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On the finite-time blow-up of solutions to a triharmonic reaction diffusion equation in variable exponent Sobolev spaces

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Abstract. In this work, we consider a triharmonic reaction diffusion equation with a viscoelastic term. Such problems arise in diverse fields: in mathematics, in physics, phase transition, in engineering and in biology. Our primary focus is on establish upper bounds for the blow-up of solutions.

1. Introduction

In this paper, we investigate the initial boundary value equation

$$\begin{cases} u_t - \Delta^3 u + \int\limits_0^t g(t-z)\,\Delta^3 u(x,z)\,dz - \Delta^3 u_t = |u|^{q(x)-2}\,u\ln|u|, & \Omega\times(0,T), \\ u(x,t) = \Delta u(x,t) = \Delta^2 u(x,t) = 0, & \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) \neq 0, & x\in\Omega, \end{cases} \tag{1}$$

here Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $(n \geq 1)$, $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded $C^1(\mathbb{R}^+)$ function and the initial value $u_0 \in W_0^{3,q(.)}(\Omega)$, the exponent $q(.)$ is given measurable function on Ω satisfying

$$2 < q_1 \leq q(x) \leq q_2 < q^*(x) = \begin{cases} \frac{nq(x)}{(n-q(x))_2}, & n > q_2, \\ \infty & n \leq q_2, \end{cases} \tag{2}$$

and

$$q_2 = ess\sup_{x\in\Omega} q(x), \quad q_1 = ess\inf_{x\in\Omega} q(x), \tag{3}$$

with observe the following Zhikov-Fan uniform local continuity condition.

For all points a and b in the bounded domain Ω with $|a-b| < \frac{1}{2}$, there exists a constant $M > 0$ such that the following inequality holds:

$$\left|q(a) - q(b)\right| \leq -\frac{M}{\ln|a-b|}. \tag{4}$$

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Di et al. [1] has been studied the following initial-boundary value problem

$$u_t - v\Delta u_t - \operatorname{div}\left(|\nabla u|^{m(x)-2}\nabla u\right) = |u|^{p(x)-2}u, \tag{5}$$

with Dirichlet boundary condition. The authors obtained both an upper bound and a lower bound for blow-up. It is clear that when $v = 1$, $m(x) = 2$, $p(x) = p$, the expression (5) reduces to the following pseudo-parabolic equation

$$u_t - \Delta u - \Delta u_t = |u|^{p-2}u. \tag{6}$$

As for (6), there are many results concerning asymptotic behavior [2, 3] the existence and uniqueness [4, 5] of solutions, blow-up [3, 6] property and so on.

Qu et al. [7] studied the following fourth-order parabolic equation

$$u_t + \Delta^2 u = u^{p(x)}.$$

They established the asymptotic behavior of solutions. Also, Liu [8] proved both the local existence and blow up of solutions the same equation.

Messaoudi and Talahmeh [10] considered the following pseudo-parabolic problem

$$u_t - \Delta u - \Delta u_t + \int_0^t g(t-\tau)\Delta u(x,\tau)\,d\tau = |u|^{p(x)-2}u.$$

They demonstrated that every weak solution, with initial data at arbitrary energy level, blows up in finite time.

Peng and Zhou, [11] examined the initial boundary value problem associated with a semilinear equation logarithmic nonlinearity

$$u_t - \Delta u = |u|^{p-2}u\ln|u|.$$

The authors obtained that existence of global solutions and finite-time blow up solutions.

Pişkin and Butakın, [12] studied the following parabolic-type Kirchhoff equation with the variable exponents

$$\left(1 + |u|^{p(x)-2}\right)u_t + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u = |u|^{q(x)-2}u.$$

They demonstrated the global existence of solutions using the Faedo-Galerkin method.

The problem (1) arise in diverse fields: in mathematics blow-up analysis, stability of PDE models, in physics thin film flows, phase transition, in engineering heat transfer, nuclear reactor models, and in biology morphogen driven pattern formation, population dispersal. The interested readers may refer to [[9, 13–17, 20–22]] and the references therein.

Here is the outline of this paper. In Section 2, we present various materials, including notations, hypotheses, and auxiliary formulas. In Section 3, we establish a lower bound for the finite-time blow-up T of the weak solution $u(x,t)$ of (1) blows up in finite time T. In Section 4, we utilize the technique of differential inequalities to derive an upper bound for the blow-up time of weak solutions for the problem described in equation (1).

2. Preliminaries

Consider a measurable function $q : \Omega \to [1, \infty]$. We define $L^{q(.)}(\Omega)$ as the set of real measurable functions u defined on Ω, satisfying the condition that

$$\int_\Omega |\lambda u(x)|^{q(x)}\,dx < \infty \text{ for some } \lambda > 0.$$

The space of variable-exponent functions $L^{q(.)}(\Omega)$ endowed with the Luxemburg-type norm

$$\|u\|_{q(.)} = \inf\left\{\lambda > 0, \int\limits_{\Omega} \left|\frac{u(x)}{\lambda}\right|^{q(x)} dx \leq 1\right\},$$

is a Banach space.

