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Nonlinear maps preserving the left bi-skew Jordan product on $*$-algebras

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Abstract. Let $\mathscr{R}$ and $\mathscr{C}$ represent two unital $*$-algebras such that $\mathscr{R}$ possesses a nontrivial projection. In the present article, we study, under certain mild assumptions, if a bijective map $\zeta : \mathscr{R} \to \mathscr{C}$ preserves the left biskew Jordan product, then ζ is $\mathbb{R}$-linear ring $*$-isomorphism. Furthermore, this result is investigated on factor von Neumann algebras, prime $*$-algebras.

1. Introduction

Throughout the entire paper, an algebra is tacitly assumed as an associative algebra over the complex field $\mathbb{C}$. Assume $\mathscr{R}$ and $\mathscr{C}$ be two $*$-algebras. Then a ring isomorphism $\zeta : \mathscr{R} \to \mathscr{C}$ is known as $*$-preserving if $\zeta(\mathscr{K}^*) = \zeta(\mathscr{K})^*$ for all $\mathscr{K} \in \mathscr{R}$.

Let $\mathscr{R}$ be a $*$-algebra. For $\mathscr{K}, \mathscr{N} \in \mathscr{R}$, we represent the product $\mathscr{K}\mathscr{N} + \mathscr{N}\mathscr{K}$ by $\mathscr{K} \circ \mathscr{N}$, $\mathscr{K}\mathscr{N} + \mathscr{N}\mathscr{K}^*$ by $\mathscr{K} \bullet \mathscr{N}$, $\mathscr{K}\mathscr{N} - \mathscr{N}\mathscr{K}$ by $[\mathscr{K}, \mathscr{N}]$, $\mathscr{K}\mathscr{N}^* - \mathscr{N}\mathscr{K}$ by ${}_*[\mathscr{K}, \mathscr{N}]$, $\mathscr{K}\mathscr{N} - \mathscr{N}\mathscr{K}^*$ by $[\mathscr{K}, \mathscr{N}]_*$, $\mathscr{K}\mathscr{N}^* + \mathscr{N}\mathscr{K}^*$ by $\mathscr{K} \odot \mathscr{N}$, $\mathscr{K}^*\mathscr{N} + \mathscr{N}^*\mathscr{K}$ by $\mathscr{K} \circledast \mathscr{N}$, $\mathscr{K}^*\mathscr{N} + \mathscr{N}\mathscr{K}^*$ by $\mathscr{K} \diamond \mathscr{N}$. The essence of these products is growing in certain research domains and in recent years, numerous authors have turned their attention to studying them. Particular attention has been devoted to understanding maps that act like derivations on such products or preserve them (see [3, 6–8, 11, 12, 17–19, 23]). For instance, Cui and Li [4] demonstrated that any bijective map that preserves the product $[\mathscr{K}, \mathscr{N}]_*$ in factor von Neumann algebras must either be a linear $*$-isomorphism or a conjugate linear $*$-isomorphism. Bai and Du [1] extended this result to von Neumann algebras. Taghavi et al. [16], demonstrated that if $\mathscr{R}$ and $\mathscr{C}$ are prime $*$-algebras and $\zeta : \mathscr{R} \to \mathscr{C}$ is a map which preserves the triple product $\mathscr{K} \diamond \mathscr{N} \diamond \mathscr{W}$ and $\zeta(I)$ is idempotent, then ζ is a ring $*$-isomorphism. Li et al. [9] demonstrated that any bijective map preserving the product $\mathscr{K} \diamond \mathscr{N}$ on von Neumann algebras lacking central abelian projections can be represented as the combination of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism.

Recently, there has been a lot of interest in nonlinear maps that preserve mixed products (for details see [13–15, 21, 24]). For example, Yang and Zhang [20, 22], investigated nonlinear map that preserve the mixed

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products $[[\mathscr{K},\mathscr{N}],\mathscr{W}]_*$ and $[[\mathscr{K},\mathscr{N}]_*,\mathscr{W}]$ on factor von Neumann algebras. Zhao et al. [25], demonstrated that if $\mathscr{R}$ and $\mathscr{C}$ are factor von Neumann algebras such that $\dim(\mathscr{R}) > 4$ and $\zeta : \mathscr{R} \to \mathscr{C}$ is a bijective map such that $\zeta([\mathscr{K} \bullet \mathscr{N}, \mathscr{W}]) = [\zeta(\mathscr{K}) \bullet \zeta(\mathscr{N}), \zeta(\mathscr{W})]$, for all $\mathscr{K},\mathscr{N},\mathscr{W} \in \mathscr{R}$, then there exists $\alpha \in \{-1, 1\}$ such that $\zeta(\mathscr{K}) = \alpha\Psi(\mathscr{K})$ for all $\mathscr{K} \in \mathscr{R}$, where $\Psi : \mathscr{R} \to \mathscr{C}$ is a linear $*$-isomorphism or conjugate linear $*$-isomorphism. Bhat et al. [2], demonstrated that any bijective map preserving the mixed product $_*[\mathscr{K} \odot \mathscr{N}, \mathscr{W}]$ on prime $*$-algebras is either a $*$-preserving ring isomorphism or the negative of a $*$-preserving ring isomorphism. Quite recently, Siddeeque et al. [10], proved that any bijective map preserving the mixed product $(M \circledast \mathscr{N} \circ \mathscr{W})$ on unital $*$-algebras is a ring $*$-isomorphism. Moreover, Darvish et al. [5] and Li et al. [9] respectively, studied nonlinear maps preserving the left biskew Jordan product on von Neumann algebras.

Inspired from the above mentioned results, in this paper, we are going to look at the structure of a nonlinear map that preserves $(\mathscr{K} \circledast \mathscr{N})$ on unital $*$-algebras. Before we mention our main result, we provide an example of a nontrivial maps which preserve left biskew Jordan product on $\mathscr{R}$ as below:

Example 1.1. *Consider the algebra $\mathscr{R} = \mathcal{M}_2(\mathbb{C})$, The algebra of all 2×2 matrices over the complex field $\mathbb{C}$. Define the maps $'*'$, $\zeta : \mathscr{R} \to \mathscr{R}$ as*

$$X^* = X^\theta \quad \text{and } \zeta(X) = A^\theta X A$$

respectively, where X^θ represents the conjugate transpose of matrix X and A is a fixed unitary matrix in $\mathscr{R}$. Then it is simple to see that $'*'$ is an involution on $\mathscr{R}$ and $\zeta(\mathscr{K} \circledast \mathscr{N}) = \zeta(\mathscr{K}) \circledast \zeta(\mathscr{N})$ holds for all $\mathscr{K},\mathscr{N} \in \mathscr{R}$. Also ζ is bijective and is a nontrivial map.

2. Main Result

To begin with, we introduce the lemma that is crucial for proving our main theorem.

