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Extremal problems on a unified family of k-uniformly starlike and convex functions with complex order

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Abstract. This paper introduces and studies a new subclass of analytic functions, denoted $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$, defined via the Hadamard product with a fixed function g. Governed by parameters $\delta \in [0,1]$, $\kappa \geq 0$, $b \in \mathbb{C} \setminus 0$, and $\lambda \in [0,1)$, this class generalizes the concepts of k-uniformly starlike and convex functions of complex order. We establish several foundational properties for this class, including sharp coefficient bounds, radii of starlikeness and convexity, and extreme points. Furthermore, we prove subordination results, resolve Silverman's conjecture on integral means inequalities, and investigate the invariance under integral operators. Our results unify and extend numerous established theorems in geometric function theory.

1. Introduction

Let $\mathcal{S}$ represent the family of functions analytic and univalent in the open unit disk $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, and of the form

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n. \tag{1}$$

For $\kappa \geq 0$ and $\lambda \in [0,1)$, we define $\mathcal{UST}(\kappa, \lambda)$ to be the family of k-uniformly starlike functions of order parameter λ which consists of $f \in \mathcal{S}$ satisfying the inequality

$$\operatorname{Re}\left(\frac{\zeta f'(\zeta)}{f(\zeta)} - \lambda\right) > \kappa \left|\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right| \quad (\zeta \in \mathbb{D}),$$

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Analogously, $f \in \mathcal{S}$ is said to lie in the family $\mathcal{UCV}(\kappa, \lambda)$ of k-uniformly convex functions of order parameter λ if

$$\operatorname{Re}\left(1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} - \lambda\right) > \kappa \left|\frac{\zeta f''(\zeta)}{f'(\zeta)}\right| \quad (\zeta \in \mathbb{D}).$$

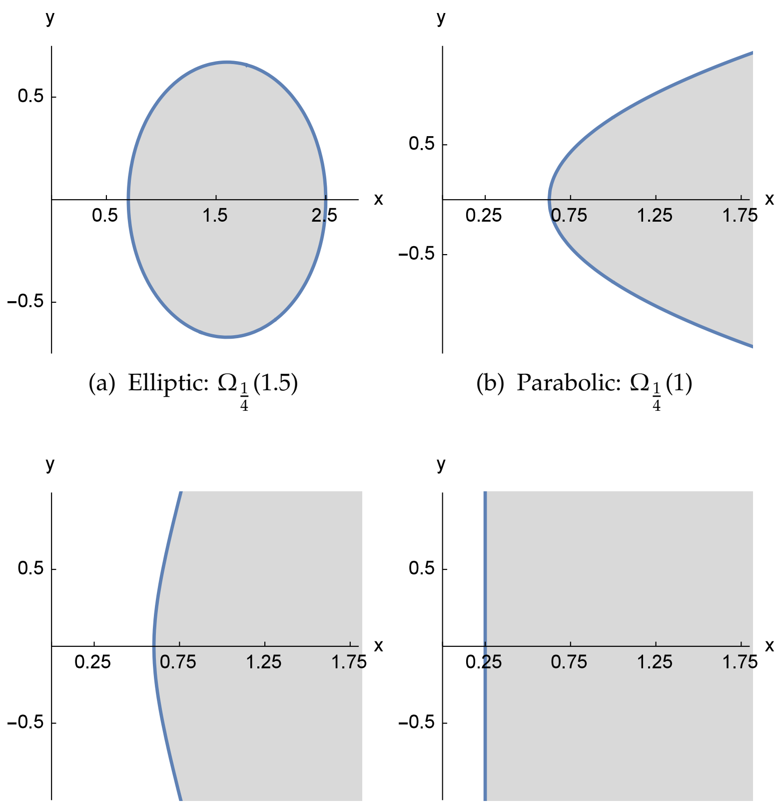
From a geometric perspective, a function f belongs to the class $\mathcal{UST}(\kappa, \lambda)$ precisely when, for every $\zeta \in \mathbb{D}$, the logarithmic derivative

$$\frac{\zeta f'(\zeta)}{f(\zeta)}$$

remains confined to the region

$$\Omega_\lambda(\kappa) := \left\{x + iy \in \mathbb{C} : x > \kappa \sqrt{(x-1)^2 + y^2} + \lambda\right\}.$$

The parameter κ governs the geometry of $\Omega_\lambda(\kappa)$: the boundary is elliptic for $\kappa > 1$, parabolic when $\kappa = 1$, hyperbolic for $0 < \kappa < 1$, and degenerates to the half-plane $\operatorname{Re}(w) > \lambda$ in the limiting case $\kappa = 0$. Despite this variety of forms, all these regions share a common feature—they are convex and lie entirely within the right half-plane. To visualize this geometric evolution, we fix $\lambda = \frac{1}{4}$ and depict in Figure 1 the regions $\Omega_\lambda(\kappa)$ corresponding to $\kappa = \frac{3}{2}, 1, \frac{1}{4}$, and 0.



(a) Elliptic: $\Omega_{\frac{1}{4}}(1.5)$ (b) Parabolic: $\Omega_{\frac{1}{4}}(1)$

(c) Hyperbolic: $\Omega_{\frac{1}{4}}(0.25)$ (d) Half-plane: $\Omega_{\frac{1}{4}}(0)$

Figure 1: Illustration of the region $\Omega_{\frac{1}{4}}(\kappa)$ for different values of κ.

These classes generalize several classical function families. Specifically, $\mathcal{UST}(0, \lambda)$ and $\mathcal{UCV}(0, \lambda)$ reduce to the well-known starlike and convex function classes of order λ, respectively, see [25]. Moreover,

$\mathcal{UST} := \mathcal{UST}(1,0)$ and $\mathcal{UCV} := \mathcal{UCV}(1,0)$ correspond to the uniformly starlike and uniformly convex functions studied in [12, 13, 26, 27, 6].

Aqlan et al. [4] introduced a broader class $\mathcal{U}(\kappa,\delta,\lambda)$ characterized by the condition

$$\mathrm{Re}\left(\frac{\zeta f'(\zeta)+\delta\zeta^2 f''(\zeta)}{(1-\delta)f(\zeta)+\delta\zeta f'(\zeta)}\right) > \kappa\left|\frac{\zeta f'(\zeta)+\delta\zeta^2 f''(\zeta)}{(1-\delta)f(\zeta)+\delta\zeta f'(\zeta)}-1\right|+\lambda,$$

where $\kappa \geq 0$, $0 \leq \lambda < 1$, and $0 \leq \delta \leq 1$. This formulation generalizes the previously discussed classes by interpolating between the k-uniformly starlike ($\delta = 0$) and convex ($\delta = 1$) subclasses.

Definition 1.1 ([19, 20, 21]). *A function $f \in \mathcal{S}$, expressed as in (1), is termed starlike of complex order parameter $b \in \mathbb{C}^* := \mathbb{C}\setminus\{0\}$ and a type parameter $\lambda \in [0,1)$ if it satisfies*

$$\mathrm{Re}\left(1+\frac{1}{b}\left(\frac{\zeta f'(\zeta)}{f(\zeta)}-1\right)\right) > \lambda \quad (\zeta \in \mathbb{D}),$$

and convex of complex order parameter b and a type parameter λ if

$$\mathrm{Re}\left(1+\frac{1}{b}\frac{\zeta f''(\zeta)}{f'(\zeta)}\right) > \lambda \quad (\zeta \in \mathbb{D}).$$

Definition 1.2 (Hadamard Product [9]). *If f is as in (1) and $g \in \mathcal{S}$ is given by*

$$g(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n\zeta^n \quad (b_n \geq 0;\ \zeta \in \mathbb{D}), \tag{2}$$

then their Hadamard product, denoted $f * g$, is defined by

$$(f*g)(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n b_n \zeta^n = (g*f)(\zeta).$$

It is well known that the convex function

$$p(\zeta) = \frac{\zeta}{1-\zeta} = \zeta + \sum_{n=2}^{\infty}\zeta^n$$

acts as the identity element under the Hadamard product, in the sense that $(f*p)(\zeta) = f(\zeta)$ for any analytic function f.

