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New inequalities involving the q-numerical radius of Hilbert space operators

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Abstract. In this article, we have established several new inequalities involving the q-numerical radius for operators and 2×2 operator matrices. By utilizing the Buzano inequality, we derived additional bounds that further extend the existing results in the literature. Comparative analysis with existing results is presented to emphasize the generality of the newly obtained inequalities.

1. Introduction

The q-numerical radius is a generalization of the numerical radius, introduced to facilitate the study of bounded linear operators with respect to different norms or scaling factors. By varying the value of q, it allows a more flexible approach for analyzing the operator's behavior, particularly in spaces or settings where the standard numerical radius is not sufficient. It can be used to explore different spectral properties of operators, particularly when analyzing the operator in non-traditional contexts, such as those involving q-norms or the spaces that are not strictly Hilbert spaces. The value of q significantly influences the behaviour of the numerical radius. For $q > 1$, the q-numerical radius, $w_q(T)$ tends to be larger than the classical numerical radius, indicating a broader representation of the operator's action. Conversely, for $0 < q < 1$, $w_q(T)$ is smaller, offering a more concentrated measure of the operator's behavior. This variability makes the q-numerical radius a powerful analytical tool in operator theory.

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. Let $\mathcal{L}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators from $\mathcal{H}$ into itself. An operator $S \in \mathcal{L}(\mathcal{H})$ is said to be positive, and denoted $S \geq 0$, if $\langle Sx, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and is called positive definite, denoted $S > 0$, if $\langle Sx, x\rangle > 0$ for all non zero vectors $x \in \mathcal{H}$. The *numerical range* of $S \in \mathcal{L}(\mathcal{H})$ is defined as $W(S) = \{\langle Sx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$ and the *numerical radius* of S, denoted by $w(S)$, is defined by $w(S) = \sup\{|z| : z \in W(S)\}$. It is known that the

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set $W(S)$ is a convex subset of the complex plane and that the numerical radius $w(\cdot)$ is a norm on $\mathcal{L}(\mathcal{H})$; being equivalent to the usual operator norm $\|S\| = \sup\{\|Sx\| : x \in \mathcal{H}, \|x\| = 1\}$. In fact, for every $S \in \mathcal{L}(\mathcal{H})$,

$$\frac{1}{2}\|S\| \leq w(S) \leq \|S\|. \tag{1}$$

These inequalities show that the norms $w(\cdot)$ and $\|\cdot\|$ are equivalent. It has been of great interest in the literature to sharpen the inequalities (1). It is a difficult task to provide a comprehensive list of contributions, so we mention just a few of the recent articles [2, 3, 25–29].

The inequalities in (1) are sharp. If $S^2 = 0$, then the first inequality becomes an equality, on the other hand, the second inequality becomes an equality if S is normal. In fact, for a nilpotent operator T with $S^n = 0$, Haagerup and Harpe [10] showed that $w(S) \leq \|S\| \cos(\pi/(n+1))$. In particular, when $n = 2$, we get the reverse inequality of the first inequality in (1). The numerical radius has some significant properties, such as the power inequality:

$$w(S^n) \leq w^n(S) \quad \text{for } n = 1, 2, \ldots. \tag{2}$$

For basic information about numerical radius, one can refer [9]. The author of [14, 16] improved the inequality (1) which is stated next. If $S \in \mathcal{L}(\mathcal{H})$, then

$$w(S) \leq \frac{1}{2}\left\||S| + |S^*|\right\| \leq \frac{1}{2}\left(\|S\| + \|S^2\|^{1/2}\right), \tag{3}$$

where $|S| = (S^*S)^{1/2}$ is the absolute value of S, and

$$\frac{1}{4}\left\|S^*S + SS^*\right\| \leq w^2(S) \leq \frac{1}{2}\left\|S^*S + SS^*\right\|. \tag{4}$$

The inequalities in (3) refines the second inequality in (1). For applications of these inequalities, one can refer [14, 15].