Lemma 2.1. *Suppose $\mathscr{R}$ and $\mathscr{C}$ are $*$-algebras, and let $\zeta : \mathscr{R} \to \mathscr{C}$ be a map such that*

$$\zeta(\mathscr{M} \circledast \mathscr{N}) = \zeta(\mathscr{M}) \circledast \zeta(\mathscr{N}), \tag{1}$$

for all $\mathscr{M},\mathscr{N} \in \mathscr{R}$. If $\zeta(\mathscr{Z}) = \zeta(\mathscr{A}) + \zeta(\mathscr{B})$ for $\mathscr{Z},\mathscr{A},\mathscr{B} \in \mathscr{R}$, then we have

(*i*) $\zeta(\mathscr{Z} \circledast \mathscr{F}) = \zeta(\mathscr{A} \circledast \mathscr{F}) + \zeta(\mathscr{B} \circledast \mathscr{F})$;
(*ii*) $\zeta(\mathscr{F} \circledast \mathscr{Z}) = \zeta(\mathscr{F} \circledast \mathscr{A}) + \zeta(\mathscr{F} \circledast \mathscr{B})$,

for all $\mathscr{F} \in \mathscr{R}$.

Proof. We are given that

$$\zeta(\mathscr{Z}) = \zeta(\mathscr{A}) + \zeta(\mathscr{B}). \tag{2}$$

Multiplying (2) by $\zeta(\mathscr{F})^*$ from left, we have

$$\zeta(\mathscr{F})^*\zeta(\mathscr{Z}) = \zeta(\mathscr{F})^*\zeta(\mathscr{A}) + \zeta(\mathscr{F})^*\zeta(\mathscr{B}). \tag{3}$$

Now from (2), we have

$$\zeta(\mathscr{Z})^* = \zeta(\mathscr{A})^* + \zeta(\mathscr{B})^*. \tag{4}$$

Multiplying (4) by $\zeta(\mathscr{F})$ from right, we obtain

$$\zeta(\mathscr{Z})^*\zeta(\mathscr{F}) = \zeta(\mathscr{A})^*\zeta(\mathscr{F}) + \zeta(\mathscr{B})^*\zeta(\mathscr{F}). \tag{5}$$

Adding (3), (5) and making use of relation (1), we have

$$\zeta(\mathscr{Z} \circledast \mathscr{F}) = \zeta(\mathscr{A} \circledast \mathscr{F}) + \zeta(\mathscr{B} \circledast \mathscr{F}).$$

In a similar way, we can demonstrate the remaining part as well. □

The key finding discussed in the paper is summarized as follows:

Theorem 2.2. *Consider $\mathscr{R}$ and $\mathscr{C}$ as unital $*$-algebras, with $\mathscr{R}$ possessing a nontrivial projection $\mathscr{E}_1$ and satisfying*

$$\mathscr{K}\mathscr{R}\mathscr{E}_1 = 0 \quad \Longrightarrow \quad \mathscr{K} = 0 \tag{6}$$

and

$$\mathscr{K}\mathscr{R}(I - \mathscr{E}_1) = 0 \quad \Longrightarrow \quad \mathscr{K} = 0, \tag{7}$$

where $\mathscr{K} \in \mathscr{R}$ and I is an identity of $\mathscr{R}$. Suppose that a unital bijective map $\zeta : \mathscr{R} \to \mathscr{C}$ satisfies $\zeta(\mathscr{K} \circledast \mathscr{N}) = \zeta(\mathscr{K}) \circledast \zeta(\mathscr{N})$ for all $\mathscr{K}, \mathscr{N} \in \mathscr{R}$. Then, ζ is an $\mathbb{R}$-linear ring $*$-isomorphism.

Proof. Let $\mathscr{E}_2 = I - \mathscr{E}_1$ and $\mathscr{R}_{ij} = \mathscr{E}_i\mathscr{R}\mathscr{E}_j$ for $i, j \in \{1, 2\}$. Then we can write $\mathscr{R}$ as $\mathscr{R} = \mathscr{R}_{11} \oplus \mathscr{R}_{12} \oplus \mathscr{R}_{21} \oplus \mathscr{R}_{22}$. Note that any $\mathscr{K} \in \mathscr{R}$ can be expressed as $\mathscr{K} = \mathscr{K}_{11} + \mathscr{K}_{12} + \mathscr{K}_{21} + \mathscr{K}_{22}$, where $\mathscr{K}_{ij} \in \mathscr{R}_{ij}$ for $i, j = 1, 2$. We will finalize the proof with the following lemmas: □

Lemma 2.3. $\zeta(0) = 0$.

Proof. For any $\mathscr{K}$ in $\mathscr{R}$, we have

$$\begin{aligned}\zeta(0) &= \zeta(\mathscr{K}^*0 + 0^*\mathscr{K})\\ &= \zeta(\mathscr{K} \circledast 0)\\ &= \zeta(\mathscr{K}) \circledast \zeta(0)\\ &= \zeta(\mathscr{K})^*\zeta(0) + \zeta(0)^*\zeta(\mathscr{K}).\end{aligned}$$

As ζ is surjective, there is $\mathscr{K}$ in $\mathscr{R}$ such that $\zeta(\mathscr{K}) = 0$. Hence $\zeta(0) = 0$. □

Lemma 2.4. *Let $\mathscr{K}_{11} \in \mathscr{R}_{11}$ and $\mathscr{U}_{12} \in \mathscr{R}_{12}$. Then*

$$\zeta(\mathscr{K}_{11} + \mathscr{U}_{12}) = \zeta(\mathscr{K}_{11}) + \zeta(\mathscr{U}_{12}).$$

Proof. As ζ is surjective, there is an element $\mathscr{Z}$ in $\mathscr{R}$ such that

$$\zeta(\mathscr{Z}) = \zeta(\mathscr{K}_{11}) + \zeta(\mathscr{U}_{12}). \tag{8}$$

Using Lemma 2.1 and (8) for $\mathscr{F} = C_{12}$, we have

$$\zeta(C_{12} \circledast \mathscr{Z}) = \zeta(C_{12} \circledast \mathscr{K}_{11}) + \zeta(C_{12} \circledast \mathscr{U}_{12}). \tag{9}$$

Again, applying Lemmas 2.1, 2.3 and (9) for $\mathscr{F} = \mathscr{E}_1$, we have

$$\begin{aligned}&\zeta(\mathscr{E}_1 \circledast (C_{12}^*\mathscr{Z} + \mathscr{Z}^*C_{12})) = \zeta(\mathscr{E}_1 \circledast (C_{12}^*\mathscr{K}_{11} + \mathscr{K}_{11}^*C_{12})) + \zeta(\mathscr{E}_1 \circledast (C_{12}^*\mathscr{U}_{12} + \mathscr{U}_{12}^*C_{12}))\\ &\Longrightarrow \zeta(\mathscr{E}_1(C_{12}^*\mathscr{Z} + \mathscr{Z}^*C_{12}) + (C_{12}^*\mathscr{Z} + \mathscr{Z}^*C_{12})^*\mathscr{E}_1) = \zeta(\mathscr{E}_1(C_{12}^*\mathscr{K}_{11} + \mathscr{K}_{11}^*C_{12})\\ &\quad + (C_{12}^*\mathscr{K}_{11} + \mathscr{K}_{11}^*C_{12})^*\mathscr{E}_1) + \zeta(\mathscr{E}_1(C_{12}^*\mathscr{U}_{12} + \mathscr{U}_{12}^*C_{12}) + (C_{12}^*\mathscr{U}_{12} + \mathscr{U}_{12}^*C_{12})^*\mathscr{E}_1)\\ &\Longrightarrow \zeta(\mathscr{E}_1\mathscr{Z}^*C_{12} + C_{12}^*\mathscr{Z}\mathscr{E}_1) = \zeta(\mathscr{K}_{11}^*C_{12} + C_{12}^*\mathscr{K}_{11}).\end{aligned}$$