Using Hadamard product, the authors in [24] presented $\mathcal{S}_g(\kappa,\lambda)$ which comprises of functions f satisfying $(f*g)(\zeta) \neq 0$ in $\mathbb{D}$ and

$$\Re\left(\frac{\zeta(f*g)'(\zeta)}{(f*g)(\zeta)}-\lambda\right) > \kappa\left|\frac{\zeta(f*g)'(\zeta)}{(f*g)(\zeta)}-1\right|,$$

This family was further extended by Aouf et al. [3]. More recently, Bukhari et al. [7] presented a more unified generalization by incorporating complex order and defining the family $\mathcal{U}_g(\kappa,b,\delta)$ as follows:

Definition 1.3 ([7]). *Suppose $f(\zeta)$ has the representation* (1). *Further, let $\kappa \geq 0$, $b \in \mathbb{C}^*$ and $\delta \in [0,1]$. Then $f \in \mathcal{U}_g(\kappa,b,\delta)$ if, for $g(\zeta)$ given by (2) such that $\Upsilon(\zeta) := (f*g)(\zeta) \neq 0$, the following inequality holds:*

$$\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{\zeta\Upsilon'(\zeta)+\delta\zeta^2\Upsilon''(\zeta)}{(1-\delta)\Upsilon(\zeta)+\delta\zeta\Upsilon'(\zeta)}-1\right)\right\} > \kappa\left|\frac{1}{b}\left(\frac{\zeta\Upsilon'(\zeta)+\delta\zeta^2\Upsilon''(\zeta)}{(1-\delta)\Upsilon(\zeta)+\delta\zeta\Upsilon'(\zeta)}-1\right)\right|.$$

In this work, we build upon these ideas to define a broader class $\mathcal{U}_g(\kappa,b,\delta,\lambda)$, which unifies and generalizes the aforementioned subclasses. We define:

Definition 1.4. *Let $f(\zeta)$ be of the form (1), and $g(\zeta)$ as in (2). Further, let $\kappa \geq 0$, $b \in \mathbb{C}^*$, $\delta \in [0,1]$, and $\lambda \in [0,1)$. Then f belongs to the class $\mathcal{U}_g(\kappa, b, \delta, \lambda)$ if the following holds*

$$\operatorname{Re}\left\{(1-\lambda) + \frac{1}{b}\left(\frac{\zeta\Upsilon'(\zeta) + \delta\zeta^2\Upsilon''(\zeta)}{(1-\delta)\Upsilon(\zeta) + \delta\zeta\Upsilon'(\zeta)} - 1\right)\right\} > \kappa\left|\frac{1}{b}\left(\frac{\zeta\Upsilon'(\zeta) + \delta\zeta^2\Upsilon''(\zeta)}{(1-\delta)\Upsilon(\zeta) + \delta\zeta\Upsilon'(\zeta)} - 1\right)\right|.$$

Remark 1.5. *By judiciously selecting $\kappa, b, \delta, \lambda$, and the function $g(\zeta)$, the family $\mathcal{U}_g(\kappa, b, \delta, \lambda)$ includes many well-known subclasses studied earlier. For instance, taking $g(\zeta) = \zeta/(1-\zeta)$, the class reduces to several prominent forms previously investigated by Altintas et al. [1], and Aqlan et al. [4]. Similarly, by choosing g as well-known operators like the Ruscheweyh derivative, Dziok-Srivastava, Salagean, or multiplier-type transforms, the class encompasses a variety of geometric function classes explored in [17, 18, 23, 8, 22, 42]. This unifying perspective illustrates the broad applicability and richness of the proposed framework.*

The subfamily $\mathcal{T} \subset \mathcal{S}$, comprised of analytic functions with negative coefficients, specifically those expressible in the form

$$f(\zeta) = \zeta - \sum_{n=2}^{\infty} a_n\zeta^n \quad (a_n \geq 0), \tag{3}$$

was introduced and studied by Silverman in [32]. In this context, we define a new class by taking the intersection of this subclass with the class $\mathcal{U}_g(\kappa, \lambda, b, \delta)$, namely,

$$\mathcal{T}\mathcal{U}_g(\kappa, \lambda, b, \delta) := \mathcal{U}_g(\kappa, \lambda, b, \delta) \cap \mathcal{T}.$$

The remainder of the paper is structured as follows. Unless stated otherwise, all results throughout the paper concern the analytic function class $\mathcal{T}\mathcal{U}_g(\kappa, b, \delta, \lambda)$. In Section 2, we establish coefficient bounds for this family. Section 3 is devoted to radii problems involving starlikeness, convexity, and close-to-convexity. Convexity aspects are further examined in Section 4, while Section 5 characterizes its extreme points. In Section 6, we investigate invariance under integral operators and address Silverman's integral means conjecture. Certain subordination problems are discussed in Section 7, and concluding remarks along with future directions are presented in Section 8.

2. Coefficient Estimates

In this section, we establish coefficient bounds for functions in the class $\mathcal{T}\mathcal{U}_g(\kappa, b, \delta, \lambda)$. These bounds serve as necessary and sufficient conditions for membership in the class and, in several instances, are shown to be sharp.

Remark 2.1. *In the results that follow, we frequently encounter the expression*

$$[(\kappa+1)(n-1) + (1-\lambda)|b|]\,[1 + \delta(n-1)]\,,$$

which plays a central role in characterizing the coefficient bounds associated with the class $\mathcal{T}\mathcal{U}_g(\kappa, b, \delta, \lambda)$. To streamline the presentation and highlight its dependence on the underlying parameters, we define

$$\Gamma_n(\kappa, b, \delta, \lambda) := [(\kappa+1)(n-1) + (1-\lambda)|b|]\,[1 + \delta(n-1)]\,.$$

This function will be used throughout the forthcoming theorems to express coefficient inequalities in a unified and concise form.

Theorem 2.2. *Let $f(\zeta)$ be expressed as in* (1) *and $g(\zeta)$ as in* (2)*. Further, let the parameters be as in Theorem 1.4. If*

$$\sum_{n=2}^{\infty} \Gamma_n(\kappa, b, \delta, \lambda)|a_n|b_n \leq (1-\lambda)|b|, \tag{4}$$

then $f \in \mathcal{U}_g(\kappa, b, \delta, \lambda)$*. Moreover,* $f \in \mathcal{T}\mathcal{U}_g(\kappa, b, \delta, \lambda)$ *if and only if the inequality*

$$\sum_{n=2}^{\infty} \Gamma_n(\kappa, b, \delta, \lambda) a_n b_n \leq (1-\lambda)|b| \tag{5}$$

holds.

Proof. We divide the proof into two parts corresponding to each class.

Part I: Sufficiency for $\mathcal{U}_g(\kappa, b, \delta, \lambda)$. Given that

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n \qquad \text{and} \qquad g(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n,$$

we consider the Hadamard product of f and g defined as

$$(f * g)(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n b_n \zeta^n = \Upsilon(\zeta).$$

Differentiating $\Upsilon(\zeta)$ twice, we obtain

$$\Upsilon'(\zeta) = 1 + \sum_{n=2}^{\infty} n a_n b_n \zeta^{n-1} \qquad \text{and} \qquad \Upsilon''(\zeta) = \sum_{n=2}^{\infty} n(n-1) a_n b_n \zeta^{n-2}.$$

Now $f \in \mathcal{U}_g(\kappa, b, \delta, \lambda)$ if

$$\mathrm{Re}\left\{1 - \lambda + \frac{1}{b}\left(\frac{\zeta\Upsilon'(\zeta) + \delta\zeta^2\Upsilon''(\zeta)}{(1-\delta)\Upsilon(\zeta) + \delta\zeta\Upsilon'(\zeta)} - 1\right)\right\} > \kappa \left|\frac{1}{b}\left(\frac{\zeta\Upsilon'(\zeta) + \delta\zeta^2\Upsilon''(\zeta)}{(1-\delta)\Upsilon(\zeta) + \delta\zeta\Upsilon'(\zeta)} - 1\right)\right|. \tag{6}$$

