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α-limit of scalar-valued functions and related Banach spaces

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Abstract. In this paper, for a set X, a scalar valued function $f : X \to \mathbb{F}$, where $\mathbb{F}$ is assumed to be the set of all real or complex numbers, and for a cardinal number α, we define the α-limit of f as a generalization of the limit of a sequence and denote it by $\lim^{\alpha} f$. Indeed, if we assume that $\alpha = \aleph_0$ and $X = \mathbb{N}$, then $\lim^{\alpha} f$ is equal to $\lim_{n\to\infty} f(n)$. If X is an infinite set and α an infinite cardinal number, then we define two classes of Banach spaces $\mathfrak{c}_0^{\alpha}(X)$ and $\mathfrak{c}^{\alpha}(X)$ containing all bounded functions $f : X \to \mathbb{F}$ which satisfy $\lim^{\alpha} f = 0$ and that $\lim^{\alpha} f$ exists, respectively. Then we show that $\mathfrak{c}_0(X) \subseteq \mathfrak{c}_0^{\alpha}(X) \subseteq \ell^{\infty}(X)$ and $\mathfrak{c}(X) \subseteq \mathfrak{c}^{\alpha}(X) \subseteq \ell^{\infty}(X)$. Also, whenever the cardinal number α increases these spaces become larger such that the equality hold on the left sides for $\alpha = \aleph_0$ and on the right sides for all $\alpha > \mathrm{card}(X)$.

1. Introduction and Preliminaries

In mathematics, there are various generalizations of the usual limit of sequences: for bounded sequences [13, 15] (which is called a Banach limit), for vector-valued functions defined on the topological spaces [16] and as defined in the following definition, for real (or complex) valued functions on infinite "sets".

Definition 1.1. *([1, 3, 4]) Let X be an infinite set (equipped with the discrete topology). Then $l \in \mathbb{F}$ is called the limit of a function $f : X \to \mathbb{F}$, which is denoted by $\lim f = l$, if for each $\epsilon > 0$ there is a finite subset $F \subseteq X$ such that $|f(x) - l| < \epsilon$, for all $x \in X \setminus F$.*

In recent years, there is a big success toward introducing new and developing existing generalizations of limits [13, 14]. Some new generalized limits of the matrix sequences are discussed in [12]. The interest in the theory of Banach limits is renewed and has led to certain applications which have opened new vistas in the structure of Banach spaces. Some new approaches, thoughts and perspectives are presented in [9]. The necessary and sufficient condition for an element $x \in \ell^{\infty}$ to have fixed value Bx for all Cesàro invariant Banach limits B and related results are provided by Semenov and Sukochev in [10] (see also [11]). We refer the reader to recently published monograph [15] about Banach limits and the similar limit generalizations.

In this paper, using cardinal numbers, we extend the concept of the limit presented in Definition 1.1. For our purpose, we need to recall the following preliminaries about the cardinal numbers. Through the paper, we assume that ZFC is consistent.

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The notion of cofinality is widely used in literature (see for example [5, Chapter 9, Definition 2.7]). However, we present below an alternative definition of cofinality of α, which is appropriate and fruitful for our consideration.

Definition 1.2. *([8] Let α be an infinite cardinal number. The cardinality of the smallest set of strictly smaller cardinals of α whose sum is α is called the cofinality of α, i.e.*

$$\mathrm{cf}(\alpha) = \min\{\mathrm{card}(I) : \alpha = \sum_{i\in I} \lambda_i\,,\ \lambda_i < \alpha\}$$

and it is denoted by $\mathrm{cf}(\alpha)$*.*

It is clear from the previous definition that if α is an infinite cardinal number, then $\mathrm{cf}(\alpha) \leq \alpha$.

Definition 1.3. *([6, 8]) A cardinal number α is said regular if* $\mathrm{cf}(\alpha) = \alpha$ *and is said singular if* $\mathrm{cf}(\alpha) < \alpha$*.*

Definition 1.4. *([2], Definition 8) For a cardinal α we define α^+, called the successor of α, to be the least cardinal greater than α. We say that β is a successor cardinal if there is some cardinal α such that $\beta = \alpha^+$. A cardinal $\alpha \neq 0$ is a limit cardinal if α is not a successor.*

Theorem 1.5. *([5], Theorem 2.4, Chapter 9; see also [6]) Every infinite successor cardinal number is regular.*