As ζ is injective, we get

$$\mathscr{E}_1\mathscr{Z}^*C_{12} + C_{12}^*\mathscr{Z}\mathscr{E}_1 = \mathscr{K}_{11}^*C_{12} + C_{12}^*\mathscr{K}_{11}.$$

Multiplying the previous relation by $\mathscr{E}_2$ from right, we get $\mathscr{Z}_{11}^*C_{12} = \mathscr{K}_{11}^*C_{12}$. Using (6) and (7), we have $\mathscr{Z}_{11} = \mathscr{K}_{11}$.
Now, from Lemma 2.1 and (8) for $\mathscr{F} = C_{11}$, we have

$$\zeta(C_{11} \circledast \mathscr{Z}) = \zeta(C_{11} \circledast \mathscr{K}_{11}) + \zeta(C_{11} \circledast \mathscr{U}_{12}). \tag{10}$$

Also, by applying Lemmas 2.1, 2.3 and (10) to $\mathscr{F} = \mathscr{E}_2$, we get

$$\begin{aligned}
&\zeta(\mathscr{E}_2 \circledast (C_{11}^*\mathscr{Z} + \mathscr{Z}^*C_{11})) = \zeta(\mathscr{E}_2 \circledast (C_{11}^*\mathscr{K}_{11} + \mathscr{K}_{11}^*C_{11})) + \zeta(\mathscr{E}_2 \circledast (C_{11}^*\mathscr{U}_{12} + \mathscr{U}_{12}^*C_{11}))\\
\implies &\zeta(\mathscr{E}_2(C_{11}^*\mathscr{Z} + \mathscr{Z}^*C_{11}) + (C_{11}^*\mathscr{Z} + \mathscr{Z}^*C_{11})^*\mathscr{E}_2)\\
&= \zeta(\mathscr{E}_2(C_{11}^*\mathscr{K}_{11} + \mathscr{K}_{11}^*C_{11}) + (C_{11}^*\mathscr{K}_{11} + \mathscr{K}_{11}^*C_{11})^*\mathscr{E}_2)\\
&+ \zeta(\mathscr{E}_2(C_{11}^*\mathscr{U}_{12} + \mathscr{U}_{12}^*C_{11}) + (C_{11}^*\mathscr{U}_{12} + \mathscr{U}_{12}^*C_{11})^*\mathscr{E}_2)\\
\implies &\zeta(\mathscr{Z}_{12}^*C_{11} + C_{11}^*\mathscr{Z}_{12}) = \zeta(\mathscr{U}_{12}^*C_{11} + C_{11}^*\mathscr{U}_{12}).
\end{aligned}$$

By the injectivity of ζ, we have

$$\mathscr{Z}_{12}^*C_{11} + C_{11}^*\mathscr{Z}_{12} = \mathscr{U}_{12}^*C_{11} + C_{11}^*\mathscr{U}_{12}.$$

Solving the last relation, we get $\mathscr{Z}_{12}^*C_{11} = \mathscr{U}_{12}^*C_{11}$. Finally using (6) and (7), we obtain $\mathscr{Z}_{12} = \mathscr{U}_{12}$.
Now, from Lemmas 2.1, 2.3 and (8) for $\mathscr{F} = \mathscr{E}_2$, we have

$$\begin{aligned}
\zeta(\mathscr{E}_2 \circledast \mathscr{Z}) &= \zeta(\mathscr{E}_2 \circledast \mathscr{K}_{11}) + \zeta(\mathscr{E}_2 \circledast \mathscr{U}_{12})\\
&= 0.
\end{aligned}$$

By injectivity, we obtain $\mathscr{E}_2\mathscr{Z} + \mathscr{Z}^*\mathscr{E}_2 = 0$. Multiplying on the right by $\mathscr{E}_1$, we obtain $\mathscr{Z}_{21} = 0$.
Similarly, by applying Lemmas 2.1, 2.3 and (8) for $\mathscr{F} = \mathscr{F}_{21}$, we have

$$\begin{aligned}
\zeta(\mathscr{F}_{21} \circledast \mathscr{Z}) &= \zeta(\mathscr{F}_{21} \circledast \mathscr{K}_{11}) + \zeta(\mathscr{F}_{21} \circledast \mathscr{U}_{12}).\\
&= 0.
\end{aligned}$$

By injectivity of ζ, we write $\mathscr{F}_{21}^*\mathscr{Z} + \mathscr{Z}^*\mathscr{F}_{21} = 0$. Multiplying the last relation on the left by $\mathscr{E}_2$, we obtain $\mathscr{Z}_{22}^*\mathscr{F}_{21} = 0$ for every $\mathscr{F}_{21} \in \mathscr{R}_{21}$. Using (6) and (7), we obtain $\mathscr{Z}_{22} = 0$. □

Lemma 2.5. *For every $\mathscr{K}_{11} \in \mathscr{R}_{11}$, $\mathscr{U}_{12} \in \mathscr{R}_{12}$, $\mathscr{V}_{21} \in \mathscr{R}_{21}$ and $\mathscr{Y}_{22} \in \mathscr{R}_{22}$, we have*

(i) $\zeta(\mathscr{K}_{11} + \mathscr{U}_{12} + \mathscr{V}_{21}) = \zeta(\mathscr{K}_{11}) + \zeta(\mathscr{U}_{12}) + \zeta(\mathscr{V}_{21})$;
(ii) $\zeta(\mathscr{U}_{12} + \mathscr{V}_{21} + \mathscr{Y}_{22}) = \zeta(\mathscr{U}_{12}) + \zeta(\mathscr{V}_{21}) + \zeta(\mathscr{Y}_{22})$.

