invertible, we obtain that A_1 and D_1 are invertible and $A_3 = 0$.

(ii)⇒(iii) is clear.

(iii)⇒(i) Given (iii), we have

$$DA = U\begin{bmatrix} D_1A_1 & D_1A_2 + D_2A_4 \\ 0 & 0 \end{bmatrix} U^*,$$

Since D_1A_1 is invertible, the matrix DA is group invertible and

$$(DA)^{\#} = U\begin{bmatrix} A_1^{-1}D_1^{-1} & A_1^{-1}D_1^{-1}A_1^{-1}D_1^{-1}(D_1A_2 + D_2A_4) \\ 0 & 0 \end{bmatrix} U^*.$$

Hence we can verify $D = DA(DA)^{\#}D$, which implies $\mathcal{R}(D) \subseteq \mathcal{R}(DA)$. In addition, note that the basic fact $\mathcal{R}(DA) \subseteq \mathcal{R}(D)$ gives $\mathcal{R}(D) = \mathcal{R}(DA)$. Therefore, $A^{\|D}$ exists.

By a direct calculation, we have $(DA)^{\#}D = U\begin{bmatrix} A_1^{-1} & A_1^{-1}D_1^{-1}D_2 \\ 0 & 0 \end{bmatrix} V^*$, so $A^{\|D} = (DA)^{\#}D = U\begin{bmatrix} A_1^{-1} & A_1^{-1}D_1^{-1}D_2 \\ 0 & 0 \end{bmatrix} V^*$. □

Several scholars stated several reflections concerning the perturbation bound of a perturbation matrix E under two-sided conditions. For instance, Ben-Israel [2] established an upper bound for $\frac{\|(A+E)^{\dagger}-A^{\dagger}\|}{\|A^{\dagger}\|}$ under the condition $AA^{\dagger}E = EAA^{\dagger} = E$. Wei et al. [27] investigated perturbation bounds for the Drazin inverse under the condition $E = AA^DE = EAA^D$.

Let $T \in \mathbb{C}_{n,m}$ be a perturbation matrix. We prove that $A^{\|D+T}$ exists and $A^{\|D+T} = A^{\|D}$ under the condition that $T = AA^{\|D}T = TAA^{\|D}$ and T is small enough. To simplify presentation and to make the results more readable, we denote $AA^{\|D}D^{\dagger}T$ by Θ_T.

Proposition 2.7. *Let $A \in \mathbb{C}_{m,n}$, $D \in \mathbb{C}_{n,m}$, and $T \in \mathbb{C}_{m,m}$ such that $A^{\|D}$ exists. If $T = AA^{\|D}T$ and $\|\Theta_T\| < 1$, then $A^{\|D+T}$ exists. In this case, $A^{\|D+T} = A^{\|D}$ if and only if $TAA^{\|D} = T$.*

Proof. By Theorem 2.6, we have

$$A = V\begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix} U^*, A^{\|D} = U\begin{bmatrix} A_1^{-1} & A_1^{-1}D_1^{-1}D_2 \\ 0 & 0 \end{bmatrix} V^*, D = U\begin{bmatrix} D_1 & D_2 \\ 0 & 0 \end{bmatrix} V^*$$

and

$$(DA)^{\#} = U\begin{bmatrix} A_1^{-1}D_1^{-1} & A_1^{-1}D_1^{-1}A_1^{-1}D_1^{-1}(D_1A_2 + D_2A_4) \\ 0 & 0 \end{bmatrix} U^*.$$

So one has

$$AA^{\|D} = V\begin{bmatrix} I_r & D_1^{-1}D_2 \\ 0 & 0 \end{bmatrix} V^* \text{ and } P_D = U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Thus

$$AA^{\|D}D^{\dagger} = A(DA)^{\#}P_D = V\begin{bmatrix} D_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Suppose that $T = U\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} V^*$, where $T_1 \in \mathbb{C}_{r,r}$. It follows by $T = AA^{\|D}T$ that $T_3 = 0$ and $T_4 = 0$. Therefore, we have

$$D + T = U\begin{bmatrix} D_1 + T_1 & D_2 + T_2 \\ 0 & 0 \end{bmatrix} V^*.$$

By $\|\Theta_T\| < 1$, we obtain that

$$I_m + \Theta_T = V\begin{bmatrix} I_r + D_1^{-1}T_1 & D_1^{-1}T_2 \\ 0 & I_{m-r} \end{bmatrix} V^*$$

is invertible, which implies that $I_r + D_1^{-1}T_1$ is invertible, so we can deduce that $D_1 + T_1$ is invertible and

$$(I_m + \Theta_T)^{-1} = V\begin{bmatrix}(I_r + D_1^{-1}T_1)^{-1} & -(D_1 + T_1)^{-1}T_2\\ 0 & I_{m-r}\end{bmatrix}V^*.$$

