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Uncertainty principle and applications of the biquaternion windowed linear canonical transform

Yi-Ting Zhu[a], Wen-Biao Gao[a,*]

[a]*School of Mathematics, Yangzhou University, Yangzhou 225002 , China*

Abstract. In this paper, the windowed linear canonical transform associated with biquaternion is given, which is called the biquaternion windowed linear canonical transforms (BiQWLCT). Then, the inversion transform and Plancherel formula of the BiQWLCT are derived. Next, Donoho-Stark's uncertainty principle for the BiQWLCT is obtained. Finally, using Donoho-Stark's uncertainty principle of the BiQWLCT, the signal recovery and potential applications are studied.

1. Introduction

The quaternionic linear canonical transform (QLCT)has been widely applied in different fields [1–5] . Many papers [6–9] have studied some important properties of the QLCT, such as linearity, Plancherel formula, Poisson summation formula, boundedness, inversion transform and convolution theorem. Uncertainty principle has aroused the research interest of several scholars. According to the properties of the QLCT, different types of uncertainty principles associated with the QLCT have been studied, for example, Donoho-Stark, Heisenberg-Weyl, Hardy, Beurling, Lieb and logarithmic uncertainty principles [8–13].

On the other hand, biquaternion is a more general form of quaternion [14, 15]. The research history of biquaternion has been a long time. Tracing back to 1853, Hamilton [14] discovered the biquaternion algebra. Subsequently, some properties of the biquaternion algebra are studied[16–19]. According to the biquaternion algebra, Said et al. [15] studied the Fourier transform and proposed the biquaternion Fourier transform (BiQFT). The biquaternion Z transform were defined by Bi et al. [20]. Moreover, we[21] studied the LCT associated with the biquaternion algebra and obtained the biquaternion linear canonical transform (BiQLCT). Some properties of the BiQLCT were obtained, such as inversion transform, Parseval theorem, convolution and Heisenberg uncertainty principle [21].

The BiQLCT is a useful tool for signal processing and analysis. The BiQLCT takes functions from the time domain to the frequency domain, but it cannot be used to show the relationship between time and

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* Corresponding author: Wen-Biao Gao
Email addresses: zyttmath@163.com (Yi-Ting Zhu), wenbiaogao@outlook.com (Wen-Biao Gao)
ORCID iDs: https://orcid.org/0009-0009-2727-1058 (Yi-Ting Zhu), https://orcid.org/0000-0002-9995-4626 (Wen-Biao Gao)

frequency or to perform localization for time and frequency simultaneously as a result of its global kernel. To overcome such a problem, the study of the linear canonical transform in the biquaternion algebra setting motivates the interest of extending the quaternion windowed linear canonical transform to biquaternion algebra[22]. Therefore, in the current paper, our aim is to utilize the advantages of biquaternion algebra to propose a new signal transform method, which we call the biquaternion windowed linear canonical transform (BiQWLCT). In some sense, the technique can be viewed as an extension of a previous work related to the quaternion windowed linear canonical transform. The complex numbers are extended to higher dimensions, which are more flexible. The BiQWLCT can perform localization for time and frequency simultaneously, it can be used to observe signals and perform the analysis of time-frequency localization by a fixed time-frequency window. Then the inversion transform and Parseval theorem of the BiQWLCT are obtained. according to the properties of the BiQWLCT, Donoho-Stark's uncertainty principles for the BiQWLCT is generated. Finally, the signal recovery and potential applications are discussed through Donoho-Stark's uncertainty principle for the BiQWLC.

The paper is organized as follows: Section 2 presents a brief introduction to some general definitions and properties of biquaternions algebra. The definition and properties of the BiQWLCT are obtained in Section 3. Section 4 studied Donoho-Stark's uncertainty principle of the BiQWLCT . The applications of the BiQWLCT is discussed in Section 5. Section 6 presented some conclusions.

2. Preliminary

In this section, we reviewed the concept of biquaternion algebra. Biquaternion algebra is complexified quaternion algebra [14, 23].

The biquaternion algebra is defined by $\mathbb{H}_\mathbb{C}$ [16, 19]. A biquaternion algebra $q \in \mathbb{H}_\mathbb{C}$ can be represented by

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \tag{1}$$

where $q_0, q_1, q_2, q_3 \in \mathbb{C}$ are complex numbers. When $q_0 = 0$, then q is the pure biquaternion [21].

In addition, q can be written the following form [17]:

$$q = S(q) + V(q), \tag{2}$$

where $S(q) = q_0$ is the scalar part of q and $V(q) = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is its vector part.

Let $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = S(q) + V(q)$ and $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} = S(p) + V(p)$, then

$$q + p = (S(q) + S(p)) + (V(q) + V(p)), \tag{3}$$

$$\lambda q = \lambda S(q) + \lambda V(q), \quad \lambda \in \mathbb{R}, \tag{4}$$

$$qp = S(q)S(p) - \langle V(q), V(p)\rangle + S(q)V(p) + S(p)V(q) + V(q) \wedge V(p), \tag{5}$$

$$\langle V(q), V(p)\rangle = q_1p_1 + q_2p_2 + q_3p_3,$$

$$V(q) \wedge V(p) = (q_2p_3 - q_3p_2)\mathbf{i} - (q_1p_3 - q_3p_1)\mathbf{j} + (q_1p_2 - q_2p_1)\mathbf{k}.$$

When q is orthogonal to p, then $\langle q, p\rangle = 0$.

There is a relationship between complex imaginary unit $\mathbf{I}$ and quaternion imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as follows[15]

$$\mathbf{iI} = \mathbf{Ii}, \mathbf{jI} = \mathbf{Ij}, \mathbf{kI} = \mathbf{Ik}. \tag{6}$$

There are two basic ways of conjugating a biquaternion. Quaternion conjugation is related to the imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and complex conjugation to $\mathbf{I}$ [17].

The quaternion conjugate of a biquaternion $q \in \mathbb{H}_{\mathbb{C}}$ is given by [17, 21]

$$\widetilde{q} = S(q) - V(q). \tag{7}$$

The complex conjugate of a biquaternion $q \in \mathbb{H}_{\mathbb{C}}$ is represented by [15, 21]

$$\check{q} = \check{q}_0 + \check{q}_1\mathbf{i} + \check{q}_2\mathbf{j} + \check{q}_3\mathbf{k}, \tag{8}$$

where $\check{q}_0, \check{q}_1, \check{q}_2, \check{q}_3$ are called as the complex conjugates of the complex coefficients of q.