Throughout the paper, we utilize $\|\cdot\|_q$ to represent the L^q norm where $1 \leq q \leq +\infty$. The space $H_0^3(\Omega)$ corresponds to the closure of $C_0^\infty(\Omega)$ under the norm defined as follows:

$$\|u\|_{H^3(\Omega)}^2 = \|u\|_2^2 + \|\nabla u\|_2^2 + \|\Delta u\|_2^2 + \|\nabla\Delta u\|_2^2 .$$

It is established that of $H_0^3(\Omega)$ can be defined through the following expression

$$\|u\|_{H_0^3(\Omega)} = \|\nabla\Delta u\|_2^2 = \left(\int\limits_{\Omega} |\nabla\Delta u(x)|^2 dx\right)^{\frac{1}{2}} .$$

Lemma 2.1. *[18, 19]*

(1) *If q satisfies condition (4), then for all $u \in W_0^{3,q(.)}$ where Ω is bounded, we have $\|u\|_{q(.)} \leq C\|\nabla\Delta u\|_{q(.)}$. Especially, the space $W_0^{3,q(.)}$ has a norm defined as $\|u\|_{3,q(.)} = \|\nabla\Delta u\|_{q(.)}$ for all $u \in W_0^{3,q(.)}$.*

(2) *If $q \in C\left(\overline{\Omega}\right)$, $p : \Omega \to [1,\infty)$ is a measurable function and*

$$ess \inf_{x\in\Omega} (q^*(x) - p(x)) > 0,$$

where $q^*(x) = \frac{nq(x)}{(n-q(x))_2}$, then

$$W_0^{3,q(.)} \hookrightarrow\hookrightarrow L^{p(.)}(\Omega).$$

We make the following assumptions:

(**A**): The memory kernel $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a $C^1(\mathbb{R}^+)$ function that satisfies

$$g(t) \geq 0,\ g'(t) \leq 0,\ 1 - \int\limits_0^\infty g(z)\,dz = l > 0.$$

(**B**): Given a measurable function $q(.)$ on $\overline{\Omega}$ such that:

$$\begin{cases} 2 < q_1 \leq q_2 < 8, & n = 1,2,3,4,5,6 \\ 2 < q_1 \leq q(x) \leq q_2 < \frac{2n+12}{n}, & n \geq 7. \end{cases}$$

If direct calculation is made

$$\int\limits_0^t g(t-z)(\nabla\Delta u(z), \nabla\Delta u_t(t))\,dz = -\frac{1}{2}g(t)\|\nabla\Delta u(t)\|_2^2 + \frac{1}{2}(g' \diamond \nabla\Delta u)(t)$$

$$-\frac{1}{2}\frac{d}{dt}\left\{(g \diamond \nabla\Delta u)(t) - \left(\int\limits_0^t g(z)\,dz\right)\|\nabla\Delta u(t)\|_2^2\right\},$$

we get

$$(g \diamond \nabla\Delta u)(t) = \int_0^t g(t-z) \int_\Omega |\nabla\Delta u(t) - \nabla\Delta u(z)|^2 \, dxdz$$
$$= \int_0^t g(t-z) \|\nabla\Delta u(t) - \nabla\Delta u(z)\|_2^2 \, dz.$$

3. Lower bound for the blow-up time

In this section, we establish a lower bound for T when the weak solution $u(x,t)$ of equation (1) experiences finite-time blow-up at T. We commence by deriving a local existence theorem for the problem described in equation (1), a result that can be

Theorem 3.1. *For any $u_0 \in W_0^{3,q(.)}(\Omega)$, the equation (1) endowed with a weak solution u that fulfills*

$$u \in C\left([0,T]; H_0^3(\Omega)\right) \cap C^2\left([0,T]; L^2(\Omega)\right) \cap C^2\left([0,T]; H^{-3}(\Omega)\right),$$

with

$$\langle u_t, v\rangle + \langle \nabla\Delta u_t, \nabla\Delta v\rangle - \int_0^t g(t-z) \langle \nabla\Delta u(x,z), \nabla\Delta v\rangle \, dz + \langle \nabla\Delta u, \nabla\Delta v\rangle$$
$$= \int_\Omega |u|^{q(.)-2} u \ln(|u|) v dx,$$

where $v \in H_0^3(\Omega)$, and for almost every t in the interval $[0,T]$. With our goal, we begin by introducing the subsequent lemma which establishes the energy of the solution.