Proof. Since ζ is surjective, there exists $\mathscr{Z} \in \mathscr{R}$ such that

$$\zeta(\mathscr{Z}) = \zeta(\mathscr{K}_{11}) + \zeta(\mathscr{U}_{12}) + \zeta(\mathscr{V}_{21}). \tag{11}$$

By applying Lemmas 2.1, 2.3 and (11) for $\mathscr{E}_2$, we have

$$\begin{aligned}
\zeta(\mathscr{E}_2 \circledast \mathscr{Z}) &= \zeta(\mathscr{E}_2 \circledast \mathscr{K}_{11}) + \zeta(\mathscr{E}_2 \circledast \mathscr{U}_{12}) + \zeta(\mathscr{E}_2 \circledast \mathscr{V}_{21})\\
&= \zeta(\mathscr{E}_2\mathscr{K}_{11} + \mathscr{K}_{11}^*\mathscr{E}_2) + \zeta(\mathscr{E}_2\mathscr{U}_{12} + \mathscr{U}_{12}^*\mathscr{E}_2) + \zeta(\mathscr{E}_2\mathscr{V}_{21} + \mathscr{V}_{21}^*\mathscr{E}_2)\\
&= \zeta(\mathscr{V}_{21} + \mathscr{V}_{21}^*).
\end{aligned}$$

Thus, by the injectivity of ζ, we have

$$\mathscr{Z}_{21} + \mathscr{Z}_{22} + \mathscr{Z}_{21}^* + \mathscr{Z}_{22}^* = \mathscr{V}_{21} + \mathscr{V}_{21}^*.$$

Multiplying the above equation by $\mathscr{E}_1$ from the left side, we obtain $\mathscr{Z}_{21}^* = \mathscr{V}_{21}^*$. Hence $\mathscr{Z}_{21} = \mathscr{V}_{21}$.
On the other hand, by applying Lemmas 2.1, 2.3 and (11) for $\mathscr{F} = \mathscr{F}_{21}$, we have

$$\begin{aligned}
\zeta(\mathscr{F}_{21} \circledast \mathscr{Z}) &= \zeta(\mathscr{F}_{21} \circledast \mathscr{K}_{11}) + \zeta(\mathscr{F}_{21} \circledast \mathscr{U}_{12}) + \zeta(\mathscr{F}_{21} \circledast \mathscr{V}_{21})\\
&= \zeta(\mathscr{F}_{21}^*\mathscr{K}_{11} + \mathscr{K}_{11}^*\mathscr{F}_{21}) + \zeta(\mathscr{F}_{21}^*\mathscr{U}_{12} + \mathscr{U}_{12}^*\mathscr{F}_{21}) + \zeta(\mathscr{F}_{21}^*\mathscr{V}_{21} + \mathscr{V}_{21}^*\mathscr{F}_{21})\\
&= \zeta(\mathscr{F}_{21}^*\mathscr{V}_{21} + \mathscr{V}_{21}^*\mathscr{F}_{21}).
\end{aligned}$$

Since ζ is injective, we write

$$\mathscr{F}_{21}^*\mathscr{Z}_{21} + \mathscr{F}_{21}^*\mathscr{Z}_{22} + \mathscr{Z}_{22}^*\mathscr{F}_{21} + \mathscr{Z}_{21}^*\mathscr{F}_{21} = \mathscr{F}_{21}^*\mathscr{V}_{21} + \mathscr{V}_{21}^*\mathscr{F}_{21}. \tag{12}$$

Using $\mathscr{Z}_{21} = \mathscr{V}_{21}$, (12) can be written as

$$\mathscr{F}_{21}^*\mathscr{Z}_{22} + \mathscr{Z}_{22}^*\mathscr{F}_{21} = 0.$$

Multiplying the above equation by $\mathscr{E}_2$ from the left side, we have $\mathscr{Z}_{22}^*\mathscr{F}_{21} = 0$, for every $\mathscr{F}_{21} \in \mathscr{R}_{21}$. By using (6) and(7), we arrive at $\mathscr{Z}_{22} = 0$. Similarly, applying Lemmas 2.1, 2.3 and (11) to $\mathscr{F} = \mathscr{E}_1$ and taking Lemma 2.4 into account, we have

$$\begin{aligned}\zeta(\mathscr{E}_1\mathscr{Z} + \mathscr{Z}^*\mathscr{E}_1) &= \zeta(\mathscr{E}_1(\mathscr{K}_{11} + \mathscr{U}_{12}) + (\mathscr{K}_{11} + \mathscr{U}_{12})^*\mathscr{E}_1) + \zeta(\mathscr{E}_1\mathscr{V}_{21} + \mathscr{V}_{21}^*\mathscr{E}_1)\\ &= \zeta(\mathscr{K}_{11} + \mathscr{U}_{12} + \mathscr{K}_{11}^* + \mathscr{U}_{12}^*).\end{aligned}$$

As ζ is injective, we obtain

$$\mathscr{Z}_{11} + \mathscr{Z}_{12} + \mathscr{Z}_{11}^* + \mathscr{Z}_{12}^* = \mathscr{K}_{11} + \mathscr{U}_{12} + \mathscr{K}_{11}^* + \mathscr{U}_{12}^*.$$

Multiplying the previous relation by $\mathscr{E}_2$ from the right side, we deduce that $\mathscr{Z}_{12} = \mathscr{U}_{12}$. Also, by applying Lemma 2.4 and (11) for $\mathscr{F} = \mathscr{F}_{12}$, a similar proof as above shows that $\mathscr{Z}_{11} = \mathscr{K}_{11}$. Thus, we got the desired result.

In a similar way, we can also demonstrate the other part . □

Lemma 2.6. *For any $\mathscr{K}_{11} \in \mathscr{R}_{11}, \mathscr{U}_{12} \in \mathscr{R}_{12}, \mathscr{V}_{21} \in \mathscr{R}_{21}, \mathscr{W}_{22} \in \mathscr{R}_{22}$, we have*

$$\zeta(\mathscr{K}_{11} + \mathscr{U}_{12} + \mathscr{V}_{21} + \mathscr{Y}_{22}) = \zeta(\mathscr{K}_{11}) + \zeta(\mathscr{U}_{12}) + \zeta(\mathscr{V}_{21}) + \zeta(\mathscr{W}_{22}).$$

Proof. In view of surjectiveness of ζ, there exists $\mathscr{Z} \in \mathscr{R}$ such that

$$\zeta(\mathscr{Z}) = \zeta(\mathscr{K}_{11}) + \zeta(\mathscr{U}_{12}) + \zeta(\mathscr{V}_{21}) + \zeta(\mathscr{W}_{22}). \tag{13}$$

By applying Lemmas 2.1, 2.3 and (13) for $\mathscr{E}_1$ and taking Lemma 2.5 into account, we have

$$\begin{aligned}\zeta(\mathscr{E}_1 \circledast \mathscr{Z}) &= \zeta(\mathscr{E}_1 \circledast \mathscr{K}_{11}) + \zeta(\mathscr{E}_1 \circledast \mathscr{U}_{12}) + \zeta(\mathscr{E}_1 \circledast \mathscr{V}_{21}) + \zeta(\mathscr{E}_1 \circledast \mathscr{W}_{22})\\ &= \zeta(\mathscr{K}_{11} + \mathscr{K}_{11}^*) + \zeta(\mathscr{U}_{12} + \mathscr{U}_{12}^*)\\ &= \zeta(\mathscr{K}_{11} + \mathscr{K}_{11}^* + \mathscr{U}_{12} + \mathscr{U}_{12}^*).\end{aligned}$$

Since ζ is injective, we obtain

$$\mathscr{E}_1\mathscr{Z} + \mathscr{Z}^*\mathscr{E}_1 = \mathscr{K}_{11} + \mathscr{K}_{11}^* + \mathscr{U}_{12} + \mathscr{U}_{12}^*. \tag{14}$$

Similarly, applying Lemmas 2.1, 2.3 and (13) for $i\mathscr{E}_1$ and taking Lemma 2.5 into account, we obtain

$$\mathscr{E}_1\mathscr{Z} - \mathscr{Z}^*\mathscr{E}_1 = \mathscr{K}_{11} - \mathscr{K}_{11}^* + \mathscr{U}_{12} - \mathscr{U}_{12}^*. \tag{15}$$

