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Screen pseudo semi-slant lightlike submanifolds of golden semi-Riemannian manifolds

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Abstract. In this note, we define a new class of lightlike submanifolds of golden semi-Riemannian manifolds which include anti-invariant, invariant, and slant distributions within a same submanifold of the ambient manifold. Moreover, we prove the necessary and sufficient conditions for the distributions of such submanifolds to be integrable, and also the cases when they are geodesic. At last, we give a non-trivial example of Screen pseudo semi-slant lightlike submanifolds using partial differential equations.

1. Introduction

The study of submanifolds has long been central to differential geometry, as it illuminates the interaction between intrinsic invariants and extrinsic curvature of ambient spaces. In almost Hermitian geometry, slant submanifolds, introduced by Chen [4], provide a unifying generalization of both complex (holomorphic) and totally real submanifolds: a slant submanifold is characterized by a constant angle (the slant angle) between the image of any tangent vector under the almost complex structure and the tangent space itself. This notion has been generalized in several directions–semi-slant, bi-slant, pseudo-slant, and related concepts–by allowing tangent bundles to decompose into distributions with different slant behaviours.

In [17], the geometers Duggal and Bejancu studied lightlike (null) submanifolds of semi-Riemannian manifolds, and these submanifolds have attracted intense interest [18, 19], partly because of their fundamental role in mathematical relativity and partly because they present distinctive geometric challenges: the induced metric on a lightlike submanifold is degenerate, so standard orthogonal decompositions fail and one must introduce auxiliary structures (notably the radical and screen distributions) to formulate Gauss, Weingarten, and integrability relations using partial differential equations. The interplay between slant-type decompositions and lightlike geometry produces rich and subtle phenomena not present in the nondegenerate setting.

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Among structured ambient spaces, golden semi-Riemannian manifolds furnish a natural arena for such investigations. A golden structure is a (1,1)-tensor field ϕ satisfying the polynomial relation $\phi^2 = \phi + I$, an algebraic identity motivated by the golden ratio (see [5], the work of Hretcanu and Crasmareanu); when compatible with a semi-Riemannian metric, it yields new curvature and integrability features that affect submanifold theory. Combining golden structures with lightlike geometry yields a promising framework to explore generalized slant behaviours.

In [1], Blaga and Hretcanu characterize anti-invariant, invariant and slant submanifolds of metallic Riemannian manifolds. An invariant submanifold is a submanifold whose tangent spaces are preserved by a given linear map or structure on the larger manifold. Anti-invariant submanifolds are a type of submanifold, specifically where the structure of the ambient manifold is mapped to the normal space of the submanifold.

In [3], Chen proposed the notion of slant submanifolds as follows:

Definition 1.1. [3] *A submanifold N of an almost Hermitian manifold (M, J, g) is named slant if the Wirtinger angle $\theta(X)$ is independent of the choice of $X \in T_pN$ and of $p \in N$. The angle $\theta(X)$ of a slant submanifold is called the slant angle. A slant submanifold with slant angle θ is simply called θ-slant.*

Slant submanifolds, which include complex and totally real submanifolds, represent the simplest and most natural class of submanifolds in an almost Hermitian manifold. The initial results on slant submanifolds were presented in [4]. Since then, the study of slant submanifolds and slant submersions has attracted increasing attention, yielding many important results over the past thirty years. In [2], the author define screen pseudo slant lightlike submanifolds of golden semi-Riemannian manifolds and studied the integrability conditions of the distributions and the cases when these distributions become geodesic. The integrability conditions and geodesic nature of the distributions for different manifolds have been studied in [6–16].

Inspired by above work, we got motivation to write this paper. In this work, we introduce and study screen pseudo semi-slant lightlike submanifolds of golden semi-Riemannian manifolds. Such a submanifold admits the standard lightlike decomposition (radical and screen distributions), while the screen distribution further decomposes into three complementary distributions: one invariant under ϕ, one anti-invariant (totally ϕ-transverse), and one proper slant distribution with a constant slant angle. This unifies and extends several existing notions (anti-invariant, invariant, slant, semi-slant, and pseudo-slant lightlike submanifolds) within a single ambient structure.

2. Abbreviations

We used the following abbreviations in this paper:
1. Lightlike submanifolds: *LS*,
2. Scree pseudo semi-slant lightlike submanifolds: *SPSSLS*,
3. Golden semi-Riemannian manifolds: *GSRM*,
4. Semi-Riemannian manifold: *SRM*,

3. Preliminaries

If ϕ is a non-null (1,1) type tensor satisfying

$$\phi^2 = \phi + \mathtt{I}, \tag{1}$$

where $\mathtt{I}$ represents the identity transformation. Then, ϕ is named a golden structure and if there is a SRM $(\bar{\mathtt{F}}, d)$ endowed with this kind of structure, then, it is called a GSRM. Also, the metric d, satisfying

$$d(\phi\mathtt{A}, \mathtt{B}) = d(\mathtt{A}, \phi\mathtt{B}), \tag{2}$$

$\forall$ A,B vector fields on $\overline{\mathrm{F}}$, is called ϕ-compatible.
From (2), we have

$$d(\phi\mathrm{A}, \phi\mathrm{B}) = d(\phi\mathrm{A}, \mathrm{B}) + d(\mathrm{A}, \mathrm{B}). \tag{3}$$

If $\overline{\nabla}$, the Levi-Civita connection respecting to d satisfies

$$\overline{\nabla}_{\mathrm{A}}\phi\mathrm{B} = \phi\, \overline{\nabla}_{\mathrm{A}}\mathrm{B}, \tag{4}$$