From the series representations of $\Upsilon(\zeta)$, $\Upsilon'(\zeta)$, and $\Upsilon''(\zeta)$, we obtain

$$\begin{aligned}\frac{\zeta\Upsilon'(\zeta) + \delta\zeta^2\Upsilon''(\zeta)}{(1-\delta)\Upsilon(\zeta) + \delta\zeta\Upsilon'(\zeta)} - 1 &= \frac{\zeta + \sum_{n=2}^{\infty} n\big[1 + \delta(n-1)\big] a_n b_n \zeta^n}{\zeta + \sum_{n=2}^{\infty}\big[1 + \delta(n-1)\big] a_n b_n \zeta^n} - 1 \\ &= \frac{\sum_{n=2}^{\infty}(n-1)\big[1 + \delta(n-1)\big] a_n b_n \zeta^n}{\zeta + \sum_{n=2}^{\infty}\big[1 + \delta(n-1)\big] a_n b_n \zeta^n}.\end{aligned}$$

Define the auxiliary function

$$H(\zeta) = \frac{\sum_{n=2}^{\infty}(n-1)\big[1 + \delta(n-1)\big] a_n b_n \zeta^n}{\zeta + \sum_{n=2}^{\infty}\big[1 + \delta(n-1)\big] a_n b_n \zeta^n}.$$

Consequently, the defining condition (6) for $f \in \mathcal{U}_g(\kappa, b, \delta, \lambda)$ reduces to

$$\mathrm{Re}\left\{1 - \lambda + \frac{1}{b}H(\zeta)\right\} > \kappa\left|\frac{1}{b}H(\zeta)\right|. \tag{7}$$

To ensure (7), we estimate the real part from below. Since, for any $w \in \mathbb{C}$,

$$\mathrm{Re}(w) \geq -|w|,$$

it follows that

$$\mathrm{Re}(w) - \kappa|w| \geq -(\kappa + 1)|w|.$$

Applying this inequality with $w = \frac{1}{b}H(\zeta)$, we deduce that (7) is satisfied provided that

$$(\kappa+1)\left|\frac{1}{b}H(\zeta)\right| \leq 1-\lambda,$$

or, equivalently,

$$|H(\zeta)| \leq \frac{(1-\lambda)|b|}{\kappa+1}. \tag{8}$$

Next, by an application of the triangle inequality, we obtain

$$|H(\zeta)| \leq \frac{\sum_{n=2}^{\infty}(n-1)\Big[1+\delta(n-1)\Big]|a_n|\,|b_n|\,|\zeta|^n}{|\zeta| - \sum_{n=2}^{\infty}\Big[1+\delta(n-1)\Big]|a_n|\,|b_n|\,|\zeta|^n}.$$

Letting $|\zeta| \to 1^-$ and combining the above estimate with (8), we conclude that (7) holds whenever

$$\sum_{n=2}^{\infty}(n-1)\Big[1+\delta(n-1)\Big]|a_n|\,|b_n| \leq \frac{(1-\lambda)|b|}{\kappa+1}\left(1-\sum_{n=2}^{\infty}\Big[1+\delta(n-1)\Big]|a_n|\,|b_n|\right).$$

Rearranging the terms and using the fact that $b_n \geq 0$, we finally arrive at

$$\sum_{n=2}^{\infty}\left[(\kappa+1)(n-1)+(1-\lambda)|b|\right]\Big[1+\delta(n-1)\Big]|a_n|\,b_n \leq (1-\lambda)|b|,$$

which is precisely inequality (4).

Part II: Characterization of $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$. Suppose now that $f \in \mathcal{TU}_g(\kappa, b, \delta, \lambda)$, i.e., the function satisfies the inequality

$$\mathrm{Re}\left\{1-\lambda+\frac{1}{b}\left(\frac{\zeta\Upsilon'(\zeta)+\delta\zeta^2\Upsilon''(\zeta)}{(1-\delta)\Upsilon(\zeta)+\delta\zeta\Upsilon'(\zeta)}-1\right)\right\} > \kappa\left|\frac{1}{b}\left(\frac{\zeta\Upsilon'(\zeta)+\delta\zeta^2\Upsilon''(\zeta)}{(1-\delta)\Upsilon(\zeta)+\delta\zeta\Upsilon'(\zeta)}-1\right)\right|$$

for all $\zeta \in \mathbb{D}$.

To obtain a necessary condition, we consider the behavior as $\zeta \to 1^-$ along the real axis.

We note that the functions $\Upsilon(\zeta)$, $\Upsilon'(\zeta)$, and $\Upsilon''(\zeta)$ are real and positive in a neighborhood approaching $\zeta = 1$, provided the coefficients a_n, b_n are real and non-negative. Thus, evaluating the inequality at the boundary point gives

$$1-\lambda+\frac{1}{b}\left(\frac{\zeta\Upsilon'(\zeta)+\delta\zeta^2\Upsilon''(\zeta)}{(1-\delta)\Upsilon(\zeta)+\delta\zeta\Upsilon'(\zeta)}-1\right) > \kappa\left|\frac{1}{b}\left(\frac{\zeta\Upsilon'(\zeta)+\delta\zeta^2\Upsilon''(\zeta)}{(1-\delta)\Upsilon(\zeta)+\delta\zeta\Upsilon'(\zeta)}-1\right)\right|. \tag{9}$$

As $\zeta \to 1^-$, let us define the quantity

$$\Theta := \frac{\zeta\Upsilon'(\zeta)+\delta\zeta^2\Upsilon''(\zeta)}{(1-\delta)\Upsilon(\zeta)+\delta\zeta\Upsilon'(\zeta)} - 1.$$

Since $\Theta \to \Theta_0 \in \mathbb{R}$, the inequality (9) reduces to

$$1-\lambda+\frac{1}{b}T_0 > \kappa\left|\frac{1}{b}T_0\right|.$$

This further simplifies to

$$1-\lambda > \left(\kappa+\frac{1}{|b|}\right)\cdot\frac{\sum_{n=2}^{\infty}(n-1)(1+\delta(n-1))a_n b_n}{1-\sum_{n=2}^{\infty}(1+\delta(n-1))a_n b_n}.$$

Solving this inequality yields

$$\sum_{n=2}^{\infty} \Gamma_n(\kappa, b, \delta, \lambda) a_n b_n \leq (1-\lambda)|b|,$$

which is precisely condition (5).

Conversely, if this inequality holds, then substituting it back into the expression for Θ, we can verify that the defining inequality of $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$ is satisfied.

Therefore, condition (5) is both necessary and sufficient for $f \in \mathcal{TU}_g(\kappa, b, \delta, \lambda)$. □

Remark 2.3. *An immediate consequence of Theorem 2.2, the following result not only establishes tight bounds but also underscores their sharpness through the corresponding extremal function. This serves as a foundational element for the deeper analysis and results presented in the subsequent sections.*

Corollary 2.4. *If $f(\zeta)$ defined by (3) lies in the class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$, then*

$$a_n \leq \frac{(1-\lambda)|b|}{\Gamma_n(\kappa, b, \delta, \lambda) b_n} \quad (n \geq 2).$$

Equality is attained for the extremal function

$$f(\zeta) = \zeta - \frac{(1-\lambda)|b|}{\Gamma_n(\kappa, b, \delta, \lambda) b_n} \zeta^n \quad (n \geq 2). \tag{10}$$

3. Radius Problems

Definition 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two subfamilies of $\mathcal{A}$. Then the $\mathcal{A}$-radius of $\mathcal{B}$ is the largest number σ $(0 < \sigma < 1)$ such that $r^{-1} f(r\zeta) \in \mathcal{A}$ for all $f \in \mathcal{B}$, where $0 < r \leq \sigma$. The problem of finding the number σ is called a radius problem. Further, if we can find an $f_0 \in \mathcal{B}$ such that $r^{-1} f_0(r\zeta) \notin \mathcal{A}$ whenever $r > \sigma$, then the number σ is said to be sharp.*

Goodman [11, Chapter 13] provided a comprehensive compilation of radius results involving various classical subclasses of the family $\mathcal{S}$.

In the forthcoming theorems, we aim to determine the radii associated with close-to-convexity, starlikeness, and convexity for the class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$.