Solving (14) and (15), we get $\mathscr{Z}_{11} = \mathscr{K}_{11}, \ \ \mathscr{Z}_{12} = \mathscr{U}_{12}$.
Similarly, by applying Lemmas 2.1, 2.3 and (13) for $\mathscr{E}_2$ and the same computation as above, we can easily obtain that $\mathscr{Z}_{21} = \mathscr{V}_{21}$ and $\mathscr{Z}_{22} = \mathscr{W}_{22}$. □

Lemma 2.7. *For any $\mathscr{K}_{ij}, \mathscr{N}_{ij} \in \mathscr{R}_{ij}, \ i, j \in \{1, 2\}$, with $i \neq j$, we have*

$$\zeta(\mathscr{K}_{ij} + \mathscr{N}_{ij}) = \zeta(\mathscr{K}_{ij}) + \zeta(\mathscr{N}_{ij}).$$

Proof. Since ζ is surjective, given $\zeta(\mathscr{K}_{ij}) + \zeta(\mathscr{N}_{ij}) \in \mathscr{C}$ and $\zeta(\mathscr{K}_{ij}^*) + \zeta(\mathscr{N}_{ij}^*) \in \mathscr{C}$, there exist $\mathscr{M}, \mathscr{N} \in \mathscr{R}$ such that $\zeta(\mathscr{M}) = \zeta(\mathscr{K}_{ij}) + \zeta(\mathscr{N}_{ij})$ and $\zeta(\mathscr{N}) = \zeta(\mathscr{K}_{ij}^*) + \zeta(\mathscr{N}_{ij}^*)$. From $\mathscr{K}_{ij} \circledast \mathscr{E}_j = 0 = \mathscr{N}_{ij} \circledast \mathscr{E}_j$ and from Lemmas 2.1 and 2.3, we have

$$\begin{aligned}\zeta(\mathscr{M} \circledast \mathscr{E}_j) &= \zeta(\mathscr{K}_{ij} \circledast \mathscr{E}_j) + \zeta(\mathscr{N}_{ij} \circledast \mathscr{E}_j)\\ &= 0.\end{aligned}$$

By injectiveness of ζ, we obtain $\mathscr{M} \circledast \mathscr{E}_j = 0$, that is, $\mathscr{M}^*\mathscr{E}_j + \mathscr{E}_j\mathscr{M} = 0$. Similarly using $i\mathscr{E}_j$ instead of $\mathscr{E}_j$ in the previous calculation, we get $\mathscr{M}^*\mathscr{E}_j - \mathscr{E}_j\mathscr{M} = 0$. From the last two relations, we get $\mathscr{E}_j\mathscr{M} = 0$. Consequently, $\mathscr{E}_j\mathscr{M}\mathscr{E}_j = \mathscr{E}_j\mathscr{M}\mathscr{E}_i = 0$. Now, repeating the above procedure with $\mathscr{N}$ in place of $\mathscr{M}$ and $\mathscr{E}_i$ instead of $\mathscr{E}_j$, one can see that $\mathscr{E}_i\mathscr{N} = 0$. Hence $\mathscr{E}_i\mathscr{N}\mathscr{E}_i = \mathscr{E}_i\mathscr{N}\mathscr{E}_j = 0$.Therefore, $\mathscr{N} = \mathscr{N}_{ji} + \mathscr{N}_{jj}$ and $\mathscr{M} = \mathscr{M}_{ii} + \mathscr{M}_{ij}$.

Now by using Lemmas 2.3 and 2.6, we have

$$\begin{aligned}\zeta(\mathscr{K}_{ij} + \mathscr{N}_{ij} + \mathscr{K}_{ij}^* + \mathscr{N}_{ij}^*) &= \zeta((\mathscr{E}_i + \mathscr{K}_{ij}^*) \circledast (\mathscr{E}_j + \mathscr{N}_{ij}))\\ &= \{\zeta(\mathscr{E}_i) + \zeta(\mathscr{K}_{ij}^*)\} \circledast \{\zeta(\mathscr{E}_j) + \zeta(\mathscr{N}_{ij})\}\\ &= \zeta(\mathscr{K}_{ij}) + \zeta(\mathscr{K}_{ij}^*) + \zeta(\mathscr{N}_{ij}) + \zeta(\mathscr{N}_{ij}^*)\\ &= \zeta(\mathscr{M}_{ii} + \mathscr{M}_{ij} + \mathscr{N}_{ji} + \mathscr{N}_{jj}).\end{aligned}$$

Since ζ is injective, we have

$$\mathscr{K}_{ij} + \mathscr{N}_{ij} + \mathscr{K}_{ij}^* + \mathscr{N}_{ij}^* = \mathscr{M}_{ii} + \mathscr{M}_{ij} + \mathscr{N}_{ji} + \mathscr{N}_{jj}.$$

Consequently, $\mathscr{K}_{ij} + \mathscr{N}_{ij} = \mathscr{M}_{ij}$ and $\mathscr{M}_{ii} = 0$. □

Lemma 2.8. *For any $\mathscr{K}_{ii}, \mathscr{N}_{ii} \in \mathscr{R}_{ii}$, $i \in \{1, 2\}$, we have*

$$\zeta(\mathscr{K}_{ii} + \mathscr{N}_{ii}) = \zeta(\mathscr{K}_{ii}) + \zeta(\mathscr{N}_{ii}).$$

Proof. Since ζ is onto, there exists $\mathscr{Z} \in \mathscr{R}$ such that

$$\zeta(\mathscr{Z}) = \zeta(\mathscr{K}_{ii}) + \zeta(\mathscr{N}_{ii}). \tag{16}$$

Using Lemmas 2.1, 2.3 and (16) for $\mathscr{E}_j$, we have

$$\begin{aligned}\zeta(\mathscr{E}_j \circledast \mathscr{Z}) &= \zeta(\mathscr{E}_j \circledast \mathscr{K}_{ii}) + \zeta(\mathscr{E}_j \circledast \mathscr{N}_{ii})\\ &= 0.\end{aligned}$$

Injectiveness of ζ gives us that

$$\mathscr{E}_j\mathscr{Z} + \mathscr{Z}^*\mathscr{E}_j = 0. \tag{17}$$

Similarly, using Lemmas 2.1, 2.3 and (16) for $i\mathscr{E}_j$ and solve as above, we get

$$\mathscr{E}_j\mathscr{Z} - \mathscr{Z}^*\mathscr{E}_j = 0. \tag{18}$$

From (17) and (18), we get $\mathscr{Z}_{ji} = \mathscr{Z}_{jj} = 0$.
Applying Lemma 2.1 and (16) for $\mathscr{E}_i$, we get

$$\zeta(\mathscr{E}_i \circledast \mathscr{Z}) = \zeta(\mathscr{E}_i \circledast \mathscr{K}_{ii}) + \zeta(\mathscr{E}_i \circledast \mathscr{N}_{ii}). \tag{19}$$

Also from Lemmas 2.1, 2.3 and (19) for $\mathscr{E}_j$, we have

$$\zeta(\mathscr{E}_j(\mathscr{E}_i \circledast \mathscr{Z}) + (\mathscr{E}_i \circledast \mathscr{Z})^*\mathscr{E}_j) = \zeta(\mathscr{E}_j(\mathscr{E}_i \circledast \mathscr{K}_{ii}) + (\mathscr{E}_i \circledast \mathscr{K}_{ii})^*\mathscr{E}_j) + \zeta(\mathscr{E}_j(\mathscr{E}_i \circledast \mathscr{N}_{ii}) + (\mathscr{E}_i \circledast \mathscr{N}_{ii})^*\mathscr{E}_j).$$

Solving it and using injectivity of ζ, we get $\mathscr{Z}_{ij} = 0$.