From Theorem 2.6 (ii), one can obtain that $A^{\|D+T}$ exists and

$$A^{\|D+T} = U\begin{bmatrix}A_1^{-1} & A_1^{-1}(D_1 + T_1)^{-1}(D_2 + T_2)\\ 0 & 0\end{bmatrix}V^*.$$

We prove that $A^{\|D+T} = A^{\|D}$ if and only if $TAA^{\|D} = T$. The equality $(D_1 + T_1)^{-1} = (I_r + D_1^{-1}T_1)^{-1}D_1^{-1}$ leads to

$$A^{\|D+T} = U\begin{bmatrix}A_1^{-1} & A_1^{-1}(I_r + D_1^{-1}T_1)^{-1}D_1^{-1}(D_2 + T_2)\\ 0 & 0\end{bmatrix}V^*.$$

By a direct calculation, we have

$$A^{\|D}(I_m + \Theta_T)^{-1}AA^{\|D}D^\dagger(D + T) = U\begin{bmatrix}A_1^{-1} & A_1^{-1}(I_r + D_1^{-1}T_1)^{-1}D_1^{-1}(D_2 + T_2)\\ 0 & 0\end{bmatrix}V^*.$$

Note that $A^{\|D}D^\dagger D = (DA)^\# DD^\dagger D = (DA)^\# D = A^{\|D}$. Then

$$\begin{aligned} A^{\|D+T} &= A^{\|D}(I_m + \Theta_T)^{-1}AA^{\|D}D^\dagger(D + T)\\ &= A^{\|D}(I_m + \Theta_T)^{-1}AA^{\|D}(I_m + \Theta_T). \end{aligned} \tag{8}$$

For the converse part, as $A^{\|D+T} = A^{\|D}$, then one has $(D + T)AA^{\|D} = D + T$, which combines with $DAA^{\|D} = D$ yield $TAA^{\|D} = T$. □

Remark 2.8. In Proposition 2.7 above, the condition $\|\Theta_T\|_2 < 1$ is only to ensure that $D_1 + T_1$ is invertible. Now if this condition is replaced by $\|A^{\|D}D^\dagger TA\|_2 < 1$, then $I_n + A^{\|D}D^\dagger TA$ is invertible. It is not difficult to obtain that

$$A^{\|D}D^\dagger = (DA)^\# P_D = U\begin{bmatrix}A_1^{-1}D_1^{-1} & 0\\ 0 & 0\end{bmatrix}U^*.$$

So we can deduce that

$$I_n + A^{\|D}D^\dagger TA = U\begin{bmatrix}I_r + A_1^{-1}D_1^{-1}T_1A_1 & A_1^{-1}D_1^{-1}T_1A_2 + A_1^{-1}D_1^{-1}T_2A_4\\ 0 & I_n - r\end{bmatrix}U^*$$

is invertible, hence $I_r + A_1^{-1}D_1^{-1}T_1A_1$ is invertible. In addition, we have

$$D + T = U\begin{bmatrix}D_1 + T_1 & D_2 + T_2\\ 0 & 0\end{bmatrix}V^*.$$

By $D_1A_1(I_r + A_1^{-1}D_1^{-1}T_1A_1) = D_1A_1 + T_1A_1$, then $D_1A_1 + T_1A_1$ is invertible. Again by Theorem 2.6 (iii), $A^{\|D+T}$ exists.

For $A \in \mathbb{C}_{n,n}$, it is known that $A^\#$ exists if and only if $I_n^{\|A}$ exists. In this case, $AA^\# = I_n^{\|A}$. Applying Proposition 2.7, we have the following corollary. It should be noted that $I_n^{\|A}A^\dagger T = AA^\# A^\dagger AA^\# T = A^\# T$.

Corollary 2.9. [27] *Suppose* $A \in \mathbb{C}_{n,n}$ *with* $\mathrm{ind}(A) = 1$ *and perturbation matrix* T *satisfies* $AA^\# T = TAA^\# = T$ *and* $\|A^\# T\|_2 < 1$, *then* $(A + T)^\#$ *exists and* $(A + T)(A + T)^\# = AA^\#$.