We will use the notation $\mathfrak{c}_0(X)$ for the set of all functions $f : X \to \mathbb{R}$ with $\lim f = 0$ and the notation $\mathfrak{c}(X)$ for the set of all functions $f : X \to \mathbb{R}$ such that $\lim f$ exists. It is verified in [3] that $\mathfrak{c}_0(X) \subseteq \mathfrak{c}(X) \subseteq \ell^\infty(X)$. Also, $\mathfrak{c}(X)$ and $\mathfrak{c}_0(X)$ are both Banach spaces under the "d-norm" that is defined by $\|f\|_d := \sup(f) - \inf(f)$. Further, on these spaces, $\|\cdot\|_d$ is equivalent to $\|\cdot\|_\infty$.

We organize this paper as follows: Section 2 discusses about the α-limit of functions $f : X \to \mathbb{F}$ as a generalization of the limit of sequences, where α is assumed to be an arbitrary cardinal number. Section 3 is devoted to the existence of α-limit of functions $f : X \to \mathbb{F}$ and its relationship to α-constant functions, where the cardinality α is assumed to be infinite. In Section 4 we try to introduce two Banach spaces $\mathfrak{c}_0^\alpha(X)$ and $\mathfrak{c}^\alpha(X)$ which are defined directly based on the notion of α-limit.

2. α-limit of scalar functions and α-constant functions

In this section, using cardinal numbers, we generalize the concept of the limit of sequences for functions $f : X \to \mathbb{F}$. For a scalar l and a sequence $f = (a_n)$ we know that $\lim_{n\to\infty} a_n = l$ is equivalent to the statement

$$\forall\epsilon > 0\ \ \exists F \subseteq \mathbb{N} : \quad \mathrm{card}(F) < \aleph_0 \ \text{ and } \ \left|f|_{F^c} - l\right| < \epsilon.$$

The natural extension of above presented definition of limit is to enable that the set F may be choosen to be an infinite set. This idea motivate us to introduce the notion α-limit as extension of the notion of the (standard) limit, where cardinal number α is a upper bound of $\mathrm{card}(F)$.

Definition 2.1. *Suppose that α is a cardinal number. Then for each $f : X \to \mathbb{F}$, $l \in \mathbb{F}$ is said to be an α-limit of f, and denoted by $\lim^\alpha f = l$, if for each $\epsilon > 0$ there is $A \subseteq X$ such that* $\mathrm{card}(A) < \alpha$ *and*

$$\left|f|_{A^c} - l\right| < \epsilon.$$

Notice that from the previous definition, for a given sequence $f = (a_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} a_n$ exists if and only if $\lim^{\aleph_0} f$ exists. Also, in this case $\lim_{n\to\infty} a_n = \lim^{\aleph_0} f$.

Remark 2.2. *Let X be an infinite set and $f : X \to \mathbb{F}$. For $\alpha \leq$* $\mathrm{card}(X)$*, as a consequence of triangular inequality, it implies that if the α-limit of $f : X \to \mathbb{F}$ exists, then it is unique. However, if $\alpha >$* $\mathrm{card}(X)$*, then the uniqueness does not hold. In fact, every $l \in \mathbb{F}$ is an α-limit of any given function $f : X \to \mathbb{F}$.*

Definition 2.3. *Let $f : X \to Y$ and $\alpha \leq \operatorname{card}(X)$. Then f is said to be α-constant if there exists $A \subseteq X$ with $\operatorname{card}(A) = \alpha$ such that f is constant on A^c; i.e., $f|_{A^c}$, the restriction of f to A^c is constant. We denote by $\operatorname{cst}^\alpha(X)$ for the set of all bounded and α-constant functions $f : X \to \mathbb{F}$.*

We recall here that a function $f : X \to Y$ is called simple if the image of f is a finite subset of Y. Using Definition 2.3, it follows directly that for a function $f : X \to Y$ we have that

(i) if f is an α-constant function for some $\alpha \in \{0, 1, 2, \ldots\}$, then f is a simple function;

(ii) f is always an α-constant function whenever $\alpha = \operatorname{card}(X)$;

(iii) if $\alpha \leq \beta \leq \operatorname{card}(X)$, then $\operatorname{cst}^\alpha(X) \subseteq \operatorname{cst}^\beta(X)$.