Now, applying Lemma 2.1 and (16) for C_{ij}, we obtain

$$\begin{aligned}
&\zeta(\mathscr{Z} \circledast C_{ij}) = \zeta(\mathscr{K}_{ii} \circledast C_{ij}) + \zeta(\mathscr{N}_{ii} \circledast C_{ij}) \\
&\zeta(\mathscr{Z}^* C_{ij} + C_{ij}^* \mathscr{Z}) = \zeta(\mathscr{K}_{ii}^* C_{ij} + C_{ij}^* \mathscr{K}_{ii}) + \zeta(\mathscr{N}_{ii}^* C_{ij} + C_{ij}^* \mathscr{N}_{ii})
\end{aligned} \tag{20}$$

Again, by taking help of Lemma 2.1, (20) for $\mathscr{E}_j$, we can write that

$$\begin{aligned}
\zeta((\mathscr{Z}^* C_{ij} + C_{ij}^* \mathscr{Z}) \circledast \mathscr{E}_j) &= \zeta((\mathscr{K}_{ii}^* C_{ij} + C_{ij}^* \mathscr{K}_{ii}) \circledast \mathscr{E}_j) + \zeta((\mathscr{N}_{ii}^* C_{ij} + C_{ij}^* \mathscr{N}_{ii}) \circledast \mathscr{E}_j) \\
&= \zeta(\mathscr{K}_{ii}^* C_{ij} + C_{ij}^* \mathscr{K}_{ii}) + \zeta(\mathscr{N}_{ii}^* C_{ij} + C_{ij}^* \mathscr{N}_{ii}).
\end{aligned}$$

By Lemma 2.7 and the injectivity of ζ, we have

$$C_{ij}^* \mathscr{Z} P_j + \mathscr{Z}^* C_{ij} + P_j \mathscr{Z}^* C_{ij} + C_{ij}^* \mathscr{Z} = \mathscr{K}_{ii}^* C_{ij} + C_{ij}^* \mathscr{K}_{ii} + \mathscr{N}_{ii}^* C_{ij} + C_{ij}^* \mathscr{N}_{ii}.$$

Multiplying the above equation by $\mathscr{E}_i$ on the left and by $\mathscr{E}_j$ on the right, we find that $\mathscr{Z}_{ii}^* C_{ij} = \mathscr{K}_{ii}^* C_{ij} + \mathscr{N}_{ii}^* C_{ij}$ for all $C_{ij} \in \mathscr{R}_{ij}$. Using (6) and (7), we get $\mathscr{Z}_{ii} = \mathscr{K}_{ii} + \mathscr{N}_{ii}$. □

Lemma 2.9. *ζ is additive.*

Proof. For any $\mathscr{K}, \mathscr{N} \in \mathscr{R}$, we write $\mathscr{K} = \mathscr{K}_{11} + \mathscr{U}_{12} + \mathscr{V}_{21} + \mathscr{W}_{22}$ and $\mathscr{N} = \mathscr{K}_{11}' + \mathscr{U}_{12}' + \mathscr{V}_{21}' + \mathscr{W}_{22}'$. Invoking Lemmas 2.6 – 2.8, we get

$$\begin{aligned}
\zeta(\mathscr{K} + \mathscr{N}) &= \zeta(\mathscr{K}_{11} + \mathscr{U}_{12} + \mathscr{V}_{21} + \mathscr{W}_{22} + \mathscr{K}_{11}' + \mathscr{U}_{12}' + \mathscr{V}_{21}' + \mathscr{W}_{22}') \\
&= \zeta(\mathscr{K}_{11} + \mathscr{K}_{11}') + \zeta(\mathscr{U}_{12} + \mathscr{U}_{12}') + \zeta(\mathscr{V}_{21} + \mathscr{V}_{21}') + \zeta(\mathscr{W}_{22} + \mathscr{W}_{22}') \\
&= \zeta(\mathscr{K}_{11}) + \zeta(\mathscr{K}_{11}') + \zeta(\mathscr{U}_{12}) + \zeta(\mathscr{U}_{12}') + \zeta(\mathscr{V}_{21}) + \zeta(\mathscr{V}_{21}') \\
&\quad + \zeta(\mathscr{W}_{22}) + \zeta(\mathscr{W}_{22}') \\
&= \zeta(\mathscr{K}_{11} + \mathscr{U}_{12} + \mathscr{V}_{21} + \mathscr{W}_{22}) + \zeta(\mathscr{K}_{11}' + \mathscr{U}_{12}' + \mathscr{V}_{21}' + \mathscr{W}_{22}') \\
&= \zeta(\mathscr{K}) + \zeta(\mathscr{N}).
\end{aligned}$$

Thus ζ is additive. □

Lemma 2.10. *ζ preserves '$*$', that is, $\zeta(\mathscr{K}^*) = \zeta(\mathscr{K})^*$ for all $\mathscr{K} \in \mathscr{R}$.*

Proof. From Lemma 2.9 and given hypothesis, we have

$$\begin{aligned}
\zeta(\mathscr{K} \circledast I) &= \zeta(\mathscr{K}) \circledast \zeta(I) \\
\zeta(\mathscr{K}^* I + I\mathscr{K}) &= \zeta(\mathscr{K})^* \zeta(I) + \zeta(I)^* \zeta(\mathscr{K}) \\
\zeta(\mathscr{K}^* + \mathscr{K}) &= \zeta(\mathscr{K})^* + \zeta(\mathscr{K}) \\
\zeta(\mathscr{K}^*) + \zeta(\mathscr{K}) &= \zeta(\mathscr{K})^* + \zeta(\mathscr{K})
\end{aligned}$$

for all $\mathscr{K} \in \mathscr{R}$. Hence, we deduce that $\zeta(\mathscr{K}^*) = \zeta(\mathscr{K})^*$ for all $\mathscr{K} \in \mathscr{R}$. □

Lemma 2.11. *$\zeta(iI)^* = -\zeta(iI)$ and $\zeta(iI)^2 = -I$.*

Proof. From Lemma 2.3 and given hypothesis, we have

$$\begin{aligned}
0 &= \zeta(0) \\
&= \zeta(I(iI) + (iI)^* I) \\
&= \zeta(I)^* \zeta(iI) + \zeta(iI)^* \zeta(I) \\
&= \zeta(iI)^* + \zeta(iI).
\end{aligned}$$

Hence, $\zeta(iI)^* = -\zeta(iI)$.