The biquaternion conjugate of q is defined as [17, 21]

$$\overline{q} = \widetilde{\check{q}} = \check{\widetilde{q}} = \check{q}_0 - \check{q}_1\mathbf{i} - \check{q}_2\mathbf{j} - \check{q}_3\mathbf{k}, \tag{9}$$

and we have $\overline{pq} = \overline{q}\,\overline{p}$.

The norm of a biquaternion q is given as follows [18]

$$\|q\|^2 = |q_0|^2 + |q_1|^2 + |q_2|^2 + |q_3|^2.$$

When $\|q\| = 1$, then q is called unit biquaternion. The modulus of a biquaternion q is $|q| = \sqrt{\|q\|}$. Biquaternions are not a normed algebra. So the norm is not multiplicative, $|pq| \neq |p||q|$ [18].

A biquaternion $\boldsymbol{\mu} \in \mathbb{H}_{\mathbb{C}}$ is a biquaternion root of -1 iff $\boldsymbol{\mu}^2 = -1$ [17].

Any three mutually orthogonal roots of can be used as a basis to decompose a biquaternion. For any biquaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, q can be represented by [15]

$$\begin{aligned} q &= (q_0' + q_1'\boldsymbol{\mu}) + (q_2' + q_3'\boldsymbol{\mu})\boldsymbol{\nu} \\ &= q_0' + q_1'\boldsymbol{\mu} + q_2'\boldsymbol{\nu} + q_3'\boldsymbol{\mu}\boldsymbol{\nu}, \end{aligned} \tag{10}$$

where $\boldsymbol{\nu}$ is a biquaternion root of -1 orthogonal to $\boldsymbol{\mu}$ and q_0', q_1', q_2', q_3' are complex numbers. Let $Simp(q) = (q_0' + q_1'\boldsymbol{\mu})$ be simplex part, $Perp(q) = (q_2' + q_3'\boldsymbol{\mu})\boldsymbol{\nu}$ is perplex part, then $q = Simp(q) + Perp(q)$[21].

A biquaternion-valued function $f(\mathbf{x})$ is defined as follows [21]

$$\begin{aligned} f(\mathbf{x}) &= f_0(\mathbf{x}) + f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k} \\ &= Simp(f) + Perp(f) \\ &= S(f) + V(f), \end{aligned} \tag{11}$$

where f_0, f_1, f_2, f_3 are complex-valued functions. The significance of the biquaternion signal itself have been given in[15].

An inner product of biquaternion functions f, g can be defined on $L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$ by[21]

$$(f, g)_{L^2(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}})} = \int_{\mathbb{R}^2} f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x},$$

where $\mathbf{x} = (x_1, x_2)$, if $f = g$, then

$$\|f\|_{L^2(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}})} = \Big(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}\Big)^{\frac{1}{2}}.$$

3. Biquaternion windowed linear canonical transform

Because of the non-commutativity of the biquaternion algebra, the multiplication in the next integral is done from left to right. The definition of the BiQLCT is represented as follows[21]:

Definition 1. *Let $A_i = (a_i, b_i; c_i, d_i) \in \mathbb{R}^{2\times 2}$ be a matrix parameter satisfying* $\det(A_i) = 1$, *for $i = 1, 2$. The BiQLCT of a function $f \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$ is defined by*

$$\mathcal{L}^{B}_{A_1,A_2}\{f\}(\mathbf{w}) = \int_{\mathbb{R}^2} f(\mathbf{x}) K^{\boldsymbol{\mu}}_{A_1}(x_1, w_1) K^{\boldsymbol{\mu}}_{A_2}(x_2, w_2) d\mathbf{x}. \tag{12}$$

Based on the definition of the BiQLCT, we denote the following definition.

Definition 2. *Let a matrix parameter $A_i = (a_i, b_i; c_i, d_i) \in \mathbb{R}^{2\times 2}$ satisfy $a_i d_i - b_i c_i = 1$, for $i = 1, 2$, and $\phi \in L^2(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}})\backslash\{0\}$ be a window function. Then the BiQWLCT of the function $f \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$ with respect to ϕ is defined by*

$$\mathcal{S}^{B,\phi}_{A_1,A_2}\{f\}(\mathbf{w}, \mathbf{u}) = \int_{\mathbb{R}^2} f(\mathbf{x})\overline{\phi(\mathbf{x}-\mathbf{u})} K^{\boldsymbol{\mu}}_{A_1}(x_1, w_1) K^{\boldsymbol{\mu}}_{A_2}(x_2, w_2) d\mathbf{x}, \tag{13}$$

where $\boldsymbol{\mu}$ is the pure unit biquaternions that are orthogonal to each other, $\mathbf{x} = (x_1, x_2), \mathbf{u} = (u_1, u_2), \mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$ and the kernel functions

$$K^{\boldsymbol{\mu}}_{A_i}(x_i, w_i) := \begin{cases} \frac{1}{\sqrt{2\pi|b_i|}} e^{\boldsymbol{\mu}\left(\frac{a_i}{2b_i}x^2 - \frac{x_i}{b_i}w_i + \frac{d_i}{2b_i}w_i^2 - \frac{\pi}{4}\right)}, & b_i \neq 0 \\ \sqrt{d_i} e^{\boldsymbol{\mu}(\frac{c_i d_i}{2}w_i^2 + w_i n_i)} \delta(x_i - d_i w_i), & b_i = 0 \end{cases}, \tag{14}$$

$\delta(\cdot)$ representing the Dirac function.

Lemma 1. *Let a function $f \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$. Then the inversion formula of the BiQLCT[21],*

$$f(\mathbf{x}) = \int_{\mathbb{R}^2} \mathcal{L}^{B}_{A_1,A_2}\{f\}(\mathbf{w}) K^{\boldsymbol{\mu}}_{A_1^{-1}}(w_1, x_1) K^{\boldsymbol{\mu}}_{A_2^{-1}}(w_2, x_2) d\mathbf{w}, \tag{15}$$

where $A_i^{-1} = (d_i, -b_i; -c_i, a_i) \in \mathbb{R}^{2\times 2}$, for $i = 1, 2$.

