



Characterization of ultradistributional wavefront sets via Moritoh's wavelet transform

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Abstract. In the present paper, we employ the special wavelet transform introduced by Moritoh [13] within ultradistributional spaces. Employing this wavelet transform, we define and characterise wave front sets for ultradistributions. In addition, we present several examples and determine their corresponding wave front sets.

1. Introduction

Around 1970, Hörmander [8] introduced the concept of the wave front to describe the singularities of generalised functions. He precisely characterised these singularities by identifying the directions in which they occur. In 1983, Bruce [4] explored the evolution of wave fronts, particularly focusing on how they transform when propagating from an initially smooth wave front in Euclidean space.

In 1989, Murenzi [14] introduced a wavelet transform in \mathbb{R}^n that incorporates an additional rotational parameter, alongside the usual dilation and translation. Building upon this framework, Moritoh [13] proposed a refined approach that preserved dilation and translation while employing a specific rotation. This rotation, depending on a variable ζ , maps the unit vector in the direction of ζ to the canonical unit vector e_n . This formulation laid the foundation for analysing wave front sets via the wavelet transform. Moritoh further demonstrated that the wave front set obtained through this method is equivalent to the classical Hörmander wave front set. This insight led to a new definition of wave front sets for tempered distributions, which was later advanced by Pilipović and Vuletić [21]. Expanding upon this foundational work, Pilipović and collaborators introduced a wave front sets for tempered distributions associated with

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special rotational matrices that satisfy Moritoh's rotational conditions. They characterised these wave front sets using a variety of tools, including the wavelet transform [21], the short-time Fourier transform [20], and the Stockwell transform [2]. Furthermore, Grohs, Kutyniok, and Labate [7, 12] investigated the resolution of wave front sets using shearlet transforms.

More recently, Johansson, Pilipović, Teofanov, and Toft [10] extended these investigations to the wave front sets of Roumieu-type ultradistributions. Notable ultradistribution spaces—such as the Pak–Kang space $\mathcal{H}_\kappa^s(\mathbb{R}^n)$ [16], and the Björck space $\mathcal{B}_{2,k}^\kappa$ [3] underscore the significance of wave front set analysis in this broader context.

Further developments include the study of wave front sets in Fourier–Lebesgue spaces with sub-exponential weights [5, 11], as well as investigations into the wavelet transforms of Beurling–Björck-type ultradistributions by Pathak and Singh [18].

The Moritoh Wavelet Transform (MWT) plays a pivotal role in characterising wave front sets in both ultradistributional and generalised Sobolev settings. Its formulation, grounded in rotation-invariant properties, enables a refined directional analysis of singularities, extending the classical framework to more general spaces. The MWT not only captures the localisation of singularities in space and frequency but also respects the nuanced structure of ultradistributions. This makes it an effective tool for precise microlocal analysis. A particularly noteworthy application of the MWT is its use in the study of the Schrödinger equation for a free particle, where it allows for an explicit and detailed description of the wave front set of the solution.

The paper is organised as follows: Section 2 presents the special orbital transform, which serves as a basis for defining the wavelet transform in ultradistributional spaces. Section 3 reviews the construction of ultradistributional spaces that serve as a framework for defining and analyzing wavelet-based microlocal properties that we need to study wave front sets. Section 4 formally defines the Moritoh wavelet transform (MWT) for ultradistributions using admissible wavelets in \mathcal{S}_κ , with support away from the origin and compactly supported Fourier transforms. The section derives its Fourier domain representation, proves continuity properties, and discusses its covariance under partitioned scaling and rotation. Furthermore, it establishes a Parseval-type identity and provides an explicit inverse formula. Section 5 provides a characterisation of the wave front set via the MWT in both ultradistributional and generalised Sobolev contexts. Section 6 presents significant examples and explicitly determines their wave front sets.

2. Special Orbital Transform

$GL(n, \mathbb{R})$ is a commonly used abbreviation for the general linear group of all $n \times n$ invertible matrices of real numbers. $O(n)$ is the standard orthogonal group of degree n , and $SO(n)$ is a subgroup of $O(n)$ whose elements have a determinant equal to 1. In this section, we discuss the application of the matrix-valued mapp $\mathcal{O} : \zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{R}^n \setminus \{0\} \mapsto \mathcal{O}_\zeta \in GL(n, \mathbb{R})$, for $n = 1, 2, 4, 8$, which represents a special orbital transform defined as

- $n = 1, \mathcal{O}_\zeta^{-1} = [\zeta];$
- $n = 2, \mathcal{O}_\zeta^{-1} = \begin{bmatrix} \zeta_1 & \zeta_2 \\ -\zeta_2 & \zeta_1 \end{bmatrix};$
- $n = 4, \mathcal{O}_\zeta^{-1} = \begin{bmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\ -\zeta_2 & \zeta_1 & \zeta_4 & -\zeta_3 \\ -\zeta_3 & -\zeta_4 & \zeta_1 & \zeta_2 \\ -\zeta_4 & \zeta_3 & -\zeta_2 & \zeta_1 \end{bmatrix};$

- $n = 8,$

$$\mathcal{O}_\zeta^{-1} = \begin{bmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 & \zeta_7 & \zeta_8 \\ -\zeta_2 & \zeta_1 & \zeta_4 & -\zeta_3 & \zeta_6 & -\zeta_5 & -\zeta_8 & \zeta_7 \\ -\zeta_3 & -\zeta_4 & \zeta_1 & \zeta_2 & \zeta_7 & \zeta_8 & -\zeta_5 & -\zeta_6 \\ -\zeta_4 & \zeta_3 & -\zeta_2 & \zeta_1 & \zeta_8 & -\zeta_7 & \zeta_6 & -\zeta_5 \\ -\zeta_5 & -\zeta_6 & -\zeta_7 & -\zeta_8 & \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\ -\zeta_6 & \zeta_5 & -\zeta_8 & \zeta_7 & -\zeta_2 & \zeta_1 & -\zeta_4 & \zeta_3 \\ -\zeta_7 & \zeta_8 & \zeta_5 & -\zeta_6 & -\zeta_3 & \zeta_4 & \zeta_1 & -\zeta_2 \\ -\zeta_8 & -\zeta_7 & \zeta_6 & \zeta_5 & -\zeta_4 & -\zeta_3 & \zeta_2 & \zeta_1 \end{bmatrix}.$$

Here, $|\zeta|^{-1}\mathcal{O}_\zeta^{-1} \in SO(n), \zeta \in \mathbb{R}^n \setminus \{0\}$ and

$$|\zeta|\mathcal{O}_\zeta^{-1} \begin{pmatrix} \zeta \\ |\zeta| \end{pmatrix} = e_r, \tag{1}$$

where e_r represents a standard vector with 1 at the r^{th} position and zeros everywhere else. The limitation of $n = 1, 2, 4, 8$ comes from the requirement $\mathbb{R}^n \setminus \{0\} \ni \zeta \mapsto |\zeta|^{-1}\mathcal{O}_\zeta^{-1} \in O(n)$ to be continuous, which is crucial when transforming sets by changes in variables that involve these matrices, [21]. Note that matrices satisfying the above properties are not uniquely determined. More details on the properties of these matrix-valued mappings can be found in [2, 21]. We state some results needed for our theorems.

Proposition 2.1. [2, Prop.1.5] *There exists a C^∞ map $\zeta \mapsto \tilde{\mathcal{O}}_\zeta, \mathbb{R}^n \setminus \{0\} \rightarrow GL(n, \mathbb{R})$, such that*

$$\mathcal{O}_\zeta^{-1}(\tau) = \tilde{\mathcal{O}}_\tau^{-1}(\zeta), \text{ for all } \zeta, \tau \in \mathbb{R}^n \setminus \{0\}, \tag{2}$$

and $|\zeta|^{-1}\tilde{\mathcal{O}}_\zeta^{-1} \in O(n), \forall \zeta \in \mathbb{R}^n \setminus \{0\}$. In particular, the maps $\zeta \mapsto |\zeta|^{-1}\tilde{\mathcal{O}}_\zeta^{-1}, \mathbb{R}^n \setminus \{0\} \rightarrow GL(n, \mathbb{R}), \zeta \mapsto |\zeta|^{-1}\mathcal{O}_\zeta^{-1}$ and $\zeta \mapsto \tilde{\mathcal{O}}_\zeta^{-1}, \mathbb{R}^n \setminus \{0\} \rightarrow O(n)$, are also C^∞ .

Remark 2.2. *If we choose $\eta = \frac{\mathcal{O}_\tau^{-1}(\zeta)}{|\zeta|^2}, \zeta, \eta \in \mathbb{R}^n \setminus \{0\}$, we get $\frac{d\eta}{|\eta|^n} = \frac{d\zeta}{|\zeta|^n}$*

Remark 2.3. *It holds*

$$\mathcal{O}_\zeta^t(\tau) = |\zeta|^{-2}\mathcal{O}_\zeta^{-1}(\tau), \text{ for all } \tau \in \mathbb{R}^n, \xi \in \mathbb{R}^n \setminus \{0\}. \tag{3}$$

Moreover, $|\zeta|\mathcal{O}_\zeta^{-1} \begin{pmatrix} \zeta \\ |\zeta| \end{pmatrix} = e_r$, implies $\mathcal{O}_\zeta^t(\zeta) = e_r$.