$\forall$ $\mathrm{A}, \mathrm{B} \in \Gamma(\mathrm{TF})$, then, $(\overline{\mathrm{F}}, \phi, d)$ is called locally GSRM.
We follow [17] for the basic definitions and the terminologies that are used in this paper. We have

$$\mathrm{TF} = Rad(\mathrm{TF}) \perp S(\mathrm{TF}),$$

$$tr(\mathrm{TF}) = ltr(\mathrm{TF}) \perp S(\mathrm{TF}^{\perp}),$$

$$\mathrm{T}\overline{\mathrm{F}}|_{\mathrm{F}} = S(\mathrm{TF}) \perp [Rad(\mathrm{TF}) \oplus ltr(\mathrm{TF})] \perp S(\mathrm{TF}^{\perp}).$$

The following two equations

$$\overline{\nabla}_{\mathrm{A}}\mathrm{B} = \nabla_{\mathrm{A}}\mathrm{B} + II(\mathrm{A}, \mathrm{B}), \;\; \forall \;\; \mathrm{A}, \mathrm{B} \in \Gamma(\mathrm{TF}), \tag{5}$$

$$\overline{\nabla}_{\mathrm{A}}\mathrm{C} = -A_{\mathrm{C}}\mathrm{A} + \nabla^{t}_{\mathrm{A}}\mathrm{C}, \;\; \forall \;\; \mathrm{A} \in \Gamma(\mathrm{TF}), \mathrm{C} \in \Gamma tr(\mathrm{TF}), \tag{6}$$

where the linear connections on $\overline{\mathrm{F}}$, F and $tr(\mathrm{TF})$ are represented by $\overline{\nabla}$, ∇ and ∇^{t} respectively, are called Gauss and Weingarten formulae.
$\{\nabla_{\mathrm{A}}\mathrm{B}, A_{\mathrm{C}}\mathrm{A}\} \in \Gamma(\mathrm{TF})$ and $\{II(\mathrm{A}, \mathrm{B}), \nabla^{t}_{\mathrm{A}}\mathrm{C}\} \in \Gamma(ltr(\mathrm{TF}))$, respectively. The linear connection on the vector bundle $ltr(\mathrm{TF})$ is represented by ∇^{t}_{A}.
$II(\mathrm{A}, \mathrm{B})$ represents the 2nd fundamental form which is symmetric $\rho(\mathrm{F})$-bilinear form on $\Gamma(\mathrm{TF})$ having values in $\Gamma(tr(\mathrm{TF}))$.
A_{C} is the shape operator which is a linear endomorphism of $\Gamma(\mathrm{TF})$. Then,

$$\overline{\nabla}_{\mathrm{A}}\mathrm{B} = \nabla_{\mathrm{A}}\mathrm{B} + II^{l}(\mathrm{A}, \mathrm{B}) + II^{s}(\mathrm{A}, \mathrm{B}), \tag{7}$$

$$\overline{\nabla}_{\mathrm{A}}\mathrm{E} = -A_{\mathrm{E}}\mathrm{A} + \nabla^{l}_{\mathrm{A}}(\mathrm{E}) + \mathrm{D}^{s}(\mathrm{A}, \mathrm{E}), \tag{8}$$

$$\overline{\nabla}_{\mathrm{A}}\mathrm{G} = -A_{\mathrm{G}}\mathrm{A} + \nabla^{s}_{\mathrm{A}}(\mathrm{G}) + \mathrm{D}^{l}(\mathrm{A}, \mathrm{G}), \;\; \forall \;\; \mathrm{A}, \mathrm{B} \in \Gamma(\mathrm{TF}), \mathrm{E} \in \Gamma(ltr(\mathrm{TF})) \tag{9}$$

and $\mathrm{G} \in \Gamma(S(\mathrm{TF}^{\perp}))$. Mark the TF−projection onto $S(\mathrm{TF})$ by $\overline{Y}$.
$\overline{\nabla}$ being a metric connection and by using (1),(3)-(5), we obtain

$$\overline{d}(II^{s}(\mathrm{A}, \mathrm{B}), \mathrm{G}) + \overline{d}(\mathrm{B}, \mathrm{D}^{l}(\mathrm{A}, \mathrm{G})) = d(A_{\mathrm{G}}\mathrm{A}, \mathrm{B}), \tag{10}$$

$$\overline{d}(\mathrm{D}^{s}(\mathrm{A}, \mathrm{E}), \mathrm{G}) = \overline{d}(\mathrm{E}, A_{\mathrm{G}}\mathrm{A}). \tag{11}$$

We set

$$\nabla_{\mathrm{A}}\overline{Y}\mathrm{B} = \nabla^{*}_{\mathrm{A}}\overline{Y}\mathrm{B} + II^{*}(\mathrm{A}, \overline{Y}\mathrm{B}), \tag{12}$$

$$\nabla_{\mathrm{A}}\eta = -A^{*}_{\eta}\mathrm{A} + \nabla^{*t}_{\mathrm{A}}\eta \tag{13}$$

for $\mathrm{A}, \mathrm{B} \in \Gamma(\mathrm{TF})$ and $\eta \in \Gamma(Rad\mathrm{TF})$.
By using above equations, we get

Definition 3.1. [20] *A LS (F, d) of a SRM $(\overline{F}, \overline{d})$ is named totally umbilical in $\overline{F}$ if there is a smooth transversal vector field $J \in \Gamma(tr(TF))$ on F named the transversal curvature vector field of F, such that for any $A, B \in \Gamma(TF)$*

$$II(A, B) = d(A, B)J. \tag{14}$$

If $J = 0$, then F is totally geodesic. From (7) and (14), we conclude that F is totally umbilical if and only if there exists smooth vector fields $J^{l} \in \Gamma(ltr(TF))$ and $J^{s} \in \Gamma(S(TF^{\perp}))$ such that

$$II^{l}(A, B) = d(A, B)J^{l}, II^{s}(A, B) = d(A, B)J^{s} \text{ and } D^{l}(A, G), \tag{15}$$

for any $A, B \in \Gamma(TF)$ and $G \in \Gamma(S(TF^{\perp}))$.