A function $f(\zeta)$, analytic in the open unit disk $\mathbb{D}$, is called close-to-convex of order $\epsilon \in [0, 1)$ if we can find a convex function $c(\zeta)$ satisfying

$$\mathrm{Re}\left(\frac{f'(\zeta)}{c'(\zeta)}\right) > \epsilon \quad \forall\, \zeta \in \mathbb{D}.$$

We denote this class by $\mathcal{K}(\epsilon)$. In particular, since the function $c(\zeta) = \zeta$ is convex, it follows that any analytic f satisfying

$$\mathrm{Re}\,(f'(\zeta)) > \epsilon \qquad (\zeta \in \mathbb{D}) \tag{11}$$

also belongs to $\mathcal{K}(\epsilon)$.

Theorem 3.2. *Let $f(\zeta)$ be a function of the form* (3), *and suppose that $f \in \mathcal{TU}_g(\kappa, b, \delta, \lambda)$, where the generating function $g(\zeta)$ is given by* (2). *Then the function $f(\zeta)$ is close-to-convex of order ϵ in the open disk $|\zeta| < r_{cc}$, where*

$$r_{cc} = \inf_{n \geq 2} \left\{ \frac{(1-\epsilon)\Gamma_n(\kappa, b, \delta, \lambda) b_n}{n(1-\lambda)|b|} \right\}^{\frac{1}{n-1}}. \tag{12}$$

Moreover, this result is sharp, with equality attained for the extremal function defined in (10).

Proof. To show that $f(\zeta) \in \mathcal{K}(\epsilon)$, it is sufficient to verify that the condition

$$\mathrm{Re}\{f'(\zeta)\} > \epsilon$$

holds in the disk $|\zeta| < r_{cc}$. This is guaranteed if the following inequality is satisfied

$$|f'(\zeta) - 1| \leq 1 - \epsilon.$$

From the series representation (3), we compute the derivative

$$f'(\zeta) = 1 + \sum_{n=2}^{\infty} na_n\zeta^{n-1},$$

which implies

$$|f'(\zeta) - 1| \leq \sum_{n=2}^{\infty} na_n|\zeta|^{n-1}.$$

Thus, to ensure $\mathrm{Re}\{f'(\zeta)\} > \epsilon$, it suffices that

$$\sum_{n=2}^{\infty}\left(\frac{n}{1-\epsilon}\right)a_n|\zeta|^{n-1} \leq 1. \tag{13}$$

Now, since $f \in \mathcal{TU}_g(\kappa, b, \delta, \lambda)$, it follows from Theorem 2.2 that

$$\sum_{n=2}^{\infty}\frac{\Gamma_n(\kappa, b, \delta, \lambda)a_nb_n}{(1-\lambda)|b|} < 1.$$

To derive a sufficient condition ensuring that (13) holds, it is adequate to compare corresponding terms in the two inequalities. That is, if for each $n \geq 2$, we ensure

$$\left(\frac{n}{1-\epsilon}\right)|\zeta|^{n-1} \leq \frac{\Gamma_n(\kappa, b, \delta, \lambda)b_n}{(1-\lambda)|b|},$$

then summing over all $n \geq 2$ yields (13). Solving the above inequality for $|\zeta|$, we obtain the radius bound

$$|\zeta| < \left\{\frac{(1-\epsilon)\Gamma_n(\kappa, b, \delta, \lambda)b_n}{n(1-\lambda)|b|}\right\}^{\frac{1}{n-1}}.$$

Taking the infimum over $n \geq 2$ yields the radius r_{cc} given in (12).

Finally, the sharpness of the result follows from the extremal function presented in (10), which attains equality in the bound. □

Theorem 3.3. *Let $f(\zeta)$ and $g(\zeta)$ be as in (3) and (2), and suppose that $f \in \mathcal{TU}_g(\kappa, b, \delta, \lambda)$. Then*

(i) *The function $f(\zeta)$ is starlike of order ϵ in the disk $|\zeta| < r_{st}$, where*

$$r_{st} = \inf_{n\geq 2}\left\{\frac{(1-\epsilon)\Gamma_n(\kappa, b, \delta, \lambda)b_n}{(n-\epsilon)(1-\lambda)|b|}\right\}^{\frac{1}{n-1}}. \tag{14}$$

(ii) *The function $f(\zeta)$ is convex of order ϵ in the disk $|\zeta| < r_{cv}$, where*

$$r_{cv} = \inf_{n\geq 2}\left\{\frac{(1-\epsilon)\Gamma_n(\kappa, b, \delta, \lambda)b_n}{n(n-\epsilon)(1-\lambda)|b|}\right\}^{\frac{1}{n-1}}. \tag{15}$$

Both bounds are sharp, with extremality attained by the function given in (10).

Proof. **(i)** To prove that f is starlike of order ϵ, we aim to establish the inequality

$$\left|\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right| \leq 1 - \epsilon \quad \text{for } |\zeta| < r_{st}.$$

From the representation in (3), we have

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n, \quad \text{and} \quad f'(\zeta) = 1 + \sum_{n=2}^{\infty} n a_n \zeta^{n-1}.$$

Thus,

$$\frac{\zeta f'(\zeta)}{f(\zeta)} = \frac{1 + \sum_{n=2}^{\infty} n a_n \zeta^{n-1}}{1 + \sum_{n=2}^{\infty} a_n \zeta^{n-1}}.$$

Subtracting 1 from both sides yields

$$\frac{\zeta f'(\zeta)}{f(\zeta)} - 1 = \frac{\sum_{n=2}^{\infty}(n-1) a_n \zeta^{n-1}}{1 + \sum_{n=2}^{\infty} a_n \zeta^{n-1}}.$$

Taking modulus and applying the triangle inequality, we get for $r = |\zeta|$,

$$\left|\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1) a_n r^{n-1}}{1 - \sum_{n=2}^{\infty} a_n r^{n-1}}.$$

This is bounded by $1 - \epsilon$ if

$$\frac{\sum_{n=2}^{\infty}(n-1) a_n r^{n-1}}{1 - \sum_{n=2}^{\infty} a_n r^{n-1}} \leq 1 - \epsilon.$$

Cross-multiplying (since the denominator is positive for small r) gives

$$\sum_{n=2}^{\infty}(n-1) a_n r^{n-1} \leq (1-\epsilon)\left(1 - \sum_{n=2}^{\infty} a_n r^{n-1}\right).$$

This, on rearranging and simplifying, yields

$$\sum_{n=2}^{\infty}(n-\epsilon) a_n r^{n-1} \leq 1 - \epsilon.$$

Using the coefficient inequality $a_n \leq \frac{(1-\lambda)|b|}{\Gamma_n(\kappa,b,\delta,\lambda)b_n}$ from the definition of $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$, the condition holds when

$$\sum_{n=2}^{\infty} \frac{(n-\epsilon)(1-\lambda)|b|}{\Gamma_n(\kappa, b, \delta, \lambda) b_n} r^{n-1} \leq 1 - \epsilon.$$

This is satisfied if for each $n \geq 2$,

$$\frac{(n-\epsilon)(1-\lambda)|b|}{\Gamma_n(\kappa, b, \delta, \lambda) b_n} r^{n-1} \leq 1 - \epsilon,$$

i.e.,

$$r \le \left(\frac{(1-\epsilon)\Gamma_n(\kappa, b, \delta, \lambda) b_n}{(n-\epsilon)(1-\lambda)|b|} \right)^{\frac{1}{n-1}}, \quad n \ge 2.$$

Hence, the largest disk ensuring starlikeness of order ϵ has radius

$$r_{st} = \inf_{n\ge 2} \left\{ \frac{(1-\epsilon)\Gamma_n(\kappa, b, \delta, \lambda) b_n}{(n-\epsilon)(1-\lambda)|b|} \right\}^{\frac{1}{n-1}}.$$

(ii) To establish the convexity radius, we invoke Alexander's theorem:

$$f \text{ is convex of order } \epsilon \iff \zeta f'(\zeta) \text{ is starlike of order } \epsilon.$$

Define $h(\zeta) := \zeta f'(\zeta) = \zeta + \sum_{n=2}^{\infty} n a_n \zeta^n$. To show h is starlike of order ϵ, we require

$$\left| \frac{\zeta h'(\zeta)}{h(\zeta)} - 1 \right| \le 1 - \epsilon.$$

Noting that $h'(\zeta) = 1 + \sum_{n=2}^{\infty} n^2 a_n \zeta^{n-1}$, we have

$$\frac{\zeta h'(\zeta)}{h(\zeta)} - 1 = \frac{\sum_{n=2}^{\infty}(n^2 - n) a_n \zeta^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n \zeta^{n-1}}.$$