3. Ultradistributions

3.1. Ultradistributional Spaces

The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is defined by $\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-ix\xi} dx, \xi \in \mathbb{R}^n$. If one replaces $-i$ with i , then one obtains the inverse Fourier transform.

We start with the Björck approach to ultradistribution theory via weight functions [3]. Let $\kappa(\xi)$ be a real-valued function that can be represented as $\kappa(\xi) = r(|\xi|)$, where $r(t)$ is an increasing, continuous, and concave function on $[0, \infty]$ such that

1. $0 \leq \kappa(\xi + \tau) \leq \kappa(\xi) + \kappa(\tau), \forall \xi, \tau \in \mathbb{R}^n$
2. $\int_{\mathbb{R}^+} (1 + \xi^2)^{-1} r(\xi) d\xi < \infty,$
3. for real ϱ and positive real ς

$$\varrho + \varsigma \log(1 + |\xi|) < r(\xi). \tag{4}$$

The Björck-space $\mathcal{S}_\kappa(\mathbb{R}^n)$ is the set of all complex-valued infinitely differentiable functions ϕ on \mathbb{R}^n such that $\hat{\phi}$ are also infinitely differentiable functions where the following holds:

$$p_{\beta,\alpha}(\phi) = \sup_t |e^{\alpha\kappa(t)} D^\beta \phi(t)| < \infty,$$

and

$$\pi_{\beta,\alpha}(\phi) = \sup_\xi |e^{\alpha\kappa(\xi)} D^\beta \hat{\phi}(\xi)| < \infty,$$

for all multi-indices β and all positive numbers α . The locally convex topology of $\mathcal{S}_\kappa(\mathbb{R}^n)$ is defined using the semi-norms $p_{\beta,\alpha}$ and $\pi_{\beta,\alpha}$. The topological dual of the Björck-space \mathcal{S}_κ is the so-called space of (κ) -tempered ultradistributions \mathcal{S}'_κ . \mathcal{S}_κ is a Fréchet space, and the Fourier transform and its inverse are one-to-one mappings from \mathcal{S}_κ onto itself and from \mathcal{S}'_κ onto itself. For $\kappa(\xi) = \log(1 + |\xi|)^s, s > 0$ Björck-space \mathcal{S}_κ reduces to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ [24]. Moreover, $\mathcal{S}' \subset \mathcal{S}'_\kappa$, which means that they generalize the usual tempered distributions. For more information on Björck-space \mathcal{S}_κ see [3, 23].

Remark 3.1. Naturally, one may also define the spaces $\mathcal{S}_{\{\kappa\}}(\mathbb{R}^n)$ and $\mathcal{S}'_{\{\kappa\}}(\mathbb{R}^n)$ of class $\{\kappa\}$ by switching the universal quantifier \forall over α to an existential one. When considering these spaces, we shall always assume that κ satisfies a stronger condition than (4), namely

$$\lim_{|\xi| \rightarrow \infty} \frac{\kappa(\xi)}{\log(1 + |\xi|)} = \infty.$$

Remark 3.2. The conditions imposed on the weighed functions κ , allow us to have the properties needed to define our wave front sets and in order for our results to work. We refer to the papers [5, 6, 19] for more details. Note that in our case, [19, Prop.1.1] is true, since r is increasing. Indeed, for $\lambda > 1$ and $n \in \mathbb{N}$ such that $n \leq \lambda < n + 1$

$$\kappa(\lambda\xi) = r(\lambda|\xi|) \leq r((n + 1)|\xi|) = \kappa((n + 1)\xi) \leq (n + 1)\kappa(\xi) \leq (\lambda + 1)\kappa(\xi) \leq 2\lambda\kappa(\xi).$$

So, $\frac{\kappa(\lambda\xi)}{\lambda\kappa(\xi)} \leq 2$. [19] also implies that composition of ultradifferentiable functions of class $\{\kappa\}$ is also ultradifferentiable functions of class $\{\kappa\}$.

In order for our theory to hold, we impose the following conditions on the function ω , stated in the lemma below.

Lemma 3.3. Let $\omega : \mathbb{R}^n \rightarrow (0, \infty)$ be continuous function

a) If there are $E_0, c_0 > 1$ such that

$$\omega(c_0x) \leq E_0\omega(x), x \in \mathbb{R}^n,$$

then there are $C, l > 0$ such that $\omega(x) \leq C(1 + |x|)^l, x \in \mathbb{R}^n$.

b) If there are $0 < \varepsilon_0 < 1$ and $E_0 > 1$ such that

$$\omega(\varepsilon x) \leq E_0\omega(x), x \in \mathbb{R}^n,$$

then there are $C, l > 0$ such that $\omega(x) \geq C(1 + |x|)^{-l}, x \in \mathbb{R}^n$.

Proof. We first proof a). Set $M := \sup_{|x| \leq 1} \omega(x)$. Let $x \in \mathbb{R}^n$ satisfy $|x| > 1$ and denote $k := \lfloor \log_{c_0} |x| \rfloor + 1$. Then,

$$\omega(x) = \omega(C_0^k C_0^{-k} x) \leq E_0 \omega(C_0^{k-1} C_0^{-k} x) \leq \dots \leq E_0^k \omega(C_0^{-k} x).$$

Notice that $C_0^k \geq C_0^{\log_{c_0} |x|} = |x|$; consequently, the above implies

$$\omega(x) \leq E_0^k M \leq C_0^{(\log_{c_0} E_0)(\log_{c_0} |x| + 1)} M = ME_0 |x|^{\log_{c_0} E_0}.$$

Hence $\omega(x) \leq ME_0(1 + |x|)^{\log_{c_0} E_0}, \forall x \in \mathbb{R}^n$ (this trivially holds when $|x| \leq 1$), and the proof of *a*) is complete.

To show *b*) denote $m := \inf_{|x| \leq 1} \omega(x) > 0$. Set $C_0 = \frac{1}{\varepsilon} > 1$. For $|x| > 1$ denote $k := \lceil \log_{c_0} |x| \rceil + 1$ and notice that

$$\omega(\varepsilon_0^k x) \leq E_0 \omega(\varepsilon_0^{k-1} x) \leq \dots \leq E_0^k \omega(x).$$

As above $C_0^k \geq |x|$ and consequently $\varepsilon_0^k |x| \leq 1$. Hence

$$\omega(x) \geq E_0^{-k} m \geq C_0^{-(\log_{c_0} E_0)(\log_{c_0} |x| + 1)} m = \frac{m}{E_0} |x|^{-\log_{c_0} E_0}.$$

So, $\omega(x) \geq \frac{m}{E_0} (1 + |x|)^{-\log_{c_0} E_0}, x \in \mathbb{R}^n$ (this trivially holds when $|x| \leq 1$) which proves *b*). \square

The generalized Sobolev space $\mathcal{H}_\omega^\kappa(\mathbb{R}^n)$ is a subspace of \mathcal{S}'_κ such that ultradistribution f belongs to the generalized Sobolev space if

$$\int_{\mathbb{R}^n} |\hat{f}(u)|^2 \omega(u) du < \infty,$$

with the norm defined as

$$\|f\|_\omega = (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} |\hat{f}(u)|^2 \omega(u) du \right)^{1/2}.$$

The space investigated by Hörmander [9] can be recovered by choosing $\kappa(u) = \log(1 + |u|)$ and letting $w(u)$ be a temperate weight function associated with $\kappa(u)$. In this context, ultradistribution spaces provide a highly general framework that encompasses a wide range of important function spaces and facilitates the analysis of their structural properties.

4. Wavelet transform

We will define the multidimensional wavelet transform under the assumption that $\psi \in L^2(\mathbb{R}^n)$ satisfies the admissibility condition.

$$C_\psi = (2\pi)^n \int_{\mathbb{R}^n} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|^n} < \infty. \tag{5}$$

As in [2, 18], we will consider the window function $\psi \in \mathcal{S}_\kappa(\mathbb{R}^n)$ satisfying $\hat{\psi} \in \mathcal{D}_\kappa(\mathbb{R}^n) = \mathcal{S}_\kappa(\mathbb{R}^n) \cap \mathcal{D}(\mathbb{R}^n)$, $\hat{\psi}(e_r) \neq 0$, $\hat{\psi} \geq 0$, and $\Omega = \text{supp } \hat{\psi}$ does not contain 0.

