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Turán type inequalities for the four-parameter Mittag-Leffler type supertrigonometric functions

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Abstract. In this paper, we aim to investigate Turán-type inequalities for the four-parameter Mittag-Leffler type supertrigonometric functions by employing a newly formulated version of the generalized Cauchy–Bunyakovsky–Schwarz (CBS) inequality. Furthermore, several significant results concerning Turán-type inequalities for the generalized Wright's hypergeometric type supertrigonometric functions are derived and established.

1. Introduction and Preliminaries

The Bagley's supersine and supercosine function [3], constructed utilizing the framework of the one-parameter Mittag-Leffler function, is specified by the following definitions:

$$\mathrm{Supersin}_{\alpha}(t^{\alpha}) = \frac{E_{\alpha}(it^{\alpha}) - E_{\alpha}(-it^{\alpha})}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}, \tag{1}$$

$$\mathrm{Supercos}_{\alpha}(t^{\alpha}) = \frac{E_{\alpha}(it^{\alpha}) + E_{\alpha}(-it^{\alpha})}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n\alpha}}{\Gamma(2n\alpha+1)}, \tag{2}$$

where $i = \sqrt{-1}, \alpha, t \in C, \Re(\alpha) > 0$ and $E_{\alpha}(t)$ is the Mittag-Leffler function [12].

The supersine and supercosine function is defined by the following definition in the framework of two-parameter Mittag-Leffler function (Wiman function):

$$\mathrm{Supersin}_{\alpha,\beta}(t^{\alpha}) = \frac{t^{\beta-1}}{2i}[E_{\alpha,\beta}(it^{\alpha}) - E_{\alpha,\beta}(-it^{\alpha})] = \sum_{n=0}^{\infty} \frac{(-1)^n t^{(2n+1)\alpha+\beta-1}}{\Gamma((2n+1)\alpha+\beta)}, \tag{3}$$

$$\mathrm{Supercos}_{\alpha,\beta}(t^{\alpha}) = \frac{t^{\beta-1}}{2}[E_{\alpha,\beta}(it^{\alpha}) + E_{\alpha,\beta}(-it^{\alpha})] = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n\alpha+\beta-1}}{\Gamma(2n\alpha+\beta)}, \tag{4}$$

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where $\alpha, \beta, t \in C, \Re(\alpha) > 0, \Re(\beta) > 0$ and $E_{\alpha,\beta}(t)$ is the generalized Mittag-Leffler function [21]. This result was established by Yang in 2019 [19].

The supersine and supercosine function can be defined in the framework of three-parameter Mittag-Leffler function (Prabhakar function) as,

$$\text{Supersin}^{\gamma}_{\alpha,\beta}(t^{\alpha}) = \frac{t^{\beta-1}}{2i}[E^{\gamma}_{\alpha,\beta}(it^{\alpha}) - E^{\gamma}_{\alpha,\beta}(-it^{\alpha})] = \sum_{n=0}^{\infty} \frac{(-1)^n(\gamma)_{(2n+1)}}{\Gamma((2n+1)\alpha+\beta)} \frac{t^{(2n+1)\alpha+\beta-1}}{(2n+1)!}, \tag{5}$$

$$\text{Supercos}^{\gamma}_{\alpha,\beta}(t^{\alpha}) = \frac{t^{\beta-1}}{2}[E^{\gamma}_{\alpha,\beta}(it^{\alpha}) + E^{\gamma}_{\alpha,\beta}(-it^{\alpha})] = \sum_{n=0}^{\infty} \frac{(-1)^n(\gamma)_{2n}}{\Gamma(2n\alpha+\beta)} \frac{t^{2n\alpha+\beta-1}}{(2n)!}, \tag{6}$$

where $\alpha, \beta, \gamma, t \in C, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$ and $E^{\gamma}_{\alpha,\beta}(t)$ is the Prabhakar function [13]. This result was established by Yang in 2019 [19].

The supersine and cosine function via Wright's generalized hypergeometric function [20] can be expressed as,

$$\begin{aligned} {}_p\sin_q(z) &= \frac{1}{2i}\left[{}_p\Psi_q(iz) - {}_p\Psi_q(-iz)\right] \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(r_1 + R_1(2k+1))\ldots\Gamma(r_p + R_p(2k+1)}{\Gamma(s_1 + S_1(2k+1))\ldots\Gamma(s_q + S_q(2k+1))} \frac{z^{(2k+1)}}{(2k+1)!}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} {}_p\cos_q(z) &= \frac{1}{2}\left[{}_p\Psi_q(iz) + {}_p\Psi_q(-iz)\right] \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(r_1 + R_1(2k))\ldots\Gamma(r_p + R_p(2k))}{\Gamma(s_1 + S_1(2k))\ldots\Gamma(s_q + S_q(2k))} \frac{z^{(2k)}}{(2k)!}, \end{aligned} \tag{8}$$

where $r_i, s_j \in \mathbf{C}, i = 1, 2, \ldots, p; j = 1, 2, \ldots, q$, and the coefficients $R_i, S_j \in \mathbf{R}^+$ satisfying the condition

$$\sum_{j=1}^{q} S_j - \sum_{i=1}^{p} R_i > -1. \tag{9}$$

In particular, when $R_i = S_j = 1\,(i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q)$, equation (7) and (8) reduces to

$${}_p\sin_q\left[\begin{matrix}(r_1,1),\ldots,(r_p,1)\\(s_1,1),\ldots,(s_q,1)\end{matrix}\middle|z\right] = \frac{\prod_{i=1}^{p}\Gamma(r_i)}{\prod_{j=1}^{q}\Gamma(s_j)}\,{}_p\text{Supersin}_q\left[\begin{matrix}r_1,\ldots,r_p\\s_1,\ldots,s_q\end{matrix}\middle|z\right], \tag{10}$$