Next, we obtain the inversion formula associated with the BiQWLCT

Theorem 3.1. *Let a function $f \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$ and $\phi \in L^2(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}})\backslash\{0\}$ be a window function. Then the inversion formula of the BiQWLCT,*

$$f(\mathbf{x}) = \frac{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{S}^{B,\phi}_{A_1,A_2}\{f\}(\mathbf{w}, \mathbf{u}) K^{\boldsymbol{\mu}}_{A_1^{-1}}(w_1, x_1) K^{\boldsymbol{\mu}}_{A_2^{-1}}(w_2, x_2) \phi(\mathbf{x}-\mathbf{u}) d\mathbf{w} d\mathbf{u}}{\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}}. \tag{16}$$

Moreover, the Plancherel theorem of the BiQWLCT is derived:

Theorem 3.2. *Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}})\backslash\{0\}$ be a window function and two functions $f, g \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$, then we obtain the following result:*

$$(\mathcal{S}^{B,\phi}_{A_1,A_2}\{f\}(\mathbf{w}, \mathbf{u}), \mathcal{S}^{B,\phi}_{A_1,A_2}\{g\}(\mathbf{w}, \mathbf{u}))_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})} = \|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})} (f, g)_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}. \tag{17}$$

Proof. According to the definition of the BiQWLCT, we have

$$
\begin{aligned}
&(\mathcal{S}^{B,\phi}_{A_1,A_2}\{f\}(\mathbf{w},\mathbf{u}),\mathcal{S}^{B,\phi}_{A_1,A_2}\{g\}(\mathbf{w},\mathbf{u}))_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{S}^{B,\phi}_{A_1,A_2}\{f\}(\mathbf{w},\mathbf{u})\overline{\mathcal{S}^{B,\phi}_{A_1,A_2}\{g\}(\mathbf{w},\mathbf{u})}d\mathbf{w}d\mathbf{u}\\
&=\int_{\mathbb{R}^4}\mathcal{S}^{B,\phi}_{A_1,A_2}\{f\}(\mathbf{w},\mathbf{u})\left(\overline{\int_{\mathbb{R}^2}g(\mathbf{x})\overline{\phi(\mathbf{x}-\mathbf{u})}K^{\boldsymbol{\mu}}_{A_1}(x_1,w_1)K^{\boldsymbol{\mu}}_{A_2}(x_2,w_2)d\mathbf{x}}\right)d\mathbf{w}d\mathbf{u}\\
&=\int_{\mathbb{R}^6}\mathcal{S}^{B,\phi}_{A_1,A_2}\{f\}(\mathbf{w},\mathbf{u})K^{\boldsymbol{\mu}}_{A_2^{-1}}(w_2,x_2)K^{\boldsymbol{\mu}}_{A_1^{-1}}(w_1,x_1)d\mathbf{w}\phi(\mathbf{x}-\mathbf{u})\overline{g(\mathbf{x})}d\mathbf{x}d\mathbf{u}\\
&=\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}\int_{\mathbb{R}^2}f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x}\\
&=\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}(f,g)_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}.
\end{aligned}
\tag{18}
$$

□

Corollary 1. *Let* $f,g\in L^2\left(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}}\right)$*, if* $f(\mathbf{x})=g(\mathbf{x})$*, then*

$$
\int_{\mathbb{R}^4}|\mathcal{S}^{B,\phi}_{A_1,A_2}\{f\}(\mathbf{w})|^2d\mathbf{w}d\mathbf{u}=\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}. \tag{19}
$$

4. Donoho-Stark uncertainty principle for the BiQWLCT

Kou et al. [25] have studied the Donoho-Stark uncertainty principle in the quaternion Fourier transform setting, Achak et al.[26] have obtained the Donoho-Stark uncertainty principle for the QLCT. In this section, we derive the Donoho-Stark uncertainty principle associated with the BiQWLCT.

Assume that $\uplus$ and $\coprod$ are two measurable sets on $\mathbb{R}^2$, a function $f\in L^2\left(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}}\right)$. We give the definitions of the operators $M_{\uplus}$ and $N_{\coprod}$, respectively :

$$
M_{\uplus}f=\chi_{\uplus}f, \tag{20}
$$

and

$$
N_{\coprod}f=\mathcal{S}^{B,\phi,-1}_{A_1,A_2}\left\{\chi_{\coprod}\mathcal{S}^{B,\phi}_{A_1,A_2}(f)\right\}, \tag{21}
$$

where $\chi_{\coprod}$ is the characteristic function of $\coprod$.

The norm of $M_{\uplus}$ is defined by:

$$
\|M_{\uplus}\|=\sup_{f\in L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}\frac{\|M_{\uplus}(f)\|_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}}{\|f\|_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}}, \tag{22}
$$

and the norm of $N_{\coprod}$ is presented as following:

$$
\|N_{\coprod}\|=\sup_{f\in L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}\frac{\|N_{\coprod}(f)\|_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}}{\|f\|_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}}. \tag{23}
$$

Let the function $f\in L^2\left(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}}\right)$ and $f\neq 0$, if the following equation holds:

$$
\|M_{\uplus^c}(f)\|_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}\leq \curlyvee_{\uplus}\|f\|_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}, \tag{24}
$$

then f is $\curlyvee_{\uplus}$-concentrated on $\uplus$. And if

$$\|N_{\coprod^c}(f)\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \le \curlyvee_{\coprod}\|f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}, \tag{25}$$

then f is $\curlyvee_{\coprod}$-concentrated on $\coprod$.