Lemma 3.2. *The energy associated with equation (1) is expressed as*

$$E(u) = \frac{1}{2}(g \diamond \nabla\Delta u) + \frac{1}{2}\left(1 - \int_0^t g(z) \, dz\right) \|\nabla\Delta u\|_2^2$$
$$- \int_\Omega \frac{1}{q(x)} |u(x,t)|^{q(x)} \ln(|u|) \, dx + \int_\Omega \frac{1}{q^2(x)} |u(x,t)|^{q(x)}, \tag{7}$$

moreover

$$\frac{dE(u)}{dt} = -\frac{1}{2} g(t) \|\nabla\Delta u\|_2^2 + \frac{1}{2}(g' \diamond \nabla\Delta u)$$
$$- \int_\Omega u_t^2 dx - \int_\Omega |\nabla\Delta u_t|^2 \, dx$$
$$\leq 0, \tag{8}$$

and the inequality $E(u) \leq E(u_0)$ follows directly from assumption ($\boldsymbol{A}$), where

$$E(u_0) = \frac{1}{2} \|\nabla\Delta u_0\|_2^2 - \int_\Omega \frac{1}{q(x)} |u_0|^{q(x)} \ln(|u_0|) \, dx$$
$$+ \int_\Omega \frac{1}{q^2(x)} |u_0|^{q(x)} \, dx. \tag{9}$$

Our primary main outcome is established by selecting σ a non-negative constant sufficiently small, such that

$$\begin{cases} 0<\sigma<8-q_2 \text{ as } n=1,2,3,4,5,6 \\ 0<\sigma<\frac{2n+12}{n}-q_2, \text{ as } n\geq 7 \end{cases} \tag{10}$$

is as follows.

Theorem 3.3. *Given the conditions ($\boldsymbol{A}$) and ($\boldsymbol{B}$), let's consider an initial function $u_0 \in H_0^3(\Omega)$ such that its norm $\|u_0\|_{H_0^3(\Omega)} \neq 0$. Additionally, consider the weak solution $u(x,t)$ to the problem described in reference (1) occurs a finite-time blow-up, occurring before a certain time denoted as T.*

(1) *When $n \geq 7$, a lower bound for T is established by*

$$\int_{\|u_0\|^2_{H_0^3(\Omega)}}^{\infty} \frac{dy}{C_1+C_3y^{C_2}+C_5y^{C_4}},$$

in which

$$C_1=\max\left\{\frac{1}{l}E(0),\ 0\right\}\geq 0,\ C_2=\frac{2\left(q_2+\sigma\right)-n\left(q_2+\sigma-2\right)}{8-n\left(q_2+\sigma-2\right)}>1,$$
$$C_4=\frac{2\left(q_1+\sigma\right)-n\left(q_1+\sigma-2\right)}{8-n\left(q_1+\sigma-2\right)},$$
$$0<C_3=\frac{2lq_1+1}{e\sigma lq_1}\frac{8-n\left(q_2+\sigma-2\right)}{8}\left(B^2\frac{n\left(q_2+\sigma-2\right)\left(2lq_1+1\right)}{e\sigma lq_1\left(8l^2+1\right)}\right)^{\frac{n\left(q_2+\sigma-2\right)}{8-n\left(q_2+\sigma-2\right)}},$$
$$0<C_5=\frac{2lq_1+1}{e\sigma lq_1}\frac{8-n\left(q_1+\sigma-2\right)}{8}\left(B^2\frac{n\left(q_1+\sigma-2\right)\left(2lq_1+1\right)}{e\sigma lq_1\left(8l^2+1\right)}\right)^{\frac{n\left(q_1+\sigma-2\right)}{8-n\left(q_1+\sigma-2\right)}}, \tag{11}$$

where B represents the optimal constant of the Sobolev embedding $H_0^3(\Omega) \hookrightarrow L^{\frac{2n}{n-6}}(\Omega)$ yielding the inequality

$$\|u\|_{\frac{2n}{n-6}} \leq B\,\|\nabla\Delta u\|_2 .$$

(2) *When $n=1,2,3,4,5,6$ a lower bound for T is established by*

$$\int_{\|u_0\|^2_{H_0^3(\Omega)}}^{\infty} \frac{dy}{C_1+C_6y^{C_5}+C_8y^{C_7}},$$

where C_1 denotes the constant as defined in reference (11),

$$C_5=\frac{2}{8-\left(q_2+\sigma\right)}>1,\quad C_7=\frac{2}{8-\left(q_1+\sigma\right)}>1,$$
$$C_6=\frac{2lq_1+1}{2e\sigma lq_1}\frac{8-\left(q_2+\sigma\right)}{2}\left(2B_*^2\frac{n\left(q_2+\sigma-2\right)\left(2lq_1+1\right)}{e\sigma lq_1\left(8l^2+1\right)}\right)^{\frac{q_2+\sigma-2}{8-\left(q_2+\sigma\right)}}>0,$$
$$C_8=\frac{2lq_1+1}{2e\sigma lq_1}\frac{8-\left(q_1+\sigma\right)}{2}\left(2B_*^2\frac{n\left(q_1+\sigma-2\right)\left(2lq_1+1\right)}{e\sigma lq_1\left(8l^2+1\right)}\right)^{\frac{q_1+\sigma-2}{8-\left(q_1+\sigma\right)}}>0,$$

and where B_*signifies the optimal constant of the Sobolev embedding $H_0^3(\Omega) \hookrightarrow L^\infty(\Omega)$ which provides the inequality

$$\|u\|_\infty \leq B_* \|\nabla\Delta u\|_2 .$$

Proof. *Let's introduce the function,*

$$\varphi(t) = \int_\Omega u(x,t)^2 \, dx + \int_\Omega |\Delta u(x,t)|^2 \, dx + \int_\Omega |\nabla\Delta u(x,t)|^2 \, dx = \|u(t)\|^2_{H_0^3(\Omega)} . \tag{12}$$