and

$${}_p\cos_q\left[\begin{matrix}(r_1,1),\ldots,(r_p,1)\\(s_1,1),\ldots,(s_q,1)\end{matrix}\middle|z\right] = \frac{\prod_{i=1}^{p}\Gamma(r_i)}{\prod_{j=1}^{q}\Gamma(s_j)}\,{}_p\text{Supercos}_q\left[\begin{matrix}r_1,\ldots,r_p\\s_1,\ldots,s_q\end{matrix}\middle|z\right], \tag{11}$$

where ${}_p\text{Supersin}_q(\cdot)$ and ${}_p\text{Supercos}_q(\cdot)$ denotes the generalized hypergeometric supersine and supercosine functions [20] respectively.

2. Four-parameter Mittag-Leffler type supertrigonometric function

In this section, we introduce the supertrigonometric function (supersine and supercosine) constructed through the four parameter Mittag-Leffler function, which are defined as follows:

$$\text{Supersin}_{\alpha,\beta}^{\gamma,q}(t^{\alpha}) = \frac{t^{\beta-1}}{2i}\left[E_{\alpha,\beta}^{\gamma,q}(it^{\alpha}) - E_{\alpha,\beta}^{\gamma,q}(-it^{\alpha})\right] = \sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{(2n+1)q}}{\Gamma((2n+1)\alpha+\beta)}\frac{t^{(2n+1)\alpha+\beta-1}}{(2n+1)!}, \tag{12}$$

$$\text{Supercos}_{\alpha,\beta}^{\gamma,q}(t^{\alpha}) = \frac{t^{\beta-1}}{2}\left[E_{\alpha,\beta}^{\gamma,q}(it^{\alpha}) + E_{\alpha,\beta}^{\gamma,q}(-it^{\alpha})\right] = \sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{2nq}}{\Gamma(2n\alpha+\beta)}\frac{t^{2n\alpha+\beta-1}}{(2n)!}, \tag{13}$$

where $\alpha, \beta, \gamma, t \in C$; $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup \mathbf{N}$. The $E_{\alpha,\beta}^{\gamma,q}(t)$ is the well-known generalized Mittag-Leffler function due to Shukla and Prajapati [16].

2.1. *Convergence analysis of the function* $\text{Supersin}_{\alpha,\beta}^{\gamma,q}(t^{\alpha})$ *&* $\text{Supercos}_{\alpha,\beta}^{\gamma,q}(t^{\alpha})$

We have,

$$\text{Supersin}_{\alpha,\beta}^{\gamma,q}(t^{\alpha}) = \sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{(2n+1)q}}{\Gamma((2n+1)\alpha+\beta)}\frac{t^{(2n+1)\alpha+\beta-1}}{(2n+1)!}.$$

Consider

$$u_n = \frac{(-1)^n(\gamma)_{(2n+1)q}}{\Gamma((2n+1)\alpha+\beta)}\frac{t^{(2n+1)\alpha+\beta-1}}{(2n+1)!},$$
$$u_{n+1} = \frac{(-1)^{n+1}(\gamma)_{(2n+3)q}}{\Gamma((2n+3)\alpha+\beta)}\frac{t^{(2n+3)\alpha+\beta-1}}{(2n+3)!}.$$

Using ratio test, we calculate

$$\begin{aligned}\left|\frac{u_{n+1}}{u_n}\right| &= \left|\frac{(\gamma)_{(2n+3)q}}{(\gamma)_{(2n+1)q}}\frac{\Gamma((2n+1)\alpha+\beta)}{\Gamma((2n+3)\alpha+\beta)}\frac{1}{(2n+2)(2n+3)}\frac{t^{(2n+3)\alpha}}{t^{(2n+1)\alpha}}\right| \\ &= \left|\frac{(\gamma+(2n+1)q)_{2q}}{(2n+2)(2n+3)}\frac{\Gamma((2n+1)\alpha+\beta)}{\Gamma((2n+3)\alpha+\beta)}t^{2\alpha}\right| \\ &= \left|\frac{\Gamma((\gamma+(2n+1)q)+2q)}{\Gamma(\gamma+(2n+1)q)}\frac{\Gamma((2n+1)\alpha+\beta)}{\Gamma((2n+3)\alpha+\beta)}\right|\frac{|t^{2\alpha}|}{|(2n+2)(2n+3)|} \\ &\to 0 < 1, \text{as } n \to \infty.\end{aligned}$$

Therefore, $\text{Supersin}_{\alpha,\beta}^{\gamma,q}(t^{\alpha})$ is convergent for all t.

Similarly, we have

$$\text{Supercos}_{\alpha,\beta}^{\gamma,q}(t^{\alpha}) = \sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{2nq}}{\Gamma(2n\alpha+\beta)}\frac{t^{2n\alpha+\beta-1}}{(2n)!}.$$

Consider

$$u_n = \frac{(-1)^n(\gamma)_{2nq}}{\Gamma(2n\alpha+\beta)}\frac{t^{2n\alpha+\beta-1}}{(2n)!},$$
$$u_{n+1} = \frac{(-1)^{n+1}(\gamma)_{(2n+2)q}}{\Gamma((2n+2)\alpha+\beta)}\frac{t^{(2n+2)\alpha+\beta-1}}{(2n+2)!}.$$