By the triangle inequality, for $r = |\zeta|$,

$$\left| \frac{\zeta h'(\zeta)}{h(\zeta)} - 1 \right| \le \frac{\sum_{n=2}^{\infty} n(n-1) a_n r^{n-1}}{1 - \sum_{n=2}^{\infty} n a_n r^{n-1}}.$$

This is bounded by $1-\epsilon$ if

$$\frac{\sum_{n=2}^{\infty} n(n-1) a_n r^{n-1}}{1 - \sum_{n=2}^{\infty} n a_n r^{n-1}} \le 1 - \epsilon,$$

or equivalently,

$$\sum_{n=2}^{\infty} n(n-\epsilon) a_n r^{n-1} \le 1 - \epsilon.$$

Using the coefficient inequality $a_n \le \frac{(1-\lambda)|b|}{\Gamma_n(\kappa,b,\delta,\lambda) b_n}$, the condition holds when

$$\sum_{n=2}^{\infty} \frac{n(n-\epsilon)(1-\lambda)|b|}{\Gamma_n(\kappa, b, \delta, \lambda) b_n} r^{n-1} \le 1 - \epsilon.$$

This will hold if for each $n \ge 2$,

$$\frac{n(n-\epsilon)(1-\lambda)|b|}{\Gamma_n(\kappa, b, \delta, \lambda) b_n} r^{n-1} \le 1 - \epsilon,$$

i.e.,

$$r \le \left(\frac{(1-\epsilon)\Gamma_n(\kappa, b, \delta, \lambda) b_n}{n(n-\epsilon)(1-\lambda)|b|} \right)^{\frac{1}{n-1}}, \quad n \ge 2.$$

Hence, the largest disk ensuring the starlikeness of $h(\zeta) := \zeta f'(\zeta)$ of order ϵ has radius

$$r_{cv} = \inf_{n\geq 2}\left\{\frac{(1-\epsilon)\Gamma_n(\kappa,b,\delta,\lambda)b_n}{n(n-\epsilon)(1-\lambda)|b|}\right\}^{\frac{1}{n-1}}.$$

By Alexander's theorem, this radius guarantees that f is convex of order ϵ.

Sharpness. For any fixed $n \geq 2$, consider the extremal function

$$f_n(\zeta) = \zeta - \frac{(1-\lambda)|b|}{\Gamma_n(\kappa,b,\delta,\lambda)b_n}\zeta^n,$$

which belongs to $\mathcal{TU}_g(\kappa,b,\delta,\lambda)$ (as given in (10)). For this function, the inequalities leading to the expressions for r_{st} and r_{cv} become equalities at the corresponding radii, showing that the order of starlikeness (resp. convexity) is exactly ϵ on the boundary $|\zeta| = r_{st}$ (resp. $|\zeta| = r_{cv}$). Hence the radii cannot be improved. □

Remark 3.4. *The radii derived in this section provide precise bounds on the domains within which functions in $\mathcal{TU}_g(\kappa,b,\delta,\lambda)$ exhibit close-to-convex, starlike, or convex behavior of a given order. These sharp results, substantiated by extremal examples, offer valuable insights into the geometric structure of the class and extend several classical results in geometric function theory.*

4. Convexity

In this section, we investigate the convexity properties of the analytic function class $\mathcal{TU}_g(\kappa,b,\delta,\lambda)$. Convexity, in the context of geometric function theory, is often studied through two fundamental operations: convex combinations and dilations. We aim to establish that this class is closed under both these operations, thereby highlighting its geometric robustness and structural stability.

Closure under Convex Combinations

We begin by examining the behavior of the class $\mathcal{TU}_g(\kappa,b,\delta,\lambda)$ under convex combinations of its members.

Theorem 4.1. *Let $f_i(\zeta) = \zeta - \sum_{n=2}^{\infty} a_{n,i}\zeta^n$, with $a_{n,i} \geq 0$, be functions in the class $\mathcal{TU}_g(\kappa,b,\delta,\lambda)$ for $i = 1,2,\ldots,m$. Then the convex combination*

$$h(\zeta) = \sum_{i=1}^{m} \tau_i f_i(\zeta), \quad \tau_i \geq 0, \quad \sum_{i=1}^{m} \tau_i = 1,$$

also belongs to the class $\mathcal{TU}_g(\kappa,b,\delta,\lambda)$.

Proof. As each f_i is a member of$\mathcal{TU}_g(\kappa,b,\delta,\lambda)$, by Theorem 2.2, they satisfy the coefficient condition

$$\sum_{n=2}^{\infty} \frac{\Gamma_n(\kappa,b,\delta,\lambda)}{(1-\lambda)|b|} a_{n,i} b_n \leq 1.$$

Define the sequence $e_n := \sum_{i=1}^{m} \tau_i a_{n,i}$, which inherits the non-negativity from the $a_{n,i}$ and τ_i. Then the convex combination function becomes

$$h(\zeta) = \zeta - \sum_{n=2}^{\infty} e_n \zeta^n.$$

Using linearity and non-negativity of the weights τ_i, we evaluate the defining sum for h as follows:

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{\Gamma_n(\kappa,b,\delta,\lambda)}{(1-\lambda)|b|}e_nb_n &= \sum_{n=2}^{\infty}\left(\sum_{i=1}^{m}\tau_i a_{n,i}\right)\cdot\frac{\Gamma_n(\kappa,b,\delta,\lambda)}{(1-\lambda)|b|}b_n\\ &= \sum_{i=1}^{m}\tau_i\sum_{n=2}^{\infty}\frac{\Gamma_n(\kappa,b,\delta,\lambda)}{(1-\lambda)|b|}a_{n,i}b_n\\ &\le \sum_{i=1}^{m}\tau_i\\ &= 1.\end{aligned}$$

Hence, $h \in \mathcal{TU}_g(\kappa,b,\delta,\lambda)$, completing the proof. □

Corollary 4.2. *If $f_1, f_2 \in \mathcal{TU}_g(\kappa,b,\delta,\lambda)$, then for any real number $0 \le \beta \le 1$, the convex combination*

$$(1-\beta)f_1(\zeta) + \beta f_2(\zeta)$$

also belongs to the class $\mathcal{TU}_g(\kappa,b,\delta,\lambda)$.

Closure under Dilations

Next, we demonstrate that the class $\mathcal{TU}_g(\kappa,b,\delta,\lambda)$ remains invariant under the operation of dilation. This means that scaling the input of a function in the class, followed by appropriate normalization, yields a function still within the same class.

Theorem 4.3. *Suppose $f(\zeta) \in \mathcal{TU}_g(\kappa,b,\delta,\lambda)$ be defined as*

$$f(\zeta) = \zeta - \sum_{n=2}^{\infty} a_n\zeta^n.$$

Then for every $0 \le \epsilon \le 1$, the dilated function defined by

$$f_\epsilon(\zeta) := \frac{f(\epsilon\zeta)}{\epsilon}$$

also belongs to the class $\mathcal{TU}_g(\kappa,b,\delta,\lambda)$.

Proof. Consider the function $f(\zeta) = \zeta - \sum_{n=2}^{\infty} a_n\zeta^n \in \mathcal{TU}_g(\kappa,b,\delta,\lambda)$. Then the dilated function becomes

$$f_\epsilon(\zeta) = \frac{f(\epsilon\zeta)}{\epsilon} = \zeta - \sum_{n=2}^{\infty} a_n\epsilon^{n-1}\zeta^n.$$

From Theorem 2.2, the coefficient sequence $\{a_n\}$ satisfies

$$\sum_{n=2}^{\infty}\frac{\Gamma_n(\kappa,b,\delta,\lambda)}{(1-\lambda)|b|}a_nb_n \le 1.$$

Since $0 \le \epsilon \le 1$, we have $\epsilon^{n-1} \le 1$, and hence

$$\sum_{n=2}^{\infty}\frac{\Gamma_n(\kappa,b,\delta,\lambda)}{(1-\lambda)|b|}a_n\epsilon^{n-1}b_n \le \sum_{n=2}^{\infty}\frac{\Gamma_n(\kappa,b,\delta,\lambda)}{(1-\lambda)|b|}a_nb_n \le 1.$$

Thus, the transformed function f_ϵ also satisfies the defining condition for membership in $\mathcal{TU}_g(\kappa,b,\delta,\lambda)$, completing the proof. □

5. Extreme Points

Extreme points are fundamental in the study of extremal problems in geometric function theory. In compact convex classes, extremal values of functionals are typically attained at extreme points. This section identifies and characterizes such extremal functions for the class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$.