If we assume that $n = 1, 2, 4, 8$, the wavelet transform of a distribution $f \in \mathcal{S}'_\kappa(\mathbb{R}^n)$ is defined by

$$W_\psi f(x, \zeta) = \int_{\mathbb{R}^n} f(t) |\zeta|^{n/2} \overline{\psi(\tilde{\mathcal{O}}_{t-x}^{-1}(\zeta))} dt, \quad (x, \zeta) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}),$$

with \mathcal{O} described in Subsection 2; of course, the integral is considered in the ultradistributional sense. The Fourier transform of $W_\psi f(x, \zeta)$ with respect to x is given by

$$\begin{aligned} \widehat{W_\psi f}(x, \zeta)(\tau, \zeta) &= (2\pi)^{n/2} \hat{f}(\tau) |\zeta|^{-n/2} \hat{\psi}(|\zeta|^{-2} \mathcal{O}_\zeta^{-1}(\tau)) \\ &= (2\pi)^{n/2} \hat{f}(\tau) |\zeta|^{-n/2} \hat{\psi}(\mathcal{O}_\zeta^t(\tau)), \quad (\tau, \zeta) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}), \end{aligned} \tag{6}$$

where we have used Proposition 2.1 and (3). It is straightforward to verify that $W_\psi f(x, \zeta)$ is a well-defined continuous function on $\mathbb{R}_x^n \times (\mathbb{R}_\zeta^n \setminus \{0\})$. Moreover, for a bounded subset B of $\mathcal{S}'_\kappa(\mathbb{R}^n)$ and $K \subseteq \mathbb{R}^n \setminus \{0\}$ compact, there exists $C > 0$ such that $|W_\psi f(x, \zeta)| \leq Ce^{k(x)}$, for all $x \in \mathbb{R}^n, \zeta \in K, f \in B$. Thus, for each fixed

$\zeta \in \mathbb{R}^n \setminus \{0\}$, the mapping $f \mapsto W_\psi f(\cdot, \zeta), \mathcal{S}'_\kappa(\mathbb{R}^n) \rightarrow \mathcal{S}'_\kappa(\mathbb{R}^n)$ maps bounded sets into bounded sets, and hence it is continuous, [1, 15, 17, 18].

We will utilize the following partition $I = \{I_1, I_2, \dots, I_\alpha\}, \nu_i = |I_i|, i = 1, 2, \dots, \alpha$ of the set $\{1, 2, \dots, n\}$. For such a partition I we denote $\mathcal{O}_{I, \zeta} = \bigoplus_{i=1}^\alpha \mathcal{O}_{\zeta_i} = \mathcal{O}_{\zeta^1} \oplus \dots \oplus \mathcal{O}_{\zeta^\alpha} \in GL(n, \mathbb{R}), \zeta = (\zeta^1, \dots, \zeta^\alpha) \in \prod_{i=1}^\alpha (\mathbb{R}^{\nu_i} \setminus \{0\})$ (the direct sum of $\mathcal{O}_{\zeta_i}, i = 1, \dots, \alpha$); of course, $\mathcal{O}_{I, \zeta}^{-1} = \bigoplus_{i=1}^\alpha \mathcal{O}_{\zeta_i}^{-1}$ and $\mathcal{O}_{I, \zeta}^t = \bigoplus_{i=1}^\alpha \mathcal{O}_{\zeta_i}^t$. The wavelet transform of an ultradistribution $f \in \mathcal{S}'_\kappa(\mathbb{R}^n)$ with respect to partition I and window $\psi = \psi_1 \oplus \psi_2 \oplus \dots \oplus \psi_\alpha \in \mathcal{S}_\kappa(\mathbb{R}^n) \cap D(\mathbb{R}^n)$ is defined as

$$W_\psi f(x, \zeta) = \int_{\mathbb{R}^n} f(t) \prod_{i=1}^\alpha |\zeta^i|^{\nu_i/2} \overline{\psi(\mathcal{O}_{t-x}^{-1}(\zeta))} dt, \quad (x, \zeta) \in \mathbb{R}^n \times \prod_{i=1}^\alpha (\mathbb{R}^{\nu_i} \setminus \{0\}),$$

where the distributional concept of integration is used. Clearly, we receive various wavelet transforms for each alternative factorization of a wavelet into a product of wavelets. One can verify that $W_\psi f(x, \zeta)$ is a well-defined continuous function on $\mathbb{R}^n \times \prod_{i=1}^\alpha (\mathbb{R}^{\nu_i} \setminus \{0\})$ with subexponential growth with respect to x uniformly when ζ varies in compact subsets of $\prod_{i=1}^\alpha (\mathbb{R}^{\nu_i} \setminus \{0\})$, and the mapping $f \mapsto W_\psi f(\cdot, \zeta), \mathcal{S}'_\kappa(\mathbb{R}^n) \rightarrow \mathcal{S}'_\kappa(\mathbb{R}^n)$, is continuous for each fixed ζ , [18].

The Fourier transform of $W_\psi f(x, \zeta)$ with respect to x is given by

$$(W_\psi \widehat{f(x, \zeta)})(\tau, \zeta) = (2\pi)^{n/2} \widehat{f}(\tau) \prod_{i=1}^\alpha |\zeta^i|^{-\nu_i/2} \widehat{\psi}(\mathcal{O}_\zeta^t(\tau)), \quad (\tau, \zeta) \in \mathbb{R}^n \times \prod_{i=1}^\alpha (\mathbb{R}^{\nu_i} \setminus \{0\}).$$

We observe that $W_\psi f$ is an ultradistribution with respect to the variable x since f is an ultradistribution, and ξ belongs to $\prod_{i=1}^\alpha (\mathbb{R}^{\nu_i} \setminus \{0\})$.

In $\mathbb{R}^n \times \mathbb{R}^n$, the complement of $\mathbb{R}^n \times \prod_{i=1}^\alpha (\mathbb{R}^{\nu_i} \setminus \{0\})$ is of zero measure. The following four assertions are thus obtained by omitting this set from the integration domain and using [13].

Proposition 4.1. (Parseval’s identity) Let $f, g \in L^2(\mathbb{R}^n)$. Then $W_\psi f \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W_\psi f(x, \zeta) \overline{W_\psi g(x, \zeta)} dx d\zeta = \mathcal{C}_\psi \int_{\mathbb{R}^n} f(t) \overline{g(t)} dt,$$

where

$$\mathcal{C}_\psi = (2\pi)^n \int_{\mathbb{R}^n} \frac{|\widehat{\psi}(\zeta)|^2}{\prod_{i=1}^\alpha |\zeta^i|^{\nu_i}} d\zeta < \infty.$$

Corollary 4.2. The wavelet transform W_ψ is an isometric transform of $L^2(\mathbb{R}^n, dt)$ to $L^2(\mathbb{R}^n \times \mathbb{R}^n), \mathcal{C}_\psi^{-1} dx d\zeta$.

Proposition 4.3. (Inverse formula) Let $f \in L^2(\mathbb{R}^n)$ then,

$$f(t) = \mathcal{C}_\psi^{-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W_\psi f(x, \zeta) \prod_{i=1}^\alpha |\zeta^i|^{\nu_i/2} \overline{\psi(\mathcal{O}_{t-x}^{-1}(\zeta))} dx d\zeta, \quad t \in \mathbb{R}^n,$$

We refer to [18] for the wavelet transform on ultradistributional spaces.

We will make use of the following proposition when considering generalized Sobolev spaces.

Proposition 4.4. $f \in \mathcal{H}_\omega^\kappa(\mathbb{R}^n)$ iff

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |W_\psi f(x, \zeta)|^2 \omega(|\zeta|) dx d\zeta < \infty.$$

Proof. By using the definition of $W_\psi f(x, \zeta)$, $\hat{\psi}_i \in \mathcal{D}(\mathbb{R}^{v_i})$ and change of variable as $\eta = |\zeta|^{-2} \tilde{\mathcal{O}}_\tau^{-1}(\zeta)$, we get

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |W_\psi f(x, \zeta)|^2 \omega(|\zeta|) dx d\zeta &= (2\pi)^n \int_{\mathbb{R}^n} d\tau |\hat{f}(\tau)|^2 \\ &\quad \times \int_{\mathbb{R}^n} \omega(|\zeta|) \prod_{i=1}^\alpha \frac{d\zeta}{|\zeta^i|^{v_i}} (\hat{\psi}(|\zeta|^{-2} \tilde{\mathcal{O}}_\tau^{-1}(\zeta)))^2 \\ &= (2\pi)^n \int_{\mathbb{R}^n} d\tau |\hat{f}(\tau)|^2 \\ &\quad \times \int_{\mathbb{R}^n} \omega \left(\left(\sum_{i=1}^\alpha \frac{|\tau^i|^2}{|\eta^i|^2} \right)^{1/2} \right) \prod_{i=1}^\alpha \frac{d\eta}{|\eta^i|^{v_i}} (\hat{\psi}(\eta))^2 \\ &\approx \mathcal{C}_\psi \int_{\mathbb{R}^n} |\hat{f}(\tau)|^2 \omega(|\tau|) d\tau. \end{aligned}$$

Here, the notation \approx indicates that both directions are valid, according to Lemma 3.3. \square

For dimensions 1, 2, 4, and 8, as in [21?], we will consider estimates of wave fronts by introducing a parameterized wavelet transform and by an intrinsic analysis of transformations of variables in the frequency domain. For $0 < \lambda < 1$, we define ψ_λ by

$$\psi_\lambda(x) = \lambda^n e^{ixe_r(1-\lambda)} \psi(\lambda x), \quad x \in \mathbb{R}^n.$$

Then $\mathcal{F}(\psi_\lambda)(\zeta) = \mathcal{F}(\psi)(\lambda^{-1}\zeta + (1 - \lambda^{-1})e_r)$ and thus $\Omega_\lambda = \text{supp } \mathcal{F}(\psi_\lambda) \subseteq B_{\lambda r}(e_r)$. Notice that $\mathcal{F}(\psi_\lambda)(e_r) = \mathcal{F}(\psi)(e_r) \neq 0$.