Using ratio test, we have

$$\begin{aligned}\left|\frac{u_{n+1}}{u_n}\right| &= \left|\frac{(\gamma)_{(2n+2)q}}{(\gamma)_{2nq}}\frac{\Gamma(2n\alpha+\beta)}{\Gamma((2n+2)\alpha+\beta)}\frac{1}{(2n+2)(2n+1)}\frac{t^{(2n+2)\alpha}}{t^{2n\alpha}}\right| \\ &= \left|\frac{(\gamma)_{2nq}(\gamma+2nq)_{2q}}{(\gamma)_{2nq}}\frac{\Gamma(2n\alpha+\beta)}{\Gamma((2n+2)\alpha+\beta)}\frac{t^{2\alpha}}{(2n+2)(2n+1)}\right| \\ &= \left|\frac{\Gamma(\gamma+2nq+2q)}{\Gamma(\gamma+2nq)}\frac{\Gamma(2n\alpha+\beta)}{\Gamma((2n+2)\alpha+\beta)}\right|\frac{|t^{2\alpha}|}{|(2n+2)(2n+1)|} \\ &\to 0<1, \text{as } n\to\infty.\end{aligned}$$

Therefore, $\text{Supercos}_{\alpha,\beta}^{\gamma,q}(t^\alpha)$ is convergent for all t.

2.2. *Elementary properties*

Theorem 2.1. *Let $\alpha,\beta,\gamma,t\in\mathbf{C}$; $\Re(\alpha)>0, \Re(\beta)>0, \Re(\gamma)>0; q\in\mathbf{N}$. Then,*

$$\text{Supercos}_{\alpha,\beta}^{\gamma,q}(t^\alpha) + i\,\text{Supersin}_{\alpha,\beta}^{\gamma,q}(t^\alpha) = t^{\beta-1}E_{\alpha,\beta}^{\gamma,q}(it^\alpha), \tag{14}$$

and

$$\text{Supercos}_{\alpha,\beta}^{\gamma,q}(t^\alpha) - i\,\text{Supersin}_{\alpha,\beta}^{\gamma,q}(t^\alpha) = t^{\beta-1}E_{\alpha,\beta}^{\gamma,q}(-it^\alpha). \tag{15}$$

Proof. From (12) and (13), we have

$$\begin{aligned}&\text{Supercos}_{\alpha,\beta}^{\gamma,q}(t^\alpha) + i\,\text{Supersin}_{\alpha,\beta}^{\gamma,q}(t^\alpha) \\ &\quad= \frac{t^{\beta-1}}{2}\left[E_{\alpha,\beta}^{\gamma,q}(it^\alpha)+E_{\alpha,\beta}^{\gamma,q}(-it^\alpha)\right] + i\frac{t^{\beta-1}}{2i}\left[E_{\alpha,\beta}^{\gamma,q}(it^\alpha)-E_{\alpha,\beta}^{\gamma,q}(-it^\alpha)\right] \\ &\quad= t^{\beta-1}E_{\alpha,\beta}^{\gamma,q}(it^\alpha),\end{aligned}$$

which is the proof of (14). Similarly, one can demonstrate the proof of (15). □

Theorem 2.2. *Let $\alpha,\beta,\gamma,t\in\mathbf{C}$; $\Re(\alpha)>0, \Re(\beta)>0, \Re(\gamma)>0; q\in\mathbf{N}$. Then,*

$$\text{Supersin}_{\alpha,\beta}^{\gamma,q}(-t^\alpha) = -\text{Supersin}_{\alpha,\beta}^{\gamma,q}(t^\alpha), \tag{16}$$

and

$$\text{Supercos}_{\alpha,\beta}^{\gamma,q}(-t^\alpha) = \text{Supercos}_{\alpha,\beta}^{\gamma,q}(t^\alpha). \tag{17}$$

Proof. Using (12), we have

$$\begin{aligned}\text{Supersin}_{\alpha,\beta}^{\gamma,q}(-t^\alpha) &= \sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{(2n+1)q}}{\Gamma((2n+1)\alpha+\beta)}\frac{(-1)^{2n+1}t^{(2n+1)\alpha+\beta-1}}{(2n+1)!} \\ &= -\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{(2n+1)q}}{\Gamma((2n+1)\alpha+\beta)}\frac{t^{(2n+1)\alpha+\beta-1}}{(2n+1)!} \\ &= -\text{Supersin}_{\alpha,\beta}^{\gamma,q}(t^\alpha),\end{aligned}$$

which is the proof of (16). Similarly, we can demonstrate the proof of (17). □

2.3. Integral expressions of the function $\mathrm{Supersin}^{\gamma,q}_{\alpha,\beta}(t^{\alpha})$ *&* $\mathrm{Supercos}^{\gamma,q}_{\alpha,\beta}(t^{\alpha})$

We represent here four-parameter Mittag-Leffler type supersine and supercosine function via generalized Wright's hypergeometric function in the framework of supersine and supercosine function.

Theorem 2.3. *The four-parameter Mittag-Leffler type supersine function holds the following representation:*

$$\int_0^1 (1-t)^{\alpha-1}\mathrm{Supersin}^{\gamma,\alpha}_{\alpha,\beta}(u^{\alpha}t^{\alpha})dt = \frac{1}{\Gamma(\gamma)}\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\sin_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^{\alpha}t^{\alpha}\right]dt. \tag{18}$$

Moreover,

$$\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\sin_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^{\alpha}t^{\alpha}\right]dt = \frac{\Gamma(\gamma)\Gamma(\alpha)}{u^{\beta+\alpha-1}}\,\mathrm{Supersin}^{\gamma,\alpha}_{\alpha,\beta+\alpha}(u^{\alpha}), \tag{19}$$

where $\alpha,\beta,\gamma,u,t\in\mathbf{C}; \Re(\alpha)>0, \Re(\beta)>0, \Re(\gamma)>0.$

