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Periodic solutions of Kolmogorov systems on quantum time scales

Halis Can Koyuncuoğlu[a,*], Marko Kostić[b], Öznur Öztunç Kaymak[c], Tuğçe Katıcan[d]

[a] *Department of Engineering Sciences, Izmir Katip Celebi University, Cigli, Izmir, 35620, Turkey*
[b] *Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, Novi Sad, 21125, Serbia*
[c] *Department of Mathematics, Izmir Democracy University, Karabaglar, Izmir, 35140, Turkey*
[d] *Department of Mathematics, Izmir University of Economics, Balcova, Izmir, 35330, Turkey*

Abstract. Kolmogorov systems are one of the important models in applied mathematics which are used for modelling various biological processes. In the present work, we concentrate on the coupled functional dynamic equations as Kolmogorov systems on quantum time scales and study the existence of their periodic solutions. As it is well-known, quantum time scales are not translation invariant, and this results in using multiplicative periodicity perceptions as an alternative to conventional periodicity. In the presentation of the main outcomes, we focus on Kolmogorov systems with/without delay term and we employ coincidence degree theory and fixed point theory for investigating the sufficient conditions for the existence of positive periodic solutions. This research provides an alternative approach to the related literature since it turns the spotlight on Kolmogorov models constructed on non-translation invariant time domains and serves a better understanding for the complicated dynamics of Kolmogorov systems involving periodic structures.

1. Introduction

In the recent years, analyzing periodic dynamics of real-life models has been a very hot topic for scholars. It is surely beyond doubt that there is a vast literature about the existence of periodic solutions of dynamic equations and dynamic systems which can be used in various disciplines such as biology, physics, economics, chemistry, etc. In the last three decades, researchers have extremely studied biological models by different mathematical tools, and this results in appearance of a fruitful research area in the applied mathematics so-called biomathematics. By a quick literature review, it is possible to find numerous papers which focus on mathematical aspects of biological models based on population interactions, diseases, epidemics, infections, prey-predator interactions, production of blood cells, etc. It is clear that the existence of periodic solutions for these particular models has been studied by several mathematicians under specific conditions. Kolmogorov systems can be regarded as one of those biological models which can be used for explaining several biological processes involving predator-prey dynamics in the nature. Unlike some biological models which are very popular in recent days, Kolmogorov systems have not taken so much

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* Corresponding author: Halis Can Koyuncuoğlu
Email addresses: haliscan.koyuncuoglu@ikcu.edu.tr (Halis Can Koyuncuoğlu), marco.s@verat.net (Marko Kostić)
ORCID iDs: https://orcid.org/0000-0002-8880-1552 (Halis Can Koyuncuoğlu), https://orcid.org/0000-0002-0392-4976 (Marko Kostić), https://orcid.org/0000-0003-3832-9947 (Öznur Öztunç Kaymak), https://orcid.org/0000-0003-1186-6750 (Tuğçe Katıcan)

attention, and they are kept in the background. We shall refer to the papers [11–13, 15, 22, 23] as important papers on Kolmogorov type systems.

Investigating the sufficient conditions for the existing of periodic solutions of Kolmogorov systems is actually an important research direction for researchers, and we refer to [20] as an inspiring paper which shares this objective. In [20], Kaufmann focuses on a Kolmogorov predator-prey dynamic system with periodic structures, and proposes the sufficient conditions for the existence of periodic solutions on time scales. Also in the brand new paper [3], authors concentrate on phytoplankton models with several delays on hybrid time domains and study existence of periodic solutions. It should be pointed out that one has to use a translation invariant time domain for defining periodic structures on the model. A time domain $\mathbb{T}$ is said to be translation invariant if there exists a fixed $T > 0$ so that $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ (see [19]). Consequentially, any translation invariant domain must be unbounded from below and above. Quantum difference equations (q-difference equations) have taken prominent interest in the last years. Quantum difference equations are used as an alternative to ordinary difference equations in the applied sciences since they serve better approximations to differential equations in particular cases. A q-difference equation is a difference equation which is defined on a quantum domain (quantum time scale) $\overline{q^{\mathbb{Z}}} := \{q^t,\ t \in \mathbb{Z}\} \cup \{0\}, q > 1$. It is obvious that quantum domains are not translation invariant since they are not closed under addition. Thus the excellent papers [20] and [3] do not cover quantum Kolmogorov (q-Kolmogorov) systems despite the fact that they provide a unified approach to Kolmogorov systems. Motivating from this issue, in this paper we aim to bring q-Kolmogorov systems to the light and fill the above-mentioned gap by studying their periodic solutions. In our approach, we consider the time domain must contain an element at each forward and backward step, and these steps are characterized by multiplication on quantum domains. We refer to the papers [1, 4–9, 16, 17, 21] which focus on periodic solutions of q-difference equations.

In this work, we introduce q-Kolmogorov systems with/without delay and propose the sufficient conditions for the existence of positive periodic solutions by coincidence degree theory and Schauder's fixed point theorem. The organization of the paper is as follows: In the next section, we exhibit quantum calculus initials and periodicity perceptions in q-calculus, and we provide a brief content about the construction of the model. In the sequel, we present our main results in Section 4 and Section 5.

2. Quantum Calculus Essentials

In this part, we aim to present a short summary for quantum calculus for the readers who are not familiar with quantum time scales. First of all, we refer to the monograph [18] for an elaborative reading on quantum calculus constructed on $\overline{q^{\mathbb{Z}}} := \{q^t,\ t \in \mathbb{Z}\} \cup \{0\}, q > 1$. We start by introducing q-derivative and q-integral.

Definition 2.1. *Let $f : q^{\mathbb{Z}} \to \mathbb{R}$ be a function. Then quantum derivative (q-derivative) of f is given by*

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t},\ t \in q^{\mathbb{Z}}.$$

Definition 2.2. *Suppose that $f : q^{\mathbb{Z}} \to \mathbb{R}$ and $a, b \in q^{\mathbb{Z}}$ with $a < b$. Then, the definite quantum integral (q-integral) is introduced by*

$$\int_a^b f(t)\,\Delta t = \sum_{t \in [a,b) \cap q^{\mathbb{Z}}} (q-1)\, t f(t). \tag{1}$$

If one sets $a = q^n$ and $b = q^m$, $n, m \in \mathbb{Z}$ with $n < m$, then (1) turns into the explicit form

$$\int_{q^n}^{q^m} f(t)\,\Delta t = \sum_{j=n}^{m-1} (q-1)\, q^j f\left(q^j\right).$$

In order to introduce quantum exponential (q-exponential) functions, first we define regressive functions on $q^{\mathbb{Z}}$.

Definition 2.3. *A function $f : q^{\mathbb{Z}} \to \mathbb{R}$ is said to be regressive if $1 + (q-1)\,tf(t) \neq 0$ for all $t \in q^{\mathbb{Z}}$.*

Definition 2.4. *Let $f : q^{\mathbb{Z}} \to \mathbb{R}$ be regressive function. For $t, s \in q^{\mathbb{Z}}$ with $t > s$, the q-exponential function of f is given by*

$$e_f(t,s) = \prod_{j=\log_q s}^{\log_q t-1} \left(1 + (q-1)\,q^j f\left(q^j\right)\right). \tag{2}$$

By setting $s = q^n$ and $t = q^m$ with $n < m$, we rewrite (2) as

$$e_f(q^m, q^n) = \prod_{j=n}^{m-1} \left(1 + (q-1)\,q^j f\left(q^j\right)\right).$$

In the next result, we list important properties of q-exponential functions. We shall emphasize that these results are the particular case of [10, Theorem 2.36] when $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$.

Theorem 2.5. *Let $f, g : q^{\mathbb{Z}} \to \mathbb{R}$ be regressive functions. Then,*

i. $e_0(t,s) = 1$ *and* $e_f(t,t) = 1$;

ii. $e_f(qt,s) = (1 + (q-1)\,tf(t))\,e_f(t,s)$;

iii. $\frac{1}{e_f(t,s)} = e_f(s,t) = e_{\ominus f}(t,s)$, *where*

$$\ominus f = \frac{-f(t)}{1 + (q-1)\,tf(t)}; \tag{3}$$

iv. $e_f(t,s)\,e_f(s,w) = e_f(t,w)$,

v. $e_f(t,s)\,e_g(t,s) = e_{f\oplus g}(t,s)$, *where*

$$f \oplus g = f(t) + g(t) + (q-1)\,tf(t)\,g(t). \tag{4}$$

Now, we are ready to introduce periodicity concepts on quantum calculus. We should emphasize that there are two main periodicity perceptions in quantum calculus. First, we give the following definition which is introduced and utilized in the pioneering papers [4–9].

Definition 2.6. *Fix $T \in q^{\mathbb{N}}$. A function $f : q^{\mathbb{Z}} \to \mathbb{R}$ is said to be periodic with period T if*

$$f(Tt)\,T = f(t) \text{ for all } t \in q^{\mathbb{Z}}.$$

A function which is periodic in the sense of Definition 2.6 possesses periodicity property in a geometric meaning. More explicitly, a function that is periodic with respect to Definition 2.6 has equal areas below its graph at each period. One of the most well-known examples of periodic functions due to Definition 2.6 is $f(t) = \frac{1}{t}$ with period $T = q$ since

$$f(qt)\,q = \frac{1}{qt}q = \frac{1}{t} = f(t).$$

We continue our review with the next periodicity definition.