Theorem 5.1. *Let $f_1(\zeta) = \zeta$, and for $n \geq 2$, define*

$$f_n(\zeta) = \zeta - \frac{(1-\lambda)|b|}{\Gamma_n(\kappa, b, \delta, \lambda)\, b_n}\zeta^n. \tag{16}$$

Then a function $f(\zeta)$ belongs to the class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$ if and only if it admits the representation

$$f(\zeta) = \sum_{n=1}^{\infty} \tau_n f_n(\zeta), \quad \tau_n \geq 0, \quad \sum_{n=1}^{\infty} \tau_n = 1. \tag{17}$$

Proof. Suppose $f(\zeta)$ has the representation in (17). Each function f_n satisfies the defining inequality of the class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$, and by Theorem 4.1, the convex combination of such functions also satisfies the defining condition. Hence, $f \in \mathcal{TU}_g(\kappa, b, \delta, \lambda)$.

Conversely, suppose $f \in \mathcal{TU}_g(\kappa, b, \delta, \lambda)$. Then from Theorem 2.2, the coefficients a_n satisfy

$$a_n \leq \frac{(1-\lambda)|b|}{\Gamma_n(\kappa, b, \delta, \lambda)\, b_n}, \quad n \geq 2.$$

For $n \geq 2$, define the weight coefficients

$$\tau_n := \frac{\Gamma_n(\kappa, b, \delta, \lambda) \cdot a_n b_n}{(1-\lambda)|b|}$$

and

$$\tau_1 := 1 - \sum_{n=2}^{\infty} \tau_n.$$

Then it follows that $f(\zeta)$ can be written in the form (17), which completes the proof. □

Corollary 5.2. *The set of extreme points of the class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$ is given by the functions*

$$f_1(\zeta) = \zeta, \quad \textit{and} \quad f_n(\zeta) = \zeta - \frac{(1-\lambda)|b|}{\Gamma_n(\kappa, b, \delta, \lambda)\, b_n}\zeta^n, \quad n \geq 2.$$

Proof. A function is an extreme point in a convex set if it cannot be written as a strict convex combination of two distinct functions in the set. Each function $f_n(\zeta)$ contains only one higher-order term beyond the linear part, and thus cannot be expressed as a convex combination of any other two distinct functions in the class. Therefore, these functions form the extreme boundary of the class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$. □

Remark 5.3. *These extremal characterizations simplify the analysis of extremal problems. Rather than searching over the entire class, one can focus on its extreme points, significantly reducing computational and conceptual complexity in applications involving sharp bounds and optimality.*

6. Invariance and Integral Means

According to Theorem 2.2, if a function $f(\zeta) = \zeta - \sum_{n=2}^{\infty} a_n \zeta^n$ belongs to $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$, then any function h of the form

$$h(\zeta) = \zeta - \sum_{n=2}^{\infty} e_n \zeta^n, \qquad e_n \geq 0,$$

with coefficients satisfying $e_n \leq a_n$ for all $n \geq 2$, also belongs to $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$. This coefficient dominance property provides a direct mechanism for establishing the invariance of the class under various integral transforms. Once an operator is shown to produce coefficients e_n dominated by the original coefficients a_n, invariance follows immediately.

Theorem 6.1. *Let $c > -1$ and $p \geq 0$. If $f \in \mathcal{TU}_g(\kappa, b, \delta, \lambda)$, then the Komatu operator*

$$F_{p,c}(\zeta) = \frac{(c+1)^p}{\zeta^c \Gamma(p)} \int_0^{\zeta} s^{c-1} \left(\log \frac{\zeta}{s}\right)^{p-1} f(s)\, ds$$

also belongs to $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$.

Proof. Let $f(\zeta) = \zeta - \sum_{n=2}^{\infty} a_n \zeta^n \in \mathcal{TU}_g(\kappa, b, \delta, \lambda)$. Substituting into the definition of $F_{p,c}$, we get

$$\begin{aligned} F_{p,c}(\zeta) &= \frac{(c+1)^p}{\zeta^c \Gamma(p)} \int_0^{\zeta} s^{c-1} \left(\log \frac{\zeta}{s}\right)^{p-1} \left(s - \sum_{n=2}^{\infty} a_n s^n\right) ds \\ &= \frac{(c+1)^p}{\zeta^c \Gamma(p)} \left[\int_0^{\zeta} s^{c} \left(\log \frac{\zeta}{s}\right)^{p-1} ds - \sum_{n=2}^{\infty} a_n \int_0^{\zeta} s^{c+n-1} \left(\log \frac{\zeta}{s}\right)^{p-1} ds\right]. \end{aligned}$$

Applying the integral identity

$$\int_0^{\zeta} s^{c+n-1} \left(\log \frac{\zeta}{s}\right)^{p-1} ds = \frac{\zeta^{c+n}}{(c+n)^p} \Gamma(p), \qquad n \geq 1,$$

we obtain the series representation

$$F_{p,c}(\zeta) = \zeta - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^p a_n \zeta^n. \tag{18}$$

Since $c > -1$ and $n \geq 2$, we have $0 < \frac{c+1}{c+n} < 1$, which implies

$$\left(\frac{c+1}{c+n}\right)^p a_n < a_n, \qquad n \geq 2.$$

By the definition of the class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$, the coefficients satisfy

$$a_n \leq \frac{(1-\lambda)|b|}{\Gamma_n(\kappa, b, \delta, \lambda) b_n},$$

and therefore

$$\left(\frac{c+1}{c+n}\right)^p a_n < \frac{(1-\lambda)|b|}{\Gamma_n(\kappa, b, \delta, \lambda) b_n}.$$

Invoking Theorem 2.2, this inequality guarantees that the function defined by (18) belongs to $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$. This completes the proof. □

Corollary 6.2. *The class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$ is invariant under the following classical integral operators, each of which is a special case of the Komatu operator $F_{p,c}$:*

(a) ***Alexander operator:*** $F_{1,0}(\zeta) = \int_0^\zeta \frac{f(s)}{s}\, ds.$

(b) ***Libera operator:*** $F_{1,1}(\zeta) = \frac{2}{\zeta}\int_0^\zeta f(s)\, ds.$

(c) ***Bernardi operator:*** $F_{1,c}(\zeta) = \frac{c+1}{\zeta^c}\int_0^\zeta s^{c-1} f(s)\, ds,\ c > -1.$

(d) ***Jung–Kim–Srivastava operator:*** $F_{p,1}(\zeta) = \frac{2^p}{\zeta\Gamma(p)}\int_0^\zeta \left(\log\frac{\zeta}{s}\right)^{p-1} f(s)\, ds,\ p \geq 0.$

Remark 6.3. *The integral operators presented in Theorem 6.1 play a fundamental role in geometric function theory. They provide systematic methods for generating new subclasses of analytic and univalent functions from existing ones, while preserving fundamental geometric properties such as univalence, starlikeness, convexity, and close-to-convexity.*

These operators unify numerous classical integral and fractional operators studied in the literature. In particular, they establish a natural connection between geometric function theory and the theory of fractional differential equations.

The fact that the class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$ remains invariant under these operators demonstrates its structural robustness. This invariance property also confirms that the class generalizes several well-known subclasses of univalent functions, making it a versatile and extensible framework for further research.