Proof. Using the definition of four-parameter Mittag-Leffler type supersine function (12), we have

$$\begin{aligned}
&\int_0^1 (1-t)^{\alpha-1}\mathrm{Supersin}^{\gamma,\alpha}_{\alpha,\beta}(u^{\alpha}t^{\alpha})dt\\
&\qquad= \int_0^1 (1-t)^{\alpha-1}\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+\beta)}\frac{u^{(2n+1)\alpha}t^{(2n+1)\alpha+\beta-1}}{(2n+1)!}dt\\
&\qquad= \frac{1}{\Gamma(\gamma)}\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\gamma+(2n+1)\alpha)}{\Gamma((2n+1)\alpha+\beta)}\frac{u^{(2n+1)\alpha}t^{(2n+1)\alpha}}{(2n+1)!}dt\\
&\qquad= \frac{1}{\Gamma(\gamma)}\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\sin_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^{\alpha}t^{\alpha}\right]dt,
\end{aligned}$$

which is the proof of (18).

Also, we have,

$$\begin{aligned}
&\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\sin_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^{\alpha}t^{\alpha}\right]dt\\
&\qquad= \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\gamma+(2n+1)\alpha)}{\Gamma((2n+1)\alpha+\beta)}\frac{u^{(2n+1)\alpha}}{(2n+1)!}\int_0^1 t^{(2n+1)\alpha+\beta-1}(1-t)^{\alpha-1}dt\\
&\qquad= \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\gamma+(2n+1)\alpha)}{\Gamma((2n+1)\alpha+\beta)}\frac{u^{(2n+1)\alpha}}{(2n+1)!}\,\mathrm{B}((2n+1)\alpha+\beta,\alpha)\\
&\qquad= \Gamma(\gamma)\Gamma(\alpha)\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+\beta+\alpha)}\frac{u^{(2n+1)\alpha}}{(2n+1)!}\\
&\qquad= \frac{\Gamma(\gamma)\Gamma(\alpha)}{u^{\beta+\alpha-1}}\,\mathrm{Supersin}^{\gamma,\alpha}_{\alpha,\beta+\alpha}(u^{\alpha}),
\end{aligned}$$

which completes the proof of (19). □

Corollary 2.4. *The generalized Wright's hypergeometric supersine function holds the following representation:*

$$\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\sin_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^{\alpha}t^{\alpha}\right]dt = \Gamma(\alpha)\,{}_1\sin_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta+\alpha,\alpha)\end{matrix}\middle| u^{\alpha}\right]. \tag{20}$$

Theorem 2.5. *The four-parameter Mittag-Leffler type supercosine function holds the following representation:*

$$\int_0^1 (1-t)^{\alpha-1}\mathrm{Supercos}_{\alpha,\beta}^{\gamma,\alpha}(u^\alpha t^\alpha)dt = \frac{1}{\Gamma(\gamma)}\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^\alpha t^\alpha\right]dt. \tag{21}$$

Moreover,

$$\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^\alpha t^\alpha\right]dt = \frac{\Gamma(\gamma)\Gamma(\alpha)}{u^{\beta+\alpha-1}}\mathrm{Supercos}_{\alpha,\beta+\alpha}^{\gamma,\alpha}(u^\alpha), \tag{22}$$

where $\alpha,\beta,\gamma,u,t\in\mathbf{C};\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0.$

Proof. Using the definition of four-parameter Mittag-Leffler type supercosine function (13), we have

$$\begin{aligned}
&\int_0^1 (1-t)^{\alpha-1}\mathrm{Supercos}_{\alpha,\beta}^{\gamma,\alpha}(u^\alpha t^\alpha)dt\\
&\qquad= \int_0^1 (1-t)^{\alpha-1}\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{2n\alpha}}{\Gamma(2n\alpha+\beta)}\frac{u^{2n\alpha}t^{2n\alpha+\beta-1}}{(2n)!}dt\\
&\qquad= \frac{1}{\Gamma(\gamma)}\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\gamma+2n\alpha)}{\Gamma(2n\alpha+\beta)}\frac{u^{2n\alpha}t^{2n\alpha}}{(2n)!}dt\\
&\qquad= \frac{1}{\Gamma(\gamma)}\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^\alpha t^\alpha\right]dt,
\end{aligned}$$

which is the proof of (21).

Also, we have,

$$\begin{aligned}
&\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^\alpha t^\alpha\right]dt\\
&\qquad= \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\gamma+2n\alpha)}{\Gamma(2n\alpha+\beta)}\frac{u^{2n\alpha}}{(2n)!}\int_0^1 t^{2n\alpha+\beta-1}(1-t)^{\alpha-1}dt\\
&\qquad= \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\gamma+2n\alpha)}{\Gamma(2n\alpha+\beta)}\frac{u^{2n\alpha}}{(2n)!}\,\mathrm{B}(2n\alpha+\beta,\alpha)\\
&\qquad= \Gamma(\gamma)\Gamma(\alpha)\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{2n\alpha}}{\Gamma(2n\alpha+\beta+\alpha)}\frac{u^{2n\alpha}}{(2n)!}\\
&\qquad= \frac{\Gamma(\gamma)\Gamma(\alpha)}{u^{\beta+\alpha-1}}\mathrm{Supercos}_{\alpha,\beta+\alpha}^{\gamma,\alpha}(u^\alpha),
\end{aligned}$$

which completes the proof of (22). □

Corollary 2.6. *The generalized Wright's hypergeometric supercosine function holds the following relation:*

$$\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^\alpha t^\alpha\right]dt = \Gamma(\alpha)\;{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta+\alpha,\alpha)\end{matrix}\middle| u^\alpha\right]. \tag{23}$$

3. Turán type inequalities

In the field of mathematics, inequalities of the type

$$f_n(x)f_{n+2}(x) - [f_{n+1}(x)]^2 \leq 0,\ n = 0, 1, 2, \ldots \tag{24}$$

continue to hold substantial importance in contemporary mathematical research and applications. These inequalities were designated as Turán-type inequalities by Szegö [17], following the pioneering work of the Hungarian mathematician P. Turán [18], who first introduced such an inequality in 1950 during his study of the zeros of Legendre polynomials.