*By multiplying both sides of Eq. (1) with u and applying integration by parts, in accordance with the given condition (**A**),*

$$\varphi'(t) = 2\int_\Omega u(x,t)\, u_t(x,t) \, dx + 2\int_\Omega \Delta u(x,t)\, \Delta u_t(x,t) \, dx + +2\int_\Omega \nabla\Delta u(x,t)\, \nabla\Delta u_t(x,t) \, dx. \tag{13}$$

Thus, we obtain

$$\begin{aligned}
\varphi'(t) =& 2\int_\Omega u(x,t)\, u_t(x,t) \, dx + 2\int_\Omega \Delta u(x,t)\, \Delta u_t(x,t) \, dx + +2\int_\Omega \nabla\Delta u(x,t)\, \nabla\Delta u_t(x,t) \, dx \\
=& 2\int_\Omega u(x,t) \begin{pmatrix} \nabla\Delta u - \int_0^t g(t-z)\, \nabla\Delta u(x,z)\, dz \\ +\nabla\Delta u_t + |u|^{q(x)-2} u \ln|u| \end{pmatrix} dx + 2\int_\Omega \Delta u \Delta u_t dx + +2\int_\Omega \nabla\Delta u \nabla\Delta u_t dx \\
=& -2\left(1 - \int_0^t g(z)\, dz\right) \|\nabla\Delta u(t)\|_2^2 + 2\int_\Omega |u|^{q(x)} \ln|u| \, dx + I_1 \\
\leq& -2l \|\nabla\Delta u(t)\|_2^2 + 2\int_\Omega |u|^{q(x)} \ln|u| \, dx + I_1,
\end{aligned}$$

in which

$$I_1 = 2\int_0^t g(t-z) \int_\Omega \nabla\Delta u(t) \left(\nabla\Delta u(z) - \nabla\Delta u(t)\right) dx dz. \tag{14}$$

*Utilizing Hölder's inequality along with condition (**A**), it is evident that*

$$\begin{aligned}
|I_1| \leq& 2l \|\nabla\Delta u(t)\|_2^2 \int_0^t g(z)\, dz + \frac{1}{2l}(g \diamond \nabla\Delta u) \\
\leq& 2l(1-l) \|\nabla\Delta u(t)\|_2^2 + \frac{1}{2l}(g \diamond \nabla\Delta u) .
\end{aligned} \tag{15}$$

Through the utilization of the inequality $\ln\chi \leq \frac{1}{e\sigma}\chi^\sigma$ for $\chi \geq 1$ and $\sigma > 0$, it can be inferred that

$$\begin{aligned}
\int_\Omega |u(x,t)|^{q(x)} \ln(|u(x,t)|) \, dx \leq& \int_{0<|u|<1} |u(x,t)|^{q(x)} \ln(|u(x,t)|) \, dx \\
&+ \int_{|u|\geq 1} |u(x,t)|^{q(x)} \ln(|u(x,t)|) \, dx \\
\leq& \frac{1}{e\sigma} \int_\Omega |u(x,t)|^{q(x)+\sigma} \, dx.
\end{aligned} \tag{16}$$

By combining equations (12),(15) and (16), we obtain

$$\varphi'(t) \leq -2l^2 \|\nabla\Delta u(t)\|_2^2 + \frac{2}{e\sigma}\int_\Omega |u|^{q(x)+\sigma}\,dx + \frac{1}{2l}(g \diamond \nabla\Delta u). \tag{17}$$

*Based on condition (**A**), equation (7) and Lemma 3.2, it follows that*

$$\begin{aligned}\frac{1}{2l}(g \diamond \nabla\Delta u) =& \frac{1}{l}\left[\begin{array}{c} E(t) - \frac{1}{2}\|\nabla\Delta u(t)\|_2^2\left(1-\int_0^t g(z)\,dz\right) \\ +\int_\Omega \frac{1}{q(x)}|u(x,t)|^{q(x)}\ln|u|\,dx \\ -\int_\Omega \frac{1}{q^2}|u(x,t)|^{q(x)}\,dx \end{array}\right] \\ \leq& \frac{1}{l}E(0) - \frac{1}{2}\|\nabla\Delta u(t)\|_2^2 \\ &+ \frac{1}{e\sigma l q_1}\int_\Omega |u(x,t)|^{q(x)+\sigma}\,dx. \end{aligned} \tag{18}$$

Subsequently, equations (17) and (18) indicate that

$$\begin{aligned}\varphi'(t) \leq& -\left(2l^2 + \frac{1}{2}\right)\|\Delta u(t)\|_2^2 + \frac{1}{l}E(0) \\ &+ \frac{1}{e\sigma}\left(2 + \frac{1}{q_1 l}\right)\int_\Omega |u|^{q(x)+\sigma}\,dx. \end{aligned} \tag{19}$$