6.1. Integral Means

It was established in [5] that for any function $f \in \mathcal{S}$ and the Koebe function $K(\zeta) = \frac{\zeta}{(1-\zeta)^2}$, the following integral inequality holds

$$\int_0^{2\pi} |f(se^{i\varphi})|^\eta\, d\varphi \leq \int_0^{2\pi} |K(se^{i\varphi})|^\eta\, d\varphi,$$

for each $0 < s < 1$ and $\eta > 0$. Later, Silverman [32] conjectured that for functions $f \in \mathcal{T}$, the extremal function is $f_2(\zeta) = \zeta - \frac{\zeta^2}{2}$, and proposed the problem

$$\int_0^{2\pi} |f(se^{i\varphi})|^\eta\, d\varphi \leq \int_0^{2\pi} |f_2(se^{i\varphi})|^\eta\, d\varphi \quad \text{for each } 0 < s < 1,\ \eta > 0.$$

This was subsequently confirmed in [31]. We now extend this result to the more general class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$.

Definition 6.4 (Subordination [14]). *A function f is said to be subordinate to a function g, written as $f \prec g$, if both f and g are analytic in the unit disk $\mathbb{D}$, and there exists an analytic function $\omega(\zeta)$ in $\mathbb{D}$ satisfying $\omega(0) = 0$ and $|\omega(\zeta)| < 1$ for all $\zeta \in \mathbb{D}$, such that*

$$f(\zeta) = g(\omega(\zeta)).$$

Lemma 6.5 (Littlewood, [14]). *If $f \prec g$, then for each $0 < s < 1$ and $\eta > 0$,*

$$\int_0^{2\pi} |f(se^{i\varphi})|^\eta\, d\varphi \leq \int_0^{2\pi} |g(se^{i\varphi})|^\eta\, d\varphi.$$

Theorem 6.6. *Suppose $f \in \mathcal{TU}_g(\kappa, b, \delta, \lambda)$ be as in (3). Cconsider the function*

$$f_2(\zeta) = \zeta - \frac{(1-\lambda)|b|}{\Gamma_2(\kappa, b, \delta, \lambda)\, b_2}\zeta^2.$$

Then, for each $\eta > 0$ and $0 < s < 1$, we have

$$\int_0^{2\pi} |f(se^{i\varphi})|^\eta\, d\varphi \leq \int_0^{2\pi} |f_2(se^{i\varphi})|^\eta\, d\varphi.$$

Proof. To prove the assertion, it suffices to show the subordination

$$1-\sum_{n=2}^{\infty} a_n\zeta^{n-1} \prec 1-\frac{(1-\lambda)|b|}{\Gamma_2(\kappa,b,\delta,\lambda)\, b_2}\zeta.$$

Define the auxiliary function

$$w(\zeta)=\sum_{n=2}^{\infty}\frac{\Gamma_2(\kappa,b,\delta,\lambda)\, b_2}{(1-\lambda)|b|}a_n\zeta^{n-1},$$

so that

$$1-\sum_{n=2}^{\infty} a_n\zeta^{n-1} = 1-\frac{(1-\lambda)|b|}{\Gamma_2(\kappa,b,\delta,\lambda)\, b_2}\cdot w(\zeta).$$

Due to the coefficient condition on a_n (as guaranteed by the definition of the class), it follows that $|w(\zeta)|<1$ for each $\zeta\in\mathbb{D}$, thereby establishing the subordination. The result then follows from Littlewood's Lemma. □

7. Subordination Results

Subordination is a fundamental concept in geometric function theory, offering a framework for comparing the growth and behavior of analytic functions. Here, we present subordination results for the members of $\mathcal{TU}_g(\kappa,b,\delta,\lambda)$.

Definition 7.1 ([14]). *A complex sequence $\{t_n\}_{n=1}^{\infty}$ is referred to as a* subordinating factor sequence *if the following condition is met: for any convex univalent function Φ expressed as*

$$\Phi(\zeta)=\zeta+\sum_{n=2}^{\infty} e_n\zeta^n, \quad (\zeta\in\mathbb{D}), \tag{19}$$

with $e_1=1$, the series

$$\sum_{n=1}^{\infty} t_n e_n\zeta^n$$

is subordinate to $\Phi(\zeta)$, that is,

$$\sum_{n=1}^{\infty} t_n e_n\zeta^n \prec \Phi(\zeta) \quad \textit{for all } \zeta\in\mathbb{D}.$$

Lemma 7.2 (Wilf [41, Theorem 2]). *A sequence $\{t_n\}_{n=1}^{\infty}$ qualifies as a subordinating factor sequence iff the function*

$$\zeta\mapsto 1+2\sum_{n=1}^{\infty} t_n\zeta^n$$

has positive real part throughout the open unit disk $\mathbb{D}$; that is,

$$\mathrm{Re}\left(1+2\sum_{n=1}^{\infty} t_n\zeta^n\right)>0 \quad (\zeta\in\mathbb{D}).$$

For additional discussions and generalizations related to subordinating factor sequences, we refer to [24, 28, 29, 39].

Theorem 7.3. *Let the function $f(\zeta)$, represented as in (1), satisfy the coefficient condition (4). Define*

$$\gamma = \frac{\Gamma_2(\kappa, b, \delta, \lambda)\, b_2}{2\,\{\Gamma_2(\kappa, b, \delta, \lambda)\, b_2 + (1-\lambda)|b|\}}.$$

Then, for any convex univalent function $\Phi(\zeta)$, the subordination

$$\gamma(f * \Phi)(\zeta) \prec \Phi(\zeta) \quad (\zeta \in \mathbb{D}) \tag{20}$$

holds. Additionally, we have the lower bound

$$\mathrm{Re}\,(f(\zeta)) > -\frac{1}{2\gamma} \quad (\zeta \in \mathbb{D}). \tag{21}$$

The constant γ here is the best possible.

Proof. Let $\Phi(\zeta)$ be defined as in (19). Then the Hadamard product of f with Φ, scaled by γ, can be expressed as

$$\gamma(f * \Phi)(\zeta) = \sum_{n=1}^{\infty} a_n e_n \zeta^n,$$

where the coefficients t_n are given by $t_1 = \gamma$ and $t_n = \gamma a_n$ for $n \geq 2$. Our goal is to prove the subordination

$$\gamma(f * \Phi)(\zeta) \prec \Phi(\zeta), \quad (\zeta \in \mathbb{D}),$$

which, according to Lemma 7.2, holds if and only if the analytic function

$$\zeta \mapsto 1 + 2\sum_{n=1}^{\infty} t_n \zeta^n$$

has positive real part in $\mathbb{D}$, i.e.,

$$\mathrm{Re}\left\{1 + 2\sum_{n=1}^{\infty} t_n \zeta^n\right\} > 0, \quad (\zeta \in \mathbb{D}). \tag{22}$$

To verify this, we consider $|\zeta| = \rho < 1$. Utilizing the coefficient bound (4) for a_n, we estimate the real part of the function as follows:

$$\mathrm{Re}\left\{1 + 2\sum_{n=1}^{\infty} t_n \zeta^n\right\} \geq 1 - 2\gamma|\zeta| - 2\sum_{n=2}^{\infty} \gamma |a_n||\zeta|^n.$$

Let $\rho = |\zeta|$, then

$$\mathrm{Re}\left\{1 + 2\sum_{n=1}^{\infty} t_n \zeta^n\right\} \geq 1 - 2\gamma\rho - 2\gamma\sum_{n=2}^{\infty} |a_n|\rho^n.$$

Given the condition (4) ensures sufficient control over the coefficients $|a_n|$, we may assert that the right-hand side remains strictly greater than zero:

$$1 - r = 1 - \left(2\gamma\rho + 2\gamma\sum_{n=2}^{\infty} |a_n|\rho^n\right) > 0.$$

This confirms the inequality (22) and thereby the subordination in (20).