Turán-type inequalities have been the subject of considerable investigation within the mathematical literature. Notable examples include the contribution by Joshi and Bissu [8], who in 1991 established specific two-sided inequalities pertaining to the ratio of modified Bessel functions of the first kind. Subsequently, Segura [15] advanced this area in 2011 by deriving bounds for these ratios, explicitly linked to the structure of Turán-type inequalities. Further extending the scope, Baricz [5] developed analogous inequalities applicable to q-hypergeometric functions in 2013.

Throughout this paper, let $\mathbf{C}, \mathbf{R}, \mathbf{N}$ denotes the set of complex, real and natural numbers. The discrete version of the well-known Cauchy-Schwarz inequality [1, 11] can be expressed in the form

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2,\ a_i, b_i \in \mathbf{R}, \tag{25}$$

and its general representation in the space of continuous real-valued functions $\mathbf{C}([a, b], \mathbf{R})$, i.e. the Cauchy-Bunyakovsky-Schwarz (CBS) inequality [1, 11],

$$\left(\int_a^b [u(t)]^{\frac{1}{2}} [v(t)]^{\frac{1}{2}} dt\right)^2 \leq \left(\int_a^b u(t) dt\right)\left(\int_a^b v(t) dt\right), \tag{26}$$

plays an important role in the different branches of modern mathematics.

In 2006, Laforgia and Natalini [9] employed a particular formulation of the Cauchy-Bunyakovsky-Schwarz (CBS) inequality

$$\left(\int_a^b g(t)[f(t)]^{\frac{m+n}{2}} dt\right)^2 \leq \left(\int_a^b g(t)[f(t)]^m dt\right)\left(\int_a^b g(t)[f(t)]^n dt\right), \tag{27}$$

to derive novel Turán-type inequalities associated with various special functions [2, 4, 6, 10], including the Gamma function, Polygamma functions, and the Riemann zeta function. In this context, f and g denote non-negative real-valued functions, and $m, n \in \mathbf{R}$ such that the involved integrals in (27) exist.

In 2018, a new form of generalized Cauchy-Bunyakovsky-Schwarz (CBS) inequality [7] was presented by Bhandari and Bissu,

$$\begin{aligned}&\left(\int_a^b [u(t)]^{\mu} [v(t)]^{\mu} [w(t)]^{\mu} dt\right)^2 \\ &\qquad \leq \left(\int_a^b [u(t)]^{\mu-l} [v(t)]^{\mu-m} [w(t)]^{\mu-n} dt\right)\left(\int_a^b [u(t)]^{\mu+l} [v(t)]^{\mu+m} [w(t)]^{\mu+n} dt\right),\end{aligned} \tag{28}$$

where $\mu, l, m, n \in \mathbf{R}$ and u, v and w are real integrable functions such that the involved integrals in (28) exist.

Motivated by the significant research on Turán-type inequalities, the present paper investigates corresponding inequalities for the class of four-parameter Mittag-Leffler type supertrigonometric functions. This investigation utilizes a novel formulation of the Cauchy-Bunyakovsky-Schwarz (CBS) inequality, denoted by (28), which is established based on the results presented in Theorem 3.1 and Theorem 3.2.

Theorem 3.1. *Let $\gamma > |p_3|, \alpha > |p_2|$ and $|u| \leq 1$. Then, the Turán type inequality for the four-parameter Mittag-Leffler type supersine function holds the following relation:*

$$\begin{aligned}\left[\mathrm{Supersin}^{\gamma,\alpha}_{\alpha,\beta+\alpha}(u^{\alpha})\right]^2 \leq{}& \frac{\mathrm{B}(\gamma-p_3,\gamma+p_3)}{\mathrm{B}(\gamma,\gamma)} \times \frac{\mathrm{B}(\alpha-p_2,\alpha+p_2)}{\mathrm{B}(\alpha,\alpha)}\\ &\times \mathrm{Supersin}^{\gamma-p_3,\alpha-p_2}_{\alpha-p_2,\alpha+\beta-p_1-p_2}(u^{\alpha-p_2})\\ &\times \mathrm{Supersin}^{\gamma+p_3,\alpha+p_2}_{\alpha+p_2,\alpha+\beta+p_1+p_2}(u^{\alpha+p_2}).\end{aligned} \tag{29}$$

Proof. Let $u(t) = t^{\beta-1}, v(t) = (1-t)^{\alpha-1}$ and $w(t) = {}_1\sin_1\left[\left.\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\right| u^{\alpha}t^{\alpha}\right]$. Then, from (19), the integral form of four-parameter Mittag-Leffler type supersine function can be expressed as,

$$\begin{aligned}\int_0^1 u(t)v(t)w(t)dt &= \int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\sin_1\left[\left.\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\right| u^{\alpha}t^{\alpha}\right]dt\\ &= \frac{\Gamma(\gamma)\Gamma(\alpha)}{u^{\beta+\alpha-1}}\mathrm{Supersin}^{\gamma,\alpha}_{\alpha,\beta+\alpha}(u^{\alpha}).\end{aligned} \tag{30}$$