4. Screen pseudo semi slant lightlike submanifolds

Definition 4.1. *Let F be a LS of a GSRM $\overline{F}$. Then, F is said to be a SPSSL of $\overline{F}$ iff*
(i) the radical distribution $RadTF$ is an invariant distribution with respect to ϕ, i.e.,

$$\phi(RadTF) = RadTF;$$

(ii) there exists non degenerate orthogonal distributions D, $D^{'}$ and $D^{''}$ on F such that

$$S(TF) = D \perp D^{'} \perp D^{''};$$

(iii) the distribution D is anti-invariant, i.e.,

$$\phi(D) \subset S(TF^{\perp});$$

(iv) the distribution $D^{'}$ is invariant, i.e.,

$$\phi(D^{'}) \subset D^{'},$$

(v) the distribution $D^{''}$ is slant with angle $\theta(\neq \frac{\pi}{2})$, i.e., for each $a \in F$ and each non-zero vector $A \in (D^{''})_a$, the angle θ between ϕA and the vector subspace $(D^{''})_a$ is a constant $(\neq \frac{\pi}{2})$ which is independent of the choice of $a \in F$ and $A \in (D^{''})_a$.
This constant angle θ is named the slant angle of the distribution $D^{''}$.
A SPSSL is said to be proper if $D, D^{'}$ and $D^{''}$ (all the three distributions $\neq \{0\}$) and $\theta \neq 0$.

From the above definition, we conclude that

$$TF = Rad(TF) \perp S(TF),$$

$$TF = Rad(TF) \perp D \perp D^{'} \perp D^{''}. \tag{16}$$

The screen transversal bundle $S(TF^{\perp})$ has the following decomposition

$$S(TF^{\perp}) = M(S(TF)) \perp v_1 \perp v_2,$$

For any $A \in \Gamma(TF)$, we have

$$\phi A = T_1 A + T_2 A, \tag{17}$$

where $T_2 A$ and $T_1 A$ are the transversal and tangential parts of ϕA, respectively.
Also, for any $U \in \Gamma(tr(TF))$, we have

$$\phi U = K_1 U + K_2 U, \tag{18}$$

where $K_2 U$ and $K_1 U$ are the transversal and tangential parts of ϕU, respectively.
Let L_1 and L_2 are the projections on $Rad(TF)$ and $S(TF)$ in TF respectively.
In the same manner, let M_1, M_2, M_3 and M_4 are the projections of $tr(TF)$ on $ltrTF$, $M(S(TF))$, v_1 and v_2 respectively.
Then, for any $A \in \Gamma(TF)$, we get

$$A = L_1 A + L_2 A. \tag{19}$$

Applying ϕ to (19), we have

$$\phi A = \phi L_1 A + \phi L_2 A, \tag{20}$$

which gives

$$\phi A = \phi L_1 A + T_1 L_2 A + T_2 L_2 A, \tag{21}$$

where T_1L_2A is the tangential and T_2L_2A is the transversal part of ϕL_2A. Thus, we get $\phi L_1A \in \Gamma Rad(TF)$, $T_1L_2A \in \Gamma(S(TF))$ and $T_2L_2A \in \Gamma(M(S(TF))) \in \Gamma(S(TF^{\perp}))$.
Also, for any $G \in \Gamma(tr(TF))$, we have

$$G = M_1G + M_2G + M_3G + M_4G. \tag{22}$$

Now, applying ϕ to (22), we obtain

$$\phi G = \phi M_1G + \phi M_2G + \phi M_3G + \phi M_4G, \tag{23}$$

which gives

$$\phi G = \phi M_1G + K_1M_2G + K_2M_2G + \phi M_3G + \phi M_4G, \tag{24}$$

where K_1M_2G is the tangential and K_2M_3G is the transversal part of ϕM_2G. Thus, we get $\phi M_1G \in \Gamma\phi(ltr(TF))$, $K_1M_2G \in \Gamma(S(TF))$, $K_2M_2G \in \Gamma(F(S(TF))) \in \Gamma(S(TF^{\perp}))$, $\phi M_3G \in \Gamma(\phi(v_1)) \in \Gamma(v_1)$ and $\phi M_4G \in \Gamma(v_2)$.

Proposition 4.2. [8] *Let F be a SPSSL of a GSRM $(\overline{F}, \overline{d}, \overline{\phi})$. Then, the distribution v_2 is invariant with respect to $\overline{\phi}$.*

Lemma 4.3. *Let (F, d) be a SPSSL of a GSRM $(\overline{F}, \overline{d}, \overline{\phi})$. Then, we have*

$$(\nabla_A T_1)B = A_{T_2B} + K_1 II(A, B), \tag{25}$$

$$(\nabla^t_A T_2)B = K_2 II(A, B) - II(A, T_1B), \tag{26}$$

$$T_1^2 A = T_1A + A - K_1T_2A, \tag{27}$$

$$T_2A = T_2T_1A + K_2T_2A, \tag{28}$$

$$d(T_1A, B) - d(A, T_1B) = d(A, T_2B) - d(T_2A, B), \tag{29}$$

$$d(T_1A, T_1B) = d(T_1A, B) + d(A, B) + d(T_2A, B) - d(T_1A, T_2B) - d(T_2A, T_1B) - d(T_2A, T_2B), \tag{30}$$

where

$$(\nabla_A T_1)B = \nabla_A T_1B - T_1\nabla_A B,$$

and

$$(\nabla^t_A T_2)B = \nabla^t_A T_2B - T_2\nabla_A B,$$

for all $A, B \in \Gamma(TF)$.