Corollary 2.10. *Suppose* $A \in \mathbb{C}_{n,n}$ *with* $\mathrm{ind}(A) = 1$ *and perturbation matrix* T *satisfies* $AA^\# T = T$ *and* $\|A^\# T\|_2 < 1$, *then* $(A + T)^\#$ *exists and* $(A + T)(A + T)^\# = AA^\#(I_n + A^\# T)^{-1}(AA^\# + A^\# T)$. *In this case,* $(A + T)(A + T)^\# = AA^\#$ *if and only if* $TAA^\# = T$.

Next, we present the upper bound for $\|A^{\|D+T} - A^{\|D}\|_2$ provided that $T = AA^{\|D}T$ and $\|\Theta_T\| < 1$.

Theorem 2.11. *Let $A \in \mathbb{C}_{m,n}$ and $D, T \in \mathbb{C}_{n,m}$ such that $A^{\|D}$ exists. If $T = AA^{\|D}T$ and $\|\Theta_T\| < 1$, then $A^{\|D+T}$ exists and $\|A^{\|D+T} - A^{\|D}\|_2 \leq \frac{\|A^{\|D}D^{\dagger}T\|_2(1+\|\Theta_T\|_2)\|AA^{\|D}\|_2}{1-\|\Theta_T\|_2}$.*

Proof. By $(I_m + \Theta_T)(I_m + \Theta_T)^{-1} = I_m$, we have

$$(I_m + \Theta_T)^{-1} = I_m - \Theta_T(I_m + \Theta_T)^{-1}.$$

Again by (8) and $A^{\|D}D^{\dagger}D = A^{\|D}$, one has

$$A^{\|D+T} = A^{\|D} + A^{\|D}D^{\dagger}T - A^{\|D}D^{\dagger}T(I_m + \Theta_T)^{-1}AA^{\|D}(I_m + \Theta_T).$$

So that

$$A^{\|D+T} - A^{\|D} = A^{\|D}D^{\dagger}T(I_m - (I_m + \Theta_T)^{-1}AA^{\|D}(I_m + \Theta_T)).$$

Note that $I_m = (I_m + \Theta_T)^{-1}(I_m + \Theta_T)$, then

$$\begin{aligned}\|A^{\|D+T} - A^{\|D}\|_2 &= \|A^{\|D}D^{\dagger}T(I_m + \Theta_T)^{-1}(I_m - AA^{\|D})(I_m + \Theta_T)\|_2 \\ &\leq \|A^{\|D}D^{\dagger}T\|_2\|(I_m + \Theta_T)^{-1}\|_2\|I_m - AA^{\|D}\|_2\|I_m + \Theta_T\|_2 \\ &\leq \frac{\|A^{\|D}D^{\dagger}T\|_2(1 + \|\Theta_T\|_2)\|AA^{\|D}\|_2}{1 - \|\Theta_T\|_2}. \end{aligned} \tag{9}$$

The proof is completed. □

Applying the aforementioned theorem, we get the result for the bound of $\|(A+T)(A+T)^{\#} - AA^{\#}\|_2$ under the condition that $T = AA^{\#}T$ and $\|A^{\#}T\|_2 < 1$.

Corollary 2.12. *Suppose $A \in \mathbb{C}_{n,n}$ with $\mathrm{ind}(A) = 1$ and perturbation matrix T satisfies $T = AA^{\#}T$ and $\|A^{\#}T\|_2 < 1$, then $\|(A + T)(A + T)^{\#} - AA^{\#}\|_2 \leq \frac{\|A^{\#}T\|_2(1+\|A^{\#}T\|_2)\|AA^{\#}\|_2}{1-\|A^{\#}T\|_2}$.*

If E_s ($s \in \mathbb{N}$) is a sequence of $m \times n$ matrices such that $\|E_s\|_2 \to 0$ ($s \to \infty$), then by Theorem 2.4, we have $(A + E_s)^{\|D} \to A^{\|D}$. Now let T_s be a sequence of $n \times m$ matrices. Note that $\|T_s\|_2 \to 0$ ($s \to \infty$) implies $\|AA^{\|D}D^{\dagger}T_s\|_2 \to 0$. If T_s satisfies the condition $T_s = AA^{\|D}T_s$, then by Theorem 2.11, we have $A^{\|D+T_s} \to A^{\|D}$.

As a corollary, we have the following result.