For an arbitrary n , let ψ_i be the analysis wavelet and $\mu \in (0, 1)$ be such that $\text{supp } \hat{\psi}_i \subseteq B_\mu(e_{r_i})$, $i \in \{1, 2, \dots, \alpha\}$. For $0 < \lambda < 1$ we define $\psi_\lambda(x) = \prod_{i=1}^\alpha \psi_{i,\lambda}(x^i)$, $x \in \mathbb{R}^n$ where

$$\psi_{i,\lambda}(x^i) = \lambda^{v_i} e^{ix^i e_{r_i}(1-\lambda)} \psi_i(\lambda x^i), \quad x^i \in \mathbb{R}^{v_i}.$$

Put $\Lambda_{i,\lambda} = \text{supp } \hat{\psi}_{i,\lambda}$ and $\Lambda_\lambda = \prod_{i=1}^\alpha \Lambda_{i,\lambda}$ then definitely $\Lambda_{i,\lambda} \subseteq B_{\lambda\mu}(e_{r_i})$.

5. Characterisation of the wave front set via wavelet transform

The key results of the paper establish the principles of directional smoothness within ultradistributional spaces, offering criteria for identifying regular directed points through the wavelet transform of ultradistributions in $\mathcal{S}'_x(\mathbb{R}^n)$. It is important to note that the wave front set, as determined by the wavelet transform, depends on the chosen wavelet [21]. In the continuation, we refer to $\Gamma(\zeta_0)$ as an arbitrarily conic neighborhood around the point ζ_0 . For $\zeta_0 \in \mathbb{R}^n \setminus \{0\}$ and $\mu \in (0, 1)$, $\Gamma(\zeta_0, \mu)$ will denote the conic neighborhood of ζ_0 defined by

$$\Gamma(\zeta_0, \mu) = \left\{ \zeta \in \mathbb{R}^n \mid \left| \frac{\zeta}{|\zeta|} - \frac{\zeta_0}{|\zeta_0|} \right| < \mu \right\}. \tag{7}$$

On a couple of occasions, the following alternative description of $\Gamma(\zeta_0, \mu)$ will prove useful. Denote by $\Gamma_{\zeta,\theta}$ the open convex cone with direction $\zeta \in \mathbb{S}^{n-1}$, where \mathbb{S}^{n-1} is the unit sphere, and opening angle $\theta \in (0, \pi/2)$, i.e. $\Gamma_{\zeta,\theta} = \{x \in \mathbb{R}^n \mid |x| \cos \theta < x\zeta\}$. Let $\theta \in (0, \pi/2)$ be the (unique) point such that $\sin(\theta/2) = \mu/2$. Then a straightforward computation yields $\Gamma(\zeta_0, \mu) = \Gamma_{\zeta_0/|\zeta_0|,\theta}$ (and thus $\Gamma(\zeta_0, \mu)$ is convex). Incidentally, one can easily verify that there exists $\theta_0 \in (0, \pi/2)$ such that for any $\zeta_0 \in \mathbb{S}^{n-1}$ and $0 < \theta \leq \theta_0$ it holds

$$|\zeta + \mu| \geq \max\{|\mu|, |\zeta|\}, \quad \text{for all } \zeta, \omega \in \Gamma_{\zeta_0,\theta}. \tag{8}$$

Remark 5.1. Notice that the above considerations concerning the cones are valid in any dimension $n \in \mathbb{Z}_+$, not just for $n = 1, 2, 4$ or 8.

5.1. The C^∞ wave front set

Definition 5.2. For a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$, a point $(x, \zeta) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ is called a regular directed point of f if and only if there exist: (i) a function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\varphi(x) = 1$ and (ii) a closed conic neighborhood $\Gamma(\zeta) \in \mathbb{R}^n$ of ξ , such that for every $N \in \mathbb{N}, \exists C > 0$

$$|\widehat{f\varphi}(\zeta)| \leq C e^{-N\kappa(\zeta)} \text{ for all } \zeta \in \Gamma(\xi). \tag{9}$$

From the definition, it is evident that the decay condition imposed in (9) relates primarily to the behavior of the high-frequency components. The wave front set $WF(f)$ is the complement of the set of regular directed points.

Definition 5.3. Let $f \in \mathcal{S}'_k(\mathbb{R}^n)$, and $n = 1, 2, 4$ and 8 . The set $WF_{W,\psi}(f) \subseteq \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ is defined in the following way: $(x_0, \zeta_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ does not belong to $WF_{W,\psi}(f)$ if and only if

$$(\exists \varphi \in \mathcal{D}_k(\mathbb{R}^n)) (\varphi(x_0) \neq 0) (\exists \Gamma(\zeta_0)) (\exists \lambda \in (0, 1)) (\forall N > 0) (\exists C_N > 0) \\ \left| W_{\psi,\lambda}(\widehat{\varphi f})(x, \zeta)(\tau, \zeta) \right| \leq C_N e^{-N\kappa(\zeta)}, \tau \in \mathbb{R}^n, \zeta \in \Gamma(\zeta_0), |\zeta| \geq 1.$$

Because of (6), $(x_0, \zeta_0) \notin WF_{W,\psi}(f)$ is equivalent to:

$$(\exists \varphi \in \mathcal{D}_k(\mathbb{R}^n)) (\phi(x_0) \neq 0) (\exists \Gamma(\zeta_0)) (\exists \lambda \in (0, 1)) (\forall N > 0) (\exists C_N > 0) \\ \left| \mathcal{F}(\varphi f)(\tau) \mathcal{F}(\psi_{\lambda_0})(\mathcal{O}_\zeta^t(\tau)) \right| \leq C_N e^{-N\kappa(\zeta)}, \tau \in \mathbb{R}^n, \zeta \in \Gamma(\zeta_0), |\zeta| \geq 1.$$

Theorem 5.4. Let $f \in \mathcal{S}'_k(\mathbb{R}^n)$. Then $WF_{W,\lambda}(f) = WF(f)$.

Proof. Let $(x_0, \zeta_0) \notin WF(f)$, with $\zeta_0 \in \mathbb{S}^{n-1}$. There exist $\varphi \in \mathcal{D}_k(\mathbb{R}^n)$, $\varphi(x_0) \neq 0$, and a conic neighborhood $\Gamma(\zeta_0)$ of ζ_0 such that for each $N > 0$, there exists $C_N > 0$ such that

$$|\mathcal{F}(\varphi f)(\zeta)| \leq C_N e^{-N\kappa(\zeta)}, \zeta \in \Gamma(\zeta_0). \tag{10}$$

Let $\mu_0 < 1/2$ and $\lambda_0 < 1$ be such that $\Gamma(\zeta_0, 2\mu_0) \subseteq \Gamma(\zeta_0)$. We fix $\lambda_0 \in (0, 1)$ in order to

$$\text{supp } \hat{\psi}_{\lambda_0} = \Omega_0 \subseteq B_{\lambda_0}(e_r) \subseteq \Gamma(e_r, \mu_0). \tag{11}$$

We will also use that for every $\zeta \in \Gamma(\zeta_0, \mu_0)$,

$$(\mathcal{O}_\zeta^t)^{-1}(\Omega_0) \subseteq \Gamma(\zeta_0, 2\mu_0) \subseteq \Gamma(\zeta_0). \tag{12}$$

Now, let $\zeta \in \Gamma(\zeta_0, \mu_0)$. When $\tau \in (\mathcal{O}_\zeta^t)^{-1}(\Omega_0)$, we avail ourselves of (11) and (12) to deduce

$$|\mathcal{F}(\varphi f)(\tau) \mathcal{F}(\psi_{\lambda_0})(\mathcal{O}_\zeta^t(\tau))| \leq C_N |\mathcal{F}(\varphi f)((\mathcal{O}_\zeta^t)^{-1})| \leq C_N e^{-N\kappa(\zeta)}$$

If $\tau \notin (\mathcal{O}_\zeta^t)^{-1}(\Omega_0)$, the estimate holds true trivially. We conclude $(x_0, \zeta_0) \notin WF_{W,\psi}(f)$.