Proof. Using (4), (5), (6), (18) and (19) and by comparing transversal and tangential parts of the resulting relation, we get (26) and (25). Now, applying ϕ to (18), by the use of (1) and (19) and on taking transversal and tangential part of the obtained equation, we get (28) and (27). At last using (2), (3) and (18), we get (29) and (30). □

Proposition 4.4. *Let F be a SPSSL of a GSRM $(\overline{F}, \overline{d}, \overline{\phi})$. Then, ϕ is a golden structure on F iff $T_2A = 0$.*

Proof. By using the same steps as in the proof of Proposition 3.5 given on page no. 7 of [8], the result follows. □

Theorem 4.5. *Let F be a SPSSL of a GSRM $\overline{F}$. Then, the following statements are equivalent:*
(i) $RadTF$ is integrable,
(ii) $\overline{d}(\nabla_A\overline{\phi}B - \nabla_B\overline{\phi}A, T_1C) + \overline{d}(II^s(A, \overline{\phi}B) - II^s(B, \overline{\phi}A), T_2C) = \overline{d}(\nabla_A\overline{\phi}B - \nabla_B\overline{\phi}A, C)$,
(iii) $\overline{d}(II^l(B, T_1C) - II^l(B, C) + D^l(B, T_2C), \overline{\phi}A) = \overline{d}(II^l(A, T_1C) - II^l(A, C) + D^l(A, T_2C), \overline{\phi}B)$,
for all $A, B \in \Gamma(RadTF)$ and $C \in \Gamma(S(TF))$.

Proof. Let F be a SPSSL of a GSRM $\overline{F}$. Then, the distribution $Rad TF$ is integrable if and only if

$$\overline{d}([A, B], C) = 0, \tag{31}$$

for all $A, B \in \Gamma(Rad TF)$ and $C \in \Gamma(S(TF))$.
(*i*) ⇒ (*ii*) From (3), (4), (7), (21) and (31), we have

$$\overline{d}(\overline{\phi}[A, B], \overline{\phi}C) - \overline{d}(\overline{\phi}[A, B], C) = 0$$

$$\overline{d}(\overline{\phi}(\nabla_A B - \nabla_B A), T_1C + T_2C) - \overline{d}(\overline{\phi}(\nabla_A B - \nabla_B A), C) = 0$$

$$\overline{d}(\nabla_A\overline{\phi}B - \nabla_B\overline{\phi}A, T_1C) + \overline{d}(II^s(A, \overline{\phi}B) - II^s(B, \overline{\phi}A), T_2C) = \overline{d}(\nabla_A\overline{\phi}B - \nabla_B\overline{\phi}A, C). \tag{32}$$

(*ii*) ⇒ (*iii*) Since $\overline{\nabla}$ is a metric connection and using (7), (21) in (32), we obtain

$$-\overline{d}(\overline{\phi}B, \overline{\nabla}_A\overline{\phi}C) + \overline{d}(\overline{\phi}A, \overline{\nabla}_B\overline{\phi}C) = -\overline{d}(\overline{\phi}B, \overline{\nabla}_A C) + \overline{d}(\overline{\phi}A, \overline{\nabla}_B C). \tag{33}$$

Now, from (7), (8), (21) and (33), we obtain

$$\overline{d}(II^l(B, T_1C) - II^l(B, C) + D^l(B, T_2C), \overline{\phi}A) = \overline{d}(II^l(A, T_1C) - II^l(A, C) + D^l(A, T_2C), \overline{\phi}B), \tag{34}$$

for all $A, B \in \Gamma(Rad TF)$ and $C \in \Gamma(S(TF))$.
(*iii*) ⇒ (*i*) Using (7), (8), (21) in (34), we get (33). Now, since $\overline{\nabla}$ is a metric connection, from (3), (4) and (33), we obtain (31), which completes the proof. □

Theorem 4.6. *Let F be a SPSSL of a GSRM $\overline{F}$. Then, the following statements are equivalent:*
(i) $S(TF)$ is integrable,
(ii) $\overline{d}(\nabla_A T_1B - A_{T_2B}A - \nabla_B T_1A + A_{T_2A}B, \overline{\phi}E) = \overline{d}(\nabla_A T_1B - A_{T_2B}A - \nabla_B T_1A + A_{T_2A}B, E)$,
(iii) $\overline{d}(A_E B - A_{\overline{\phi}E}B, T_1A) + \overline{d}(A_{\overline{\phi}E}A - A_E A, T_1A) = \overline{d}(D^s(B, E) - D^s(B, \overline{\phi}E), T_2A) + \overline{d}(D^s(A, \overline{\phi}E) - D^s(A, E), T_2B)$,
for all $A, B \in \Gamma(S(TF))$ and $E \in \Gamma(ltr(TF))$.