Let $\dagger$ be an integral operator and be denoted by

$$\dagger f(\mathbf{l}) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\mathbf{x})\,\ddagger(\mathbf{l},\mathbf{x},\mathbf{u})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{u}, \tag{26}$$

where $\mathbf{l} = (l_1, l_2) \in \mathbb{R}^2$,

$$\ddagger(\mathbf{l},\mathbf{x},\mathbf{u}) = \chi_{\uplus}(\mathbf{l})\mathcal{S}^{B,\phi,-1}_{A_1,A_2}\Big(\chi_{\coprod}(\mathbf{w})\overline{\phi(\mathbf{x}-\mathbf{u})}K^{\boldsymbol{\mu}}_{A_1}(x_1,w_1)K^{\boldsymbol{\mu}}_{A_2}(x_2,w_2)\Big)(\mathbf{x}), \tag{27}$$

is the kernel function and the Hilbert-Schmidt norm of $\dagger$ is defined by

$$\|\dagger\|_H = \Big(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\ddagger(\mathbf{l},\mathbf{x},\mathbf{u})|^2\mathrm{d}\mathbf{l}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{u}\Big)^{\frac{1}{2}}. \tag{28}$$

Lemma 2. *If $\uplus$ and H are sets of finite measure on $\mathbb{R}^2$, then*

$$\|N_{\coprod}M_{\uplus}\|_H = \|M_{\uplus}N_{\coprod}\|_H. \tag{29}$$

Proof. According to the formulas (20) and (21), we have

$$\begin{aligned}
&N_{\coprod}M_{\uplus}f(\mathbf{x})\\
&= \mathcal{S}^{B,\phi,-1}_{A_1,A_2}\Big(\chi_{\coprod}(\mathbf{w})\mathcal{S}^{B,\phi}_{A_1,A_2}(M_{\uplus}f)\Big)(\mathbf{x})\\
&= \int_{\mathbb{R}^2}\int_{\coprod}\mathcal{S}^{B,\phi}_{A_1,A_2}(M_{\uplus}f)(\mathbf{w},\mathbf{u})K^{-\boldsymbol{\mu}}_{A_2}(x_2,w_2)K^{-\boldsymbol{\mu}}_{A_1}(x_1,w_1)\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{u}\\
&= \int_{\mathbb{R}^2}\int_{\coprod}\Big(\int_{\uplus} f(\mathbf{y})\overline{\phi(\mathbf{y}-\mathbf{u})}K^{\boldsymbol{\mu}}_{A_1}(y_1,w_1)K^{\boldsymbol{\mu}}_{A_2}(y_2,w_2)\mathrm{d}\mathbf{y}\Big)\\
&\quad\times K^{-\boldsymbol{\mu}}_{A_2}(x_2,w_2)K^{-\boldsymbol{\mu}}_{A_1}(x_1,w_1)d\mathrm{w}d\mathrm{u}\\
&= \int_{\mathbb{R}^2}\int_{\uplus} f(\mathbf{y})\Big(\int_{\coprod}\overline{\phi(\mathbf{y}-\mathbf{u})}K^{\boldsymbol{\mu}}_{A_1}(y_1,w_1)K^{\boldsymbol{\mu}}_{A_2}(y_2,w_2)\\
&\quad\times K^{-\boldsymbol{\mu}}_{A_2}(x_2,w_2)K^{-\boldsymbol{\mu}}_{A_1}(x_1,w_1)d\mathrm{w}\Big)dyd\mathrm{u}\\
&= \int_{\mathbb{R}^2}\int_{\uplus} f(\mathbf{y})\,\ddagger(\mathbf{y},\mathbf{x},\mathbf{u})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{u}.
\end{aligned} \tag{30}$$

Hence, using the formulas (26) and (28), we obtain

$$\|N_{\coprod}M_{\uplus}\|_H = \Big(\int_{\uplus}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\ddagger(\mathbf{y},\mathbf{x},\mathbf{u})|^2\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{u}\Big)^{\frac{1}{2}}. \tag{31}$$

On the other hand, according to the definition of the BiQWLCT, we obtain

$$\begin{aligned}
&M_{\uplus}N_{\coprod}f(\mathbf{x})\\
&= \chi_{\uplus}(\mathbf{x})N_{\coprod}f(\mathbf{x})\\
&= \int_{\mathbb{R}^2}\chi_{\uplus}(\mathbf{x})\int_{\coprod}\mathcal{S}^{B,\phi}_{A_1,A_2}(f)K^{-\boldsymbol{\mu}}_{A_2}(x_2,w_2)K^{-\boldsymbol{\mu}}_{A_1}(x_1,w_1)\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{u}\\
&= \int_{\mathbb{R}^2}\chi_{\uplus}(\mathbf{x})\int_{\coprod}\Big(\int_{\mathbb{R}^2} f(\mathbf{y})\overline{\phi(\mathbf{y}-\mathbf{u})}K^{\boldsymbol{\mu}}_{A_1}(y_1,w_1)K^{\boldsymbol{\mu}}_{A_2}(y_2,w_2)\mathrm{d}\mathbf{y}\Big)\\
&\quad\times K^{-\boldsymbol{\mu}}_{A_2}(x_2,w_2)K^{-\boldsymbol{\mu}}_{A_1}(x_1,w_1)d\mathrm{w}d\mathrm{u}\\
&= \int_{\mathbb{R}^2}\chi_{\uplus}(\mathbf{x})\int_{\mathbb{R}^2} f(\mathbf{y})\,\ddagger(\mathbf{y},\mathbf{x},\mathbf{u})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{u}.
\end{aligned} \tag{32}$$

Hence,

$$\|M_{\biguplus}N_{\coprod}\|_H = \Big(\int_{\biguplus}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\ddagger(\mathbf{y},\mathbf{x},\mathbf{u})|^2 \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{u}\Big)^{\frac{1}{2}}. \tag{33}$$

We obtain the results. □

Lemma 3. *Let the norm of* $\dagger$ *be presented by:*

$$\|\dagger\| = \sup_{f\in L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})} \frac{\|\dagger f\|_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}}{\|f\|_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}}, \tag{34}$$

then

$$\|\dagger\| \le \|\dagger\|_H \le \frac{\sqrt{|\biguplus\|\coprod|}}{2\pi\sqrt{|b_1b_2|}}. \tag{35}$$

Proof. According to Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|\dagger f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})} &= \int_{\mathbb{R}^2} |\dagger f(\mathbf{l})|^2 \mathrm{d}\mathbf{l} \\
&= \int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)\ddagger(l,x,u)\mathrm{d}x\mathrm{d}u \Big|^2 d\mathrm{l} \\
&\le \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\ddagger(\mathbf{l},\mathbf{x},\mathbf{u})|^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{l} \\
&= \|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}\|\dagger\|^2_S.
\end{aligned} \tag{36}$$

So,

$$\|\dagger\| \le \|\dagger\|_H. \tag{37}$$

In addition, using the formula (27), we have

$$\mathcal{S}^{B,\phi}_{A_1,A_2}\{\ddagger(\mathbf{l},\mathbf{x})\}(\mathbf{w},\mathbf{u}) = \chi_{\biguplus}(\mathbf{l})\Big(\chi_{\coprod}(\mathbf{w})\overline{\phi(\mathbf{x}-\mathbf{u})}K^{\boldsymbol{\mu}}_{A_1}(x_1,w_1)K^{\boldsymbol{\mu}}_{A_2}(x_2,w_2)\Big)(\mathbf{x}). \tag{38}$$