Now, let's consider the scenario where $n \geq 7$. By applying Hölder's inequality with positive values $a > 0$ and $b > 0$, we can observe that

$$\begin{aligned}\int_\Omega |u|^{q(x)+\sigma}\,dx \leq& \int_\Omega |u|^{q_1+\sigma}\,dx + \int_\Omega |u|^{q_2+\sigma}\,dx = \|u\|_{q_1+\sigma}^{q_1+\sigma} + \|u\|_{q_2+\sigma}^{q_2+\sigma} \\ \leq& \left(a\|u\|_2^{\frac{2n-(q_2+\sigma)(n-6)}{2}}\right)\left(a^{-1}\|u\|_{\frac{2n}{n-6}}^{\frac{n(q_2+\sigma-2)}{2}}\right) \\ &+ \left(b\|u\|_2^{\frac{2n-(q_1+\sigma)(n-6)}{2}}\right)\left(b^{-1}\|u\|_{\frac{2n}{n-6}}^{\frac{n(q_1+\sigma-2)}{2}}\right). \end{aligned} \tag{20}$$

Conversely, let's consider the subsequent inequality:

$$b^r c^s \leq rb + sc, \text{ for all } r, s, b, c > 0,\ r + s = 1. \tag{21}$$

The condition $2 < q_1 + \sigma \leq q_2 + \sigma < 2 + \frac{8}{n}$ leads to the implication $0 < \frac{n(q_1+\sigma-2)}{8} < \frac{n(q_2+\sigma-2)}{8} < 1$. Subsequently, by *employing equation (21) within equation (20), we can deduce that for all* $d > 0$,

$$\begin{aligned}
\|u\|_{q_2+\sigma}^{q_2+\sigma} \leq & \left(d\,\|u\|_2^{\frac{2n-(q_2+\sigma)(n-6)}{2}}\right)^{\frac{8}{8-n(q_2+\sigma-2)}\frac{8-n(q_2+\sigma-2)}{8}} \\
& \times\left(d^{-1}\,\|u\|_{\frac{2n}{n-6}}^{\frac{n(q_2+\sigma-2)}{2}}\right)^{\frac{n(q_2+\sigma-2)}{8}\frac{8}{n(q_2+\sigma-2)}} \\
\leq & \frac{8-n(q_2+\sigma-2)}{8}\left(d\,\|u\|_2^{\frac{2n-(q_2+\sigma)(n-6)}{2}}\right)^{\frac{8}{8-n(q_2+\sigma-2)}} \\
& + \frac{n(q_2+\sigma-2)}{8}\left(d^{-1}\,\|u\|_{\frac{2n}{n-6}}^{\frac{n(q_2+\sigma-2)}{2}}\right)^{\frac{8}{n(q_2+\sigma-2)}} \\
\leq & \frac{8-n(q_2+\sigma-2)}{8} d^{\frac{8}{8-n(q_2+\sigma-2)}} \varphi^{\frac{2(q_2+\sigma)-n(q_2+\sigma-2)}{8-n(q_2+\sigma-2)}}(t) \\
& + \frac{n(q_2+\sigma-2)}{8} d^{\frac{-8}{n(q_2+\sigma-2)}} \|u\|_{\frac{2n}{n-6}}^2 . \qquad (22)
\end{aligned}$$

Similarly, for any $b > 0$ *using a comparable approach,*

$$\begin{aligned}
\|u\|_{q_1+\sigma}^{q_1+\sigma} \leq & \frac{8-n(q_1+\sigma-2)}{8} b^{\frac{8}{8-n(q_1+\sigma-2)}} \varphi^{\frac{2(q_1+\sigma)-n(q_1+\sigma-2)}{8-n(q_1+\sigma-2)}}(t) \\
& + \frac{n(q_1+\sigma-2)}{8} b^{\frac{-8}{n(q_1+\sigma-2)}} \|u\|_{\frac{2n}{n-2}}^2 . \qquad (23)
\end{aligned}$$

By combining equations (19), (22) and (23), for all positive values of a and b,we obtain

$$\begin{aligned}
\varphi'(t) \leq & \frac{1}{l} E(0) \\
& + \begin{pmatrix} \frac{1}{e\sigma} B^2 \frac{2lq_1+1}{lq_1} \frac{n(q_2+\sigma-2)}{8} a^{\frac{-8}{n(q_2+\sigma-2)}} \\ + \frac{1}{e\sigma} B^2 \frac{2lq_1+1}{lq_1} \frac{n(q_1+\sigma-2)}{8} a^{\frac{-8}{n(q_1+\sigma-2)}} - \left(2l^2 + \frac{1}{2}\right) \end{pmatrix} \|\nabla\Delta u(t)\|_2^2 \\
& + \frac{1}{e\sigma} \frac{2lq_1+1}{lq_1} \frac{8-n(q_2+\sigma-2)}{8} a^{\frac{8}{8-n(q_2+\sigma-2)}} \varphi^{\frac{2(q_2+\sigma)-n(q_2+\sigma-2)}{8-n(q_2+\sigma-2)}}(t) \\
& + \frac{1}{e\sigma} \frac{2lq_1+1}{lq_1} \frac{8-n(q_1+\sigma-2)}{8} a^{\frac{8}{8-n(q_1+\sigma-2)}} \varphi^{\frac{2(q_1+\sigma)-n(q_1+\sigma-2)}{8-n(q_1+\sigma-2)}}(t) . \qquad (24)
\end{aligned}$$