Example 2.4. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function defined by $f(x) = x^2\chi_{\mathbb{Q}}(x)$. Let $\epsilon > 0$ be given. By considering $A = \mathbb{Q}$ and $c = \aleph_1$, the cardinality of the continuum, we have $\operatorname{card}(A) = \aleph_0 < c$, and $|f(x)| = 0 < \epsilon$ for each $x \in \mathbb{Q}^c$. Thus $\lim^c f = 0$. This example also implies that the existence of $\lim^\alpha f$ does not necessarily follow that f is bounded.*

It is not difficult to show the following theorem which determines α-constant functions when α is finite.

Theorem 2.5. *Let X be an infinite set and $N \in \mathbb{N}$. Then the N-limit of a function $f : X \to \mathbb{F}$ exists if and only if f is $(N-1)$-constant function.*

The next results, which are well-known tools in mathematical analysis regards to standard definition of the sequence limit, will be provided for α-limit.

Theorem 2.6. *(Squeeze theorem) Let X be an infinite set and let α be a cardinal number with $\alpha \leq \operatorname{card}(X)$. Suppose that $f, g, h : X \to \mathbb{R}$ are functions which satisfy $\lim^\alpha g = l$ and $\lim^\alpha h = l$. If*

$$g(x) \leq f(x) \leq h(x) \quad \text{for each} \quad x \in X, \tag{1}$$

then

i) $\lim^\alpha f = l$ *whenever α is infinite.*

ii) $\lim^{2\alpha-1} f = l$ *whenever α is finite.*

Proof. Let $\epsilon > 0$. By definition, there are sets $A_1, A_2 \subseteq X$ such that $\operatorname{card}(A_1) < \alpha$ and $\operatorname{card}(A_2) < \alpha$, and

$$\left|g|_{A_1^c} - l\right| < \epsilon \text{ and } \left|h|_{A_2^c} - l\right| < \epsilon.$$

Let $A = A_1 \cup A_2$. We get

$$\left|g|_{A^c} - l\right| < \epsilon \text{ and } \left|h|_{A^c} - l\right| < \epsilon.$$

Hence,

$$l - \epsilon < g(x) \leq f(x) \leq h(x) < l + \epsilon$$

for each $x \in A^c$. Therefore,

$$\left|f|_{A^c} - l\right| < \epsilon.$$

i) If α is infinite, then $\operatorname{card}(A) < \alpha$, therefore $\lim^\alpha f = l$.

ii) If α is finite, then $\operatorname{card}(A_1) \leq \alpha - 1$ and $\operatorname{card}(A_2) \leq \alpha - 1$. So, $\operatorname{card}(A) = \operatorname{card}(A_1 \cup A_2) \leq 2\alpha - 2 < 2\alpha - 1$. Therefore, $\lim^{2\alpha-1} f = l$.

□

Proposition 2.7. *Let X be an infinite set and let α be a cardinal number with $\alpha \leq \operatorname{card}(X)$ and suppose that $g, h : X \to \mathbb{R}$ and $l \in \mathbb{R}$. If $g(x) \leq l \leq h(x)$ for each $x \in X$ and $\lim^\alpha(h - g) = 0$, then $\lim^\alpha h = \lim^\alpha g = l$.*

Proof. The proof may be obtained in the similar way as in the above corollary using facts that for all $x \in X$ we have $|h(x) - l| \leq |h(x) - g(x)|$ and $|g(x) - l| \leq |h(x) - g(x)|$.

□

3. Existence of α-limit of a scalar-valued function when α is infinite

In this section, we will find a relationship between Banach cardinals and α-constant functions. Here, we use the phrase"Banach Cardinal" for the cardinality of the algebraic dimension of a Banach space.

Theorem 3.1. *Let X be an infinite set and $\alpha = \alpha_0^+$ be an infinite successor cardinal number with $\alpha \leq \operatorname{card}(X)$. Then for a given function $f : X \to \mathbb{F}$ the following assertions are equivalent:*