According to the formula (19), we have

$$\begin{aligned}
&\|\dagger\|^2_H \\
&= \|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})} \int_{\biguplus}\int_{\coprod} |K^{\boldsymbol{\mu}}_{A_1}(x_1,w_1)K^{\boldsymbol{\mu}}_{A_2}(x_2,w_2)|^2 d\mathrm{v}d\mathrm{w} \\
&\le \frac{\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}}{4\pi^2|b_1b_2|} \int_{\biguplus}\int_{\coprod} d\mathrm{l}d\mathrm{w} \\
&= \frac{\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}}{4\pi^2|b_1b_2|} |\biguplus\|\coprod|.
\end{aligned} \tag{39}$$

Then

$$\|\dagger\|_H \le \frac{\|\phi\|_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})}}{2\pi\sqrt{|b_1b_2|}}\sqrt{|\biguplus\|\coprod|}. \tag{40}$$

Therefore, we have this conclusion. □

Let f be unit energy signal, then $\|f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} = 1$. Based on this assumption, we obtain the following theorem.

Theorem 4.1. *Let $\uplus, H$ be two finite measure sets on $\mathbb{R}^2$ and $\curlyvee_{\uplus} + \curlyvee_{\coprod} < 1$. If f is $\curlyvee_{\uplus}-$ concentrated on $\uplus$ and $\mathcal{S}^{B,\phi}_{A_1,A_2}\{f\}$ is $\curlyvee_{\coprod}-$ concentrated on $\coprod$, then*

$$(1 - \curlyvee_{\uplus} - \curlyvee_{\coprod})^2 \leq \frac{\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}}{4\pi^2|b_1 b_2|} |\uplus| |\coprod|. \tag{41}$$

Proof. By the formulas (24) and (25), we have

$$\begin{aligned}\|f - N_{\coprod} M_{\uplus} f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} &\leq \|f - N_{\coprod} f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} + \|N_{\coprod} f - N_{\coprod} M_{\uplus} f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \\ &\leq \curlyvee_{\uplus} + \|N_{\coprod}\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \|f - N_{\coprod} M_{\uplus} f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \\ &\leq \curlyvee_{\uplus} + \curlyvee_{\coprod}.\end{aligned} \tag{42}$$

On the other hand, from the above formula, we obtain

$$\begin{aligned}\|N_{\coprod} M_{\uplus} f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} &\geq \|f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} - \|f - N_{\coprod} M_{\uplus} f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \\ &\geq 1 - \curlyvee_{\uplus} - \curlyvee_{\coprod}.\end{aligned} \tag{43}$$

From the Lemma 2 and Lemma 3, we have the following formula

$$\|N_{\coprod} M_{\uplus} f\| \geq 1 - \curlyvee_{\uplus} - \curlyvee_{\coprod}. \tag{44}$$

Then

$$1 - \curlyvee_{\uplus} - \curlyvee_{\coprod} \leq \frac{\|\phi\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}}{2\pi\sqrt{|b_1 b_2|}} \sqrt{|\uplus| |\coprod|}. \tag{45}$$

Therefore, we have this conclusion. □

Theorem 4.2. *(Donoho-Stark uncertainty principle) Let $\uplus, H$ be two finite measure sets on $\mathbb{R}^2$, $2\pi\sqrt{|b_1 b_2|} > \|\phi\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\sqrt{|\uplus||\coprod|}$ and $\curlyvee^2_{\uplus} + \curlyvee^2_{\coprod} < 1$. If $f \in L^2\left(\mathbb{R}^2, \mathbb{H}_\mathbb{C}\right)$ is $\curlyvee_{\uplus}-$ concentrated on $\uplus$ and $\mathcal{L}^{B}_{A_1,A_2}\{f\}$ is $\curlyvee_{\coprod}-$ concentrated on $\coprod$, then*

$$1 - \frac{\|\phi\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\sqrt{|\uplus||\coprod|}}{2\pi\sqrt{|b_1 b_2|}} \leq (\curlyvee^2_{\uplus} + \curlyvee^2_{\coprod})^{\frac{1}{2}}. \tag{46}$$

Proof. Let $\uplus^c$ and $\coprod^c$ be the complementary sets of $\uplus$ and $\coprod$, respectively. Then

$$\begin{aligned}I &= M_{\uplus} + M_{\uplus^c} \\ &= M_{\uplus}(N_{\coprod} + N_{\coprod^c}) + M_{\uplus^c} \\ &= M_{\uplus} N_{\coprod} + M_{\uplus} N_{\coprod^c} + M_{\uplus^c},\end{aligned} \tag{47}$$

where I is the identity operator.

Hence,

$$f = M_{\uplus} N_{\coprod} f + M_{\uplus} N_{\coprod^c} f + M_{\uplus^c} f, \tag{48}$$

that is to say,

$$f - M_{\uplus} N_{\coprod} f = M_{\uplus} N_{\coprod^c} f + M_{\uplus^c} f. \tag{49}$$

Because of the orthogonality of $M_{\uplus}$ and $M_{\uplus^c}$, we have

$$\begin{aligned} \|f - M_{\uplus}N_{\coprod}f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} &= \|M_{\uplus}N_{\coprod^c}f + M_{\uplus^c}f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \\ &= \|M_{\uplus}N_{\coprod^c}f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} + \|M_{\uplus^c}f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}, \end{aligned} \tag{50}$$

then

$$\|f - M_{\uplus}N_{\coprod}f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \leq (\|M_{\uplus}N_{\coprod^c}f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} + \|M_{\uplus^c}f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})})^{\frac{1}{2}}. \tag{51}$$