Definition 2.7. *Suppose that $f : q^{\mathbb{Z}} \to \mathbb{R}$ be a function and $T \in q^{\mathbb{N}}$ is fixed. Then, f is said to be T-periodic if*

$$f(Tt) = f(t) \text{ for all } t \in q^{\mathbb{Z}}.$$

Obviously, the periodicity definition provided in Definition 2.7 resembles the conventional periodicity used in translation-invariant domains since a periodic function repeats it values at each forward or backward step with certain period. We borrow the following examples of T-periodic functions in the sense of Definition 2.7 from the popular paper of Adıvar (see [2]). Consider the functions $f(t) = (-1)^{\frac{\ln t}{\ln q}}$ and $g(t) = \sin\left(\frac{\ln t}{\ln q}\pi\right)$ defined on $q^{\mathbb{Z}}$. Then, f and g are q^2-periodic functions with respect to Definition 2.7 since

$$f\left(q^2 t\right) = (-1)^{\frac{\ln q^2 t}{\ln q}} = (-1)^{\frac{\ln t}{\ln q}} = f(t)$$

and

$$g\left(q^2 t\right) = \sin\left(\frac{\ln q^2 t}{\ln q}\pi\right) = \sin\left(\frac{\ln t}{\ln q}\pi\right) = g(t).$$

We shall highlight that both periodicity definitions have been remarkably used in the existing literature, and in this manuscript, we employ both of the definitions. In order to prevent any confusion between the periodicity notions given in Definition 2.6 and Definition 2.7 for the readership, we call the periodic functions in the sense of Definition 2.6 **multiplicatively periodic** throughout the paper. Besides, we refer to the paper [16] for the comparison of two periodicity perceptions. We conclude this section by providing following results.

Lemma 2.8 ([4]). *Let $f : q^{\mathbb{Z}} \to \mathbb{R}$ be a multiplicatively T-periodic function i.e. T-periodic in the sense of Definition 2.6. Then*

$$\int_1^t f(s)\,\Delta s = \int_T^{Tt} f(s)\,\Delta s, \text{ for all } t \in q^{\mathbb{Z}}.$$

Lemma 2.9 ([7]). *Suppose that $f : q^{\mathbb{Z}} \to \mathbb{R}$ be a regressive, multiplicatively T-periodic function due to Definition 2.6. Then*

$$e_f(Tt, Ts) = e_f(t, s), \text{ for all } t, s \in q^{\mathbb{Z}}.$$

3. Derivation of the Model

Dynamic equations of the form

$$x_i' = x_i h_i(x_1, x_2, ..., x_n), \ 1 \leq i \leq n$$

are called Kolmogorov equations, and they are oftenly used in real life models in which the per unit change x_i'/x_i is explained by the functions $h_i(x_1, x_2, ..., x_n)$. This model can be considered as a unified abstract model in mathematical biology since it turns into logistic, Volterra prey-predator, and Lotka-Volterra competition model in particular cases.

Kolmogorov-type systems constitute a unifying mathematical framework that encompasses a broad class of population interaction models, including logistic growth equations, Lotka–Volterra competition systems, and predator–prey dynamics. In particular, the Kolmogorov predator–prey system studied by Kaufmann [20] establishes sufficient conditions for the existence of positive periodic solutions on translation-invariant time scales. That work demonstrates how periodic environmental effects and biological interactions can be rigorously incorporated into the time-scale setting using coincidence degree theory.

A key structural assumption in [20] is that the underlying time scale is translation invariant, which allows the use of classical additive periodicity. However, this assumption excludes an important class of non-additive time domains. In particular, quantum time scales of the form $q^{\mathbb{Z}}$ are not closed under addition and therefore fall outside the scope of the framework developed in [20]. Despite their growing relevance in discrete modeling, quantum calculus, and applications where multiplicative dynamics arise naturally, Kolmogorov systems on quantum time scales have not been systematically investigated in the context of periodic solutions.

Motivated by this gap, we are inspired by [20], and concentrate on the following 2-D Kolmogorov system on quantum domains

$$\begin{cases} x^{\Delta}(t) = x(t)\,\hat{f}_1(t, x(t), y(t)) \\ y^{\Delta}(t) = y(t)\,\hat{f}_2(t, x(t), y(t)) \end{cases}, \quad t \in q^{\mathbb{Z}}, \tag{5}$$

where $q > 1$ and Δ stands for the $q-$difference operator, i.e.,

$$x^{\Delta}(t) = \frac{x(qt) - x(t)}{(q-1)t}, \; t \in q^{\mathbb{Z}}.$$

Obviously, the $q-$difference system (5) can be used to describe the dynamics of prey-predator interaction in mathematical biosciences, which can be viewed as a quantum-time-scale analogue of the Kolmogorov predator–prey model studied in [20]. The proposed formulation preserves the classical biological interpretation and Kolmogorov-type structural assumptions, while replacing additive periodicity by multiplicative periodicity, which is the natural notion of periodicity on quantum domains. This approach allows us to extend the existence theory of positive periodic solutions to a genuinely new class of non-translation-invariant time scales and to highlight qualitative differences between additive and multiplicative periodic dynamics.

Subsequently, we rewrite the system (5) as follows

$$\begin{cases} x^{\Delta}(t) = x(t)[a(t) - f_1(t, x(t), y(t))] \\ y^{\Delta}(t) = y(t)[b(t) - f_2(t, x(t), y(t))] \end{cases}, \quad t \in q^{\mathbb{Z}}; \tag{6}$$

cf. also [13]. Obviously, when (6) is interpreted as a prey-predator system then x can be considered as the population of prey and y indicates the population of the predator. For exhibiting the functional $q-$difference system (6) as a Kolmogorov prey-predator system, one needs to make the following conditions on (6) (see [20]):

(i) There is a carrying capacity τ_1 for the prey population. That is, there exist $\tau_1 > 0$ so that $f_1(t, \tau_1, 0) = a(t)$ and $a(t) - f_1(t, x, y) < 0$ whenever $x > \tau_1$.

(ii) We naturally assume that predation affects growth in prey population negatively, and this means f_1 is increasing in y.

(iii) There is a minimum value for the prey population to promote the population of predators. Equivalently, there exists $\tau_2 > 0$ so that $b(t) - f_2(t, \tau_2, 0) = 0$.

(iv) The function f_2 is decreasing with respect to x, and increasing in y.

Clearly, it is reasonable to focus on nonnegative solutions of the quantum Kolmogorov system (6). Since we are interested in the existence of periodic solutions of (6), it is natural to make some periodicity assumptions on the model. For a fixed $T \in q^{\mathbb{N}}$, we suppose that

(v) a and b are positive valued periodic functions due to Definiton 2.6, that is multiplicatively T-periodic functions

$$a(tT)T = a(t) \text{ and } b(tT)T = b(t) \text{ for all } t \in q^{\mathbb{Z}}.$$

(vi) f_1 and f_2 are positive valued periodic functions in the sense of Definiton 2.6 (multiplicatively T-periodic functions) in t; that is,

$$f_{1,2}(tT, x, y)T = f_{1,2}(t, x, y) \text{ for all } t \in q^{\mathbb{Z}}.$$

We will always assume that conditions (v-vi) hold throughout manuscript.

Next, we provide the following construction which is inevitable for investigation of the existence of periodic solutions. Fix $T \in q^{\mathbb{N}}$, and consider the set θ_T which is the set of all T-periodic couples (x, y) based on the periodicity perception in Definiton 2.7 where $x,\ y : q^{\mathbb{Z}} \longrightarrow [0, \infty)$. Then $(\theta_T, \|\cdot\|)$ is a Banach space with the norm

$$\|(x, y)\| := \max_{[1,T]\cap q^{\mathbb{Z}}} |x(t)| + \max_{[1,T]\cap q^{\mathbb{Z}}} |y(t)|.$$

4. Periodic Solutions by Coincidence Degree

First, we present an important theorem in coincidence degree theory which is fundamental for obtaining one of the main results of the manuscript. First of all, we present the following terminology by using the same notations with [14] for the sake of consistency with the already established literature.

Let X and Z be two normed spaces and the mapping $L : \operatorname{dom} L \subset X \longrightarrow Z$ be linear. L is called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L < \infty$ and $\operatorname{Im} L \subset Z$ is closed with $\operatorname{co}\dim \operatorname{Im} L < \infty$. Besides, if L is a Fredholm mapping of index zero, and there exist continuous projections $P : Z \longrightarrow Z$ and $Q : Z \longrightarrow Z$ so that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$, then the mapping $L_{\operatorname{dom} L\cap \operatorname{Ker} P} : (I - P)X \longrightarrow \operatorname{Im} L$ has the inverse $K_P : \operatorname{Im} L \longrightarrow (I - P)X$. Furthermore, if Ω is an open bounded subset of X, then the continuous mapping $N : X \longrightarrow Z$ is said to be L-compact on $\bar{\Omega}$ whenever $QN(\bar{\Omega})$ is bounded, and $K_P(I - Q)N : \bar{\Omega} \longrightarrow X$ is compact. Also, there exists an isomorphism $J : \operatorname{Im} Q \longrightarrow \operatorname{Ker} L$ since $\dim \operatorname{Im} Q = \operatorname{co}\dim \operatorname{Im} L$.

Next, we illustrate the following result to be employed in the main outcome.

Theorem 4.1. *(Continuation Theorem) Let L be a Fredholm mapping of index zero, and N be L-compact on $\bar{\Omega}$. Suppose*

(a) $L_Z = \lambda N_Z$ for each $\lambda \in (0, 1)$ and $Z \notin \partial\Omega$.

(b) $QN_Z \neq 0$ for each $Z \in \partial\Omega \cap KerL$, and Brouwer degree $\deg\{JQN, \Omega \cap KerL, 0\} \neq 0$.

Then the equation $L_Z = N_Z$ has at least one solution in $DomL \cap \bar{\Omega}$.