To establish the inequality (21), consider the specific convex function $\Phi(\zeta) = \zeta/(1-\zeta)$, which is known to map the unit disk $\mathbb{D}$ onto the half-plane $\{w \in \mathbb{C} : \text{Re}(w) > -\frac{1}{2}\}$. If

$$\gamma(f * \Phi)(\zeta) \prec \frac{\zeta}{1-\zeta},$$

then, by the principle of subordination, we deduce

$$\text{Re}\{\gamma f(\zeta)\} > -\frac{1}{2}, \quad (\zeta \in \mathbb{D}),$$

which yields the bound

$$\text{Re}\{f(\zeta)\} > -\frac{1}{2\gamma}.$$

Sharpness of γ: Consider the function

$$f_2(\zeta) = \zeta - \frac{(1-\lambda)|b|}{\Gamma_2(\kappa, b, \delta, \lambda)\, b_2}\zeta^2,$$

which satisfies the condition (4) on the coefficients. For this function, one computes

$$\gamma f_2(\zeta) = \gamma\zeta - \left[\frac{\gamma(1-\lambda)|b|}{\Gamma_2(\kappa, b, \delta, \lambda)\, b_2}\right]\zeta^2.$$

This expression is subordinate to $\Phi(\zeta) = \zeta/(1-\zeta)$, and it attains the minimum value of its real part as

$$\min_{|\zeta|\leq r} \text{Re}\{f_2(\zeta)\} = -\frac{1}{2\gamma}.$$

Hence, no smaller value of γ can satisfy the required subordination, which proves that the constant γ is indeed sharp. □

Remark 7.4. *Setting $\lambda = 0$ in Theorem 7.3 recovers the result in [7, Theorem 3.4], and for $\delta = 0$, the following result is attained.*

Corollary 7.5. *Suppose $f(\zeta)$ of the form (1) satisfy*

$$\sum_{n=2}^{\infty} \Gamma_n(\kappa, b, 0, \lambda)\, |a_n| b_n \leq (1-\lambda)|b|.$$

Then

$$\frac{\Gamma_2(\kappa, b, 0, \lambda)]b_2}{2[(\kappa+1)b_2 + (1-\lambda)(b_2+1)|b|]}(f * \Phi)(\zeta) \prec \Phi(\zeta), \tag{23}$$

for every convex $\Phi(\zeta)$, and

$$\text{Re}(f(\zeta)) > -\frac{(\kappa+1)b_2 + (1-\lambda)(b_2+1)|b|}{\Gamma_2(\kappa, b, 0, \lambda)\, b_2}.$$

Corollary 7.6. *If $f(\zeta)$ satisfies*

$$\sum_{n=2}^{\infty} n\Gamma_n(\kappa, b, 0, \lambda)\, |a_n| b_n \leq (1-\lambda)|b|,$$

then

$$\frac{\Gamma_2(\kappa, b, 0, \lambda)\, b_2}{2(\kappa+1)b_2 + (1-\lambda)(2b_2+1)|b|}(f * \Phi)(\zeta) \prec \Phi(\zeta),$$

for every convex $\Phi(\zeta)$, *and*

$$\mathrm{Re}(f(\zeta)) > -\frac{2(\kappa+1)b_2 + (1-\lambda)(2b_2+1)|b|}{\Gamma_2(\kappa, b, 0, \lambda)\, b_2}.$$

Corollary 7.7 ([23]). *Suppose* $f \in \mathcal{S}$ *and*

$$\sum_{n=2}^{\infty} \Gamma_n(0, b, 0, \lambda)\, |a_n| b_n \leq (1-\lambda)|b|.$$

Then

$$\frac{\Gamma_2(0, b, 0, \lambda)]b_2}{2[b_2 + (1-\lambda)(b_2+1)|b|]}(f * \Phi)(\zeta) \prec \Phi(\zeta),$$

for every convex $\Phi(\zeta)$, *and*

$$\mathrm{Re}(f(\zeta)) > -\frac{b_2 + (1-\lambda)(b_2+1)|b|}{\Gamma_2(0, b, 0, \lambda)\, b_2}.$$

Remark 7.8. *Prajapat [23] also obtained similar results for* $g(\zeta)$ *taken as the Dziok–Srivastava operator* $H_m^l[\lambda_1]$*; see [10].*

Remark 7.9. *Fixing* $g(\zeta) = \zeta/(1-\zeta)$ *and* $\kappa = 0$, *Theorem 7.3 yields the results for the classes*

$$\mathcal{S}_\lambda^*(b) = \left\{ f : \mathrm{Re}\left(1 + \frac{1}{b}\left(\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right)\right) > \lambda \right\} \quad \textit{and} \quad C_\lambda(b) = \left\{ f : \mathrm{Re}\left(1 + \frac{1}{b}\frac{\zeta f''(\zeta)}{f'(\zeta)}\right) > \lambda \right\}.$$

The results in this section underscore the importance of subordination techniques in exploring the structure of the class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$, highlighting their geometric robustness and analytic utility.

8. Concluding Remarks and Future Prospects

In this paper, we have introduced and investigated the class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$, which serves as a generalization of several classical classes of analytic functions. By incorporating the Hadamard product, a complex order parameter b, and type parameter λ, we have developed a unified framework that enables a thorough analysis of geometric and analytic properties of this class. Below, we summarize the main contributions and outline directions for future research.

Remark 8.1. *The function class* $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$ *serves as a unifying framework that encompasses several classical subclasses of analytic functions through appropriate choices of parameters and generating function* $g(\zeta)$*. In particular, setting* $b = 1$ *and adjusting the parameters* δ, κ, λ, *along with selecting specific forms of* $g(\zeta)$, *leads to notable simplifications:*

- *For* $\delta = 0$, *the class simplifies to the family of* κ*-uniformly starlike functions characterized by the parameters* b *and* λ.
- *For* $\delta = 1$, *it reduces to the class of* κ*-uniformly convex functions, again governed by the parameters* b *and* λ.
- *Choosing* $g(\zeta) = \zeta/(1-\zeta)$, *which is a classical mapping function, recovers the settings examined in the work of Aouf et al. [2, 3], thereby including numerous subclasses of uniformly starlike and convex functions as special cases.*

These reductions highlight the structural versatility and encompassing nature of the generalized class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$. It acts as a parent structure for various earlier established subclasses, reflecting its capacity to model diverse geometric behaviors under a unified theoretical framework.

Remark 8.2. *The analytic function class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$, introduced in this paper, offers a flexible framework within the open unit disk $\mathbb{D}$ for analyzing subclasses of analytic, multivalent, and univalent functions in the context of linear and integral operators. This framework is built upon the foundational work by H. M. Srivastava and his collaborators, who significantly advanced the theory of analytic functions through the use of convolution operators, fractional differential operators, and geometric function theory.*

The following works by Srivastava and collaborators are particularly noteworthy:

- *Srivastava et al. [34, 35] investigated subclasses of multivalent functions defined via multiplier transformations and fractional calculus operators, establishing key coefficient estimates and distortion theorems.*
- *Additional contributions in [36, 15] introduced generalized operators, including the Sălăgean differential operator, which provided insights into inclusion relations and univalence criteria.*
- *In [16], Srivastava unified fractional derivatives and convolution-type operators to study geometric properties such as subordination and convexity.*
- *The edited volume by Srivastava and Owa [38] provided a comprehensive approach to fractional differential equations, establishing the necessary tools for extending our framework.*
- *Srivastava et al. [37] also introduced generalizations in the context of q-calculus, focusing on q-analogues of differential operators and their applications to function theory.*
- *In [33], Srivastava explored q-derivatives and their impact on the geometric properties of analytic functions, laying the groundwork for further research into q-calculus and its relationship with fractional differential operators.*

Looking ahead, the class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$ opens up new possibilities for generalization and further exploration in multiple directions. Specifically, the framework can be extended to include q-calculus, where parameters can be adapted to account for q-analogues of the operators involved. This would enable a deeper investigation into the properties of functions in the context of q-analytic functions, further enhancing the breadth of geometric function theory. Moreover, as we expand on the ideas of p-valency and fractional calculus, this generalized class presents an ideal platform for continued research into fractional differential operators and their applications in higher-dimensional domains.

This flexible structure provides a natural extension to q-calculus, p-valency, and fractional theory, paving the way for the exploration of new subclasses of functions. Such advancements would not only deepen the existing theory but also enable applications in novel areas of applied and theoretical mathematics.

Remark 8.3. *The theoretical results derived herein have significant implications in both pure and applied mathematics. For instance:*

- *In geometric function theory, the sharp coefficient estimates, subordination results, and inclusion properties assist in solving extremal problems and in the classification of function behaviors.*
- *In applied sciences, the function class $\mathcal{TU}_g(\kappa, b, \delta, \lambda)$ can be utilized to model physical phenomena, particularly in areas such as fluid dynamics, elasticity theory, and boundary value problems involving complex variables.*

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