(i) $\lim^{\alpha} f$ *exists;*

(ii) f *is α_0-constant function;*

(iii) f *is β-constant for some cardinal number $0 \leq \beta < \alpha$.*

Proof. (i) ⇒ (ii) Suppose that $\lim^{\alpha} f = l \in \mathbb{F}$. Then for each $n \in \mathbb{N}$ there is $A_n \subseteq X$ such that $\operatorname{card}(A_n) < \alpha$ and $\left|f|_{A_n^c} - l\right| < \frac{1}{n}$. Since α is successor, it implies that $\operatorname{card}(A_n) \leq \alpha_0$. Now, by considering $A = \cup_{n\in\mathbb{N}} A_n \subseteq X$ we have $\operatorname{card}(A) \leq \operatorname{card}(A_1) + \operatorname{card}(A_2) + \cdots \leq \aleph_0 \alpha_0 = \alpha_0$, where the last equality holds since α_0 is infinite. Thus for each $x \in A^c$ we have

$$\left|f(x) - l\right| < \frac{1}{n}, \quad \text{for all } n \in \mathbb{N},$$

which implies $f|_{A^c} = l$; i.e., f is α_0-constant.
(ii) ⇒ (iii) Let $f \in \operatorname{cst}^{\alpha_0}(X)$. Then for $\beta = \alpha_0$ we have $0 \leq \beta = \alpha_0 < \alpha_0^+ = \alpha$. Therefore f is $\beta-$constant.
(iii) ⇒ (i) Suppose that $f \in \operatorname{cst}^{\beta}(X)$ for some $0 \leq \beta < \alpha$. Then there is $B \subseteq X$ such that $\operatorname{card}(B) = \beta$ and $f|_{B^c}$ is a constant function with constant value $l \in \mathbb{F}$. Because $\beta < \alpha \leq \operatorname{card}(X)$ it follows that $B \subsetneq X$. Now, for any given $\epsilon > 0$, if we set $A = B \subseteq X$, then $\operatorname{card}(A) = \operatorname{card}(B) = \beta < \alpha$ and $\left|f|_{A^c} - l\right| = 0 < \epsilon$. □

Notice that the previous theorem is not necessarily true when α is a limit cardinal number. For example, in the sequence (a_n) defined by $a_n = \frac{1}{n}$ we have $\lim^{\aleph_0}(a_n) = \lim a_n = 0$, whereas (a_n) is not β-constant for each $0 \leq \beta < \aleph_0$. Indeed, the next theorem shows that for some limit cardinalities, the previous theorem remains true.

Theorem 3.2. *Let X be an infinite set and α be an infinite cardinal number with $\aleph_0 < \operatorname{cf}(\alpha)$. Then for a given function $f : X \to \mathbb{F}$, $\lim^{\alpha} f$ exists if and only if f is β-constant for some cardinal number $0 \leq \beta < \alpha$.*

Proof. Let $\lim^{\alpha} f = l \in \mathbb{F}$. Similar to the proof of Theorem 3.1, for each $n \in \mathbb{N}$ there is $A_n \subseteq X$ such that $\operatorname{card}(A_n) < \alpha$ and $\left|f|_{A_n^c} - l\right| < \frac{1}{n}$. Since α is infinite, this yields $\operatorname{card}(A) = \operatorname{card}(\cup_{n\in\mathbb{N}} A_n) \leq \sum_{n=1}^{\infty} \operatorname{card}(A_n) \leq \aleph_0 \alpha = \alpha$, where $A = \cup_{n\in\mathbb{N}} A_n$. Now, if $\alpha = \sum_{n=1}^{\infty} \operatorname{card}(A_n)$, then $\operatorname{cf}(\alpha) \leq \aleph_0$, which contradicts the hypothesis. Thus, $\operatorname{card}(A) < \alpha$. Now if $x \in A^c$ then

$$\left|f(x) - l\right| < \frac{1}{n}, \quad \text{for all } n \in \mathbb{N},$$

which implies $f|_{A^c} = l$. Thus f is β-constant for $\beta = \operatorname{card}(A)$.
To prove the converse of theorem, suppose that $f \in \operatorname{cst}^{\beta}(X)$ for some $0 \leq \beta < \alpha$. Then there are $B \subseteq X$ and $l \in \mathbb{F}$ such that $\operatorname{card}(B) = \beta$ and $f|_{B^c} = l$. Now for $\epsilon > 0$ if we set $A := B \subseteq X$, then $\operatorname{card}(A) = \operatorname{card}(B) = \beta < \alpha$ and $\left|f|_{A^c} - l\right| = 0 < \epsilon$. □

Remark 3.3. *If α is an infinite cardinal number that is a successor, then it follows from Theorem 1.5 that $\operatorname{cf}(\alpha) = \alpha > \aleph_0$. So, Theorem 3.2 is a generalization of Theorem 3.1.*

Definition 3.4. *A cardinal number α is said to be a Banach cardinal if $\alpha^{\aleph_0} = \alpha$ or α is a finite cardinality.*

The next theorem shows that the algebraic dimension of a Banach space is always a Banach cardinal.