Assume now $(x_0, \zeta_0) \notin WF_{W,\psi}(f)$, $\zeta_0 \in \mathbb{S}^{n-1}$. There exist $\varphi \in \mathcal{D}_k(\mathbb{R}^n)$, $\varphi(x_0) \neq 0$, a conic neighborhood $\Gamma(\zeta_0)$ of ζ_0 and $\lambda \in (0, 1)$ such that for each $N > 0$, there exists $C'_N > 0$ satisfying

$$|\mathcal{F}(\varphi f)(\tau) \mathcal{F}(\psi_\lambda)(\mathcal{O}_\zeta^t(\tau))| \leq C'_N e^{-N\kappa(\zeta)}, \text{ for all } \tau \in \mathbb{R}^n, \zeta \in \Gamma(\zeta_0), |\zeta| \geq 1. \tag{13}$$

Since $\mathcal{O}_\zeta^t(\zeta) = e_r$, we deduce

$$\mathcal{F}(\psi_\lambda)(\mathcal{O}_\zeta^t(\tau)) = \mathcal{F}(\psi)(\mathcal{O}_\zeta^t(\tau)(\lambda^{-1}\tau - (\lambda^{-1} - 1)\zeta)).$$

As $c = |\mathcal{F}(\psi)(e_r)| > 0$, specialising the estimate (13) for $\tau = \zeta$ we deduce

$$|\mathcal{F}(\varphi f)(\zeta)| \leq c^{-1} C'_N e^{-N\kappa(\zeta)}, \text{ for all } \zeta \in \Gamma(\zeta_0), |\zeta| \geq 2,$$

which completes the proof. \square

Now, we will make a discussion of the characterization of wave front sets for an arbitrary n .

Definition 5.5. (Ultradistributional wave front set) Let $f \in \mathcal{S}'_{\kappa}(\mathbb{R}^n)$. Then the Ultradistributional wave front set $\mathcal{WF}(f) \subseteq \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ is the complement of the set of microlocal regular points $(x_0, \zeta_0) \in \mathbb{R}^n \times \prod_{i=1}^{\alpha} (\mathbb{R}^{n_i} \setminus \{0\})$, that is, $(x_0, \zeta_0) \notin \mathcal{WF}(f)$ iff

$$(\exists \varphi \in \mathcal{D}_{\kappa}(\mathbb{R}^n)) (\varphi(x_0) \neq 0) (\exists \Gamma(\zeta_0) \subseteq \mathbb{R}^n \setminus \{0\}) (\forall N > 0) (\exists c_N > 0) \\ |\widehat{(\varphi f)}(\zeta)| \leq c_N e^{-N\kappa(\zeta)}, \quad \zeta \in \Gamma(\zeta_0), \quad |\zeta| \geq 1.$$

Note that the cone $\Gamma(\zeta_0)$ is a subset of $\mathbb{R}^n \setminus \{0\}$.

Definition 5.6. Let $f \in \mathcal{S}'_{\kappa}(\mathbb{R}^n)$. Let $\tau \in \mathbb{R}^n$, $\zeta \in \Gamma(\zeta_0) \subseteq \mathbb{R}^n \setminus \{0\}$ be a conic neighborhood of $\zeta_0 \in \mathbb{R}^n \setminus \{0\}$ with $|\zeta| \geq 1$. We say $(x_0, \zeta_0) \notin \mathcal{WF}_{\psi, \lambda}(f)$ iff

$$(\exists \varphi \in \mathcal{D}_{\kappa}(\mathbb{R}^n)) (\varphi(x_0) \neq 0) (\exists \Gamma(\xi_0) \subseteq \mathbb{R}^n \setminus \{0\}) (\exists \lambda \in (0, 1)) (\forall N > 0) (\exists c_N > 0)$$

$$|W_{\psi, \lambda} \widehat{(\varphi f)}(\tau, \zeta)| \leq c_N e^{-N\kappa(\zeta)} \tag{14}$$

or, equivalently

$$(\exists \varphi \in \mathcal{D}_{\kappa}(\mathbb{R}^n)) (\varphi(x_0) \neq 0) (\exists \Gamma(\xi_0) \subseteq \mathbb{R}^n \setminus \{0\}) (\exists \lambda \in (0, 1)) (\forall N > 0) (\exists c_N > 0)$$

$$\left| \mathcal{F}(\varphi f)(\tau) \mathcal{F}(\psi_{\lambda_0})(\mathcal{O}_{\zeta}^t(\tau)) \right| \leq c_N e^{-N\kappa(\zeta)}, \quad \tau \in \mathbb{R}^n, \zeta \in \Gamma(\zeta_0), |\zeta| \geq 1.$$

Theorem 5.7. Let $f \in \mathcal{S}'_{\kappa}(\mathbb{R}^n)$. Then $\mathcal{WF}_{\psi, \lambda}(f) = \mathcal{WF}(f) \cap \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$.

Proof. Let $(x_0, \zeta_0) \notin \mathcal{WF}(f)$ and $\zeta_0 = (\zeta_0^1, \dots, \zeta_0^{\alpha}) \in \mathbb{R}^n \setminus \{0\}$. There exist $\varphi \in \mathcal{D}_{\kappa}(\mathbb{R}^n)$ satisfying $\varphi(x_0) \neq 0$ and a convex open cone $\Gamma(\xi_0)$ such that for every $N \in \mathbb{N}$ there exists $c_N > 0$ such that

$$|\mathcal{F}(\varphi f)(\zeta)| \leq c_N e^{-N\kappa(\zeta)}, \quad \forall \zeta \in \Gamma(\zeta_0); \tag{15}$$

of course, without losing generality, we can assume $\Gamma(\zeta_0) = \Gamma_{\zeta_0/|\zeta_0|, \theta} \subseteq \mathbb{R}^n \setminus \{0\}$ for small enough angle θ such that (8) holds true (cf. Remark 5.1). Thus, we can find $c_0 > 1$ such that

$$|\zeta| \leq c_0 \min_{1 \leq j \leq \alpha} |\zeta^j|, \quad \text{for all } \zeta = (\zeta^1, \dots, \zeta^{\alpha}) \in \Gamma(\zeta_0). \tag{16}$$

Pick $\mu_0 \in (0, 1/2)$ such that $\prod_{j=1}^{\alpha} B_{n_j}(\zeta_0^j, 2\mu_0) \subseteq \Gamma(\zeta_0)$ and consider the open cone

$$\tilde{\Gamma}(\zeta_0) = \left\{ \zeta \in \mathbb{R}^n \mid \exists s > 0, s\zeta \in \prod_{j=1}^{\alpha} B_{n_j}(\zeta_0^j, 2\mu_0) \right\}; \quad \text{clearly } \tilde{\Gamma}(\zeta_0) \subseteq \Gamma(\zeta_0). \tag{17}$$

Choose $\lambda_0 \in (0, \mu_0/(|\zeta_0| + 1))$ so that

$$\Omega_0 = \text{supp } \mathcal{F}(\psi_{\lambda_0}) \subseteq \prod_{j=1}^{\alpha} B_{n_j}(e_r^j, \lambda_0) \subseteq \Gamma(e, \mu_0). \tag{18}$$

In [2] it is proven that $(\mathcal{O}_{I, \zeta}^t)^{-1}(\Omega_0) \subseteq \tilde{\Gamma}(\zeta_0)$, for all $\zeta \in \Gamma(\zeta_0, \mu_0/(|\zeta_0| + 1))$, i.e. $\tau|\zeta_0|/|\zeta| \in \prod_{j=1}^{\alpha} B_{n_j}(\zeta_0^j, 2\mu_0)$ and thus $\tau \in \tilde{\Gamma}(\zeta_0)$.

Let $\zeta \in \Gamma(\zeta_0, \mu_0/(|\zeta_0| + 1))$, $|\zeta| \geq 1$; incidentally, notice that $\Gamma(\zeta_0, \mu_0/(|\zeta_0| + 1)) \subseteq \tilde{\Gamma}(\zeta_0)$. Then [15, Lemma 9], (16) and (8) imply

$$|\mathcal{F}(\psi_{\lambda_0})(\mathcal{O}_{I, \zeta}^t(\tau))| |\mathcal{F}(\varphi f)(\tau)| \leq c_0^n c_N e^{-N\kappa(\zeta)}.$$

The above holds trivially when $\tau \notin (\mathcal{O}_{I, \zeta}^t)^{-1}(\Omega_0)$. We conclude $(x_0, \zeta_0) \notin \mathcal{WF}_{\psi, \lambda}(f)$.

The proof of the converse can be done in an analogous fashion as in the proof of Theorem 5.4. \square

5.2. Sobolev wave front set

Now, we will give a similar result for the generalized Sobolev wave front set. For the wavelet transform on generalized Sobolev space, see [17].

If we assume that $n = 1, 2, 4, 8$, we use the following definition.