Subsequently, we introduce the following notations:

- $\Psi := \min\{[0, \infty) \cap q^{\mathbb{Z}}\}$
- $I_T := [\Psi, \Psi T] \cap q^{\mathbb{Z}}$
- $\bar{g} := \dfrac{1}{\Psi(T - 1)} \int_{I_T} g(s)\Delta s$
- $\theta_T^0 := \{(x, y) \in \theta_T : \bar{x} = \bar{y} = 0\}$
- $\theta_T^{k_1,k_2} := \{(x, y) \in \theta_T : \bar{x} = k_1 \text{ and } \bar{y} = k_2,\ k_{1,2} \geq 0,\ for\ all\ t \in q^{\mathbb{Z}}\}$.

Then θ_T^0 and $\theta_T^{k_{1,2}}$ are both closed and linear subspaces of θ_T. Moreover, we can simply prove that $\theta_T = \theta_T^0 \oplus \theta_T^{k_{1,2}}$ following the lines of the proof of [20, Lemma 4.1].

Remark 4.2. *In [7], it has been shown that for any regressive multiplicatively T-periodic function $f : q^{\mathbb{Z}} \longrightarrow \mathbb{R}$ in the sense of Definition 2.6, the q-exponential function satisfies*

$$e_f(tT, sT) = e_f(t, s)\ for\ all\ t,\ s \in q^{\mathbb{Z}}.$$

Moreover, it is worth underlining another interesting property of q-exponential function of a multiplicatively T-periodic function f which is presented in [21]. Set $t = q^m$, $\Psi = q^n$, and $T = q^p$, $m,\ n \in \mathbb{Z}$, and $p \in \mathbb{N}$. Also, consider such a decomposition for m, that is $m = n + \gamma p + w$, where $0 \le w \le p$ and $\gamma \in \mathbb{N}_0$. Then we have:

$$\begin{aligned} e_f(Tt, T) &= \prod_{s=n+\gamma p+w}^{n+(\gamma+1)p+w-1} (1 + (q-1)q^s f(q^s)) \\ &= \prod_{s=n+w}^{n+p+w-1} (1 + (q-1)q^{s+\gamma p} f(q^{s+\gamma p})) \\ &= \prod_{s=n+w}^{n+p+w-1} (1 + (q-1)q^s f(q^s)) \\ &= \frac{\prod_{s=n}^{n+p-1}(1 + (q-1)q^s f(q^s)) \prod_{s=n+p}^{n+p+w-1}(1 + (q-1)q^s f(q^s))}{\prod_{s=n}^{n+w-1}(1 + (q-1)q^s f(q^s))} \\ &= \prod_{s=n}^{n+p-1} (1 + (q-1)q^s f(q^s)), \end{aligned}$$

where we utilized multiplicative T-periodicity of f. This results in

$$e_f(tT, t) = e_f(\Psi T, \Psi) \text{ for all } t \in q^{\mathbb{Z}}. \tag{7}$$

We derive the next result by using the same steps of [24, Lemma 3.3] (see also [21, Lemma 3]).

Lemma 4.3. *Suppose that the pair (x, y) be a nonnegative T-periodic solution of the quantum Kolmogorov system (6) based on the Definition 2.7. Then,*

$$\begin{aligned} &\min_{t \in I_T} x(t) \ge e_{\ominus a}(\Psi T, \Psi)\|x\| \\ &\min_{t \in I_T} y(t) \ge e_{\ominus b}(\Psi T, \Psi)\|y\|. \end{aligned}$$

Proof. As an implementation of q-derivative, we write

$$x(qt) = x(t) + (q-1)tx^\Delta(t),$$

and we reconstruct the Kolmogorov system (6) as

$$\begin{aligned} x^\Delta(t) &= a(t)(x(qt) - (q-1)tx^\Delta(t)) - x(t)f_1(t, x, y) \\ y^\Delta(t) &= b(t)(y(qt) - (q-1)ty^\Delta(t)) - y(t)f_2(t, x, y). \end{aligned}$$

Subsequently, we get

$$\begin{cases} x^\Delta(t)(1 + (q-1)ta(t)) - a(t)x(qt) = -x(t)f_1(t, x, y) \\ y^\Delta(t)(1 + (q-1)tb(t)) - b(t)y(qt) = -y(t)f_2(t, x, y) \end{cases}. \tag{8}$$

Of course, the functions a and b are tacitly assumed to be regressive, namely $1 + (q-1)ta(t) \neq 0$ and $1 + (q-1)tb(t) \neq 0$ for all $t \in q^{\mathbb{Z}}$. Consequently, (8) turns into the coupled system

$$\begin{cases} x^\Delta(t) - \dfrac{a(t)}{1 + (q-1)ta(t)} x(qt) = \dfrac{-x(t)}{1 + (q-1)ta(t)} f_1(t, x, y) \\ y^\Delta(t) - \dfrac{b(t)}{1 + (q-1)tb(t)} y(qt) = \dfrac{-y(t)}{1 + (q-1)tb(t)} f_2(t, x, y) \end{cases}. \tag{9}$$

We multiply the equations in (9) with $e_{\ominus a}(t,\Psi)$ and $e_{\ominus b}(t,\Psi)$, respectively. Then we get

$$\begin{cases} (e_{\ominus a}(t,\Psi)x(t))^{\Delta} = \dfrac{-e_{\ominus a}(t,\Psi)x(t)}{1+(q-1)ta(t)}f_1(t,x,y) \\ (e_{\ominus b}(t,\Psi)y(t))^{\Delta} = \dfrac{-e_{\ominus b}(t,\Psi)y(t)}{1+(q-1)tb(t)}f_2(t,x,y) \end{cases}. \tag{10}$$

We integrate the equations in (10) from t to tT, and obtain

$$\begin{cases} x(tT)e_{\ominus a}(tT,\Psi) - x(t)e_{\ominus a}(t,\Psi) = \displaystyle\int_t^{tT} \frac{-e_{\ominus a}(u,\Psi)x(u)}{1+(q-1)ua(u)}f_1(u,x,y)\Delta u \\ y(tT)e_{\ominus b}(tT,\Psi) - y(t)e_{\ominus b}(t,\Psi) = \displaystyle\int_t^{tT} \frac{-e_{\ominus b}(u,\Psi)y(u)}{1+(q-1)ub(u)}f_2(u,x,y)\Delta u \end{cases}. \tag{11}$$

By T-periodicity of the couple (x,y), (11) can be alternatively represented as

$$\begin{cases} x(t)e_{\ominus a}(t,\Psi)(e_{\ominus a}(tT,t)-1) = \displaystyle\int_t^{tT} \frac{-e_{\ominus a}(u,\Psi)x(u)}{1+(q-1)ua(u)}f_1(u,x,y)\Delta u \\ y(t)e_{\ominus b}(t,\Psi)(e_{\ominus b}(tT,t)-1) = \displaystyle\int_t^{tT} \frac{-e_{\ominus b}(u,\Psi)y(u)}{1+(q-1)ub(u)}f_2(u,x,y)\Delta u \end{cases}, \tag{12}$$

where we used [Appendix, Theorem 2.5, iii-iv]. At this stage, we use (7), and write

$$e_{\ominus a}(tT,t) = e_{\ominus a}(\Psi T,\Psi) \text{ and } e_{\ominus b}(tT,t) = e_{\ominus b}(\Psi T,\Psi).$$

This yields to

$$x(t) = \int_t^{tT} \frac{-x(u)}{1+(q-1)ua(u)}\frac{e_{\ominus a}(u,t)}{e_{\ominus a}(\Psi T,\Psi)-1}f_1(u,x,y)\Delta u$$

$$y(t) = \int_t^{tT} \frac{-y(u)}{1+(q-1)ub(u)}\frac{e_{\ominus b}(u,t)}{e_{\ominus b}(\Psi T,\Psi)-1}f_2(u,x,y)\Delta u.$$

We set

$$G_1(t,u) := \frac{e_{\ominus a}(u,t)}{1-e_{\ominus a}(\Psi T,\Psi)}$$

and

$$G_2(t,u) := \frac{e_{\ominus b}(u,t)}{1-e_{\ominus b}(\Psi T,\Psi)},$$

for all $u \in [t,tT]\cap q^{\mathbb{Z}}$. Accordingly, we have the following inequalities

$$\frac{e_{\ominus a}(\Psi T,\Psi)}{1-e_{\ominus a}(\Psi T,\Psi)} \le G_1(t,u) \le \frac{1}{1-e_{\ominus a}(\Psi T,\Psi)}$$

and

$$\frac{e_{\ominus b}(\Psi T,\Psi)}{1-e_{\ominus b}(\Psi T,\Psi)} \le G_2(t,u) \le \frac{1}{1-e_{\ominus b}(\Psi T,\Psi)},$$

for all $u \in [t, tT] \cap q^{\mathbb{Z}}$. Consequently, we obtain

$$\begin{cases} \|x\| \leq \dfrac{1}{1 - e_{\ominus a}(\Psi T, \Psi)} \displaystyle\int_t^{tT} \dfrac{x(u)}{1 + (q-1)ua(u)} f_1(u, x, y) \Delta u \\ \|y\| \leq \dfrac{1}{1 - e_{\ominus b}(\Psi T, \Psi)} \displaystyle\int_t^{tT} \dfrac{y(u)}{1 + (q-1)ub(u)} f_2(u, x, y) \Delta u \end{cases} \tag{13}$$

and

$$\begin{cases} \min_{t \in I_T} x(t) \geq \dfrac{e_{\ominus a}(\Psi T, \Psi)}{1 - e_{\ominus a}(\Psi T, \Psi)} \displaystyle\int_t^{tT} \dfrac{x(u)}{1 + (q-1)ua(u)} f_1(u, x, y) \Delta u \\ \min_{t \in I_T} y(t) \geq \dfrac{e_{\ominus b}(\Psi T, \Psi)}{1 - e_{\ominus b}(\Psi T, \Psi)} \displaystyle\int_t^{tT} \dfrac{y(u)}{1 + (q-1)ub(u)} f_2(u, x, y) \Delta u \end{cases}. \tag{14}$$