In addition, according to the Lemma (3),

$$\begin{aligned} \|f - M_{\uplus}N_{\coprod}f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} &\geq \|f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} - \|M_{\uplus}N_{\coprod}f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \\ &\geq \|f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}(1 - \|M_{\uplus}N_{\coprod}\|), \end{aligned} \tag{52}$$

combining the two equations above yields, we have

$$\|f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}(1 - \|M_{\uplus}N_{\coprod}f\|) \leq (\|M_{\uplus}N_{\coprod^c}f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} + \|M_{\uplus^c}f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})})^{\frac{1}{2}}, \tag{53}$$

that is to say,

$$\begin{aligned} \|f\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\Big(1 - \frac{\|\phi\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\sqrt{|\uplus\|\coprod|}}{2\pi\sqrt{|b_1b_2|}}\Big) &\leq (\|M_{\uplus}N_{\coprod^c}f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} + \|M_{\uplus^c}f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})})^{\frac{1}{2}} \\ &\leq \Big((\curlyvee^2_{\uplus} + \curlyvee^2_{\coprod})\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\Big)^{\frac{1}{2}}, \end{aligned} \tag{54}$$

hence,

$$1 - \frac{\|\phi\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\sqrt{|\uplus\|\coprod|}}{2\pi\sqrt{|b_1b_2|}} \leq (\curlyvee^2_{\uplus} + \curlyvee^2_{\coprod})^{\frac{1}{2}}. \tag{55}$$

□

5. Applications

The uncertainty principle for the BiQWLCT has been studied in this paper. In this section, to show the correctness and usefulness of the theorem, an application is given to verify the result, and potential applications are also presented to show the importance of the theorem.

Several papers [27–29] have discussed the problem of signal recovery in signal processing. In this section, we study signal recovery by Donoho-Stark's uncertainty principle of the BiQWLCT.

Assume that the original signal $f(\mathbf{x}) \in L^2\left(\mathbb{R}^2, \mathbb{H}_\mathbb{C}\right)$ is transmitted to a receiver. The received signal $R(\mathbf{x})$ is given by:

$$R(\mathbf{x}) = \begin{cases} f(\mathbf{x}) + t(\mathbf{x}), & \mathbf{x} \notin P \\ 0, & \mathbf{x} \in P \end{cases}, \tag{56}$$

where $t(\mathbf{x}) \in L^2\left(\mathbb{R}^2, \mathbb{H}_\mathbb{C}\right)$ is a noise signal.

Theorem 5.1. *Assume that an operator E and a constant F have the following relationship:*

$$\|f - ER\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \leq F\|t\|_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}, \tag{57}$$

and two finite measure sets $\uplus \in \mathbb{R}^2$ *and* $H \in \mathbb{R}^2$ *satisfy:*

$$|\biguplus||\coprod| < \frac{4\pi^2|b_1b_2|}{||\phi||^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}}. \tag{58}$$

Then the original signal $f(\mathbf{x})$ *can be stably restored from* $R(\mathbf{x})$*, and the constant F has the following condition:*

$$F \leq \Big(1 - \frac{||\phi||_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\sqrt{|\uplus||\coprod|}}{2\pi\sqrt{|b_1b_2|}}\Big)^{-1}. \tag{59}$$

Proof. Let the linear operator E be defined by $E = (I - M_{\uplus}N_{\coprod})^{-1}$. Based on the condition (58)and the Lemma 3, we have

$$||M_{\uplus}N_{\coprod}|| \leq \frac{||\phi||_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\sqrt{|\uplus||\coprod|}}{2\pi\sqrt{|b_1b_2|}} < 1, \tag{60}$$

so $I - M_{\uplus}N_{\coprod}$ is invertible.

In addition, for the original signal $f(\mathbf{x})$, we have $(I - M_{\uplus})f = (I - M_{\uplus}N_{\coprod})f$, and

$$\begin{aligned} f(\mathbf{x}) - ER(\mathbf{x}) &= f(\mathbf{x}) - E((I - M_{\uplus})f(\mathbf{x}) + t(\mathbf{x})) \\ &= f(\mathbf{x}) - E(I - M_{\uplus}N_{\coprod})f(\mathbf{x}) - Et(\mathbf{x}) \\ &= f(\mathbf{x}) - (I - M_{\uplus}N_{\coprod})^{-1}(I - M_{\uplus}N_{\coprod})f(\mathbf{x}) - Et(\mathbf{x}) \\ &= -Et(\mathbf{x}), \end{aligned} \tag{61}$$

hence,

$$\begin{aligned} ||f(\mathbf{x}) - ER(\mathbf{x})||_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} &= ||Et(\mathbf{x})||_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \\ &\leq ||(I - M_{\uplus}N_{\coprod})^{-1}||||t(\mathbf{x})||_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}. \end{aligned} \tag{62}$$

According to

$$||(I - M_{\uplus}N_{\coprod})^{-1}|| \leq (I - ||M_{\uplus}N_{\coprod}||)^{-1}, \tag{63}$$

then

$$\begin{aligned} ||f(\mathbf{x}) - ER(\mathbf{x})||_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} &\leq (I - ||M_{\uplus}N_{\coprod}||)^{-1}||t(\mathbf{x})||_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \\ &\leq \Big(1 - \frac{||\phi||_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\sqrt{|\uplus||\coprod|}}{2\pi\sqrt{|b_1b_2|}}\Big)^{-1}||t(\mathbf{x})||_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}, \end{aligned} \tag{64}$$

that is to say,

$$F \leq \Big(1 - \frac{||\phi||_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\sqrt{|\uplus||\coprod|}}{2\pi\sqrt{|b_1b_2|}}\Big)^{-1}. \tag{65}$$

□

Theorem 5.2. *Let* $R(\mathbf{x})$ *be given by the formula (56) and* $|\uplus||\coprod| < \frac{4\pi^2|b_1b_2|}{||\phi||^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}}$*. For any* $\varepsilon > 0$*, if* $||t(\mathbf{x})||^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \leq \frac{(4\pi^2|b_1b_2| - ||\phi||^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}|\coprod||\uplus|)\varepsilon}{8\pi^2|b_1b_2|}$ *and* $f_1(\mathbf{x})$ *is recovered with* $||R(\mathbf{x}) - M_{\uplus^c}f_1(\mathbf{x})||^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \leq \frac{(4\pi^2|b_1b_2| - ||\phi||^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}|\uplus||\coprod|)\varepsilon}{8\pi^2|b_1b_2|}$*, then*

$$||f(\mathbf{x}) - f_1(\mathbf{x})||^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})} \leq \varepsilon. \tag{66}$$