Definition 5.8. Let $f \in \mathcal{H}_\omega^\kappa(\mathbb{R}^n)$. Then the Sobolev wave front set $WF^\omega(f) \subseteq \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ is the complement of the set of microlocal regular points $(x_0, \zeta_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, that is, $(x_0, \zeta_0) \notin WF^\omega(f)$ iff

$$(\exists \varphi \in \mathcal{D}_\kappa(\mathbb{R}^n)) \quad (\varphi(x_0) \neq 0) \quad (\exists \Gamma(\zeta_0) \subseteq \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$$

$$\int_{\Gamma(\zeta_0)} |\widehat{(\varphi f)}(\tau)|^2 \omega(|\tau|) d\tau < \infty.$$

Definition 5.9. $f \in \mathcal{H}_\omega^\kappa(\mathbb{R}^n)$. $(x_0, \zeta_0) \notin WF_{\psi,\lambda}^\omega(f)$ iff

$$(\exists \varphi \in \mathcal{D}_\kappa(\mathbb{R}^n)) (\varphi(x_0) \neq 0) (\exists \Gamma(\xi_0) \subseteq (\mathbb{R}^n \setminus \{0\})) (\exists \lambda \in (0, 1))$$

$$\int_{\Gamma(\zeta_0)} \int_{\mathbb{R}^n} |W_{\psi,\lambda} \widehat{(\varphi f)}(\tau, \zeta)|^2 \omega(|\zeta|) d\tau d\zeta < \infty. \tag{19}$$

Theorem 5.10. Let $f \in \mathcal{H}_\omega^\kappa(\mathbb{R}^n)$. Then $WF^\omega(f) = WF_{\psi,\lambda}^\omega(f)$.

Proof. The proof of this theorem is contained within the proof of Theorem 5.13. \square

For the general case, we have

Definition 5.11. (Generalized Sobolev wave front set) Let $f \in \mathcal{H}_\omega^\kappa(\mathbb{R}^n)$. Then Generalized Sobolev wave front set $\mathcal{W}\mathcal{F}^\omega(f) \subseteq \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ is the complement of the set of microlocal regular points $(x_0, \zeta_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, that is, $(x_0, \zeta_0) \notin \mathcal{W}\mathcal{F}^\omega(f)$ iff

$$(\exists \varphi \in \mathcal{D}_\kappa(\mathbb{R}^n)) \quad (\varphi(x_0) \neq 0) \quad (\exists \Gamma(\zeta_0) \subseteq (\mathbb{R}^n \setminus \{0\}))$$

$$\int_{\Gamma(\zeta_0)} |\widehat{(\varphi f)}(\tau)|^2 \omega(|\tau|) d\tau < \infty.$$

Definition 5.12. $f \in \mathcal{H}_\omega^\kappa(\mathbb{R}^n)$. Consider the wavelet ψ defined in Section 5. Let $\tau \in \mathbb{R}^n$, $\zeta \in \Gamma(\zeta_0) \subseteq (\mathbb{R}^n \setminus \{0\})$ is a conic neighborhood of $\zeta_0 \in \mathbb{R}^n \setminus \{0\}$. $(x_0, \zeta_0) \notin \mathcal{W}\mathcal{F}_{\psi,\lambda}^\omega(f)$ iff

$$(\exists \varphi \in \mathcal{D}_\kappa(\mathbb{R}^n)) (\varphi(x_0) \neq 0) (\exists \Gamma(\xi_0) \subseteq \mathbb{R}^n \setminus \{0\}) (\exists \lambda \in (0, 1)) (\forall N > 0) (\exists C_N > 0)$$

$$\int_{\Gamma(\zeta_0)} \int_{\mathbb{R}^n} |W_{\psi,\lambda} \widehat{(\varphi f)}(\tau, \zeta)|^2 \omega(|\zeta|) d\tau d\zeta < \infty. \tag{20}$$

Here, we will present the result for the general case.

Theorem 5.13. Let $f \in \mathcal{H}_\omega^\kappa(\mathbb{R}^n)$. Then $\mathcal{W}\mathcal{F}_{\psi,\lambda}^\omega(f) = \mathcal{W}\mathcal{F}^\omega(f) \cap (\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$.

Proof. Assume that $(x_0, \zeta_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ does not belong to $WF^\omega(f)$. Then we have,

$$I = \int_{\Gamma(\zeta_0, \mu_0)} \int_{\mathbb{R}^n} |W_{\psi,\lambda} \widehat{(\varphi f)}(\tau, \zeta)|^2 \omega(|\zeta|) d\tau d\zeta$$

$$= (2\pi)^n \int_{\Gamma(\zeta_0, \mu_0)} \omega(|\zeta|) d\zeta \int_{\mathbb{R}^n} |\widehat{\varphi f}(\tau)|^2 \prod_{i=1}^\alpha |\zeta^i|^{-\nu_i} \hat{\psi}_\lambda^2(|\zeta|^{-2} \tilde{\theta}_\tau^{-1}(\zeta)) d\tau$$

$$= (2\pi)^n \int_{\Gamma(\zeta_0, \mu_0)} \omega(|\zeta|) d\zeta \int_{\tilde{\Gamma}(\zeta_0)} |\widehat{\varphi f}(\tau)|^2 \prod_{i=1}^\alpha |\zeta^i|^{-\nu_i} \hat{\psi}_\lambda^2(\mathcal{O}_\zeta^t(\tau)) d\tau,$$

where $\tilde{\Gamma}(\zeta_0) = \{\tau \in \mathbb{R}^n \setminus \{0\}; \exists \zeta \in \Gamma(\zeta_0, \mu_0) \text{ such that } \mathcal{O}_\zeta^t(\tau) \in \Lambda_\lambda \text{ for } i = 1, \dots, \alpha\}$, and $0 < \mu_0 < 1/2$ such that $\Gamma(\zeta_0, 2\mu_0) \subseteq \Gamma(\zeta_0)$. So it follows that,

$$I = (2\pi)^n \int_{\tilde{\Gamma}(\zeta_0)} |\widehat{\varphi f}(\tau)|^2 d\tau \int_{\Gamma(\zeta_0, \mu_0)} \omega(|\zeta|) \prod_{i=1}^{\alpha} |\zeta^i|^{-\nu_i} \hat{\psi}_\lambda^2(\mathcal{O}_\zeta^t(\tau)) d\zeta.$$

Consider the C^∞ map

$$(\zeta, \tau) \mapsto (\zeta, \mathcal{O}_\zeta^t(\tau)) = (\zeta, |\zeta|^{-2} \mathcal{O}_\zeta^{-1}(\tau)), \tilde{\Gamma}(\zeta_0) \times (\mathbb{R}^n \setminus \{0\}) \rightarrow \tilde{\Gamma}(\zeta_0) \times (\mathbb{R}^n \setminus \{0\}). \tag{21}$$

It is a bijection with the inverse given by $(\zeta, \tau) \mapsto (\zeta, |\zeta|^2 \mathcal{O}_\zeta(\omega))$; consequently, it is a diffeomorphism. By this change of variable we get,

$$I = (2\pi)^n \int_{\tilde{\Gamma}(\zeta_0)} |\widehat{\varphi f}(\tau)|^2 d\tau \int_{\Lambda'(\tau)} \omega \left(\left(\sum_{i=1}^{\alpha} \frac{|\tau^i|^2}{|\eta^i|^2} \right)^{1/2} \right) \prod_{i=1}^{\alpha} |\eta^i|^{-\nu_i} \hat{\psi}_\lambda^2(\eta) d\eta$$

where $\Lambda'(\tau) = \{\eta \in \mathbb{R}^n \setminus \{0\}; \exists \zeta \in \Gamma(\zeta_0, \mu_0) \text{ such that } \eta = (|\zeta^1|^{-2} |\tau^1| \tilde{\mathcal{O}}_{\tau^1}^{-1}(\zeta^1), \dots, |\zeta^\alpha|^{-2} |\tau^\alpha| \tilde{\mathcal{O}}_{\tau^\alpha}^{-1}(\zeta^\alpha)) \in \prod_{i=1}^{\alpha} \Lambda_{i,\lambda}\}$. From [21] Lemma 24, $\eta \in \Lambda'(\tau) \subseteq \prod_{i=1}^{\alpha} \Lambda_{i,\lambda}$ so for $\tau \in \tilde{\Gamma}(\zeta_0)$ and $\mu_0 = \mu/2, \mu_{\Lambda_\lambda} < \mu/2$ gives $\tilde{\Gamma}(\zeta_0) \subseteq \Gamma(\zeta_0, \mu)$. From Lemma 3.3, part a) we have

$$\begin{aligned} I &\leq C_1(2\pi)^n \int_{\tilde{\Gamma}(\zeta_0)} |\widehat{\varphi f}(\tau)|^2 w(|\tau|) d\tau \int_{\Lambda'(\tau)} \hat{\psi}_\lambda^2(\eta) d\eta \\ &\leq C_1(2\pi)^n \int_{\tilde{\Gamma}(\zeta_0)} |\widehat{\varphi f}(\tau)|^2 w(|\tau|) d\tau \int_{\prod_{i=1}^{\alpha} \Lambda_{i,\lambda}} \prod_{i=1}^{\alpha} \hat{\psi}_{i,\lambda}^2(\eta^i) d\eta \\ &\leq C \int_{\tilde{\Gamma}(\zeta_0)} |\widehat{\varphi f}(\tau)|^2 w(|\tau|) d\tau \\ &\leq C \int_{\Gamma(\zeta_0, r)} |\widehat{\varphi f}(\tau)|^2 w(|\tau|) d\tau \end{aligned}$$

where $C = C_1(2\pi)^n \int_{\prod_{i=1}^{\alpha} \Lambda_{i,\lambda}} \prod_{i=1}^{\alpha} \hat{\psi}_{i,\lambda}^2(\eta^i) d\eta$. Thus, the assumption $(x_0, \zeta_0) \notin \mathcal{W} \mathcal{F}^w(f)$ implies that (x_0, ζ_0) satisfy (20).