Considering (13) and (14) together implies that the assertion is correct. □

For the construction of the next existence result, we introduce the following conditions:

C1: There exist constants $M_2 > M_1 > 0$ so that if $x(t) \geq M_2$ for all $t \in I_T$, then

$$f_1(t, x(t), y(t)) > a(t), \ t \in I_T$$

and if $0 < x(t) \leq M_1$ for all $t \in I_T$, then

$$f_1(t, x(t), y(t)) < a(t), \ t \in I_T.$$

C2: There exist constants $M_4 > M_3 > 0$ so that if $y(t) \geq M_4$ for all $t \in I_T$, then

$$f_2(t, x(t), y(t)) > b(t), \ t \in I_T$$

and if $0 < y(t) \leq M_3$ for all $t \in I_T$, then

$$f_2(t, x(t), y(t)) < b(t), \ t \in I_T.$$

Theorem 4.4. *Suppose that* (C1-C2) *hold. Then, the Kolmogorov system* (6) *has a T-periodic solution with respect to Definiton 2.7.*

Proof. In order to utilize continuation theorem in coincidence degree theory, we fix $X = Z = \theta_T$, and present the operators as

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^{\Delta} \\ y^{\Delta} \end{pmatrix}$$

$$N\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t)[a(t) - f_1(t, x(t), y(t))] \\ y(t)[b(t) - f_2(t, x(t), y(t))] \end{pmatrix},$$

where $Ker(L) = \theta_T^{k_1,k_2}$ and $Im(L) = \theta_T^0$. Then, L is a Fredholm operator of index zero since $Im(L)$ is closed and $\dim KerL = \operatorname{codim} ImL = 1$. Besides, there exist projection $P : X \longrightarrow X$ and $Q : Z \longrightarrow Z$ so that

$$P\begin{pmatrix} x \\ y \end{pmatrix} = Q\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}.$$

It is worth emphasizing that $ImP = KerL$ and $ImL = KerQ = Im(I-Q)$. Accordingly, the inverse of L exists and it is represented by $K_P : ImL \longrightarrow KerP \cap DomL$ so that

$$K_P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \hat{x} - \bar{\hat{x}} \\ \hat{y} - \bar{\hat{y}} \end{pmatrix},$$

where

$$\hat{x} = \int_{\Psi}^{t} x(s)\Delta s.$$

In the sequel, we get

$$QN\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\Psi T - \Psi}\displaystyle\int_{\Psi}^{\Psi T} x(s)[a(s) - f_1(s, x(s), y(s))]\Delta s \\ \dfrac{1}{\Psi T - \Psi}\displaystyle\int_{\Psi}^{\Psi T} y(s)[b(s) - f_2(s, x(s), y(s))]\Delta s \end{pmatrix}$$

and

$$K_P(I-Q)N\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} \displaystyle\int_{\Psi}^{t} \mathcal{K}_1(s)\Delta s - \frac{1}{\Psi T - \Psi}\int_{\Psi}^{\Psi T}\int_{\Psi}^{t} \mathcal{K}_1(s)\Delta s\Delta t - (t - \Psi - \frac{1}{\Psi T - \Psi}\int_{\Psi}^{\Psi T}(t-\Psi)\Delta t)\overline{\mathcal{K}_1(t)} \\ \int_{\Psi}^{t} \mathcal{K}_2(s)\Delta s - \displaystyle\frac{1}{\Psi T - \Psi}\int_{\Psi}^{\Psi T}\int_{\Psi}^{t} \mathcal{K}_2(s)\Delta s\Delta t - (t - \Psi - \frac{1}{\Psi T - \Psi}\int_{\Psi}^{\Psi T}(t-\Psi)\Delta t)\overline{\mathcal{K}_2(t)} \end{pmatrix}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are chosen as

$$\begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{pmatrix} = \begin{pmatrix} x(t)[a(t) - f_1(t, x(t), y(t))] \\ y(t)[b(t) - f_2(t, x(t), y(t))] \end{pmatrix}$$

for the sake of brevity.

Noting that X is a Banach space, $\overline{P(I-Q)N(\bar{\Omega})}$ is compact for any open subset $\Omega \subset X$ by Arzela-Ascoli theorem. On the other hand, $QN(\bar{\Omega})$ is bounded and this yields to L-compactness of N for any open subset $\Omega \subset X$.

For $\alpha \in (0,1)$, consider the equation $Lx = \alpha Nx$, that is

$$\begin{pmatrix} x^{\Delta} \\ y^{\Delta} \end{pmatrix} = \begin{pmatrix} \alpha x(t)[a(t) - f_1(t, x(t), y(t))] \\ \alpha y(t)[b(t) - f_2(t, x(t), y(t))] \end{pmatrix} \tag{15}$$

and suppose that the pair (x, y) is an arbitrary solution of (15). Then, we have

$$\min_{t\in I_T} x(t) \geq e_{\ominus\alpha a}(\Psi T, \Psi)\|x\| \geq e_{\ominus a}(\Psi T, \Psi)\|x\|$$

$$\min_{t\in I_T} y(t) \geq e_{\ominus\alpha b}(\Psi T, \Psi)\|y\| \geq e_{\ominus b}(\Psi T, \Psi)\|y\|$$

as a consequence of Lemma 4.3. Additionally, we integrate (5) from Ψ to ΨT and get

$$\int_{\Psi}^{\Psi T} \alpha x(s)[a(s) - f_1(s, x(s), y(s))]\Delta s = 0 \tag{16}$$

and

$$\int_{\Psi}^{\Psi T} \alpha y(s)[b(s) - f_2(s, x(s), y(s))]\Delta s = 0. \tag{17}$$

We pursue this part of the proof by establishing contradictions. First, we aim to show that $\|x\| < \dfrac{M_2}{e_{\ominus a}(\Psi T, \Psi)}$ and $\|y\| < \dfrac{M_4}{e_{\ominus b}(\Psi T, \Psi)}$. Assume the opposite. If $\|x\| \geq \dfrac{M_2}{e_{\ominus a}(\Psi T, \Psi)}$ and $\|y\| \geq \dfrac{M_4}{e_{\ominus b}(\Psi T, \Psi)}$, then

$$\min_{t \in q^{\mathbb{Z}_1}} x(t) = \min_{t \in I_T} x(t) \geq e_{\ominus a}(\Psi T, \Psi)\|x\| \geq M_2$$

and

$$\min_{t \in q^{\mathbb{Z}_1}} y(t) = \min_{t \in I_T} y(t) \geq e_{\ominus b}(\Psi T, \Psi)\|y\| \geq M_4.$$

This indicates

$$f_1(t, x(t), y(t)) > a(t), \ t \in I_T$$

and

$$f_2(t, x(t), y(t)) > b(t) \ t \in I_T$$

which contradicts with (16) and (17), respectively. In a similar fashion, suppose that

$$\min_{t \in I_T} x(t) < e_{\ominus a}(\Psi T, \Psi)M_1$$

and

$$\min_{t \in I_T} y(t) < e_{\ominus b}(\Psi T, \Psi)M_3.$$

Then, we obtain the inequalities

$$M_1 e_{\ominus a}(\Psi T, \Psi) > \min_{t \in I_T} x(t) \geq e_{\ominus a}(\Psi T, \Psi)\|x\|$$

and

$$M_3 e_{\ominus b}(\Psi T, \Psi) > \min_{t \in I_T} y(t) \geq e_{\ominus b}(\Psi T, \Psi)\|y\|$$

in the light of Lemma 4.3. These contradict with (16) and (17) again. Consequentially, we deduce that

$$e_{\ominus a}(\Psi T, \Psi)M_1 < x(t) < \frac{M_2}{e_{\ominus a}(\Psi T, \Psi)} \tag{18}$$

and

$$e_{\ominus b}(\Psi T, \Psi)M_3 < y(t) < \frac{M_4}{e_{\ominus b}(\Psi T, \Psi)} \tag{19}$$

for all $t \in I_T$.

Next, we introduce the set Ω from the couples $(x, y) \in X$ so that (18)-(19) hold. Let us underline that (a) of continuation theorem, Theorem 4.1, is satisfied. Also, suppose that $(x, y) \in \partial\Omega \cap KerL$. Then, x and y must be constant with

$$QN\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\Psi T - \Psi}\displaystyle\int_{\Psi}^{\Psi T} x(s)[a(s) - f_1(s, x(s), y(s))]\Delta s \\ \dfrac{1}{\Psi T - \Psi}\displaystyle\int_{\Psi}^{\Psi T} y(s)[b(s) - f_2(s, x(s), y(s))]\Delta s \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moreover, $J : ImQ \longrightarrow KerL$ and $ImQ = KerL$, thus $J = L$. We introduce the homotopy

$$H_{\lambda^*}\begin{pmatrix} x \\ y \end{pmatrix} := \alpha^* \begin{pmatrix} \dfrac{1}{2}(e_{\ominus a}(\Psi T, \Psi)M_1 + \dfrac{M_2}{e_{\ominus a}(\Psi T, \Psi)}) - x \\ \dfrac{1}{2}(e_{\ominus b}(\Psi T, \Psi)M_3 + \dfrac{M_4}{e_{\ominus b}(\Psi T, \Psi)}) - y \end{pmatrix} + (1 - \alpha^*)QN\begin{pmatrix} x \\ y \end{pmatrix}$$

for $\alpha^* \in [0, 1]$. Obviously,

$$H_{\lambda^*}\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for any $\alpha^* \in [0, 1]$ and $(x, y) \in \partial\Omega \cap KerL$. Then evaluate the degree

$$\begin{aligned} \deg\{JQN, \Omega \cap KerL, 0\} &= \deg\{QN, \Omega \cap KerL, 0\} \\ &= \deg\left\{\begin{pmatrix} \dfrac{1}{2}(e_{\ominus a}(\Psi T, \Psi)M_1 + \dfrac{M_2}{e_{\ominus a}(\Psi T, \Psi)}) - x \\ \dfrac{1}{2}(e_{\ominus b}(\Psi T, \Psi)M_3 + \dfrac{M_4}{e_{\ominus b}(\Psi T, \Psi)}) - y, \end{pmatrix}\Omega \cap KerL, 0\right\} \\ &\neq 0. \end{aligned}$$