Proof. Let F be a SPSSL of a GSRM $\overline{F}$. Then, the distribution $S(TF)$ is integrable if and only if

$$\overline{d}([A, B], E) = 0, \tag{35}$$

for all $A, B \in \Gamma(S(TF))$ and $E \in \Gamma(ltr(TF))$.
(*i*) ⇒ (*ii*) From (3), (4), (7), (21) and (35), we have

$$\overline{d}(\overline{\phi}[A, B], \overline{\phi}E) - \overline{d}(\overline{\phi}[A, B], E) = 0$$

$$\overline{d}(\overline{\phi}(\nabla_A B - \nabla_B A), \overline{\phi}E) - \overline{d}(\overline{\phi}(\nabla_A B - \nabla_B A), E) = 0$$

$$\overline{d}(\nabla_A\overline{\phi}B - \nabla_B\overline{\phi}A, \overline{\phi}E) = \overline{d}(\nabla_A\overline{\phi}B - \nabla_B\overline{\phi}A, E)$$

$$\overline{d}(\nabla_A T_1B - A_{T_2B}A - \nabla_B T_1A + A_{T_2A}B, \overline{\phi}E) = \overline{d}(\nabla_A T_1B - A_{T_2B}A - \nabla_B T_1A + A_{T_2A}B, E). \tag{36}$$

(*ii*) ⇒ (*iii*) Since $\overline{\nabla}$ is a metric connection and using (7), (9), (21) in (36), we obtain

$$-\overline{d}(\overline{\phi}B, \overline{\nabla}_A\overline{\phi}E) + \overline{d}(\overline{\phi}A, \overline{\nabla}_B\overline{\phi}E) = -\overline{d}(\overline{\phi}B, \overline{\nabla}_A E) + \overline{d}(\overline{\phi}A, \overline{\nabla}_B E). \tag{37}$$

Using (8), (21) in above (37), we get

$$\overline{d}(A_E B - A_{\overline{\phi}E}B, T_1A) + \overline{d}(A_{\overline{\phi}E}A - A_E A, T_1A) = \overline{d}(D^s(B, E) - D^s(B, \overline{\phi}E), T_2A) + \overline{d}(D^s(A, \overline{\phi}E) - D^s(A, E), T_2B), \tag{38}$$

for all $A, B \in \Gamma(S(TF))$ and $E \in \Gamma(ltr(TF))$.
(*iii*) ⇒ (*i*) Using (8) and (21) in (38), we get (37). Now, since $\overline{\nabla}$ is a metric connection, from (3), (4) and (37), we obtain (35), which completes the proof. □

Theorem 4.7. *Let F be a SPSSL of a GSRM $\overline{F}$. Then, the induced connection ∇ is a metric connection if and only if $T_1L_2\nabla_A B + K_1M_2II^s(A,B) + \overline{\phi}M_3\nabla_A B = 0$, for all $A \in \Gamma(TF)$ and $B \in \Gamma(Rad(TF))$.*

Proof. Let F be a SPSSL of a GSRM $\overline{F}$. Then, the induced connection ∇ is a metric connection if and only if $Rad TF$ is a parallel distribution with respect to ∇ (see [17]). From (4), (7), (21) and (24), we have

$$\overline{\nabla}_A\overline{\phi}B = \overline{\nabla}_A(\overline{\phi}L_1B + T_1L_2B + T_2L_2B)$$

$$= \overline{\phi}\,\overline{\nabla}_A L_1B + \overline{\nabla}_A T_1L_2B + \overline{\nabla}_A T_2L_2B$$

$$= \overline{\phi}L_1\overline{\nabla}_A B + T_1L_2\overline{\nabla}_A B + T_2L_2\overline{\nabla}_A B$$

$$= \overline{\phi}L_1\nabla_A B + T_1L_2\nabla_A B + T_2L_2\nabla_A B + \overline{\phi}L_1II^l(A,B) + T_1L_2II^l(A,B) + K_1M_2II^s(A,B) + K_2M_2II^s(A,B) + \overline{\phi}M_3\nabla_A B + \overline{\phi}M_4II^s(A,B).$$

Comparing tangential parts of both sides of above relation, we obtain

$$\nabla_A\overline{\phi}B = \overline{\phi}L_1\nabla_A B + T_1L_2\nabla_A B + K_1M_2II^s(A,B) + \overline{\phi}M_3\nabla_A B.$$

This completes the proof. □

5. Foliations determined by distributions

This section includes the necessary and sufficient conditions for foliations determined by distributions on a SPSSL of a GSRM to be totally geodesic.

For details about foliation, totally geodesic foliations and totally geodesic submanifolds, see the definition 4.1 given on page no. 12 of [8] and for more details see [21].

Theorem 5.1. *Let F be a SPSSL of a GSRM $\overline{F}$. Then, the following statements are equivalent:*
(i) $Rad TF$ defines a totally geodesic foliation,
(ii)

$$\overline{d}(\nabla_A\overline{\phi}B + II^s(A,\overline{\phi}B), \overline{\phi}C) = \overline{d}(\nabla_A\overline{\phi}B, C),$$

(iii)

$$\overline{d}(II^l(A,T_1C) + D^l(A,T_2C), \overline{\phi}B) = \overline{d}(II^l(A,T_1C) + D^l(A,T_2C), B),$$

for all $A, B \in \Gamma(Rad(TF))$ and $C \in \Gamma(S(TF))$.

Proof. Let F be a SPSSL of a GSRM $\overline{F}$. Then, the distribution $Rad TF$ defines a totally geodesic foliation if and only if

$$\overline{d}(\nabla_A B, C) = 0, \tag{39}$$

for all $A, B \in \Gamma(Rad(TF))$ and $C \in \Gamma(S(TF))$.