Using CBS inequality (28), we find that

$$\begin{aligned}&\left(\int_0^1 t^{\mu(\beta-1)}(1-t)^{\mu(\alpha-1)}\left[{}_1\sin_1\left[\left.\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\right| u^{\alpha}t^{\alpha}\right]\right]^{\mu}dt\right)^2\\ &\qquad\leq \int_0^1 t^{(\mu-l)(\beta-1)}(1-t)^{(\mu-m)(\alpha-1)}\left[{}_1\sin_1\left[\left.\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\right| u^{\alpha}t^{\alpha}\right]\right]^{(\mu-n)}dt\\ &\qquad\times \int_0^1 t^{(\mu+l)(\beta-1)}(1-t)^{(\mu+m)(\alpha-1)}\left[{}_1\sin_1\left[\left.\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\right| u^{\alpha}t^{\alpha}\right]\right]^{(\mu+n)}dt.\end{aligned} \tag{31}$$

Now applying (30) on the consequence of (31), we have

$$\begin{aligned}&\left[\mathrm{Supersin}^{\mu\gamma,\mu(\alpha-1)+1}_{\mu(\alpha-1)+1,\mu(\beta-1)+\mu(\alpha-1)+2}(u^{\mu(\alpha-1)+1})\right]^2 \leq \left[\frac{u^{\mu(\beta-1)+\mu(\alpha-1)+1}}{\Gamma(\mu\gamma)\Gamma(\mu(\alpha-1)+1)}\right]^2\\ &\quad\times \frac{\Gamma((\mu-n)\gamma)\Gamma((\mu-m)(\alpha-1)+1)}{u^{(\mu-l)(\beta-1)+(\mu-m)(\alpha-1)+1}} \times \frac{\Gamma((\mu+n)\gamma)\Gamma((\mu+m)(\alpha-1)+1)}{u^{(\mu+l)(\beta-1)+(\mu+m)(\alpha-1)+1}}\\ &\quad\times \mathrm{Supersin}^{(\mu-n)\gamma,(\mu-m)(\alpha-1)+1}_{(\mu-m)(\alpha-1)+1,(\mu-l)(\beta-1)+(\mu-m)(\alpha-1)+2}(u^{(\mu-m)(\alpha-1)+1})\\ &\quad\times \mathrm{Supersin}^{(\mu+n)\gamma,(\mu+m)(\alpha-1)+1}_{(\mu+m)(\alpha-1)+1,(\mu+l)(\beta-1)+(\mu+m)(\alpha-1)+2}(u^{(\mu+m)(\alpha-1)+1}).\end{aligned} \tag{32}$$

Substituting $\mu(\alpha-1)+1 = x_1, \mu(\beta-1)+1 = x_2, \mu\gamma = x_3; m(\alpha-1) = y_1, l(\beta-1) = y_2, n\gamma = y_3$, and utilizing the relation between beta and gamma functions $\mathrm{B}(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$, the aforementioned inequality can be reformulated accordingly as,

$$\begin{aligned}\left[\mathrm{Supersin}^{x_3,x_1}_{x_1,x_2+x_1}(u^{x_1})\right]^2 \leq{}& \frac{\mathrm{B}(x_3-y_3,x_3+y_3)}{\mathrm{B}(x_3,x_3)} \times \frac{\mathrm{B}(x_1-y_1,x_1+y_1)}{\mathrm{B}(x_1,x_1)}\\ &\times \mathrm{Supersin}^{x_3-y_3,x_1-y_1}_{x_1-y_1,x_1+x_2-(y_1+y_2)}(u^{x_1-y_1})\\ &\times \mathrm{Supersin}^{x_3+y_3,x_1+y_1}_{x_1+y_1,x_1+x_2+(y_1+y_2)}(u^{x_1+y_1}).\end{aligned} \tag{33}$$

Moreover, by setting $\mu = 1$ in equation (32), this yields the following inequality for the four-parameter Mittag-Leffler type supersine function:

$$\begin{aligned}\left[\mathrm{Supersin}_{\alpha,\beta+\alpha}^{\gamma,\alpha}(u^\alpha)\right]^2 \leq &\left[\frac{u^{\beta+\alpha-1}}{\Gamma(\gamma)\Gamma(\alpha)}\right]^2 . \frac{\Gamma((1-n)\gamma)\Gamma(\alpha-m(\alpha-1))}{u^{\alpha+\beta-1-l(\beta-1)-m(\alpha-1)}} \frac{\Gamma((1+n)\gamma)\Gamma(\alpha+m(\alpha-1))}{u^{\alpha+\beta-1+l(\beta-1)+m(\alpha-1)}}\\ &\times \mathrm{Supersin}_{\alpha-m(\alpha-1),\alpha+\beta-l(\beta-1)-m(\alpha-1)}^{(1-n)\gamma,\alpha-m(\alpha-1)}(u^{\alpha-m(\alpha-1)})\\ &\times \mathrm{Supersin}_{\alpha+m(\alpha-1),\alpha+\beta+l(\beta-1)+m(\alpha-1)}^{(1+n)\gamma,\alpha+m(\alpha-1)}(u^{\alpha+m(\alpha-1)}). \end{aligned} \tag{34}$$