Considering the condition $q_1 > 2$, *let us choose*

$$a = \left(B^2 \frac{n(q_2+\sigma-2)(lq_1+1)}{e\sigma lq_1(8l^2+1)}\right)^{\frac{n(q_2+\sigma-2)}{8}}$$

$$b = \left(B^2 \frac{n(q_1+\sigma-2)(lq_1+1)}{e\sigma lq_1(8l^2+1)}\right)^{\frac{n(q_1+\sigma-2)}{8}}$$

in (24),we get

$$\varphi'(t) \leq C_1 + C_3\varphi^{C_2}(t) + C_5\varphi^{C_4}(t) . \qquad (25)$$

By integrating equation (25) from 0 *to* t *and introducing the substitution* $y = \varphi(t)$ *we arrive at the subsequent inequality*

$$t \geq \int_{\varphi(0)}^{\varphi(t)} \frac{dy}{C_1 + C_3 y^{C_2} + C_5 y^{C_4}} . \qquad (26)$$

Let $t \to T^-$ in (26), we have the lower bound for blow-up time T :

$$T \geq \int_{\|u_0\|^2_{H_0^2(\Omega)}}^{\infty} \frac{dy}{C_1 + C_3 y^{C_2} + C_5 y^{C_4}}. \tag{27}$$

Subsequently, where $n = 1, 2, 3, 4, 5, 6$. By applying Hölder's inequality alongside the Sobolev embedding theorem, we deduce

$$\begin{aligned}\int_\Omega |u|^{q(x)+\sigma}\,dx &\leq \|u\|^{q_1+\sigma}_{q_1+\sigma} + \|u\|^{q_2+\sigma}_{q_2+\sigma},\\ &\leq \|u\|_2^2\,\|u\|_\infty^{q_2+\sigma-2} + \|u\|_2^2\,\|u\|_\infty^{q_1+\sigma-2},\\ &\leq \begin{pmatrix} (B_*\,\|\nabla\Delta u\,(t)\|_2)^{q_2+\sigma-2} \\ +\,(B_*\,\|\nabla\Delta u\,(t)\|_2)^{q_1+\sigma-2}\end{pmatrix}\|u\|_2^2.\end{aligned} \tag{28}$$

The given condition of $2 < q_1 + \sigma \leq q_2 + \sigma < 8$ establishes $0 < \frac{q_1+\sigma-2}{6} < \frac{q_2+\sigma-2}{6} < 1$. Utilizing equations from (21) through (28), we can consequently, for any $d > 0$,

$$\begin{aligned}\|u\|^{q_2+\sigma}_{q_2+\sigma} &\leq \left(d\,\|u\|_2^2\right)^{\frac{2}{8-(q_2+\sigma)}\frac{8-(q_2+\sigma)}{2}}\\ &\quad \left(d^{-1}\,(B_*\,\|\nabla\Delta u\,(t)\|_2)^{q_2+\sigma-2}\right)^{\frac{q_2+\sigma-2}{2}\frac{2}{q_2+\sigma-2}}\\ &\leq \frac{8-(q_2+\sigma)}{2} d^{\frac{2}{8-(q_2+\sigma)}} \varphi^{\frac{2}{8-(q_2+\sigma)}}(t)\\ &\quad + \frac{q_2+\sigma-2}{2} d^{\frac{-2}{q_2+\sigma-2}} B_*^2\,\|\nabla\Delta u\,(t)\|_2^2.\end{aligned} \tag{29}$$

For every, $e > 0$ using a similar approach

$$\begin{aligned}\|u\|^{q_1+\sigma}_{q_1+\sigma} &\leq \frac{8-(q_1+\sigma)}{2} e^{\frac{2}{8-(q_1+\sigma)}} \varphi^{\frac{2}{8-(q_1+\sigma)}}(t)\\ &\quad + \frac{q_1+\sigma-2}{2} e^{\frac{-2}{q_1+\sigma-2}} B_*^2\,\|\nabla\Delta u\,(t)\|_2^2.\end{aligned} \tag{30}$$