To conclude, the Kolmogorov system (6), which is constructed on quantum domain, has at least one positive T-periodic solution due to Definiton 2.7 by Theorem 4.1. □

We update the conditions (C1) and (C2) as follows:

C3: There exist constants $M_6 > M_5 > 0$ such that if $x(t) \geq M_6$ for all $t \in I_T$, then

$$f_1(t, x(t), y(t)) < a(t), \ t \in I_T$$

and if $0 < x(t) \leq M_5$ for all $t \in I_T$, then

$$f_1(t, x(t), y(t)) > a(t), \ t \in I_T.$$

C4: There exist constants $M_8 > M_7 > 0$ such that if $y(t) \geq M_8$ for all $t \in I_T$, then

$$f_2(t, x(t), y(t)) < b(t), \ t \in I_T$$

and if $0 < y(t) \leq M_7$ for all $t \in I_T$, then

$$f_2(t, x(t), y(t)) > b(t), \ t \in I_T.$$

The following result can be proved by exactly repeating the same steps in the proof of Theorem 4.4, therefore it is trivial.

Theorem 4.5. *Suppose that* (C3) *and* (C4) *are satisfied. Then, the Kolmogorov system* (6) *has a T-periodic solution in the light of Definiton 2.7.*

We aim to highlight controllability of our technical conditions by providing the following example:

Example 4.6. *We consider the following quantum Kolmogorov system as a prey-predator model at which there does not exist a minimum value for the prey population to support predators. Let x and y stand for the populations of preys and predators and consider*

$$\begin{cases} x^{\Delta}(t) = x(t)[\frac{1}{t} - \frac{1}{t}\left(\frac{1}{128}x(t) + \frac{1}{32}\frac{y(t)}{1+y(t)}\right)] \\ y^{\Delta}(t) = y(t)[\frac{1}{t} - \frac{1}{t}\frac{y(t)}{4+\arctan(x(t))}] \end{cases}, \ t \in q^{\mathbb{Z}}, \ q > 1. \tag{20}$$

If we compare (20) with (6), then we arrive at $a(t) = b(t) = \frac{1}{t}$,

$$f_1(t, x, y) = \frac{1}{t}\left(\frac{1}{128}x + \frac{1}{32}\frac{y}{1+y}\right),$$

and

$$f_2(t, x, y) = \frac{1}{t}\frac{y}{4 + \arctan(x)}.$$

The model given in (20) can be regarded as a q-Kolmogorov prey-predator system since

- *There exists a carrying capacity $\tau_1 = 128$ for prey population so that*

  $$f_1(t, 128, 0) = \frac{1}{t} = a(t),$$

  and

  $$\frac{1}{t} - \frac{1}{t}\left(\frac{1}{128}x(t) + \frac{1}{32}\frac{y(t)}{1+y(t)}\right) < 0 \text{ for } x > 128.$$

- *f_1 is increasing in y.*
- *f_2 is decreasing with respect to x and, it is increasing in y.*

By relaxing the condition (iii) from Kolmogorov prey-predator system, we could satisfy the conditions (i, ii, iv). Also, a, b, f_1, f_2 are 2-periodic in the sense of Definition 2.6 (multiplicatively 2-periodic) in t, and consequently, the conditions (v,vi) hold. Moreover,

$$\frac{1}{t} < \frac{1}{t}\left(\frac{1}{128}x + \frac{1}{32}\frac{y}{1+y}\right) \text{ for } x \geq 128,$$

and

$$\frac{1}{t}\left(\frac{1}{128}x + \frac{1}{32}\frac{y}{1+y}\right) < \frac{1}{t} \text{ for } x \leq 123.$$

This means (C1) is satisfied. Similarly, we get

$$\frac{1}{t} < \frac{1}{t}\frac{y}{4 + \arctan(x)} \text{ for } y \geq 16,$$

and

$$\frac{1}{t}\frac{y}{4+\arctan(x)} \text{ for } y \leq 4.$$

Thus, (C2) holds. According to Theorem 4.4, q-Kolmogorov system given in (22) *has a* 2*-periodic solution based on Definition 2.7.*

Remark 4.7. *Although the present work is primarily analytical, Example 4.6 admits a natural numerical interpretation on the quantum time scale. Starting from $t_0 = 1$ and using the identity*

$$x(qt) = x(t) + (q-1)t\, x^{\Delta}(t),$$

system (20) induces a well-defined q-difference iteration on the grid $t_n = q^n$. For representative positive initial data, this iteration can be evaluated numerically in a straightforward manner.

We emphasize that the periodicity considered in this paper is understood in the sense of Definition 6, namely $x(Tt) = x(t)$ (and similarly for y). In particular, choosing $T = q^2$ (2-periodicity)*, the condition $x(Tt) = x(t)$ is equivalent on the grid $t_n = q^n$ to the discrete relation $x_{n+2} = x_n$. Numerical evaluation of the induced q-difference scheme confirms that this identity is satisfied after a small number of initial iterations, providing a numerical consistency check for the periodicity property asserted in Example 4.6.*

5. Periodic Solutions by Fixed Point Theory: Delayed Model

We introduce the function $\delta : [1,\infty)\cap q^{\mathbb{Z}} \longrightarrow [\delta(1),\infty)\cap q^{\mathbb{Z}}$ so that $\delta(t) = t/q^{\zeta}$, $\zeta \in \mathbb{N}$. It is straightforward that δ is strictly increasing, $\delta(t) < t$, and $\delta(qt) = q\delta(t)$ for all $t \in q^{\mathbb{Z}}$. Accordingly, we utilize the function δ as the delay function in this part of the manuscript and introduce the delayed Kolmogorov system on quantum domains:

$$\begin{cases} x^{\Delta}(t) = x(t)[a(t) - f_1(t, x(\delta(t)), y(t))] \\ y^{\Delta}(t) = y(t)[b(t) - f_2(t, x(t), y(\delta(t)))] \end{cases}, \ t \in q^{\mathbb{Z}}. \tag{21}$$

By a solution of the delayed quantum Kolmogorov system (21), we understand a couple of functions $(x(t), y(t))_{t\in q^{\mathbb{Z}}}$ which satisfies (21) for all $t \in q^{\mathbb{N}_0}$. We construct the initial phase space for the solutions of (21) as a nonempty set of pairs of all functions $(\eta, \xi) = (\eta(t), \xi(t))_{t\in q^{\mathbb{Z}_-}}$ so that

$$\max\left\{\sup_{t\in q^{\mathbb{Z}_-}} |\eta(t)| + \sup_{t\in q^{\mathbb{Z}_-}} |\xi(t)|\right\} < \infty.$$

Obviously, for any given couple $(\eta(t), \xi(t))_{t\in q^{\mathbb{Z}_-}}$ there exists a unique solution $(x(t), y(t))_{t\in q^{\mathbb{Z}}}$ of (21) satisfying

$$(x(t), y(t)) = (\eta(t), \xi(t)) \text{ for all } t \in q^{\mathbb{Z}_-}.$$

Subsequently, one may easily represent a solution of (21), $(x^{\eta}(t), y^{\xi}(t))$, corresponding to any pair $(\eta, \xi) \in S$ as

$$(x^{\eta}(t), y^{\xi}(t)) = \begin{cases} (\eta(t), \xi(t)), & for\ t \in q^{\mathbb{Z}_-} \\ (x(t), y(t)), & for\ t \in q^{\mathbb{N}_0} \end{cases}.$$

First of all, we assume that the conditions (i-vi) hold for (21) similar to the system (6) which does not involve a delay term.