Also from Lemmas 2.9, 2.10 and given hypothesis, we have

$$\begin{aligned} 2I &= 2\zeta(I) \\ &= \zeta((iI)^*(iI) + (iI)^*(iI)) \\ &= \zeta(iI)^*\zeta(iI) + \zeta(iI)^*\zeta(iI) \\ &= -2\zeta(iI)^2. \end{aligned}$$

Hence $\zeta(iI)^2 = -I$. $\square$

Lemma 2.12. *$\zeta(iI) \in \mathscr{Z}(\mathscr{C})$.*

Proof. From Lemmas 2.3, 2.9, 2.10 and for any $\mathscr{K} = \mathscr{K}^* \in \mathscr{C}$, there is a self-adjoint element $\mathscr{Y}$ such that $\zeta(\mathscr{Y}) = \mathscr{K}$. Then

$$\begin{aligned} \mathscr{K}\zeta(iI) - \zeta(iI)\mathscr{K} &= \mathscr{K}^*\zeta(iI) + \zeta(iI)^*\mathscr{K} \\ &= \zeta(\mathscr{Y})^*\zeta(iI) + \zeta(iI)^*\zeta(\mathscr{Y}) \\ &= \zeta(\mathscr{Y}^*(iI) + (iI)^*\mathscr{Y}) \\ &= 0. \end{aligned}$$

Hence $\mathscr{K}\zeta(iI) = \zeta(iI)\mathscr{K}$.

Also, any $\mathscr{N} \in \mathscr{C}$ can be written as $\mathscr{N} = \mathscr{N}_1 + i\mathscr{N}_2$, where $\mathscr{N}_1 = \frac{\mathscr{N}+\mathscr{N}^*}{2}$ and $\mathscr{N}_2 = \frac{\mathscr{N}-\mathscr{N}^*}{2i}$. Therefore, $\zeta(iI) \in \mathscr{Z}(\mathscr{C})$. $\square$

Lemma 2.13. *$\zeta(i\mathscr{K}) = \zeta(iI)\zeta(\mathscr{K})$ for all $\mathscr{K} \in \mathscr{R}$.*

Proof. For any $\mathscr{K} = \mathscr{K}_1 + i\mathscr{K}_2 \in \mathscr{R}$, where $\mathscr{K}_1 = \frac{\mathscr{K}+\mathscr{K}^*}{2}$ and $\mathscr{K}_2 = \frac{\mathscr{K}-\mathscr{K}^*}{2i}$, we assume that $\zeta(\mathscr{K}_1 + i\mathscr{K}_2) = \mathscr{N}_1 + i\mathscr{N}_2$ where $\mathscr{N}_1 = \frac{\mathscr{N}+\mathscr{N}^*}{2}$ and $\mathscr{N}_2 = \frac{\mathscr{N}-\mathscr{N}^*}{2i}$ for $\mathscr{N} \in \mathscr{C}$. From Lemmas 2.9, 2.10 and given hypothesis, we have

$$\begin{aligned} 2\zeta(\mathscr{K}_1) &= \zeta(2\mathscr{K}_1) \\ &= \zeta(I \circledast (\mathscr{K}_1 + i\mathscr{K}_2)) \\ &= \zeta(I) \circledast \zeta(\mathscr{K}_1 + i\mathscr{K}_2) \\ &= 2\mathscr{N}_1. \end{aligned}$$

Thus $\zeta(\mathscr{K}_1) = \mathscr{N}_1$. Also, we have

$$\begin{aligned} 2\zeta(\mathscr{K}_2) &= \zeta(2\mathscr{K}_2) \\ &= \zeta(iI \circledast (\mathscr{K}_1 + i\mathscr{K}_2)) \\ &= \zeta(iI) \circledast \zeta(\mathscr{K}_1 + i\mathscr{K}_2) \\ &= -2i\zeta(iI)\mathscr{N}_2. \end{aligned}$$

Thus $-i\zeta(iI)\zeta(\mathscr{K}_2) = \mathscr{N}_2$. It follows that $\zeta(\mathscr{K}_1 + i\mathscr{K}_2) = \zeta(\mathscr{K}_1) + \zeta(iI)\zeta(\mathscr{K}_2)$.
Now $\zeta(i\mathscr{K}) = \zeta(i\mathscr{K}_1 - \mathscr{K}_2) = \zeta(iI)\zeta(\mathscr{K}_1) - \zeta(\mathscr{K}_2) = \zeta(iI)\zeta(\mathscr{K})$. Thus $\zeta(i\mathscr{K}) = \zeta(iI)\zeta(\mathscr{K})$ for all $\mathscr{K} \in \mathscr{R}$. $\square$

Lemma 2.14. *ζ preserves product, that is, $\zeta(\mathscr{K}\mathscr{N}) = \zeta(\mathscr{K})\zeta(\mathscr{N})$ for all $\mathscr{K}, \mathscr{N} \in \mathscr{R}$.*

Proof. For any $\mathscr{K}, \mathscr{N} \in \mathscr{R}$, from Lemmas 2.9, 2.10 and 2.13, we have

$$\begin{aligned} \zeta(iI)\zeta(\mathscr{N}^*\mathscr{K} - \mathscr{K}^*\mathscr{N}) &= \zeta(i(\mathscr{N}^*\mathscr{K} - \mathscr{K}^*\mathscr{N})) \\ &= \zeta(\mathscr{N}^*(i\mathscr{K}) + (i\mathscr{K})^*\mathscr{N})) \\ &= \zeta(\mathscr{N})^*\zeta(i\mathscr{K}) + \zeta(i\mathscr{K})^*\zeta(\mathscr{N}) \\ &= \zeta(iI)(\zeta(\mathscr{N})^*\zeta(\mathscr{K}) - \zeta(\mathscr{K})^*\zeta(\mathscr{N})). \end{aligned}$$

Multiplying both sides by $-\zeta(iI)$ and use Lemma 2.11, we have

$$\zeta(\mathcal{N}^*\mathcal{K} - \mathcal{K}^*\mathcal{N}) = \zeta(\mathcal{N})^*\zeta(\mathcal{K}) - \zeta(\mathcal{K})^*\zeta(\mathcal{N}). \quad (21)$$

Also, we have

$$\zeta(\mathcal{N}^*\mathcal{K} + \mathcal{K}^*\mathcal{N}) = \zeta(\mathcal{N})^*\zeta(\mathcal{K}) + \zeta(\mathcal{K})^*\zeta(\mathcal{N}). \quad (22)$$

From (21) and (22), we get $\zeta(\mathcal{N}^*\mathcal{K}) = \zeta(\mathcal{N})^*\zeta(\mathcal{K})$ for every $\mathcal{N}, \mathcal{K} \in \mathcal{R}$. Hence $\zeta(\mathcal{N}\mathcal{K}) = \zeta(\mathcal{N})\zeta(\mathcal{K})$.