Combining the results from equations (19),(29) and (30), for all positive values of d and e it is evident that

$$\begin{aligned}\varphi'(t) &\leq \frac{1}{l} E(0)\\ &\quad + \begin{pmatrix} -\left(2l^2 + \frac{1}{2}\right) + \frac{2lq_1+1}{e\sigma l q_1}\frac{q_2+\sigma-2}{2} d^{-\frac{2}{(q_2+\sigma)-2}} B_*^2 \\ +\frac{2lq_1+1}{e\sigma l q_1}\frac{q_1+\sigma-2}{2} e^{-\frac{2}{(q_1+\sigma)-2}} B_*^2\end{pmatrix}\|\nabla\Delta u\,(t)\|_2^2\\ &\quad + \frac{2lq_1+1}{e\sigma l q_1}\frac{8-(q_2+\sigma)}{2} d^{\frac{2}{8-(q_2+\sigma)}} \varphi^{\frac{2}{8-(q_2+\sigma)}}(t)\\ &\quad + \frac{2lq_1+1}{e\sigma l q_1}\frac{8-(q_1+\sigma)}{2} e^{\frac{2}{8-(q_1+\sigma)}} \varphi^{\frac{2}{8-(q_1+\sigma)}}(t).\end{aligned} \tag{31}$$

Furthermore, by utilizing the given condition $q_1 > 2$, we get

$$\begin{aligned} d &= \left(2B_*^2 \frac{(q_2+\sigma-2)(2lq_1+1)}{e\sigma l q_1 (8l^2+1)}\right)^{\frac{q_2+\sigma-2}{2}},\\ e &= \left(2B_*^2 \frac{(q_1+\sigma-2)(2lq_1+1)}{e\sigma l q_1 (8l^2+1)}\right)^{\frac{q_1+\sigma-2}{2}}.\end{aligned}$$

From equation (31) we obtain

$$\varphi'(t) \leq C_1 + C_6\varphi^{C_5}(t) + C_8\varphi^{C_7}(t). \tag{32}$$

Following a similar approach as in equations (26) and (27), we can derive a lower bound for the blow-up time T :

$$T \geq \int_{\|u_0\|^2_{H_0^3(\Omega)}}^{\infty} \frac{dy}{C_1 + C_6 y^{C_5} + C_8 y^{C_7}}. \tag{33}$$

Hence, the proof of Theorem 3.3 is concluded. □

4. Upper bounds for the blow-up time

We will establish blow-up criterion and use a differential inequality technique to derive an upper bound for the blow-up time of weak solutions for the given problem (1). The following assumptions are also necessary.

(C) : The initial value u_0 belongs to the space $H_0^3(\Omega) \cap L^{q(.)}(\Omega)$ and fulfills the condition $E(0) < 0$, Here $E(t)$ represents the energy functional as defined in equation (7) and

$$E(u_0) = \frac{1}{2}\|\nabla\Delta u_0\|_2^2 - \int_\Omega \frac{1}{q(x)}|u_0|^{q(x)}\ln(|u_0|)\,dx + \int_\Omega \frac{1}{q^2(x)}|u_0|^{q(x)}\,dx.$$

(D) : g fulfills:

$$1 - \int_0^{+\infty} g(z)\,dz = l \in \left[\frac{1}{(q_1-2)^2}, 1\right].$$

Theorem 4.1. *Assuming the conditions* (A), (C) *and* (D) *hold,and given that* $q_1 > 2$ *and* $\|u_0\|^2_{H_0^3(\Omega)} \neq 0$, *if* $u(x,t)$ *represents a weak solution for problem (1) then* $u(x,t)$ *occures finite-time blow-up at time T. Moreover, an upper bound for the blow-up time T is provided by* $\frac{\|u_0\|^2_{H_0^3(\Omega)}}{q_1(2-q_1)E(0)}$.

Proof. Considering the definition of $\varphi(t)$ *as given in equation (12), and given that* $E(0) < 0$, *we can deduce from equations (7) and (8) that*

$$\int_\Omega \frac{1}{q(x)}|u(x,t)|^{q(x)}\ln(|u(x,t)|)\,dx > 0, \text{ for each } t > 0, \tag{34}$$

equation (13) indicates

$$\begin{aligned}\varphi'(t) = &-2\|\nabla\Delta u(t)\|_2^2\left(1 - \int_0^t g(t-z)\,dz\right)\\ &+ 2\int_\Omega |u|^{q(x)}\ln|u|\,dx + I_1,\end{aligned} \tag{35}$$

in which

$$I_1 = 2\int_0^t g(t-z)\int_\Omega \nabla\Delta u(t)\left(\nabla\Delta u(z) - \nabla\Delta u(t)\right)dxdz.$$

*Utilizing Hölder's inequality and taking into account condition (**A**), we observe*

$$
\begin{aligned}
I_1 \geq & \frac{-1}{q_1} \int_0^t g(t-z) \|\nabla\Delta u(t)\|_2^2 \, dz \\
& - q_1 \int_0^t g(t-z) \int_\Omega |\nabla\Delta u(t) - \nabla\Delta u(z)|^2 \, dxdz \\
\geq & - \frac{1-l}{q_1} \|\nabla\Delta u(t)\|_2^2 - q_1 (g \diamond \nabla\Delta u) .
\end{aligned} \tag{36}
$$