$(i) \Rightarrow (ii)$ From (3), (4), (7) and (39), we have

$$\overline{d}(\overline{\phi}\nabla_A B, \overline{\phi}C) = \overline{d}(\overline{\phi}\nabla_A B, C)$$

$$\overline{d}(\nabla_A\overline{\phi}B, \overline{\phi}C) = \overline{d}(\nabla_A\overline{\phi}B, C)$$

$$\overline{d}(\nabla_A\overline{\phi}B + II^s(A,\overline{\phi}B), \overline{\phi}C) = \overline{d}(\nabla_A\overline{\phi}B, C). \tag{40}$$

(*ii*) ⇒ (*iii*) Since, $\overline{\nabla}$ is a metric connection and using (7) in (40), we get

$$\overline{d}(\overline{\phi}\mathtt{B}, \overline{\nabla}_{\mathtt{A}}\overline{\phi}\mathtt{C}) = \overline{d}(\mathtt{B}, \overline{\nabla}_{\mathtt{A}}\overline{\phi}\mathtt{C}) \tag{41}$$

for all $\mathtt{A}, \mathtt{B} \in \Gamma(Rad(\mathtt{TF}))$ and $\mathtt{C} \in \Gamma(\mathtt{S}(\mathtt{TF}))$.

From (7), (9), (21) and (41), we get

$$\overline{d}(II^l(\mathtt{A}, \mathtt{T}_1\mathtt{C}) + \mathtt{D}^l(\mathtt{A}, \mathtt{T}_2\mathtt{C}), \overline{\phi}\mathtt{B}) = \overline{d}(II^l(\mathtt{A}, \mathtt{T}_1\mathtt{C}) + \mathtt{D}^l(\mathtt{A}, \mathtt{T}_2\mathtt{C}), \mathtt{B}) \tag{42}$$

(*iii*) ⇒ (*i*) Using (7), (9) and (21) in (42), we get (41).
Since, $\overline{\nabla}$ is a metric connection, then from (3) and (41), we get (39), which completes the proof. □

Theorem 5.2. *Let F be a SPSSL of a GSRM $\overline{F}$. Then, the following statements are equivalent:*
(i)$S(TF)$ defines a totally geodesic foliation,
(ii)

$$\overline{d}(\nabla_A T_1 B - A_{T_2 B} A, \overline{\phi} E) = \overline{d}(\nabla_A T_1 B - A_{T_2 B} A, E),$$

(iii)

$$\overline{d}(D^s(A, \overline{\phi} E), T_2 B) + \overline{d}(A_{\overline{\phi} E} A, B) = \overline{d}(A_{\overline{\phi} E} A, T_1 B)$$

for all $A, B \in \Gamma(S(TF))$ and $E \in \Gamma(ltr(TF))$.

Proof. Let $\mathtt{F}$ be a SPSSL of a GSRM $\overline{\mathtt{F}}$. Then, the distribution $\mathtt{S}(\mathtt{TF})$ defines a totally geodesic foliation if and only if

$$\overline{d}(\nabla_{\mathtt{A}}\mathtt{B}, \mathtt{E}) = 0, \tag{43}$$

for all $\mathtt{A}, \mathtt{B} \in \Gamma(\mathtt{S}(\mathtt{TF}))$ and $\mathtt{E} \in \Gamma(ltr(\mathtt{TF}))$.
(*i*) ⇒ (*ii*) From (3), (4), (7), (9), (21) and (43), we have

$$\overline{d}(\overline{\phi}\nabla_{\mathtt{A}}\mathtt{B}, \overline{\phi}\mathtt{E}) = \overline{d}(\overline{\phi}\nabla_{\mathtt{A}}\mathtt{B}, \mathtt{E})$$

$$\overline{d}(\nabla_{\mathtt{A}}\overline{\phi}\mathtt{B}, \overline{\phi}\mathtt{E}) = \overline{d}(\nabla_{\mathtt{A}}\overline{\phi}\mathtt{B}, \mathtt{E})$$

$$\overline{d}(\nabla_{\mathtt{A}}\mathtt{T}_1\mathtt{B} - A_{\mathtt{T}_2\mathtt{B}}\mathtt{A}, \overline{\phi}\mathtt{E}) = \overline{d}(\nabla_{\mathtt{A}}\mathtt{T}_1\mathtt{B} - A_{\mathtt{T}_2\mathtt{B}}\mathtt{A}, \mathtt{E}), \tag{44}$$

(*ii*) ⇒ (*iii*) Since, $\overline{\nabla}$ is a metric connection and using (7), (9), (21) in (44), we get

$$\overline{d}(\overline{\phi}\mathtt{B}, \overline{\nabla}_{\mathtt{A}}\overline{\phi}\mathtt{E}) = \overline{d}(\overline{\phi}\mathtt{B}, \overline{\nabla}_{\mathtt{A}}\mathtt{E})$$

$$\overline{d}(\overline{\phi}\mathtt{B}, \overline{\nabla}_{\mathtt{A}}\overline{\phi}\mathtt{E}) = \overline{d}(\mathtt{B}, \overline{\phi\nabla_{\mathtt{A}}}\mathtt{E})$$

$$\overline{d}(\overline{\phi}\mathtt{B}, \overline{\nabla}_{\mathtt{A}}\overline{\phi}\mathtt{E}) = \overline{d}(\mathtt{B}, \overline{\nabla}_{\mathtt{A}}\overline{\phi}\mathtt{E}) \tag{45}$$

for all $\mathtt{A}, \mathtt{B} \in \Gamma(\mathtt{S}(\mathtt{TF}))$ and $\mathtt{E} \in \Gamma(ltr(\mathtt{TF}))$.
From (8), (21) and (45), we get

$$\overline{d}(\mathtt{D}^s(\mathtt{A}, \overline{\phi}\mathtt{E}), \mathtt{T}_2\mathtt{B}) + \overline{d}(A_{\overline{\phi}\mathtt{E}}\mathtt{A}, \mathtt{B}) = \overline{d}(A_{\overline{\phi}\mathtt{E}}\mathtt{A}, \mathtt{T}_1\mathtt{B}). \tag{46}$$