Proof. Based on the following fact:

$$\begin{aligned}&\|f(\mathbf{x})-f_1(\mathbf{x})\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\\&=\|M_{\uplus^c}(f(\mathbf{x})-f_1(\mathbf{x}))\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}+\|(I-M_{\uplus^c})(f(\mathbf{x})-f_1(\mathbf{x}))\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}.\end{aligned}\tag{67}$$

We can obtain the following two inequalities:

$$\begin{aligned}&\|M_{\uplus^c}(f(\mathbf{x})-f_1(\mathbf{x}))\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\\&=\|M_{\uplus^c}(R(\mathbf{x})-t(\mathbf{x}))-M_{\uplus^c}f_1(\mathbf{x})\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\\&\leq\|M_{\uplus^c}R(\mathbf{x})-M_{\uplus^c}f_1(\mathbf{x})\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}+\|M_{\uplus^c}t(\mathbf{x})\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\\&=\|R(\mathbf{x})-M_{\uplus^c}f_1(\mathbf{x})\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}+\|t(\mathbf{x})\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\\&\leq\frac{(4\pi^2|b_1b_2|-\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}|\uplus\|\coprod|)\varepsilon}{8\pi^2|b_1b_2|}+\frac{(4\pi^2|b_1b_2|-\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}|\uplus\|\coprod|)\varepsilon}{8\pi^2|b_1b_2|}\\&=\Big(1-\frac{\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}|\uplus\|\coprod|}{4\pi^2|b_1b_2|}\Big)\varepsilon,\end{aligned}\tag{68}$$

and

$$\begin{aligned}\|(I-M_{\uplus^c})(f(\mathbf{x})-f_1(\mathbf{x}))\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}&=\|(I-M_{\uplus^c})N_{\coprod}(f(\mathbf{x})-f_1(\mathbf{x}))\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\\&=\|M_{\uplus}N_{\coprod}(f(\mathbf{x})-f_1(\mathbf{x}))\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\\&\leq\|M_{\uplus}N_{\coprod}\|^2\|f(\mathbf{x})-f_1(\mathbf{x}))\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\\&\leq\frac{\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}|\uplus\|\coprod|}{4\pi^2|b_1b_2|}\|f(\mathbf{x})-f_1(\mathbf{x}))\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}.\end{aligned}\tag{69}$$

Hence, we have

$$\begin{aligned}&\|f(\mathbf{x})-f_1(\mathbf{x})\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\\&\leq\Big(1-\frac{\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}|\uplus\|\coprod|}{4\pi^2|b_1b_2|}\Big)\varepsilon+\frac{\|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}|\uplus\|\coprod|}{4\pi^2|b_1b_2|}\|f(\mathbf{x})-f_1(\mathbf{x}))\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})},\end{aligned}\tag{70}$$

then,

$$\|f(\mathbf{x})-f_1(\mathbf{x})\|^2_{L^2(\mathbb{R}^2,\mathbb{H}_\mathbb{C})}\leq\varepsilon.\tag{71}$$

□

Theorem 5.3. *Let* $E=(1-M_{\uplus}N_{\coprod})^{-1}=\Sigma_{\lambda=0}^{\infty}(M_{\uplus}N_{\coprod})^{\lambda}$ *and* $f^{(\kappa)}(\mathbf{x})=\Sigma_{\lambda=0}^{\kappa}(M_{\uplus}N_{\coprod})^{\lambda}h(\mathbf{x})$*, then* $f(\mathbf{x})$ *can be reconstructed in the set* $\uplus$ *by the following iterated algorithm:*

$$f^{(0)}(\mathbf{x})=R(\mathbf{x}),$$

$$f^{(1)}(\mathbf{x})=R(\mathbf{x})+M_{\uplus}N_{\coprod}f^{(0)}(\mathbf{x}),$$

$$f^{(2)}(\mathbf{x})=R(\mathbf{x})+M_{\uplus}N_{\coprod}f^{(1)}(\mathbf{x}),$$

$$\dots$$

$$f^{(\kappa)}(\mathbf{x}) = R(\mathbf{x}) + M_{\uplus} N_{\coprod} f^{(\kappa-1)}(\mathbf{x}),$$

$$f^{(\kappa+1)}(\mathbf{x}) = R(\mathbf{x}) + M_{\uplus} N_{\coprod} f^{(\kappa)}(\mathbf{x}),$$

and when $\kappa \to \infty$, *we have* $f^{(\kappa)}(\mathbf{x}) \to ER(\mathbf{x})$.

In addition, the BiQWLCT is an affine transformation in the time-frequency plane. The uncertainty principles of the BiQWLCT play a very important role in the analysis of spectra in affine modulation systems[29]. For example, when we try to implement the BiQWLCT in affine modulation schemes where the spectral efficiency is crucial with ability to compress pulses within the time-frequency plane, it is necessary to discuss the effective bandwidth in the BiQWLCT domain. It is theoretically meaningful and practically feasible to discuss the spread in the BiQWLCT domain through combining the new uncertainty principle inequalities with the multichannel interpolation techniques.

Other potential applications can be found in the field of image processing. For the image processing, Donoho-Stark's uncertainty principle of the BiQWLCT can help optimize the clarity and detail representation of images, especially when dealing with high-frequency information. In audio signal processing, this uncertainty principle helps to understand and improve the quality of audio signals, especially during compression and transmission[30].

Moreover, The results of this paper can be extended to other time-frequency analysis tools, and the discrete algorithms of the transform can be further studied[31–35].

6. Conclusions

In this paper, we presented the definition of the BiQWLCT. By derived the inversion transform and Plancherel formula associated with the BiQWLCT, Donoho-Stark's uncertainty principle for the BiQWLCT can be obtained. The result has well generalized the classical results of the windowed Fourier transform. The uncertainty principle states that the temporal and frequency resolutions of a signal cannot be very small simultaneously. This research is of great significance because it enables us to better understand the mathematical principles behind signal processing. Through signal recovery of the biquaternion signal, it is proved that the Donoho-Stark's uncertainty principle of the BiQWLCT is effectiveness and accuracy.

Conflict of interests:

The author declares that they have no conflict of interest.

References

[1] K.I.Kou,J.Morais,Y.Zhang,Generalized prolate spheroidal wave functions for offset linear canonical transform in Clifford analysis, Math. Methods Appl.Sci., **36**(9)(2013)1028-1041.

[2] X.X.Hu, D.Cheng, K.I.Kou, Convolution theorems associated with quaternion linear canonical transform and applications, Signal Process., **202** (2023)1-17.