Subsequently, we introduce the function

$$
\begin{aligned}
\psi(t) = & - 2q_1 E(t) \\
= & - q_1 (g \diamond \nabla\Delta u) - q_1 \left(1 - \int_0^t g(z) \, dz\right) \|\nabla\Delta u(t)\|_2^2 \\
& + 2q_1 \int_\Omega \frac{1}{q(x)} |u(x,t)|^{q(x)} \ln(|u|) \, dx \\
& - 2q_1 \int_\Omega \frac{1}{q^2(x)} |u(x,t)|^{q(x)} \, dx.
\end{aligned} \tag{37}
$$

Using equations (35) through (37), it follows that

$$
\begin{aligned}
\varphi'(t) - \psi(t) \geq & 2 \int_\Omega |u|^{q(x)} \ln(|u|) \, dx \\
& - 2q_1 \int_\Omega \frac{1}{q(x)} |u(x,t)|^{q(x)} \ln(|u|) \, dx \\
& + \left((q_1 - 2) l - \frac{1-l}{q_1}\right) \|\nabla\Delta u(t)\|_2^2 ,
\end{aligned} \tag{38}
$$

applying equation (34), we conclude

$$
q_1 \int_\Omega \frac{1}{q(x)} |u(x,t)|^{q(x)} \ln(|u|) \, dx \leq \int_\Omega |u(x,t)|^{q(x)} \ln(|u|) \, dx. \tag{39}
$$

*By utilizing the given conditions (**A**) and (**D**), as well as referring to equations (38) and (39), we get*

$$
\varphi'(t) - \psi(t) \geq 0.
$$

*Additionally, based on condition (**A**) and equation (8), it follows that*

$$
\begin{aligned}
\psi'(t) = & - 2q_1 E'(t) = q_1 \begin{pmatrix} g(t) \|\nabla\Delta u(t)\|_2^2 - (g' \diamond \nabla\Delta u) \\ +2 \int_\Omega u_t^2(x,t) \, dx + 2 \int_\Omega |\nabla\Delta u_t(x,t)|^2 \, dx \end{pmatrix} \\
\geq & 2q_1 \left(\int_\Omega u_t^2(x,t) \, dx + \int_\Omega |\nabla\Delta u_t(x,t)|^2 \, dx \right) \geq 0.
\end{aligned} \tag{40}
$$

Utilizing (12) and (40) along with Schwarz's inequality and Young's inequality, we obtain

$$\begin{aligned}\varphi(t)\psi'(t) \geq& 2q_1\left(\int_\Omega u^2(x,t)\,dx + \int_\Omega |\nabla\Delta u(x,t)|^2\,dx\right)\\ &\times\left(\int_\Omega u_t^2(x,t)\,dx + \int_\Omega |\nabla\Delta u_t(x,t)|^2\,dx\right)\\ \geq& 2q_1\left(\int_\Omega u(x,t)\,u_t(x,t)\,dx + \int_\Omega \nabla\Delta u(x,t)\,\nabla\Delta u_t(x,t)\,dx\right)^2\\ =&\frac{q_1}{2}\left(\varphi'(t)\right)^2. \end{aligned} \tag{41}$$

*Given condition (**C**) and equation (37),we have* $\psi(0) > 0$. *When we combine equations (38) and (40), we obtain*

$$\varphi'(t) \geq \psi(t) > 0,\ \varphi(t) \geq \varphi(0) > 0,\ \forall t > 0. \tag{42}$$

Therefore, equations (41) and (42) lead to

$$\varphi(t)\psi'(t) \geq \frac{q_1}{2}\varphi'(t)\varphi'(t) \geq \frac{q_1}{2}\varphi'(t)\psi(t),$$

which can be formulated as follows

$$\frac{\psi'(t)}{\psi(0)} \geq \frac{q_1}{2}\frac{\varphi'(t)}{\varphi(t)}. \tag{43}$$

By integrating equation (43) from 0 *to* t, *we have*

$$\frac{\psi'(t)}{\psi(0)} \geq \left(\frac{\varphi(t)}{\varphi(0)}\right)^{\frac{q_1}{2}}. \tag{44}$$

Utilizing equations (42) and (43), we obtain

$$\frac{\varphi'(t)}{\varphi^{\frac{q_1}{2}}(t)} \geq \frac{\psi(0)}{\varphi^{\frac{q_1}{2}}(0)}. \tag{45}$$

By integrating the inequality given in equation (45) from 0 *to* t, *considering* $q_1 > 2$, *that is*

$$\frac{1}{\varphi^{\frac{q_1-2}{2}}(t)} \leq \frac{1}{\varphi^{\frac{q_1-2}{2}}(0)} - \frac{q_1-2}{2}\frac{\psi(0)}{\varphi^{\frac{q_1}{2}}(0)}t. \tag{46}$$

Consequently equation (46) cannot hold for for all time interval and we deduce the occurrence of a finite-time blow-up in $u(x,t)$ *at time* T. *Specifically, when we approach the limit as* t *approaches* T, *equation (46) yields*

$$T \leq \frac{\|u_0\|^2_{H_0^3(\Omega)}}{q_1(2-q_1)E(0)}.$$

The proof of Theorem 4.1 is now concluded. □

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