Theorem 3.5. *([7], Corollary 2.4) Suppose that X is a linear space. Then X is a Banach space under some norm if and only if* $\dim(X)$*, the algebraic dimension of X, is a Banach cardinal.*

Theorem 3.6. *([5] Theorems 3.8 and 3.10, Chapter 9) Assume the generalized continuum hypothesis. Suppose that* $\aleph_\gamma$ *and* $\aleph_\beta$ *are two (infinite) cardinal numbers.*

(i) *If* $\aleph_\gamma$ *is regular, then* $\aleph_\gamma^{\aleph_\beta} = \begin{cases} \aleph_\gamma, & \beta < \gamma, \\ \aleph_{\beta+1}, & \beta \geq \gamma. \end{cases}$

(ii) *If* $\aleph_\gamma$ *is singular, then* $\aleph_\gamma^{\aleph_\beta} = \begin{cases} \aleph_\gamma, & \aleph_\beta < \mathrm{cf}(\aleph_\gamma), \\ \aleph_{\gamma+1}, & \mathrm{cf}(\aleph_\gamma) \leq \aleph_\beta \leq \aleph_\gamma, \\ \aleph_{\beta+1}, & \aleph_\beta \geq \aleph_\gamma. \end{cases}$

An ordinal ([5]) is a set α with a transitive well-ordering relation $<$.

Lemma 3.7. *([5], Lemma 3.9, Chapter 9) If* $\alpha > 1$ *is a cardinal number and* γ *is an ordinal, then* $\mathrm{cf}(\alpha^{\aleph_\gamma}) > \aleph_\gamma$.

The following result gives an equivalent condition for α to be a Banach cardinal and can be obtained easily from Theorem 3.6, Lemma 3.7 and by considering two cases: α is regular and singular.

Remark 3.8. *An infinite cardinal* $\alpha = \aleph_\gamma$ *is a Banach cardinal if and only if* $\mathrm{cf}(\alpha) > \aleph_0$.

Theorem 3.9. *Let X be an infinite set,* α *be a Banach cardinal with* $\alpha \leq \mathrm{card}(X)$*. Then the* α*-limit of a function* $f : X \to \mathbb{F}$ *exists if and only if f is* β*-constant function for some* $0 \leq \beta < \alpha$.

Proof. If α is infinite, then the result obtains from Theorem 3.2 and Remark 3.8. If α is finite, then it follows from Theorem 2.5. □

Let X be an infinite set and $\alpha \leq \mathrm{card}(X)$. Then we define $\mathfrak{c}^\alpha(X)$ to be the set of all bounded functions $f : X \to \mathbb{F}$ such that $\lim^\alpha f$ exists.

It is clear that $\cup_{\beta<\alpha}\mathrm{cst}^\beta(X) \subseteq \mathfrak{c}^\alpha(X)$. Indeed, as the following theorem shows, when α is a Banach cardinal, then the equality holds.

Theorem 3.10. *Let X be an infinite set and* $\alpha \neq 0$ *be a cardinal number with* $\alpha \leq \mathrm{card}(X)$*. Then the following conditions are equivalent:*

(i) *α is a Banach cardinal;*

(ii) $\cup_{\beta<\alpha}\mathrm{cst}^\beta(X) = \mathfrak{c}^\alpha(X)$.

Proof. When α is finite, the implication (i) ⇒ (ii) follows by Theorem 2.5. When α is infinite, the above implication follows by Theorem 3.9.
Case: (ii) ⇒ (i). When α is finite, then it is a Banach cardinal by definition.

Let α be an infinite cardinal number. Suppose on the contrary, let α be an infinite cardinal number which is not a Banach cardinal. Then α must be a limit cardinal that is not a successor and $\mathrm{cf}(\alpha) = \aleph_0$. Truly, if α is an infinite cardinal number that is a successor, then it follows from Theorem 1.5 that $\mathrm{cf}(\alpha) = \alpha > \aleph_0$ which implies that α is a Banach cardinal, by Remark 3.8. Note that the cofinality of an infinite cardinal number is infinite. Thus, there are cardinal numbers $\alpha_1, \alpha_2, \ldots$ such that $\alpha_n < \alpha$ and $\alpha = \sum_{n\in\mathbb{N}} \alpha_n$. Therefore, $\alpha = \sup_{n\in\mathbb{N}}(\alpha_1 + \cdots + \alpha_n)$. Since $\alpha \leq \mathrm{card}(X)$, there is $X_1 \subseteq X$ with $\mathrm{card}(X_1) = \alpha$. Now we consider a partition $\{A_1, A_2, \ldots\}$ of X_1 such that $\mathrm{card}(A_n) = \alpha_n$, for each n. Then for the function $f : X \to \mathbb{F}$, defined by