(*iii*) ⇒ (*i*) Using (8), (21) in (46), we get (45).
Since, $\overline{\nabla}$ is a metric connection, then from (3) and (45), we get (43), which completes the proof. □

Theorem 5.3. *Let $\mathtt{F}$ be a totally umbilical SPSSL of a GSRM $\overline{\mathtt{F}}$. Then, $\mathtt{F}$ is totally geodesic if $\overline{d}(B, A_{\overline{\phi}U}A) = 0$, for any $B \in \Gamma(S(TF))$, $A \in \Gamma(TF)$ and $U \in \Gamma(v_2)$.*

Proof. Let $\mathtt{F}$ be a totally umbilical SPSSL of a GSRM $\overline{\mathtt{F}}$. Since, $\overline{\nabla}$ is a metric connection, then $(\overline{\nabla}_{\mathtt{A}}\overline{d})(\overline{\phi}\eta, \mathtt{C}) = 0$, for any $\eta \in \Gamma(Rad(\mathtt{TF}))$, $\mathtt{C} \in \Gamma(\mathtt{F}(\mathtt{S}(\mathtt{TF})))$ and for all $\mathtt{A} \in \Gamma(\mathtt{TF})$. From (2) and (4), we get

$$\overline{d}(\overline{\nabla}_{\mathtt{A}}\eta, \overline{\phi}\mathtt{C}) = -\overline{d}(\overline{\phi}\eta, \overline{\nabla}_{\mathtt{A}}\mathtt{C}) \tag{47}$$

Now, from (7), (9), (13), (24) and (47), we get

$$-\overline{d}(A^*_{\eta}\mathtt{A}, \mathtt{K}_1\mathtt{C}) + \overline{d}(II^s(\mathtt{A}, \eta), \mathtt{K}_2\mathtt{C}) = -\overline{d}(\overline{\phi}\eta, \mathtt{D}^l(\mathtt{A}, \mathtt{C})). \tag{48}$$

Using (10), (12) and (15) in (48), we get

$$\overline{\mathtt{d}}(II^l(\mathtt{A}, \mathtt{K}_1\mathtt{C}), \eta) + \overline{\mathtt{d}}(II^s(\mathtt{A}, \overline{\phi}\eta), \mathtt{C}) = 0,$$

and by using (15) in this equation, we get

$$\overline{\mathtt{d}}(II^l(\mathtt{A}, \mathtt{K}_1\mathtt{C}), \eta) = 0, \tag{49}$$

which implies $II^l(\mathtt{A}, \mathtt{K}_1\mathtt{C}) = 0$. Thus, from (15), $\mathtt{J}^l = 0$.

Let $\overline{\mathtt{d}}(\mathtt{B}, \overline{\phi}\mathtt{U}) = 0$, for all $\mathtt{B} \in \Gamma(\mathtt{S}(\mathtt{TF}))$ and $\mathtt{U} \in \Gamma(\mathtt{v}_2)$, from (2), we get $\overline{\mathtt{d}}(\overline{\phi}\mathtt{B}, \mathtt{U}) = 0$, for any $\mathtt{B} \in \Gamma(\mathtt{S}(\mathtt{TF}))$ and $\mathtt{U} \in \Gamma(\mathtt{v}_2)$.
Since $\overline{\nabla}$ is a metric connection, then $(\overline{\nabla}_{\mathtt{A}}\overline{\mathtt{d}})(\overline{\phi}\mathtt{B}, \mathtt{U}) = 0$, for any $\mathtt{B} \in \Gamma(\mathtt{S}(\mathtt{TF}))$, $\mathtt{U} \in \Gamma(\mathtt{v}_2)$ and for all $\mathtt{A} \in \Gamma(\mathtt{TF})$. From (2) and (4), we obtain

$$\overline{\mathtt{d}}(\overline{\nabla}_{\mathtt{A}}\mathtt{B}, \overline{\phi}\mathtt{U}) = -\overline{\mathtt{d}}(\mathtt{B}, \overline{\nabla}_{\mathtt{A}}\overline{\phi}\mathtt{U}). \tag{50}$$

Also, from (7), (9) and (50), we get

$$\overline{\mathtt{d}}(II^s(\mathtt{A}, \mathtt{B}), \overline{\phi}\mathtt{U}) = \overline{\mathtt{d}}(\mathtt{B}, A_{\overline{\phi}\mathtt{U}}\mathtt{A}). \tag{51}$$

Now, if $\overline{\mathtt{d}}(\mathtt{B}, A_{\overline{\phi}\mathtt{U}}\mathtt{A}) = 0$, then, $II^s(\mathtt{A}, \mathtt{B}) = 0$, then, from (15), we get $\mathtt{J}^s = 0$.
Hence, $\mathtt{F}$ is totally geodesic if $\overline{\mathtt{d}}(\mathtt{B}, A_{\overline{\phi}\mathtt{U}}\mathtt{A}) = 0$, which completes the proof. □

6. Example

Now, we are giving an example of SPSSL:

Example 6.1. Let $(\mathtt{R}^{14}_1, \overline{d})$ having signature $(-, +, +, +, +, +, +, +, +, +, +, +, +, +)$ w. r. t. the canonical basis in the term of partial differential equations $\{\partial a_1, \partial a_2, \partial a_3, \partial a_4, \partial a_5, \partial a_6, \partial a_7, \partial a_8, \partial a_9, \partial a_{10}, \partial a_{11}, \partial a_{12}, \partial a_{13}, \partial a_{14}\}$ is a semi-Riemannian space and the golden structure ϕ is as follows

$$\overline{\mathtt{F}}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14})$$
$$=(\sigma a_1, \sigma a_2, \overline{\sigma} a_3, \overline{\sigma} a_4, \overline{\sigma} a_5, \overline{\sigma} a_6, \sigma a_7, \sigma a_8, \sigma a_9, \sigma a_{10}, \overline{\sigma} a_{11}, \sigma a_{12}, \overline{\sigma} a_{13}, \sigma a_{14}),$$

where $\sigma = \frac{1+\sqrt{5}}{2}$ and $\overline{\sigma} = \frac{1-\sqrt{5}}{2}$ are the roots of $a^2 - a - \mathtt{I} = 0$. Let $\mathtt{F}$ be a 7-dimensional submanifold of $(\mathtt{R}^{14}_1, \overline{d})$ given by

$a_1 = x_1$, $a_2 = x_1$, $a_3 = \sigma\, x_2$, $a_4 = \overline{\sigma} x_2$, $a_5 = \sigma x_3$, $a_6 = \overline{\sigma} x_3$,
$a_7 = \sigma x_4$, $a_8 = \overline{\sigma} x_4$, $a_9 = \sigma x_5$, $a_{10} = \overline{\sigma} x_5$, $a_{11} = \sigma x_6$ $a_{12} = \overline{\sigma} x_6$,
$a_{13} = \sigma x_7$, $a_{14} = \overline{\sigma} x_7$.

Then, $\mathtt{TF} = span\{\mathtt{A}_1, \mathtt{A}_2, \mathtt{A}_3, \mathtt{A}_4, \mathtt{A}_5, \mathtt{A}_6, \mathtt{A}_7\}$; where

$$\mathtt{A}_1 = \partial a_1 + \partial a_2, \quad \mathtt{A}_2 = \sigma\partial a_3 + \overline{\sigma}\partial a_4, \quad \mathtt{A}_3 = \sigma\partial a_5 + \overline{\sigma}\partial a_6,$$
$$\mathtt{A}_4 = \sigma\partial a_7 + \overline{\sigma}\partial a_8, \quad \mathtt{A}_5 = \sigma\partial a_9 + \overline{\sigma}\partial a_{10}, \quad \mathtt{A}_6 = \sigma\partial a_{11} + \overline{\sigma}\partial a_{12},$$
$$\mathtt{A}_7 = \sigma\partial a_{13} + \overline{\sigma}\partial a_{14}.$$

This gives $Rad\mathtt{TF} = span\{\mathtt{A}_1\}$ and $\mathtt{S(TF)} = span\{\mathtt{A}_2, \mathtt{A}_3, \mathtt{A}_4, \mathtt{A}_5, \mathtt{A}_6, \mathtt{A}_7\}$.
Now, $ltr(\mathtt{TF}) = span\{\mathtt{C}\}$ in the term of partial differential equations where $\mathtt{C}$ is given by

$$\mathtt{C} = -\frac{1}{2}(\partial a_1 - \partial a_2).$$

Now, we see that $\phi\mathtt{A}_1 = \sigma\mathtt{A}_1$ which implies that $\phi Rad\mathtt{TF} = \sigma Rad\mathtt{TF}$.
Now, we have vectors $\mathtt{G}_1$ and $\mathtt{G}_2$ in $\mathtt{S}(\mathtt{TF}^{\perp})$ such that $\phi\mathtt{A}_2 = \mathtt{G}_1$ and $\phi\mathtt{A}_3 = \mathtt{G}_2$ where $\mathtt{G}_1$ and $\mathtt{G}_2$ in the term of partial differential equations are given by

$$\mathtt{G}_1 = -\partial a_3 + \overline{\sigma}^2\partial a_4,$$

and

$$\mathtt{G}_2 = -\partial a_5 + \overline{\sigma}^2\partial a_6.$$

We also observe that $\phi\mathtt{A}_4 = \sigma\mathtt{A}_4$ and $\phi\mathtt{A}_5 = \sigma\mathtt{A}_5$.
Now, we have $\overline{d}(\phi\mathtt{A}_6, \mathtt{A}_6) = -1$, $|\phi\mathtt{A}_6| = \sqrt{2}$,$|\mathtt{A}_6| = \sqrt{3}$ and $\overline{d}(\phi\mathtt{A}_7, \mathtt{A}_7) = -1$, $|\phi\mathtt{A}_7| = \sqrt{2}$, $|\mathtt{A}_7| = \sqrt{3}$ and obtain the slant angle $\theta = arccos\{\frac{-1}{\sqrt{6}}\}$.
Thus, we have $\mathtt{D} = span\{\mathtt{A}_2, \mathtt{A}_3\}$ is an anti-invariant distribution,
$\mathtt{D}' = span\{\mathtt{A}_4, \mathtt{A}_5\}$ is an invariant distribution, and $\mathtt{D}'' = span\{\mathtt{A}_6, \mathtt{A}_7\}$ is a slant distribution. Thus, $\mathtt{F}$ is a SPSSL of dimension 1 of $\mathtt{R}_1^{14}$.

7. Conclusion

In this paper, we have introduced and studied the notion of SPSSLS of GSRM. We established fundamental structural results such as the decompositions of the tangent and normal bundles adapted to the golden structure and the integrability conditions for the associated distributions. Furthermore, we provided characterizations of totally geodesic case under natural metric and curvature assumptions. An example is constructed to justify the existence of the proposed submanifolds. These results generalize and unify earlier findings in both lightlike geometry and slant geometry, thereby opening further avenues of research on the interplay between golden structures and lightlike geometry.

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