Now, let (x_0, ζ_0) satisfy (20) then, using Lemma 3.3 part b)

$$\int_{\Gamma(\zeta_0, \mu_0)} \int_{\mathbb{R}^n} |W_{\psi,\lambda}(\widehat{\varphi f})(\tau, \zeta)|^2 w(|\zeta|) d\tau d\zeta < \infty.$$

By [21] Lemma 25, there exist conic neighborhood $\tilde{\Gamma}'(\zeta_0) \subseteq \tilde{\Gamma}(\zeta_0)$ and $\prod_{j=1}^k \Lambda_j$ neighborhood of e_r such that for every $\tau \in \tilde{\Gamma}'(\zeta_0)$ we have $\prod_{j=1}^k \Lambda_j \subseteq \Lambda'(\tau)$ then

$$\begin{aligned} I &= (2\pi)^n \int_{\tilde{\Gamma}'(\zeta_0)} |\widehat{\varphi f}(\tau)|^2 d\tau \int_{\Lambda'(\tau)} w \left(\left(\sum_{i=1}^{\alpha} \frac{|\tau^i|^2}{|\eta^i|^2} \right)^{1/2} \right) \prod_{i=1}^{\alpha} |\eta^i|^{-\nu_i} \hat{\psi}_\lambda^2(\eta) d\eta \\ &\geq C \int_{\tilde{\Gamma}'(\zeta_0)} |\widehat{\varphi f}(\tau)|^2 w(|\tau|) d\tau. \end{aligned}$$

□

6. Examples

In this section we present examples using the Dirac delta distribution, finite sums of shifted deltas, and rapidly growing coefficient distributions to illustrate how the Moritoh Wavelet Transform framework characterizes the microlocal properties of ultradistributions and their generalized Sobolev regularity.

Example 6.1. Consider $w(x) = (1 + |x|^2)^s$ then the space of ultradistributions $\mathcal{H}'_\omega(\mathbb{R}^n)$ reduces to the Sobolev space $\mathcal{H}^s(\mathbb{R}^n)$. Let δ be the Dirac delta distribution: $\delta(\varphi) = \varphi(0)$ for $\varphi \in \mathcal{D}_\kappa(\mathbb{R}^2)$. We have $\varphi\delta = \varphi(0)\delta$ and hence $\widehat{\varphi\delta}(\zeta) = \varphi(0)$ for all $\zeta \in \mathbb{R}^2$. Consequently, it is trivial to conclude that $\text{supp}(\delta) = \text{sing supp}(\delta) = \{0\}$. Also, it is known that $\mathcal{WF}^\omega(\delta) = \{0\} \times (\mathbb{R}^2 \setminus \{0\})$ holds true.

We will show the latter fact by applying our results. Consider

$$\begin{aligned} I &= \int_{\Gamma(\zeta_0, \mu_0)} \int_{\mathbb{R}^2} |W_{\psi, \lambda}(\widehat{\varphi f})(\tau, \zeta)|^2 \omega(|\zeta|) d\tau d\zeta \\ &= 4\pi^2 \int_{\Gamma(\zeta_0, \mu_0)} (1 + |\zeta|^2)^s d\zeta \int_{\mathbb{R}^2} |\widehat{\varphi\delta}(\tau)|^2 |\zeta|^{-2} \hat{\psi}_\lambda^2(|\zeta|^{-2} \tilde{\mathcal{O}}_\tau^{-1}(\zeta)) d\tau \\ &= 4\pi^2 \int_{\Gamma(\zeta_0, \mu_0)} (1 + |\zeta|^2)^s d\zeta \int_{\tilde{\Gamma}(\zeta_0)} |\varphi(0)|^2 |\zeta|^{-2} \hat{\psi}_\lambda^2(|\zeta|^{-2} \tilde{\mathcal{O}}_\tau^{-1}(\zeta)) d\tau \end{aligned}$$

where

$$\tilde{\Gamma}(\zeta_0) = \{\tau \in \mathbb{R}^2 \setminus \{0\}; \exists \zeta \in \Gamma(\zeta_0, \mu_0) \text{ such that } |\zeta|^{-2} |\tau| \tilde{\mathcal{O}}_\tau^{-1}(\zeta) \in \Lambda_\lambda\} \tag{22}$$

so it follows that,

$$\begin{aligned} I &= 4\pi^2 \int_{\tilde{\Gamma}(\zeta_0)} |\varphi(0)|^2 d\tau \int_{\Gamma(\zeta_0, \mu_0)} (1 + |\zeta|^2)^s |\zeta|^{-2} \hat{\psi}_\lambda^2(|\zeta|^{-2} \tilde{\mathcal{O}}_\tau^{-1}(\zeta)) d\zeta \\ &= 4\pi^2 \int_{\tilde{\Gamma}(\zeta_0)} |\varphi(0)|^2 d\tau \int_{\Lambda'(\tau)} \left(1 + \frac{|\tau|^2}{|\eta|^2}\right)^s |\eta|^{-2} \hat{\psi}_\lambda^2(\eta) d\eta. \end{aligned}$$

where we use the change of variable as $\eta = |\zeta|^{-2} \tilde{\mathcal{O}}_\tau^{-1}(\zeta)$ and

$$\Lambda'(\tau) = \{\eta \in \mathbb{R}^2 \setminus \{0\}; \exists \zeta \in \Gamma(\zeta_0, \mu_0) \text{ such that } \eta = |\zeta|^{-2} \tilde{\mathcal{O}}_\tau^{-1}(\zeta) \in \Lambda_\lambda\}. \tag{23}$$

Now since $\hat{\psi}$ is of compact support $\Lambda \subseteq B_\mu(0, 1)$, $0 < \mu < 1$,

$$\left(1 + \frac{|\tau|^2}{|\eta|^2}\right)^s |\eta|^{-2} \approx (1 + |\tau|^2)^s \text{ for } \eta \in \Lambda.$$

and since from $\Lambda' \subseteq \Lambda'(\tau) \subseteq \Lambda_\lambda \subseteq \Lambda$,

$$\left(1 + \frac{|\tau|^2}{|\eta|^2}\right)^s |\eta|^{-2} \approx (1 + |\tau|^2)^s \text{ for } \eta \in \Lambda_\lambda$$

we have,

$$\begin{aligned} I &\approx 4\pi^2 \int_{\tilde{\Gamma}(\zeta_0)} |\varphi(0)|^2 (1 + |\tau|^2)^s d\tau \int_{\Lambda'(\tau)} \hat{\psi}_\lambda^2(\eta) d\eta \\ &= 4\pi^2 C^* |\varphi(0)|^2 \int_{\tilde{\Gamma}(\zeta_0)} (1 + |\tau|^2)^s d\tau \\ &= C_1^* \int_{\tilde{\Gamma}(\zeta_0)} (1 + |\tau|^2)^s d\tau. \end{aligned}$$

Thus, $\mathcal{WF}^\omega(\delta) = \emptyset$ for $s < -1$, and $\mathcal{WF}^\omega(\delta) = \{0\} \times (\mathbb{R}^2 \setminus \{0\})$ for $s \geq -1$.

Remark 6.2. Consider the distribution

$$f(x) = \sum_{m=0}^k \delta(x + 2m - k),$$

supported at the points $x = k - 2m$, for $m = 0, \dots, k, k \in \mathbb{N}$. Then, if the weight function is $w(x) = (1 + |x|^2)^{-2m}$, from Example 6.1 we have

$$\mathcal{WF}^w(f) = \begin{cases} \bigcup_{m=0}^k \{(k - 2m, \xi) : \xi \in \mathbb{R} \setminus \{0\}\}, & \text{if } m \leq \frac{1}{2}, \\ \emptyset, & \text{if } m > \frac{1}{2}. \end{cases}$$

Remark 6.3. In general wave front set of $f = g + h$ need not to be union of Wave front set of g and h but when singular support of g and h are isolated then Wave front set of f become union of Wave front set of g and h (see[9]).