$$f(x) = \begin{cases} \frac{1}{n}, & x \in A_n, \text{ for some } n \in \mathbb{N}, \\ 0, & x \in X \setminus X_1, \end{cases}$$

it is clear that f is bounded and $\lim^\alpha f = 0$. Therefore, $f \in \mathfrak{c}^\alpha(X)$. On the other hand, it is easily verified that $f \notin \mathrm{cst}^\beta(X)$ for all $\beta < \alpha$. Thus, we have $f \in \mathfrak{c}^\alpha(X) \setminus \cup_{\beta<\alpha}\mathrm{cst}^\beta(X)$, which is a contradiction with statement (ii). □

4. Banach spaces $\mathfrak{c}_0^\alpha(X)$ and $\mathfrak{c}^\alpha(X)$

The goal of this section is to prove the completeness of the norms for both spaces $\mathfrak{c}_0^\alpha(X)$ and $\mathfrak{c}^\alpha(X)$. For an infinite set X and a cardinal number α with $\alpha \leq \operatorname{card}(X)$, we mention again that

$$\mathfrak{c}_0^\alpha(X) := \{f : X \to \mathbb{F} : f \text{ is a bounded function with } \lim^\alpha f = 0\},$$

and

$$\mathfrak{c}^\alpha(X) := \{f : X \to \mathbb{F} : f \text{ is a bounded function with } \lim^\alpha f \text{ exists}\}.$$

It is clear that $\mathfrak{c}_0^\alpha(X) \subseteq \mathfrak{c}^\alpha(X)$, and for each $f \in \mathfrak{c}^\alpha(X)$, by using Remark 2.2, $\lim^\alpha f$ is unique. Also, if α and β are both cardinal numbers with $\alpha \leq \beta$, then we have

$$\mathfrak{c}_0^\alpha(X) \subseteq \mathfrak{c}_0^\beta(X), \quad \mathfrak{c}^\alpha(X) \subseteq \mathfrak{c}^\beta(X), \quad \mathfrak{c}_0^\alpha(X) \subseteq \mathfrak{c}^\alpha(X). \tag{2}$$

In the following of this paper, α is assumed an infinite cardinal number and X is an infinite set.

It is not difficult to show the following theorem which states that $\lim^\alpha$ is a linear and multiplicative functional on $\mathfrak{c}^\alpha(X)$.

Theorem 4.1. *(Algebraic properties of α-limit) If $l_1 = \lim^\alpha f$ and $l_2 = \lim^\alpha g$, then*

(i) $l_1 l_2 = \lim^\alpha (fg)$,

(ii) $l_1 + \lambda l_2 = \lim^\alpha (f + \lambda g)$, *for each* $\lambda \in \mathbb{F}$.

Remark 4.2. *It follows from* (2) *and Theorem 4.1 that* $\mathfrak{c}_0^\alpha(X)$ *and* $\mathfrak{c}^\alpha(X)$ *are two (normed) linear subspaces of* $\ell^\infty(X)$. *Indeed,*

(i) $\mathfrak{c}_0(X) \subseteq \mathfrak{c}_0^\alpha(X) \subseteq \ell^\infty(X)$, *and*

(ii) $\mathfrak{c}(X) \subseteq \mathfrak{c}^\alpha(X) \subseteq \ell^\infty(X)$.

Also, for $\alpha = \aleph_0$, the equality holds in the left hand sides of (i) *and* (ii).

We note that both of the spaces $\mathfrak{c}_0(X)$ and $\mathfrak{c}_0^\alpha(X)$ equipped with $\|\cdot\|_\infty$ are normed linear spaces, and therefore, a sequence (x_n) in each of these spaces is a Cauchy sequence if it is Cauchy in the space $(\ell^\infty(X), \|\cdot\|_\infty)$.