[3] W.B.Gao, B.Z.Li, Quaternion windowed linear canonical transform of two- dimensional signals. Adv. Appl. Clifford Algebr., **30**(16) (2020)1-18.

[4] W.B.Gao, B.Z.Li, Uncertainty principle for the two-sided quaternion windowed linear canonical transform, Circuits Syst. Signal Process., **41**(3) (2022)1324-1348.

[5] M.Kundu, A.Prasad, Uncertainty principles associated with quaternion linear canonical transform and their estimates. Math. Methods Appl. Sci., **45**(8) (2022)4772-4790.

[6] M.Y.Bhat, A.H.Dar, The two-sided short-time quaternionic offset linear canonical transform and associated convolution and correlation. Math. Methods Appl.Sci., **46**(8) (2023)8478-8495.

[7] E.Hitzer, Special affine Fourier transform for space-time algebra signals in detail. Adv. Appl. Clifford Algebr., **32**(5) (2022)1-18.

[8] Y.E.Youssef, E.Hitzer, Generalized uncertainty principles associated with the quaternionic offset linear canonical transform. Complex. Var. Elliptic., **67**(9) (2022)2102-2121.

[9] M.E.Kassimi, Y.E.Haoui and S.Fahlaoui, The Wigner-Ville distribution associated with the quaternion offset linear canonical transform. Analysis Math., **45**(4) (2019)787-802.

[10] Q.Feng, B.Z.Li, J.M.Rassias, Weighted Heisenberg-Pauli-Weyl uncertainty principles for the linear canonical transform. Signal Process., **165** (2019)7205-221.
[11] W.B.Gao, New uncertainty principles in the linear canonical transform domains based on hypercomplex functionsm. Axioms, **14**(6) (2025)1-14.
[12] D.Urynbassarova, Y.E.Haoui, F.Zhang, Uncertainty principles for Wigner-Ville distribution associated with the quaternion offset linear canonical transform. Circuits Syst. Signal Process., **42** (1) (2022)385-404.
[13] A.H.Dar, M.Y.Bhat, Donoho-stark's and hardy's uncertainty principles for the short-time quaternion offset linear canonical transform. Filomat, **37**(14) (2023) 4467-4480.
[14] W.R.Hamilton, On the geometrical interpretation of some results obtained by calculation with biquaternions, Proceedings of the Royal Irish Academy. **5** (1853) 388-390.
[15] S.Said, N.L.Bihan, S.J.Sangwine, Fast complexified quaternion fourier transform. IEEE Trans. Signal Process., **56**(4) (2008) 1522-1531.
[16] J.P.Ward, Quaternions and Cayley Numbers: Algebra and Applications of Mathematics and Its Applications. Kluwer, Dordrecht (1997).
[17] S.J.Sangwine, Biquaternion (complexified quaternion) roots of -1. Adv. Appl. Clifford Algebr., **16**(1) (2006) 63-68.
[18] S.J.Sangwine, T.A.Ell, N.L.Bihan, Fundamental representations and algebraic properties of biquaternions or complexified quaternions. Adv. Appl. Clifford Algebr., **21** (3) (2011)607-636.
[19] M.Bekar, Y.Yayli, Involutions of complexified quaternions and split quaternions. Adv. Appl. Clifford Algebr., **23** (2) (2013) 283-299.
[20] W.Bi, Z.F.Cai and K.I.Kou, Biquaternion z transform (2021) arXiv:2108.02975.
[21] W.B.Gao, B.Z.Li, Theories and applications associated with biquaternion linear canonical transform. Math. Methods Appl. Sci., **46** (12) (2023) 13124-13141.
[22] H.H.Yang, Q.Feng, X.X.Wang, D. Urynbassarova, A.A Teali, Reduced biquaternion windowed linear canonical transform: properties and applications. Mathematics, **12** (5) (2024) 743.
[23] W.B.Gao, Biquaternion Fourier transform and its applications. Filomat, **38** (28) (2024) 9851-9865.
[24] W.B.Gao, Uncertainty principles for the biquaternion offset linear canonical transform. J. Pseudo-Differ. Oper. Appl., **15** (22) (2024)1-19.
[25] K.I.Kou, Y.Yang, C.Zou, Uncertainty principle for measurable sets and signal recovery in quaternion domains. Math. Methods Appl. Sci., **40** (11) (2017) 3892-3900.
[26] A.Achak, A.Abouelaz and R.Daher and N.Safouane, Uncertainty principles for the quaternion linear canonical transform. Adv. Appl. Clifford Algebr., **29** (99) (2019) 1-19.
[27] P.Dang, W.X.Mai, W.L.Pan, Uncertainty principle in random quaternion domains. Digit. Signal Process., **136** (2023) 1-17.
[28] Z.C.Zhang, Uncertainty principle for linear canonical transform using matrix decomposition of absolute spread matrix. Digit. Signal Process., **89** (2019) 145-154.
[29] W.B.Gao, B.Z.Li, Uncertainty principles for the short-time linear canonical transform of complex signals. Digit. Signal Process., **111** (2021) 1-10.
[30] B.Ricaud, B.Torrsani, A survey of uncertainty principles and some signal processing applications, Adv. Comput. Math., **40** (2014) 629-650.
[31] S.C. Pei, S.G.Huang, Two-dimensional nonseparable discrete linear canonical transform based on CM-CC-CM-CC decomposition. J. Opt. Soc. Amer. AJ. Opt. Soc. Amer. A, **33** (2) (2016) 214-227.
[32] D.Y.Wei, Y.M.Li, Convolution and multichannel sampling for the offset linear canonical transform and their applications. IEEE Trans. Signal Process. **67** (23) (2019) 6009–6024.
[33] D.Y.Wei, Y.Shen, Fast numerical computation of two-dimensional non-separable linear canonical transform based on matrix decomposition. IEEE Trans. Signal Process. **69** (2021) 5259-5272.
[34] D.Y.Wei, Y.J.Zhang, Y.M.Li, Linear canonical Stockwell transform: theory and applications. IEEE Trans. Signal Process. **70** (2022) 1333-1347.
[35] N.Jiang, Q.Feng, X.Yang, J.R.He, B.Z.Li, The octonion linear canonical transform: Properties and applications. Chaos, Solitons Fractals. **192** (2025) 116039.