Next, we introduce the mapping

$$\Gamma(x,y) := \Gamma(x(t),y(t))_{t\in q^{\mathbb{Z}}} := \left\{ \begin{pmatrix} \Gamma_1(x(t),y(t)) \\ \Gamma_2(x(t),y(t)) \end{pmatrix}_{t\in q^{\mathbb{Z}}} \right\},$$

where

$$\Gamma_1(x(t),y(t)) = \begin{cases} \eta(t), \ t \in q^{\mathbb{Z}_-} \\ \dfrac{1}{1-e_{\ominus a}(T,1)} \displaystyle\int_t^{tT} x(s)e_{\ominus a}(qs,t)f_1(s,x(\delta(s)),y(s))\Delta s, \ t \in q^{\mathbb{N}_0} \end{cases}$$

and

$$\Gamma_2(x(t),y(t)) = \begin{cases} \xi(t), \ t \in q^{\mathbb{Z}_-} \\ \dfrac{1}{1-e_{\ominus b(T,1)}} \displaystyle\int_t^{tT} y(s)e_{\ominus b}(qs,t)f_2(s,x(s),y(\delta(s)))\Delta s, \ t \in q^{\mathbb{N}_0} \end{cases}.$$

Lemma 5.1. *The couple* $(x(t),y(t)) := \{(x^{\eta}(t),y^{\xi}(t))\}_{t\in q^{\mathbb{Z}}}$ *with an initial sequence* $\{(\eta(t),\xi(t))\}_{t\in q^{\mathbb{Z}_-}})$ *in S satisfies*

$$\Gamma(x(t),y(t)) = (x(t),y(t)) \ \ for\ all\ t \in q^{\mathbb{Z}}$$

if and only if it is a T-periodic solution of (21) *in the sense of Definition 2.7.*

Proof. It is straightforward to verify that the pair $(x^{\eta}(t),y^{\xi}(t))$ satisfying $\Gamma(x^{\eta},y^{\xi}) = (x^{\eta},y^{\xi})$ is a $T-$periodic solution of (21). Oppositely, we suppose that $(x(t),y(t)) = (x^{\eta}(t),y^{\xi}(t))$ is a $T-$periodic solution of (21) due to Definition 2.7. Then, one may easily repeat the initial steps in the proof of Lemma 4.3 and arrive at

$$x(t)e_{\ominus a}(t,1)(e_{\ominus a}(tT,t)-1) = \int_t^{tT} \frac{-x(u)e_{\ominus a}(u,1)}{1+(q-1)ua(u)} f_1(u,x(\delta(u)),y(u))\Delta u$$
$$y(t)e_{\ominus b}(t,1)(e_{\ominus b}(tT,t)-1) = \int_t^{tT} \frac{-y(u)e_{\ominus b}(u,1)}{1+(q-1)ub(u)} f_2(u,x(u),y((\delta(u)))\Delta u$$

similar to (12). Then, we write

$$x(t) = \frac{1}{1-e_{\ominus a}(tT,t)} \int_t^{tT} \frac{x(u)e_a(t,1)}{1+(q-1)ua(u)e_a(u,1)} f_1(u,x(\delta(u)),y(u))\Delta u$$
$$y(t) = \frac{1}{1-e_{\ominus b}(tT,t)} \int_t^{tT} \frac{y(u)e_b(t,1)}{1+(q-1)ub(u)e_b(u,1)} f_2(u,x(u),y(\delta(u)))\Delta u.$$

By [Bohner, Theorem 2.36, (ii)], we have

$$(1+(q-1)ua(u))e_a(u,1) = e_a(qu,1)$$
$$(1+(q-1)ub(u))e_b(u,1) = e_b(qu,1),$$

where the tacit regressiveness assumptions on a and b are taken into account . By employing (7), we obtain

$$x(t) = \frac{1}{1 - e_{\ominus a}(T,1)} \int_t^{tT} x(u) e_{\ominus a}(qu,t) f_1(u, x(\delta(u)), y(u)) \Delta u$$

and

$$y(t) = \frac{1}{1 - e_{\ominus b}(T,1)} \int_t^{tT} y(u) e_{\ominus b}(qu,t) f_2(u, x(u), y(\delta(u))) \Delta u.$$

This proves the assertion. □

In the sequel, we introduce the set $\Xi(W) := \{(x,y) \in \theta_T : \|(x,y)\| \leq W\}$ for any $(x,y) \in \theta_T$, which is a bounded, closed and convex subset of θ_T . Suppose that there exist positive constants W_1, W_2, K_1 and K_2 such that

C5: $|f_1(t, x(t), y(t))| \leq W_1$ and $|f_2(t, x(t), y(t))| \leq W_2$ for all $t \in q^{\mathbb{N}_0}$ and $(x,y) \in \Xi(W)$

C6:

$$\frac{1}{1 - e_{\ominus a}(T,1)} \int_t^{tT} e_{\ominus a}(qs,t) \Delta s \leq K_1$$

and

$$\frac{1}{1 - e_{\ominus b}(T,1)} \int_t^{tT} e_{\ominus b}(qs,t) \Delta s \leq K_2$$

for all for all $t \in q^{\mathbb{N}_0}$.

C7: $W_1 K_1 + W_2 K_2 \leq 1$.

Lemma 5.2. *Suppose that* (C5-C7) *hold. Then* Γ *maps* $\Xi(W)$ *into itself.*

Proof. First, we show that $\Gamma(x(tT), y(tT)) = \Gamma(x(t), y(t))$ for any $(x,y) \in \theta_T$. To see this, we write

$$\begin{aligned}
\Gamma_1(x(tT), y(tT)) &= \frac{1}{1 - e_{\ominus a}(T,1)} \int_{tT}^{tT^2} x(s) e_{\ominus a}(qs, tT) f_1(s, x(\delta(s)), y(s)) \Delta s \\
&= \frac{1}{1 - e_{\ominus a}(T,1)} \int_t^{tT} x(Ts) e_{\ominus a}(qsT, tT) f_1(sT, x(\delta(sT)), y(sT)) T \Delta s \\
&= \frac{1}{1 - e_{\ominus a}(T,1)} \int_t^{tT} x(s) e_{\ominus a}(qs, t) f_1(sT, x(T\delta(s)), y(sT)) T \Delta s \\
&= \frac{1}{1 - e_{\ominus a}(T,1)} \int_t^{tT} x(s) e_{\ominus a}(qs, t) f_1(s, x(\delta(s)), y(s)) \Delta s \\
&= \Gamma_1(x(t), y(t)).
\end{aligned}$$

Clearly, some steps can be repeated for Γ_2 to obtain $\Gamma_2(x(tT), y(tT)) = \Gamma_2(x(t), y(t))$. It remains to show that $\|\Gamma(x, y)\| = \|(\Gamma_1(x, y), \Gamma_2(x, y))\| \leq W$ for any $(x, y) \in \Xi(W)$. To achieve this task, we consider

$$\|\Gamma(x, y)\| = \max_{[1,T]\cap q^{\mathbb{Z}}} |\Gamma_1(x, y)| + \max_{[1,T]\cap q^{\mathbb{Z}}} |\Gamma_2(x, y)|,$$

where

$$\begin{aligned} \left|\Gamma_1(x, y)\right| &\leq \frac{1}{1 - e_{\ominus a}(T, 1)} \int_t^{tT} |x(s)| e_{\ominus a}(qs, t) |f_1(s, x(\delta(s)), y(s))| \Delta s \\ &\leq W W_1 K_1 \end{aligned}$$

and

$$\begin{aligned} |\Gamma_2(x, y)| &\leq \frac{1}{1 - e_{\ominus b}(T, 1)} \int_t^{tT} |y(s)| e_{\ominus b}(qs, t) |f_2(s, x(s), y(\delta(s)))| \Delta s \\ &\leq W W_2 K_2. \end{aligned}$$

Then we get $\|\Gamma(x, y)\| \leq W\,(W_1 K_1 + W_2 K_2) \leq W$ by C7 and the proof is complete. □

In addition to the conditions (C5-C7), we introduce

C8: There exist positive constants $E_{1,2,3,4}$ so that for any $(x_1, y_1), (x_2, y_2) \in \Xi(W)$ we have

$$|f_1(t, x_1, y_1) - f_1(t, x_2, y_2) \leq E_1 |x_1 - x_2| + E_2 |y_1 - y_2|$$

and

$$|f_2(t, x_1, y_1) - f_2(t, x_2, y_2) \leq E_3 |x_1 - x_2| + E_4 |y_1 - y_2|$$

for all $t \in q^{\mathbb{Z}}$.

Lemma 5.3. *If* (C5), (C6) *and* (C8) *hold, then the mapping* Γ *is continuous.*

Proof. Let us pick a sequence $\{(x^n, y^n)\}_{n\in\mathbb{N}_0}$ from $\Xi(W)$ so that

$$\lim_{n\longrightarrow\infty} \|(x^n, y^n) - (x, y)\| = \lim_{n\longrightarrow\infty} \left\{ \max_{t\in[1,T]\cap q^{\mathbb{Z}}} |x^n(t) - x(t)| + \max_{t\in[1,T]\cap q^{\mathbb{Z}}} |y^n(t) - y(t)| \right\} = 0.$$

By closedness of Ξ, we expect to have $(x, y) \in \Xi(W)$. Now, we consider

$$\begin{aligned} \left\|\Gamma(x^n, y^n) - \Gamma(x, y)\right\| = &\max_{t\in[1,T]\cap q^{\mathbb{Z}}} |\Gamma_1(x^n(t), y^n(t)) - \Gamma_1(x(t), y(t))| \\ &+ \max_{t\in[1,T]\cap q^{\mathbb{Z}}} |\Gamma_2(x^n(t), y^n(t)) - \Gamma_2(x(t), y(t))|, \end{aligned}$$

where

$$
\begin{aligned}
|\Gamma_1(x^n(t), y^n(t)) - \Gamma_1(x(t), y(t))| &\leq \frac{1}{1-e_{\ominus a}(T,1)} \int_t^{tT} e_{\ominus a}(qs,t) \left| x^n(s) f_1(s, x^n(\delta(s)), y^n(s)) \right. \\
&\quad \left. - x(s) f_1(s, x^n(\delta(s)), y^n(s)) \right| \Delta s \\
&\leq \frac{1}{1-e_{\ominus a}(T,1)} \int_t^{tT} e_{\ominus a}(qs,t) \left[|x^n(s) - x(s)| |f_1(s, x^n(\delta(s)), y^n(s))| \right. \\
&\quad \left. + |x(s)| |f_1(s, x^n(\delta(s)), y^n(s)) - f_1(s, x(\delta(s)), y(s))| \right] \Delta s \\
&\leq W_1 K_1 \int_t^{tT} |x^n(s) - x(s)| \Delta s + W K_1 \int_t^{tT} (E_1 |x^n(\delta(s)) - x^n(s)| \\
&\quad + E_2 |y^n(s) - y(s)|) \Delta s.
\end{aligned}
$$