Lemma 2.15. *ζ is $\mathbb{R}$-linear.*

Proof. For every rational number q, we have $\zeta(qI) = qI$. Indeed, since q is a rational number, there exist two integers r and s such that $q = \frac{r}{s}$. Since $\zeta(I) = I$ and ζ is additive, we get that

$$\zeta(qI) = \zeta(\frac{r}{s}I) = r\zeta(\frac{1}{s}I) = \frac{r}{s}\zeta(I) = q\zeta(I)$$

Let $\mathcal{X} \in \mathcal{R}$ be a positive element. Then $\mathcal{X} = \mathcal{Y}^2$ for some $\mathcal{Y}^* = \mathcal{Y} \in \mathcal{R}$. Since ζ is a ring $*$-isomorphism, we have $\zeta(\mathcal{X}) = \zeta(\mathcal{Y})^2$ and $\zeta(\mathcal{Y})$ is self-adjoint. So, $\zeta(\mathcal{X})$ is positive. This shows that ζ preserves positive elements. Let $\lambda \in \mathcal{R}$. Choose sequences a_n and b_n of rational numbers such that $a_n \leq \lambda I \leq b_n$ for all n and $lim_{n\to\infty}a_n = lim_{n\to\infty}b_n = \lambda$. It follows from

$$a_nI \leq \lambda I \leq b_nI$$

that

$$a_nI \leq \zeta(\lambda I) \leq b_nI.$$

Taking the limit, we get that $\zeta(\lambda I) = \lambda I$. Hence for all $\mathcal{X} \in \mathcal{R}$,

$$\zeta(\lambda\mathcal{X}) = \zeta((\lambda I)\mathcal{X}) = \zeta(\lambda I)\zeta(\mathcal{X}) = \lambda\zeta(\mathcal{X}).$$

Hence ζ is $\mathbb{R}$-linear. □

Thus in view of Lemmas 2.10, 2.9, 2.14 and 2.15, we conclude that ζ is $\mathbb{R}$-linear ring $*$-isomorphism. This concludes the proof. □

3. Corollaries

Let $\mathcal{R}$ be an algebra. Then $\mathcal{R}$ is prime, if for each $\mathcal{K}, \mathcal{N} \in \mathcal{R}$, $\mathcal{K}\mathcal{R}\mathcal{N} = 0$ implies that either $\mathcal{K} = 0$ or $\mathcal{N} = 0$. Therefore, a prime $*$-algebra trivially satisifies the conditions stated in (6) and (7), as discussed in the remark below.

Remark:- Let $\mathcal{R}$ be a prime algebra. Then, by definiton, for any $\mathcal{K}, \mathcal{N} \in \mathcal{R}$, $\mathcal{K}\mathcal{R}\mathcal{N} = 0$ implies that either $\mathcal{K} = 0$ or $\mathcal{N} = 0$. In particular, taking $\mathcal{N} = \mathcal{E}_1$, we obtain either $\mathcal{K} = 0$ or $\mathcal{E}_1 = 0$. Since $\mathcal{E}_1 \neq 0$, it follows that $\mathcal{K} = 0$, hence (6) holds. In a similar way, we can verify (7). Thus, the following corollary follows:

Corollary 3.1. *Consider the prime $*$-algebras $\mathcal{R}$ and $\mathcal{C}$ with unity such that $\mathcal{R}$ has a nontrivial projection. Let $\zeta : \mathcal{R} \to \mathcal{C}$ be a unital bijective map satisfying $\zeta(\mathcal{K} \circledast \mathcal{N}) = \zeta(\mathcal{K}) \circledast \zeta(\mathcal{N})$ for all $\mathcal{K}, \mathcal{N} \in \mathcal{R}$. Then, ζ is an $\mathbb{R}$-linear ring $*$-isomorphism.*

Assume that $\mathcal{H}$ is a Hilbert space. Then a von Neumann algebra $\mathcal{N}$ defined on $\mathcal{H}$ is an algebra of operators that is self-adjoint, weakly closed, and possesses the identity operator. It should be pointed out that $\mathcal{N}$ is referred to as a factor if the only elements in its center are scalar operators. Clearly every factor von Neumann algebra is prime. Thus, the following corollary follows:

Corollary 3.2. *Let $\mathcal{N}$ and $\mathcal{M}$ be factor von Neumann algebras such that $dim(\mathcal{N}) \geq 2$. Assume that a unital bijective map $\zeta : \mathcal{N} \to \mathcal{M}$ satisfies $\zeta(\mathcal{K} \circledast \mathcal{N}) = \zeta(\mathcal{K}) \circledast \zeta(\mathcal{N})$ for all $\mathcal{K}, \mathcal{N} \in \mathcal{N}$. Then, ζ is an $\mathbb{R}$-linear ring $*$-isomorphism.*

Open Problem

A natural direction for future research is to examine the extent to which the principal results of this study (Theorem 2.1 and the related Lemmas) may be generalized to wider categories of algebraic structures, with particular attention to non-associative rings and algebras, including alternative rings, alternative algebras, W^*-algebras.

Can the main results of our work be extended to non-associative settings, particularly to alternative algebras. Investigating this problem could lead to new insights into the structural properties of non-associative algebras. So we have mentioned a conjecture for future work in this direction as given below.

Conjecture 1 (Extension to alternative $*$-algebras).

Let $\mathscr{R}$ and $\mathscr{C}$ be unital alternative $*$-algebras such that $\mathscr{R}$ has nontrivial projections $\mathscr{E}_i$ $(i = 1, 2)$ satisfying

$$\mathscr{K}\mathscr{R}\mathscr{E}_i = 0 \qquad \Longrightarrow \qquad \mathscr{K} = 0.$$

Suppose that a unital bijective map $\zeta : \mathscr{R} \to \mathscr{C}$ satisfies

$$\zeta(U \circledast V) = \zeta(U) \circledast \zeta(V)$$

for all $U, V \in \mathscr{R}$. Then, ζ is an $\mathbb{R}$-linear ring $*$-isomorphism.

Similarly, can our result be extended in $*$-algebras and still the result holds. For this question, we have mentioned a conjecture as given below.

Conjecture 2 Consider $\mathscr{R}$ and $\mathscr{C}$ as unital $*$-algebras, with $\mathscr{R}$ possessing a nontrivial projection $\mathscr{E}_1$ and satisfying

$$\mathscr{K}\mathscr{R}\mathscr{E}_1 = 0 \qquad \Longrightarrow \qquad \mathscr{K} = 0$$

and

$$\mathscr{K}\mathscr{R}(I - \mathscr{E}_1) = 0 \qquad \Longrightarrow \qquad \mathscr{K} = 0,$$

where $\mathscr{K} \in \mathscr{R}$ and I is an identity of $\mathscr{R}$. Suppose that a unital bijective map $\zeta : \mathscr{R} \to \mathscr{C}$ satisfies

$$\zeta(U_1 \circledast U_2 \circledast \cdots \circledast U_{n-1} \circledast U_n) = \zeta(U_1) \circledast \zeta(U_2) \circledast \cdots \circledast \zeta(U_{n-1}) \circledast \zeta(U_n)$$

for all $U_i : i = 1, 2, \cdots, n \in \mathscr{R}$. Then, ζ is an $\mathbb{R}$-linear ring $*$-isomorphism.

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