Replace $l(\beta-1) = p_1, m(\alpha-1) = p_2, n\gamma = p_3, \gamma > |p_3|, \alpha > |p_2|$ and $|u| \leq 1$, and using the relation between beta and gamma function, above inequality becomes,

$$\begin{aligned}\left[\mathrm{Supersin}_{\alpha,\beta+\alpha}^{\gamma,\alpha}(u^\alpha)\right]^2 \leq &\frac{\mathrm{B}(\gamma-p_3,\gamma+p_3)}{\mathrm{B}(\gamma,\gamma)} \times \frac{\mathrm{B}(\alpha-p_2,\alpha+p_2)}{\mathrm{B}(\alpha,\alpha)}\\ &\times \mathrm{Supersin}_{\alpha-p_2,\alpha+\beta-p_1-p_2}^{\gamma-p_3,\alpha-p_2}(u^{\alpha-p_2})\\ &\times \mathrm{Supersin}_{\alpha+p_2,\alpha+\beta+p_1+p_2}^{\gamma+p_3,\alpha+p_2}(u^{\alpha+p_2}),\end{aligned} \tag{35}$$

which is the desired proof of (29). □

Theorem 3.2. *Let $\gamma > |p_3|, \alpha > |p_2|$ and $|u| \leq 1$. Then, the Turán type inequality for the four-parameter Mittag-Leffler type supersine function holds the following relation:*

$$\begin{aligned}\left[\mathrm{Supercos}_{\alpha,\beta+\alpha}^{\gamma,\alpha}(u^\alpha)\right]^2 \leq &\frac{\mathrm{B}(\gamma-p_3,\gamma+p_3)}{\mathrm{B}(\gamma,\gamma)} \times \frac{\mathrm{B}(\alpha-p_2,\alpha+p_2)}{\mathrm{B}(\alpha,\alpha)}\\ &\times \mathrm{Supercos}_{\alpha-p_2,\alpha+\beta-p_1-p_2}^{\gamma-p_3,\alpha-p_2}(u^{\alpha-p_2})\\ &\times \mathrm{Supercos}_{\alpha+p_2,\alpha+\beta+p_1+p_2}^{\gamma+p_3,\alpha+p_2}(u^{\alpha+p_2}).\end{aligned} \tag{36}$$

Proof. Let $u(t) = t^{\beta-1}, v(t) = (1-t)^{\alpha-1}$ and $w(t) = {}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^\alpha t^\alpha\right]$. Then, from (22), the integral representation of four-parameter Mittag-Leffler type supercosine function can be defined as,

$$\begin{aligned}\int_0^1 u(t)v(t)w(t)dt &= \int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^\alpha t^\alpha\right]dt\\ &= \frac{\Gamma(\gamma)\Gamma(\alpha)}{u^{\beta+\alpha-1}}\mathrm{Supercos}_{\alpha,\beta+\alpha}^{\gamma,\alpha}(u^\alpha).\end{aligned} \tag{37}$$

Using CBS inequality (28), we have

$$\begin{aligned}&\left(\int_0^1 t^{\mu(\beta-1)}(1-t)^{\mu(\alpha-1)}\left[{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^\alpha t^\alpha\right]\right]^\mu dt\right)^2\\ &\qquad \leq \int_0^1 t^{(\mu-l)(\beta-1)}(1-t)^{(\mu-m)(\alpha-1)}\left[{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^\alpha t^\alpha\right]\right]^{(\mu-n)} dt\\ &\qquad \times \int_0^1 t^{(\mu+l)(\beta-1)}(1-t)^{(\mu+m)(\alpha-1)}\left[{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^\alpha t^\alpha\right]\right]^{(\mu+n)} dt.\end{aligned} \tag{38}$$

Upon applying (37) to the outcome derived from (38), we arrive at

$$\begin{aligned}\Big[\text{Supercos}^{\mu\gamma,\mu(\alpha-1)+1}_{\mu(\alpha-1)+1,\mu(\beta-1)+\mu(\alpha-1)+2}(u^{\mu(\alpha-1)+1})\Big]^2 &\\ \leq \left[\frac{u^{\mu(\beta-1)+\mu(\alpha-1)+1}}{\Gamma(\mu\gamma)\Gamma(\mu(\alpha-1)+1)}\right]^2 &\\ \times \frac{\Gamma((\mu-n)\gamma)\Gamma((\mu-m)(\alpha-1)+1)}{u^{(\mu-l)(\beta-1)+(\mu-m)(\alpha-1)+1}} \times \frac{\Gamma((\mu+n)\gamma)\Gamma((\mu+m)(\alpha-1)+1)}{u^{(\mu+l)(\beta-1)+(\mu+m)(\alpha-1)+1}} &\\ \times \text{Supercos}^{(\mu-n)\gamma,(\mu-m)(\alpha-1)+1}_{(\mu-m)(\alpha-1)+1,(\mu-l)(\beta-1)+(\mu-m)(\alpha-1)+2}(u^{(\mu-m)(\alpha-1)+1}) &\\ \times \text{Supercos}^{(\mu+n)\gamma,(\mu+m)(\alpha-1)+1}_{(\mu+m)(\alpha-1)+1,(\mu+l)(\beta-1)+(\mu+m)(\alpha-1)+2}(u^{(\mu+m)(\alpha-1)+1}). \qquad (39)\end{aligned}$$

Letting $\mu(\alpha-1)+1 = x_1, \mu(\beta-1)+1 = x_2, \mu\gamma = x_3; m(\alpha-1) = y_1, l(\beta-1) = y_2, n\gamma = y_3$, and using the relationship between beta and gamma functions $B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$, the above aforementioned inequality can be written as,

$$\begin{aligned}\Big[\text{Supercos}^{x_3,x_1}_{x_1,x_2+x_1}(u^{x_1})\Big]^2 \leq \frac{B(x_3-y_3,x_3+y_3)}{B(x_3,x_3)} \times \frac{B(x_1-y_1,x_1+y_1)}{B(x_1,x_1)} &\\ \times \text{Supercos}^{x_3-y_3,x_1-y_1}_{x_1-y_1,x_1+x_2-(y_1+y_2)}(u^{x_1-y_1}) &\\ \times \text{Supercos}^{x_3+y_3,x_1+y_1}_{x_1+y_1,x_1+x_2+(y_1+y_2)}(u^{x_1+y_1}). \qquad (40)\end{aligned}$$