Example 6.4. Let $f(x) = \sum_{\alpha} a_{\alpha} \delta^{(\alpha)}(x - \alpha)$, $a_{\alpha} \in \mathbb{C}$ and $w(x) = e^{-\epsilon'|x|^d}$, $0 < d < 1$ such that for each $\epsilon > 0$, \exists constant $c_{\epsilon} > 0$ such that $|a_{\alpha}| \leq c_{\epsilon} \cdot \epsilon^{|\alpha|} (\alpha!)^{-1/d}$. The formed space of ultradistributions $\mathcal{H}_{\omega}^{\epsilon}(\mathbb{R}^n)$ contains ultradistribution which does not belong to the Sobolev space $\mathcal{H}^s(\mathbb{R}^n)$, [22], and also $|\hat{f}(\zeta)| \leq D_{\epsilon} \exp[\epsilon |\zeta|^d]$. Applying or results for generalized Sobolev wave front set of f , we have

$$\begin{aligned} I &= \int_{\Gamma(\zeta_0, \mu)} |\widehat{\varphi f}(\zeta)|^2 w(|\zeta|) d\zeta \\ &= \int_{\Gamma(\zeta_0, \mu)} |\varphi \cdot \sum_{\alpha} \widehat{a_{\alpha} \delta^{(\alpha)}}(x - \alpha)(\zeta)|^2 e^{-\epsilon' |\zeta|^d} d\zeta \\ &= \int_{\Gamma(\zeta_0, \mu)} |\sum_{\alpha} a_{\alpha} \{\widehat{\varphi \cdot \delta^{(\alpha)}}(x - \alpha)(\zeta)\}|^2 e^{-\epsilon' |\zeta|^d} d\zeta \\ &= \int_{\Gamma(\zeta_0, \mu)} |\sum_{\alpha} a_{\alpha} \{\sum_{j=0}^{\alpha} c_j \zeta^{\alpha-j} \varphi^j(\alpha) \cdot e^{-i\zeta\alpha}\}|^2 e^{-\epsilon' |\zeta|^d} d\zeta \quad \text{for some } c_j \in \mathbb{C} \\ &= \int_{\Gamma(\zeta_0, \mu)} |\sum_{j=0}^{\alpha} c_{j,\alpha} \zeta^{\alpha-j} \varphi^j(\alpha) \cdot e^{-i\zeta\alpha}|^2 e^{-\epsilon' |\zeta|^d} d\zeta \quad \text{for some } c_{j,\alpha} \in \mathbb{C} \\ &= \int_{\Gamma(\zeta_0, \mu)} |\sum_{j=0}^{\alpha} c_{j,\alpha} \zeta^{\alpha-j} \varphi^j(\alpha)|^2 e^{-\epsilon' |\zeta|^d} d\zeta \\ &= \int_{\Gamma(\zeta_0, \mu)} |\sum_{j_1 \in \{0, 1, \dots, \alpha\}} c_{j_1, \alpha} \zeta^{\alpha-j_1} \varphi^{j_1}(\alpha) \\ &\quad + i \sum_{j_2 \in \{0, 1, \dots, \alpha\}} c_{j_2, \alpha} \zeta^{\alpha-j_2} \varphi^{j_2}(\alpha)|^2 e^{-\epsilon' |\zeta|^d} d\zeta \quad \text{for some } c_{j,\alpha} \in \mathbb{R} \\ &= \int_{\Gamma(\zeta_0, \mu)} \{ |\sum_{j_1 \in \{0, 1, \dots, \alpha\}} c_{j_1, \alpha} \zeta^{\alpha-j_1} \varphi^{j_1}(\alpha)|^2 \\ &\quad + |\sum_{j_2 \in \{0, 1, \dots, \alpha\}} c_{j_2, \alpha} \zeta^{\alpha-j_2} \varphi^{j_2}(\alpha)|^2 \} e^{-\epsilon' |\zeta|^d} d\zeta \\ &= \int_{\Gamma(\zeta_0, \mu)} \{ |\sum_{j_1 \in \{0, 1, \dots, \alpha\}} c_{j_1, \alpha} \zeta^{-j_1} \varphi^{j_1}(\alpha)|^2 \\ &\quad + |\sum_{j_2 \in \{0, 1, \dots, \alpha\}} c_{j_2, \alpha} \zeta^{-j_2} \varphi^{j_2}(\alpha)|^2 \} |\zeta|^{2\alpha} e^{-\epsilon' |\zeta|^d} d\zeta. \end{aligned}$$

From [22, p. 118], $|\zeta|^{2\alpha} \leq D_{\epsilon,\alpha} \exp[2\epsilon |\zeta|^d]$ then we have,

$$I \leq \int_{\Gamma(\zeta_0,\mu)} \left\{ \left| \sum_{j_1 \in \{0,1,\dots,\alpha\}} c_{j_1,\alpha} \zeta^{-j_1} \varphi^{j_1}(\alpha) \right|^2 + \left| \sum_{j_2 \in \{0,1,\dots,\alpha\}} c_{j_2,\alpha} \zeta^{-j_2} \varphi^{j_2}(\alpha) \right|^2 \right\} D_{\epsilon,\alpha} e^{-(\epsilon'-2\epsilon)|\zeta|^d} d\zeta < \infty$$

if $\epsilon' > 2\epsilon$ for each $\epsilon > 0$. Now consider,

$$\begin{aligned} J &= \int_{\Gamma(\zeta_0,\mu_0)} \int_{\mathbb{R}^2} |W_{\psi,\lambda}(\varphi f)(\tau,\zeta)|^2 w(|\zeta|) d\tau d\zeta \\ &= 4\pi^2 \int_{\Gamma(\zeta_0,\mu_0)} e^{-\epsilon'|\zeta|^d} d\zeta \int_{\mathbb{R}^2} |\varphi \cdot \sum_{\alpha} \widehat{a_{\alpha}\delta^{(\alpha)}}(x-\alpha)(\tau)|^2 |\zeta|^{-2} \hat{\psi}_{\lambda}^2(|\zeta|^{-2} \tilde{\mathcal{O}}_{\tau}^{-1}(\zeta)) d\tau \\ &= 4\pi^2 \int_{\Gamma(\zeta_0,\mu_0)} e^{-\epsilon'|\zeta|^d} d\zeta \int_{\tilde{\Gamma}(\zeta_0)} |\varphi \cdot \sum_{\alpha} \widehat{a_{\alpha}\delta^{(\alpha)}}(x-\alpha)(\tau)|^2 |\zeta|^{-2} \hat{\psi}_{\lambda}^2(|\zeta|^{-2} \tilde{\mathcal{O}}_{\tau}^{-1}(\zeta)) d\tau \\ &= 4\pi^2 \int_{\tilde{\Gamma}(\zeta_0)} |\varphi \cdot \sum_{\alpha} \widehat{a_{\alpha}\delta^{(\alpha)}}(x-\alpha)(\tau)|^2 d\tau \int_{\Gamma(\zeta_0,\mu_0)} e^{-\epsilon'|\zeta|^d} |\zeta|^{-2} \hat{\psi}_{\lambda}^2(|\zeta|^{-2} \tilde{\mathcal{O}}_{\tau}^{-1}(\zeta)) d\zeta \\ &= 4\pi^2 \int_{\tilde{\Gamma}(\zeta_0)} |\varphi \cdot \sum_{\alpha} \widehat{a_{\alpha}\delta^{(\alpha)}}(x-\alpha)(\tau)|^2 d\tau \int_{\Lambda'(\tau)} e^{-\epsilon' \frac{|\tau|^d}{|\eta|^d}} |\eta|^{-2} \hat{\psi}_{\lambda}^2(\eta) d\eta, \end{aligned}$$

where $\tilde{\Gamma}(\zeta_0)$ and $\Lambda'(\tau)$ are as in (22) and (23) accordingly.

Now since $\hat{\psi}$ is of compact support $\Lambda \subseteq B_{\mu}(0,1)$, $0 < \mu < 1$,

$$e^{-\epsilon' \frac{|\tau|^d}{|\eta|^d}} |\eta|^{-2} \approx e^{-\epsilon'|\tau|^d} \text{ for } \eta \in \Lambda.$$

Since $\Lambda' \subseteq \Lambda'(\tau) \subseteq \Lambda_{\lambda} \subseteq \Lambda$,

$$e^{-\epsilon' \frac{|\tau|^d}{|\eta|^d}} |\eta|^{-2} \approx e^{-\epsilon'|\tau|^d} \text{ for } \eta \in \Lambda_{\lambda}$$

then,

$$\begin{aligned} I &\approx 4\pi^2 \int_{\tilde{\Gamma}(\zeta_0)} |\varphi \cdot \sum_{\alpha} \widehat{a_{\alpha}\delta^{(\alpha)}}(x-\alpha)(\tau)|^2 e^{-\epsilon'|\tau|^d} d\tau \int_{\Lambda'(\tau)} \hat{\psi}_{\lambda}^2(\eta) d\eta \\ &= 4\pi^2 C^* \int_{\tilde{\Gamma}(\zeta_0)} |\varphi \cdot \sum_{\alpha} \widehat{a_{\alpha}\delta^{(\alpha)}}(x-\alpha)(\tau)|^2 e^{-\epsilon'|\tau|^d} d\tau \\ &= C_1^* \int_{\tilde{\Gamma}(\zeta_0)} |\varphi \cdot \sum_{\alpha} \widehat{a_{\alpha}\delta^{(\alpha)}}(x-\alpha)(\tau)|^2 e^{-\epsilon'|\tau|^d} d\tau. \end{aligned}$$

Therefore,

$$\mathcal{W} \mathcal{F}^w(f) = \begin{cases} \{\cup_{\alpha} \alpha\} \times (\mathbb{R}^2 \setminus \{0\}), & \text{if } \epsilon' \leq 0 \\ \emptyset, & \text{if } \epsilon' > 0. \end{cases}$$

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No potential conflict of interest is reported by the author(s).

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