Corollary 2.13. *Let T_s ($s \in \mathbb{N}$) be a sequence of $n \times m$ matrices such that $\|T_s\|_2 \to 0$ ($s \to \infty$). If $T_s = AA^{\|D}T_s$, then $A^{\|D+T_s} \to A^{\|D}$.*

Theorem 2.14. *Let $A \in \mathbb{C}_{m,n}$ and $D \in \mathbb{C}_{n,m}$ such that $A^{\|D}$ exists. Suppose perturbation matrices $E \in \mathbb{C}_{m,n}$ and $T \in \mathbb{C}_{n,m}$ satisfy $\|A^{\|D}E\|_2 < 1$, $AA^{\|D}T = TAA^{\|D} = T$ and $\|\Theta_T\|_2 < 1$, then $(A + E)^{\|D+T}$ exists and $\|(A + E)^{\|D+T} - A^{\|D}\|_2 \leq \frac{\|A^{\|D}\|_2\|A^{\|D}E\|_2}{1-\|A^{\|D}E\|_2}$.*

Proof. By Proposition 2.7, the system of conditions $AA^{\|D}T = TAA^{\|D} = T$ and $\|\Theta_T\|_2 < 1$ implies that $A^{\|D+T}$ exists and $A^{\|D+T} = A^{\|D}$. So we have $\|A^{\|D}E\|_2 = \|A^{\|D+T}E\|_2 < 1$. From Theorem 2.4, one can obtain that $(A + E)^{\|D+T}$ exists. Furthermore,

$$\begin{aligned}\|(A + E)^{\|D+T} - A^{\|D}\|_2 &= \|(A + E)^{\|D+T} - A^{\|D+T}\|_2 \\ &\leq \frac{\|A^{\|D+T}\|_2\|A^{\|D+T}E\|_2}{1 - \|A^{\|D+T}E\|_2} \\ &= \frac{\|A^{\|D}\|_2\|A^{\|D}E\|_2}{1 - \|A^{\|D}E\|_2}. \end{aligned} \tag{10}$$

The proof is completed. □

Recall from [6] the following equivalence that $A^{\dagger}$ exists if and only if $A^{\|A^*}$ exists in the setting of complex matrices, in this case, $A^{\dagger} = A^{\|A^*}$. Let $D = A^*$ and $T = E^*$ in Theorem 2.14. The system of conditions $\|A^{\|D}E\|_2 < 1$, $AA^{\|D}T = TAA^{\|D} = T$ and $\|\Theta_T\|_2 < 1$ is equivalent to $\|A^{\dagger}E\|_2 < 1$ and $AA^{\dagger}E = EAA^{\dagger} = E$. So we have the following corollary.

Corollary 2.15. [2, Theorem 4] *Let $A \in \mathbb{C}_{m,n}$. Suppose perturbation matrix $E \in \mathbb{C}_{m,n}$ satisfies $\|A^{\dagger}E\|_2 < 1$ and $AA^{\dagger}E = EAA^{\dagger} = E$, then $\|(A+E)^{\dagger} - A^{\dagger}\|_2 \leq \frac{\|A^{\dagger}\|_2\|A^{\dagger}E\|_2}{1-\|A^{\dagger}E\|_2}$.*

Let $A, E \in \mathbb{C}_{n,n}$. We know that $A^{\#}$ exists if and only if $A^{\|A}$ exists, in this case, $A^{\|A} = A^{\#}$. We assume that $EAA^{\#} = AA^{\#}E = E$. Then one can get $AA^{\|A}A^{\dagger}E = AA^{\#}A^{\dagger}AA^{\#}E = A^{\#}E$.

Applying Theorem 2.14, we have

Corollary 2.16. [27, Theorem 3.2] *Let $A \in \mathbb{C}_{n,n}$ be of ind(A)=1 exists. Suppose perturbation matrix $E \in \mathbb{C}_{n,n}$ satisfies $\|A^{\#}E\|_2 < 1$ and $EAA^{\#} = AA^{\#}E = E$, then $(A+E)^{\#}$ exists and $\|(A+E)^{\#} - A^{\#}\|_2 \leq \frac{\|A^{\#}\|_2\|A^{\#}E\|_2}{1-\|A^{\#}E\|_2}$.*

Let $A, B \in \mathbb{C}_{m,n}$ and $D, C \in \mathbb{C}_{n,m}$ such that $A^{\|D}$ and $B^{\|C}$ exist. We close this section with the problem of bounding $\|B^{\|C} - A^{\|D}\|$ under Frobenius norm. We use a similar approach in [17].