If we take the limits as $n \longrightarrow \infty$, then we observe that $|\Gamma_1(x^n, y^n) - \Gamma_1(x, y)| \longrightarrow 0$ due to the dominated convergence theorem. In a similar fashion, one may easily obtain that $|\Gamma_2(x^n, y^n) - \Gamma_2(x, y)| \longrightarrow 0$ as $n \longrightarrow \infty$. This results in Γ is continuous. □

We represent the following fixed point theorem which is a crucial tool for the next existence result:

Theorem 5.4 (Schauder). *If $\mathbb{S}$ is a closed, bounded, convex subset of a Banach space $\mathbb{X}$ and $\Gamma : \mathbb{S} \longrightarrow \mathbb{S}$ is completely continuous, then Γ has a fixed point in $\mathbb{S}$.*

Theorem 5.5. *Suppose that the conditions* (C5-C8) *are satisfied. Then the delayed quantum Kolmogorov system has a (positive) periodic solution in the sense of Definition 2.7.*

Proof. Due to Lemma 5.2 and Lemma 5.3, it remains to show that $\Gamma\Xi(W)$ is compact. To achieve this task, we employ sequential compactness, that is $\Gamma\Xi(W)$ is said to be sequentially compact if and only if $\Gamma\Xi(W)$ has a subsequence converging to an element in $\Gamma\Xi(W)$. At this point, we pursue the diagonalization process. Let $\{(x^n, y^n)\}_{n \in \mathbb{N}_0}$ be a sequence in $\Xi(W)$, and consequentially $\{(x^n(t), y^n(t))\}$ is a bounded pair of sequences. Due to the Bolzano-Weierstrass theorem, there is a sequence $\{(x^{n_{1_k}}, y^{n_{1_k}})\}$ so that $\{(x^{n_{1_k}}(1), y^{n_{1_k}}(1))\}$ is convergent. Next, choose another sequence with index n_{2_k} which is a subsequence of $\{(x^{n_{1_k}}, y^{n_{1_k}})\}$ so that $\{(x^{n_{2_k}}(q), y^{n_{2_k}}(q))\}$ is convergent. By repeating the same procedure, we obtain the subsequence $\{(x^{n_{j_k}}, y^{n_{j_k}})\}$ which is convergent for $q^{j-1} = t_j \in q^{\mathbb{N}_0}$. By setting $n_k = n_{k_k}$, we establish a diagonal sequence $\{(x^{n_j}, y^{n_j})\}$ which is a subsequence of $\{(x^n, y^n)\}$. By continuity of Γ, $\Gamma(x^n, y^n)$ has a convergent subsequence in $\Xi(W)$. This proves the assertion. □

In the next example, we aim to provide an implementation of Theorem 5.5 for the controllability of our technical conditions:

Example 5.6. *Consider the following quantum Kolmogorov system with delay*

$$
\begin{cases}
x^{\Delta}(t) = x(t) \left[\frac{1}{t} - \frac{1}{3t} \left(2 + \sin\left(\frac{\ln t}{\ln 2} \pi \right) \left(x\left(\frac{t}{2} \right) + y(t) \right) \right) \right] \\
y^{\Delta}(t) = y(t) \left[\frac{1 + (-1)^{\frac{\ln t}{\ln 2}}}{t} - \frac{1}{8t} \left(2 + \cos(x(t)) + \sin\left(y\left(\frac{t}{2} \right) \right) \right) \right]
\end{cases}, \ t \in 2^{\mathbb{Z}}. \tag{22}
$$

By comparing (22) with (21), we obtain

$$
a(t) = \frac{1}{t}, \tag{23}
$$

$$
b(t) = \frac{1 + (-1)^{\frac{\ln t}{\ln 2}}}{t}, \tag{24}
$$

$$
f_1(t, x, y) = \frac{1}{3t} \left(2 + \sin\left(\frac{\ln t}{\ln 2} \pi \right) (x + y) \right),
$$

and

$$f_2(t,x,y) = \frac{1}{8t}\left(2+\cos(x)+\sin(y)\right).$$

It is obvious that the functions a, b, f_1, and f_2 are 4*-periodic with respesct to Definition 2.6 (multiplicatively* 4*-periodic) in t. Next, we observe that $f_1(t,0,0) = 0$, and for any $(x_{1,2}, y_{1,2}) \in \Xi(W)$ we have*

$$\begin{aligned}\left|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)\right| &\le \left|\frac{1}{3t}\left(2+\sin\left(\frac{\ln t}{\ln 2}\pi\right)\right)\right|\left(|x_1-x_2| + |y_1-y_2|\right)\\ &\le |x_1-x_2| + |y_1-y_2| \text{ for all } t \in 2^{\mathbb{N}_0}.\end{aligned}$$

Then

$$\left|f_1(t,x,y)\right| = \left|f_1(t,x,y) - f_1(t,0,0)\right| \le |x| + |y| \le W$$

for any $(x,y) \in \Xi(W)$. Also,

$$\left|f_2(t,x,y)\right| \le \frac{1}{8t}\left(2+|\cos x| + |\sin y|\right) \le \frac{1}{2} \text{ for all } t \in 2^{\mathbb{N}_0}.$$

Moreover for any $(x_{1,2}, y_{1,2}) \in \Xi(W)$, one may easily get the inequality

$$\begin{aligned}\left|f_2(t,x_1,y_1) - f_2(t,x_2,y_2)\right| &\le \frac{1}{8t}\left(|\cos x_1 - \cos x_2| + |\sin y_1 - \sin y_2|\right)\\ &\le \frac{1}{8}\left(|x_1-x_2| + |y_1-y_2|\right)\end{aligned}$$

for all $t \in 2^{\mathbb{N}_0}$. Thus the conditions (C5) and (C8) are satisfied with $W_1 = W$, $W_2 = \frac{1}{2}$, $E_1 = E_2 = 1$ and $E_3 = E_4 = \frac{1}{8}$. On the other hand, we get

$$\frac{1}{1-e_{\ominus a}(4,1)}\int_t^{4t} e_{\ominus a}(2s,t)\Delta s \le \frac{1}{2}$$

and

$$\frac{1}{1-e_{\ominus b}(4,1)}\int_t^{4t} e_{\ominus b}(2s,t)\Delta s \le \frac{8}{27},$$

where a and b are as in (23-24), respectively. Thus, the condition (C6) holds with $K_1 = \frac{1}{2}$ and $K_2 = \frac{8}{27}$. Then, the condition (C8) holds whenever $W < 1{,}7$. In the sequel, Theorem 5.5 implies that (22) *has a* 4*-periodic solution with respect to Definition 2.7.*

Remark 5.7. *Although the present work is mainly analytical, Example 5.6 naturally admits a numerical realization on the quantum grid*

$$t_n = 2^n, \qquad n \in \mathbb{Z}.$$

Setting $x_n := x(2^n)$ and $y_n := y(2^n)$, and using the identity $x(2t) = x(t) + t\,x^{\Delta}(t)$ (and similarly for y), system (22) induces a well-defined nonlinear iteration for the sequences $\{x_n\}$ and $\{y_n\}$. Since the delay satisfies $t_n/2 = t_{n-1}$, the corresponding discrete system involves the delayed terms x_{n-1} and y_{n-1}.

We emphasize that the notion of periodicity used in this paper is that of Definition 2.7. In particular, for $T = 4$ the condition

$$x(4t) = x(t), \qquad y(4t) = y(t),$$

reduces on the quantum grid to the discrete relations

$$x_{n+2} = x_n, \qquad y_{n+2} = y_n, \qquad n \in \mathbb{Z}.$$

Therefore, the periodicity of the numerical solution can be verified by monitoring the two-step recurrence of the sequences $\{x_n\}$ *and* $\{y_n\}$.

Starting from positive initial data (x_{-1}, y_{-1}) *and* (x_0, y_0)*, direct numerical evaluation shows that the solution remains positive and approaches a stable regime exhibiting a clear even–odd repeating pattern. This behavior corresponds to a* 4*-periodic solution in the sense of Definition 2.7. To quantify this observation, we introduce the periodicity defect*

$$\varepsilon_n := |x_{n+2} - x_n| + |y_{n+2} - y_n|.$$

Representative numerical values are reported in Table 1, where one observes that ε_n *becomes small after a short transient, confirming the* $T = 4$ *periodicity numerically.*

Table 1: Numerical verification of $T = 4$ periodicity for Example 2 on the quantum grid $t_n = 2^n$.

n	x_n	y_n	x_{n+2}	y_{n+2}	ε_n
10	0.7421	0.3184	0.7420	0.3185	1.6×10^{-4}
11	0.5287	0.6142	0.5288	0.6141	1.4×10^{-4}
12	0.7420	0.3185	0.7420	0.3184	9.8×10^{-5}
13	0.5288	0.6141	0.5288	0.6141	7.2×10^{-5}

The periodic behavior is also illustrated in Figure 1, where the time series of x_n *and* y_n *display a characteristic two-step (even–odd) zigzag pattern, and in Figure 2, which shows the decay of the periodicity defect* ε_n.