Lemma 4.3. *Suppose that $\alpha \leq \operatorname{card}(X)$. If (f_n) is a Cauchy sequence in $\mathfrak{c}^\alpha(X)$ and (l_n) is a sequence of scalars defined as $l_n := \lim^\alpha f_n$, then (l_n) converges.*

Proof. Let $\epsilon > 0$ be arbitrary. Since $f_n \in \mathfrak{c}^\alpha(X)$, it follows that there is $A_n \subseteq X$ with $\operatorname{card}(A_n) < \alpha$ such that

$$\left| f_n|_{A_n^c} - l_n \right| < \frac{\epsilon}{3}. \tag{3}$$

Because (f_n) is a Cauchy sequence, there is $N \in \mathbb{N}$ such that

$$\|f_m - f_n\|_\infty < \frac{\epsilon}{3}, \quad (m, n \geq N). \tag{4}$$

Now, suppose that $m, n \geq N$. It is clear that $A_m \cup A_n \subsetneq X$. Choose $x \in X \setminus (A_m \cup A_n)$, and then using (3) and (4) we obtain that

$$|l_m - l_n| \leq |f_m(x) - l_m| + |f_m(x) - f_n(x)| + |f_n(x) - l_n| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus, the sequence (l_n) in $\mathbb{F}$ is Cauchy, therefore, it is convergent. $\square$

Theorem 4.4. *The normed linear space $\mathfrak{c}^\alpha(X)$ is complete, i.e., it is a Banach space.*

Proof. If the cardinal number α satisfies $\alpha > \mathrm{card}(X)$, then $\mathfrak{c}^\alpha(X) = \ell^\infty(X)$. So, in this case the result is clear. Otherwise, suppose that $\alpha \leq \mathrm{card}(X)$. Let (f_n) be a Cauchy sequence in $\mathfrak{c}^\alpha(X)$ and $\epsilon > 0$ be arbitrary. Since $\mathfrak{c}^\alpha(X) \subseteq \ell^\infty(X)$, the sequence (f_n) converges to some $f \in \ell^\infty(X)$. If $l_n = \lim^\alpha f_n$, then from Lemma 4.3, there exists $N \in \mathbb{N}$ such that

$$\|f_n - f\|_\infty < \frac{\epsilon}{3}, \quad \text{and} \quad |l_n - l| < \frac{\epsilon}{3}, \tag{5}$$

hold for all $n \geq N$. Since $f_N \in \mathfrak{c}^\alpha(X)$, we can choose $A \subseteq X$ such that

$$\mathrm{card}(A) < \alpha, \quad \text{and} \quad \left| f_N|_{A^c} - l_N \right| < \frac{\epsilon}{3}. \tag{6}$$

If we choose $x \in A^c$, then by using the relations (5) and (6) we have

$$\begin{aligned} \left| f|_{A^c}(x) - l \right| &= |f(x) - l| \\ &\leq \left| f(x) - f_N(x) \right| + \left| f_N(x) - l_N \right| + \left| l_N - l \right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore, $\lim^\alpha f = l$. Moreover, f is bounded since $f \in \ell^\infty(X)$. Thus (f_n) converges to $f \in \mathfrak{c}^\alpha(X)$. □

The following result is obtained from the proof of the previous theorem and shows that whenever $f_n \to f$ in $\mathfrak{c}^\alpha(X)$, then the α-limit of f_n also, converges to the α-limit of f.

Corollary 4.5. *Let $\alpha \leq \mathrm{card}(X)$ and (f_n) is a sequence in $\mathfrak{c}^\alpha(X)$ converging to $f \in \mathfrak{c}^\alpha(X)$. Then*

$$\lim_{n\to\infty} \lim{}^\alpha f_n = \lim{}^\alpha f.$$

Corollary 4.6. *The normed linear space $\mathfrak{c}_0^\alpha(X)$ is a Banach space.*

Proof. Let (f_n) be a Cauchy sequence in $\mathfrak{c}_0^\alpha(X)$. Since $\mathfrak{c}_0^\alpha(X) \subseteq \mathfrak{c}^\alpha(X)$, it follows from Theorem 4.4 that there is $f \in \mathfrak{c}^\alpha(X)$ such that $f_n \to f$. On the other hand, according to Corollary 4.5 we have

$$\lim{}^\alpha f = \lim_{n\to\infty} \lim{}^\alpha f_n = \lim_{n\to\infty} 0 = 0.$$

Thus $f \in \mathfrak{c}_0^\alpha(X)$. □

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