Lemma 2.17. [8] *Let $U \in \mathbb{C}_{m,m}$ and $V \in \mathbb{C}_{n,n}$ be unitary matrices. If $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ and $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$, then for any matrix $X \in \mathbb{C}_{m,n}$, we have*

$$\|X\|_F^2 = \|U_1^*XV_1\|_F^2 + \|U_1^*XV_2\|_F^2 + \|U_2^*XV_1\|_F^2 + \|U_2^*XV_2\|_F^2,$$

where $U_1 \in \mathbb{C}_{m,s}$ and $V_1 \in \mathbb{C}_{n,t}$.

Theorem 2.18. *Let $A, B \in \mathbb{C}_{m,n}$ and $C, D \in \mathbb{C}_{n,m}$ such that $A^{\|D}$ and $B^{\|C}$ exist. Then $\|B^{\|C} - A^{\|D}\|_F \leq max\{\|A^{\|D}\|_2^2, \|B^{\|C}\|_2^2\}\|M - N\|_F$, where $M = P_{C^*}BP_C$ and $N = P_{D^*}AP_D$.*

Proof. As $A^{\|D}$ exists, then by Proposition 2.2, there exist two unitary matrices $U \in \mathbb{C}_{n,n}$ and $V \in \mathbb{C}_{m,m}$ such that

$$D = U\begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}V^*,\ A = V\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}U^* \text{ and } A^{\|D} = U\begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*,$$

where $A_1, D_1 \in \mathbb{C}_{r,r}$ are invertible. One has $D^{\dagger} = V\begin{bmatrix} D_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*$, and consequently,

$$P_{D^*}AP_D = V\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Let $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ and $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$, where $U_1 \in \mathbb{C}_{n,r}$ and $V_1 \in \mathbb{C}_{m,r}$. Then $A^{\|D} = U_1A_1^{-1}V_1^*$ and $P_{D^*}AP_D = V_1A_1U_1^*$.

Similarly, by Proposition 2.2, there exist two unitary matrices $P \in \mathbb{C}_{n,n}$ and $Q \in \mathbb{C}_{m,m}$ such that

$$C = P\begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}Q^*,\ B = Q\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}P^* \text{ and } B^{\|C} = P\begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}Q^*,$$

where $B_1, C_1 \in \mathbb{C}_{r,r}$ are invertible. One has $C^{\dagger} = Q\begin{bmatrix} C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^*$, and consequently,

$$P_{C^*}BP_C = Q\begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}P^*.$$

Assume $\mathrm{rk}(C) = t$, write $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$ and $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$, where $P_1 \in \mathbb{C}_{n,t}$ and $Q_1 \in \mathbb{C}_{m,t}$, then $B^{\|C} = P_1 B_1^{-1} Q_1^*$ and $P_{C^*} B P_C = Q_1 B_1 P_1^*$.

Note that $B^{\|C} - A^{\|D} = P_1 B_1^{-1} Q_1^* - U_1 A_1^{-1} V_1^*$. Then

$$\begin{aligned} \|B^{\|C} - A^{\|D}\|_F &= \| \begin{bmatrix} P_1^* \\ P_2^* \end{bmatrix} (B^{\|C} - A^{\|D}) \begin{bmatrix} V_1 & V_2 \end{bmatrix} \|_F \\ &= \| \begin{bmatrix} B_1^{-1} Q_1^* V_1 - P_1^* U_1 A_1^{-1} & B_1^{-1} Q_1^* V_2 \\ -P_2^* U_1 A_1^{-1} & 0 \end{bmatrix} \|_F, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \|B^{\|C} - A^{\|D}\|_F &= \| \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} (B^{\|C} - A^{\|D}) \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \|_F \\ &= \| \begin{bmatrix} U_1^* P_1^* B_1^{-1} - A_1^{-1} V_1^* Q_1 & -A_1^{-1} V_1^* Q_2 \\ U_2^* P_1 B_1^{-1} & 0 \end{bmatrix} \|_F. \end{aligned} \tag{12}$$