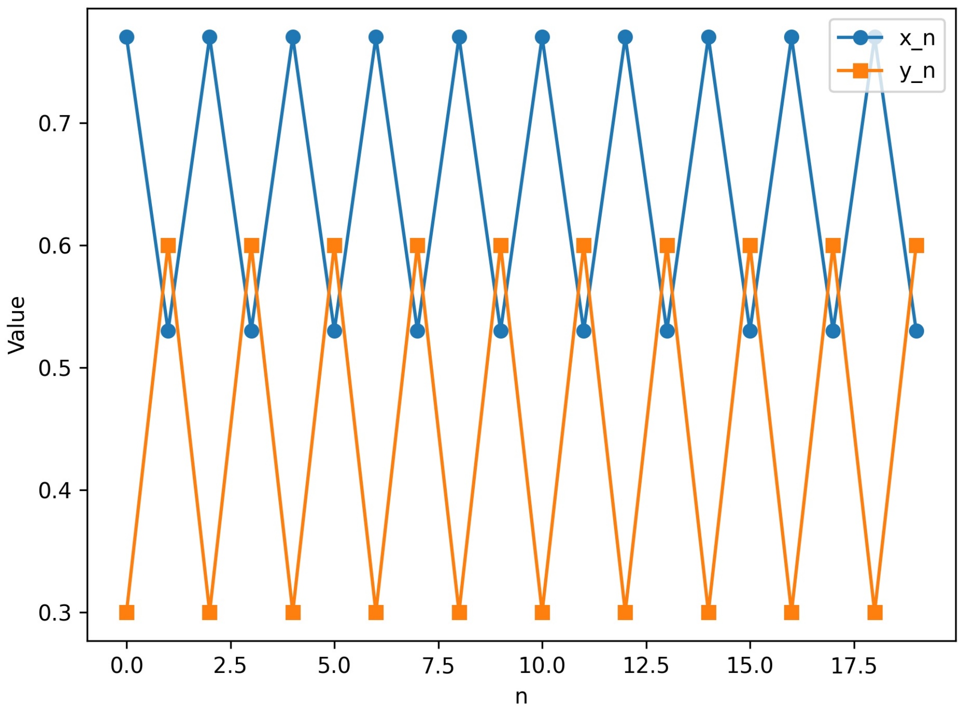

Figure 1: Time series of the numerical solution of Example 2 on the quantum grid $t_n = 2^n$. The even–odd repeating pattern of $\{x_n\}$ and $\{y_n\}$ corresponds to a $T = 4$ periodic solution, i.e., $x(4t) = x(t)$ and $y(4t) = y(t)$.

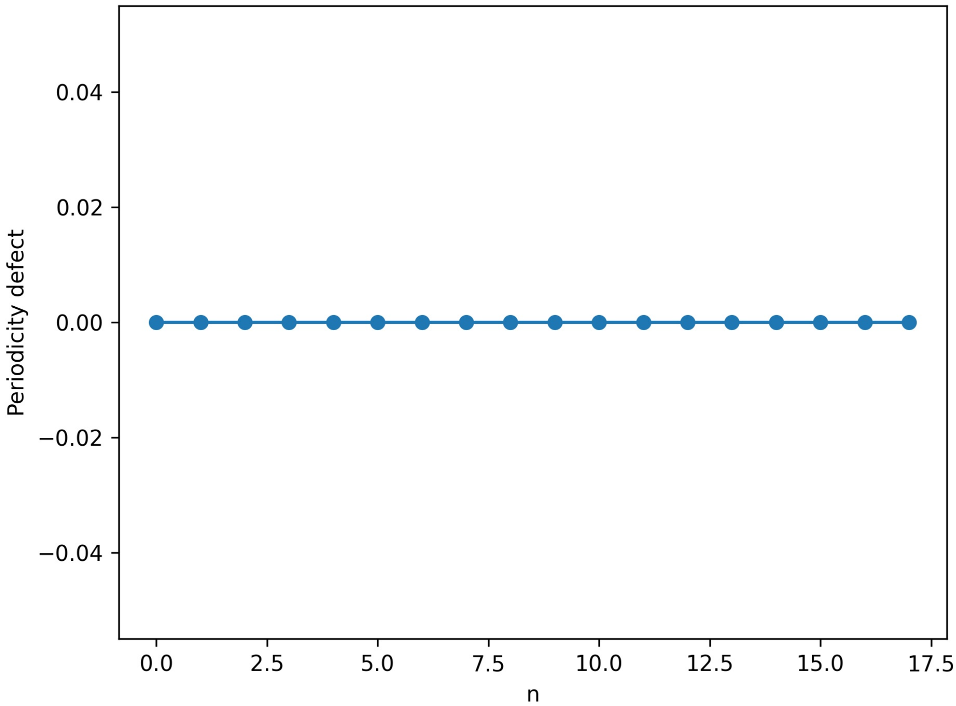

Figure 2: Periodicity defect $\varepsilon_n = |x_{n+2} - x_n| + |y_{n+2} - y_n|$ for Example 2. After a short transient, the defect remains small, providing numerical confirmation of the $T = 4$ periodicity.

6. Conclusion

In this paper, Kolmogorov systems are studied on quantum domains and the sufficient conditions for the existence of positive periodic solutions are investigated. The setup of the manuscript is two-fold: In the first stage of the paper, we concentrate on q-Kolmogorov systems due to coincidence degree theory in order to focus on the existence of positive periodic solutions. Subsequently, the second part of the manuscript is devoted to delayed q-Kolmogorov systems. As a result of having delay term in the model, we establish a phase space by initial functions and we study the existence of positive periodic solutions by Schauder's fixed point theorem for providing an alternative approach to coincidence degree theory. It is worth emphasizing that the obtained conditions in Section 4 and Section 5 are incomparable. Just to note that, one may easily handle the delayed q-Kolmogorov model by repeating the same procedure in Section 4, thus it is left to the readers. In the construction of the main outcomes, we use the periodicity definition for our solutions which is parallel to the conventional periodicity perception (see Section 2, Definition 2.7), and this enables us to employ Banach space of periodic functions endowed with the maximum norm. As the future work, the main results of this research can be unified and generalized by using the new periodicity concept based on shift operators $\delta_{\pm}$ on time scales in the light of [2].

Declarations

Authors declare that they have no conflict of interest.

References

[1] M. Adıvar, H. C. Koyuncuoğlu, *Floquet theory based on new periodicity concept for hybrid systems involving q-difference equations*, Appl. Math. Comput. **273** (2016), 1208–1233.

[2] M. Adıvar, *A new periodicity concept for time scales*, Math. Slovaca **63**(4) (2013), 817–828.
[3] D. Agarwal, S. Abbas, *Existence of periodic solutions for a class of dynamic equations with multiple time varying delays on time scales*, Qual. Theory Dyn. Syst. **23**(1) (2024).
[4] M. Bohner, R. Chieochan, *Floquet theory for q-difference equations*, Sarajev. J. Math. **8**(2) (2012), 1–12.
[5] M. Bohner, R. Chieochan, *The Beverton-Holt q-difference equation*, J. Biol. Dyn. **7**(1) (2013), 86–95.
[6] M. Bohner, J. G. Mesquita, *Periodic averaging principle in quantum calculus*, J. Math. Anal. Appl. **435**(2) (2016), 1146–1159.
[7] M. Bohner, J. G. Mesquita, *Massera's theorem in quantum calculus*, Proc. Amer. Math. Soc. **146**(11) (2018), 4755–4766.
[8] M. Bohner, S. Streipert, *Optimal harvesting policy for the Beverton-Holt quantum difference model*, Math. Morav. **20**(2) (2016), 39–57.
[9] M. Bohner, S. Streipert, *The second Cushing-Henson conjecture for the Beverton-Holt q-difference equation*, Opusc. Math. **37**(6) (2017), 795–819.
[10] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhauser, Boston, 2001.
[11] T. Caraballo, R. Colucci, X. Han, *Semi-Kolmogorov models for predation with indirect effects in random environments*, Discrete Contin. Dyn. Syst. Ser. B **21**(7) (2016), 2129–2143.
[12] E. D. Conway, J. A. Smoller, *Global analysis of a system of predator-prey equations*, SIAM J. Appl. Math. **46**(4) (1986), 630–642.
[13] J. M. Cushing, *Periodic Kolmogorov systems*, SIAM J. Math. Anal. **13**(5) (1982), 811–827.
[14] R. E. Gaines, J. L. Mahwin, *Coincidence Degree, and Nonlinear Differential Equations*, in: Lecture Notes in Mathematics. Springer, Heidelberg, 2006.
[15] L. Huang, M. Tang, J. Yu, *Qualitative analysis of Kolmogorov-type models of predator-prey systems*, Math. Biosci. **130**(1) (1995), 85–97.
[16] M. Islam, J. T. Neugebauer, *Existence of periodic solutions for a quantum Volterra equation*, Adv. Dyn. Syst. Appl. **11**(1) (2016), 67–80.
[17] M. Islam, J. T. Neugebauer, *Asymptotically p-periodic solutions of a quantum Volterra integral equation*, Sarajev. J. Math. **14**(1) (2018), 59–70.
[18] V. Kac, P. Cheung, *Quantum Calculus*. Springer, New York, 2012. doi:10.1007/978-1-4613-0071-7
[19] E. R. Kaufmann, Y. N. Raffoul, *Periodic solutions for a neutral nonlinear dynamical equation on a time scale*, J. Math. Anal. Appl. **319**(1) (2006), 315–325.
[20] E. R. Kaufmann, *A Kolmogorov predator-prey system on a time scale*, Dyn. Syst. Appl. **23**(4) (2014), 561–573.
[21] M. Kostić, H. C. Koyuncuoglu, Y. N. Raffoul, *Positive periodic solutions for certain kinds of delayed q-difference equations with biological background*, Ann. Funct. Anal. **15**(1), (2024).
[22] S. Ruan, *Absolute stability, conditional stability and bifurcation in Kolmogorov-type predator-prey systems with discrete delays*, Quart. Appl. Math. **59**(1) (2001), 159–173.
[23] H. Wang, C. Ou, B. Dai, *The existence and stability of order-*1 *periodic solutions for an impulsive Kolmogorov predator-prey model with non-selective harvesting*, J. Appl. Anal. Comput. **11**(3) (2021), 1348–1370.
[24] J. Zhang, M. Fan, H. Zhu, *Periodic solution of single population models on time scales*, Math. Comput. Model. **52**(3–4) (2010), 515–521.