Moreover, by substituting $\mu = 1$ in (39), this yields the following inequality for the four-parameter Mittag-Leffler type supercosine function:

$$\begin{aligned}\Big[\text{Supercos}^{\gamma,\alpha}_{\alpha,\beta+\alpha}(u^{\alpha})\Big]^2 \leq \left[\frac{u^{\beta+\alpha-1}}{\Gamma(\gamma)\Gamma(\alpha)}\right]^2 . \frac{\Gamma((1-n)\gamma)\Gamma(\alpha-m(\alpha-1))}{u^{\alpha+\beta-1-l(\beta-1)-m(\alpha-1)}} \frac{\Gamma((1+n)\gamma)\Gamma(\alpha+m(\alpha-1))}{u^{\alpha+\beta-1+l(\beta-1)+m(\alpha-1)}} &\\ \times \text{Supercos}^{(1-n)\gamma,\alpha-m(\alpha-1)}_{\alpha-m(\alpha-1),\alpha+\beta-l(\beta-1)-m(\alpha-1)}(u^{\alpha-m(\alpha-1)}) &\\ \times \text{Supercos}^{(1+n)\gamma,\alpha+m(\alpha-1)}_{\alpha+m(\alpha-1),\alpha+\beta+l(\beta-1)+m(\alpha-1)}(u^{\alpha+m(\alpha-1)}). \qquad (41)\end{aligned}$$

Replace $l(\beta-1) = p_1, m(\alpha-1) = p_2, n\gamma = p_3, \gamma > |p_3|, \alpha > |p_2|$ and $|u| \leq 1$, and utilizing the relationship between beta and gamma function, above inequality becomes

$$\begin{aligned}\Big[\text{Supercos}^{\gamma,\alpha}_{\alpha,\beta+\alpha}(u^{\alpha})\Big]^2 \leq \frac{B(\gamma-p_3,\gamma+p_3)}{B(\gamma,\gamma)} \times \frac{B(\alpha-p_2,\alpha+p_2)}{B(\alpha,\alpha)} &\\ \times \text{Supercos}^{\gamma-p_3,\alpha-p_2}_{\alpha-p_2,\alpha+\beta-p_1-p_2}(u^{\alpha-p_2}) &\\ \times \text{Supercos}^{\gamma+p_3,\alpha+p_2}_{\alpha+p_2,\alpha+\beta+p_1+p_2}(u^{\alpha+p_2}), \qquad (42)\end{aligned}$$

which is the required proof of (36). □

Theorem 3.3. *Let $\alpha > |p_1|$ and $|u| \leq 1$. Then, Turán type inequality for the generalized Wright's hypergeometric supersine function can be represented by the following relation*

$$\begin{aligned}\left[{}_1\sin_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta+\alpha,\alpha)\end{matrix}\middle| u^{\alpha}\right]\right]^2 \leq \frac{B(\alpha-p_1,\alpha+p_1)}{B(\alpha,\alpha)} &\\ \times {}_1\sin_1\left[\begin{matrix}(\beta-p_1,\alpha-p_1)\\(\beta+\alpha-p_1-p_2,\alpha-p_1)\end{matrix}\middle| u^{\alpha-p_1}\right] &\\ \times {}_1\sin_1\left[\begin{matrix}(\beta+p_1,\alpha+p_1)\\(\beta+\alpha+p_1+p_2,\alpha+p_1)\end{matrix}\middle| u^{\alpha+p_1}\right]. \qquad (43)\end{aligned}$$

Proof. The proof of the above theorem pursue from the integral representation of generalized Wright's hypergeometric supersine function

$$\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\sin_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^\alpha t^\alpha\right]dt = \Gamma(\alpha)\,{}_1\sin_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta+\alpha,\alpha)\end{matrix}\middle| u^\alpha\right].$$

□

Theorem 3.4. *Let $\alpha > |p_1|$ and $|u| \leq 1$. Then, Turán type inequality for the generalized Wright's hypergeometric supercosine function holds the following relation*

$$\left[{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta+\alpha,\alpha)\end{matrix}\middle| u^\alpha\right]\right]^2 \leq \frac{\mathrm{B}(\alpha-p_1,\alpha+p_1)}{\mathrm{B}(\alpha,\alpha)} \times {}_1\cos_1\left[\begin{matrix}(\beta-p_1,\alpha-p_1)\\(\beta+\alpha-p_1-p_2,\alpha-p_1)\end{matrix}\middle| u^{\alpha-p_1}\right] \times {}_1\cos_1\left[\begin{matrix}(\beta+p_1,\alpha+p_1)\\(\beta+\alpha+p_1+p_2,\alpha+p_1)\end{matrix}\middle| u^{\alpha+p_1}\right]. \quad (44)$$

Proof. The proof of the above theorem pursue from the integral representation of generalized Wright's hypergeometric supercosine function

$$\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta,\alpha)\end{matrix}\middle| u^\alpha t^\alpha\right]dt = \Gamma(\alpha)\,{}_1\cos_1\left[\begin{matrix}(\gamma,\alpha)\\(\beta+\alpha,\alpha)\end{matrix}\middle| u^\alpha\right].$$

□

Statements & Declarations

Conflicts of interest

The authors declare that they have no competing or conflict of interests.

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