The two equalities 11 and 12 above imply

$$\begin{aligned} 2\|B^{\|C} - A^{\|D}\|_F^2 &= \|B_1^{-1} Q_1^* V_1 - P_1^* U_1 A_1^{-1}\|_F^2 + \|B_1^{-1} Q_1^* V_2\|_F^2 + \|P_2^* U_1 A_1^{-1}\|_F^2 \\ &+ \|U_1^* P_1^* B_1^{-1} - A_1^{-1} V_1^* Q_1\|_F^2 + \|A_1^{-1} V_1^* Q_2\|_F^2 + \|U_2^* P_1 B_1^{-1}\|_F^2. \end{aligned} \tag{13}$$

Let $M = P_{C^*} B P_C$ and $N = P_{D^*} A P_D$. Then $M - N = Q_1 B_1 P_1^* - V_1 A_1 U_1^*$. Applying the property of unitary matrices, one has $P_1^* P_2 = 0$ and $V_1^* V_2 = 0$. Hence, it follows that

$$A_1^{-1} V_1^* (M - N) P_1 B_1^{-1} = A_1^{-1} V_1^* Q_1 - U_1^* P_1 B_1^{-1}. \tag{14}$$

$$B_1^{-1} Q_1^* (M - N) U_1 A_1^{-1} = P_1^* U_1 A_1^{-1} - B_1^{-1} Q_1^* V_1. \tag{15}$$

$$A_1^{-1} V_1^* (M - N) P_2 = -U_1^* P_2. \tag{16}$$

$$V_2^* (M - N) P_1 B_1^{-1} = V_2^* Q_1. \tag{17}$$

$$Q_2^* (M - N) U_1 A_1^{-1} = -Q_2^* V_1. \tag{18}$$

$$B_1^{-1} Q_1^* (M - N) U_2 = P_1^* U_2. \tag{19}$$

By (14)-(19), the equality (13) can be reduced to

$$\begin{aligned} 2\|B^{\|C} - A^{\|D}\|_F^2 &= \|B_1^{-1} Q_1^* (M - N) U_1 A_1^{-1}\|_F^2 + \|B_1^{-1} (V_2^* (M - N) P_1 B_1^{-1})^*\|_F^2 \\ &+ \|(A_1^{-1} V_1^* (M - N) P_2)^* A_1^{-1}\|_F^* + \|A_1^{-1} V_1^* (M - N) P_1 B_1^{-1}\|_F^2 \\ &+ \|A_1^{-1} (Q_2^* (M - N) U_1 A_1^{-1})^*\|_F^2 + \|(B_1^{-1} Q_1^* (M - N) U_2)^* B_1^{-1}\|_F^2. \end{aligned} \tag{20}$$

Let $m = \|A_1^{-1}\|_2$. Then $m = \|B^{\|C}\|_2$ since $B^{\|C} = P \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^*$. Let $n = \|B_1^{-1}\|_2$. Then one has $n = \|A^{\|D}\|_2$ similarly. Hence by Lemma 2.17, we have

$$\begin{aligned} 2\|B^{\|C} - A^{\|D}\|_F^2 &\le m^2 n^2 (\|Q_1^* (M - N) U_1\|_F^2 + \|V_1^* (M - N) P_1\|_F^2) \\ &+ m^4 (\|P_1^* (M - N)^* V_2\|_F^2 + \|U_2^* (M - N)^* Q_1\|_F^2) \\ &+ n^4 (\|P_2^* (M - N)^* V_1\|_F^2 + \|U_1^* (M - N)^* Q_2\|_F^2) \\ &\le 2 max\{m^4, n^4\} \|M - N\|_F^2 \end{aligned} \tag{21}$$

Consequently,

$$\|B^{\|C} - A^{\|D}\|_F \le max\{\|B^{\|C}\|_2^2, \|A^{\|D}\|_2^2\} \|M - N\|_F.$$

The proof is completed. □

Applying Theorem 2.18, we have the following corollaries.

Corollary 2.19. [17, Theorem 2.1] *Let* $A, E \in \mathbb{C}_{m,n}$ *and* $B = A + E$. *Then* $\|B^{\dagger} - A^{\dagger}\|_F \leq max\{\|A^{\dagger}\|_2^2, \|B^{\dagger}\|_2^2\}\|E\|_F$.

Corollary 2.20. *Let* $A, B \in \mathbb{C}_{n,n}$ *with* $\mathrm{ind}(A) = \mathrm{ind}(B) = 1$. *Then* $\|B^{\#} - A^{\#}\|_F \leq max\{\|A^{\#}\|_2^2, \|B^{\#}\|_2^2\}\|P_{B^*}BP_B - P_{A^*}AP_A\|_F$.

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