If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $T \in \mathcal{L}(\mathcal{H})$, then T can be written as a block-matrix [12]

$$T = \begin{bmatrix} I_1^*TI_1 & I_1^*TI_2 \\ I_2^*TI_1 & I_2^*TI_2 \end{bmatrix}, \tag{5}$$

where $I_j \in \mathcal{L}(\mathcal{H}_j, \mathcal{H})$, and $I_j(x) = x$.

If $T, S \in \mathcal{L}(\mathcal{H})$, then

$$w\left(\begin{bmatrix} T & O \\ O & S \end{bmatrix}\right) = \max\{w(T), w(S)\}, \tag{6}$$

and

$$\left\|\begin{bmatrix} T & O \\ O & S \end{bmatrix}\right\| = \left\|\begin{bmatrix} O & T \\ S & O \end{bmatrix}\right\| = \max\{\|T\|, \|S\|\}. \tag{7}$$

In a similar way, the q-numerical range is defined by

$$W_q(S) = \{\langle Sx, y\rangle : x, y \in \mathcal{H}, \|x\| = \|y\| = 1, \langle x, y\rangle = q\},$$

while the *q-numerical radius* is defined by

$$w_q(S) = \sup\{|z| : z \in W_q(S)\}. \tag{8}$$

One can observe that the q-numerical radius is a generalization of the classical numerical radius, for $|q| = 1$. This observation follows from the fact that equality would have to hold in the Cauchy-Schwarz inequality

$|q| = |\langle x, y\rangle| \le \|x\|\|y\| = 1$, provided $|q| = 1$. It is clear that $y = \lambda x$ would need to be true for some $\lambda \in \mathbb{C}, |\lambda| = 1$, hence $|\langle Sx, y\rangle| = |\langle Sx, x\rangle|$.

In 1977, Marcus and Andresen [20] introduced the notion of q-numerical range in the context of an n-dimensional unitary space equipped with an inner product. They showed that, for $q \in \mathbb{C}$ with $q \le 1$, the set generated by rotating the q-numerical range about the origin forms an annulus in the complex plane. Furthermore, they determined explicit expressions for the inner and outer radii of this annulus in the case of Hermitian operators. In 1984, Nam-Kiu Tsing [31], established the convexity of the q-numerical range. In 1994, C. K. Li *et al.* [18] investigated several elementary properties for the q-numerical range on finite-dimensional spaces, see also [19]. In 2002, M. T. Chien and H. Nakazato [5] described the boundary of the q-numerical range of a square matrix using its Davis-Wielandt shell. In 2005, R. Rajic [24] considered a generalization of the q-numerical range. In 2007, M. T. Chien and H. Nakazato [6] further investigated the q-numerical radius of weighted unilateral and bilateral shift operators. In particular, they computed the q-numerical radius of shift operators with periodic weight sequences. In 2012, M. T. Chien, [4] investigated the q-numerical radius of a weighted shift operator whose weights follow either a geometric sequence or a periodic sequence. Recently, Moghaddam *et al.* [21] established several upper and lower bounds for the q-numerical radius of bounded linear operators which generalize some classical numerical radius inequalities. In addition, they provided illustrative examples to demonstrate the sharpness of their bounds for various values of $q \in (0, 1)$. Very recently, Stankovic et al. [30] investigated various properties for the q-numerical radius with a theoretical perspective and presented an improved version of some earlier results established in [21]. The earlier results are proved for $q \in [0, 1]$, while in [30], some of the results are established with $q \in \overline{\mathbb{D}}$, the closed unit disc in $\mathbb{C}$. Also, they established several q-numerical radius inequalities for operator matrices defined on the direct sum of Hilbert spaces. Further generalization and refinements of q-numerical radius inequalities, one can see the recent articles [13, 17, 23]. Motivated by the work of several authors, we have presented the q-numerical radius of some special type of operators and operator matrices.

The primary objective of this article is to investigate the generalized numerical radius of operators and operator matrices, with particular focus on the q-numerical radius. The study aims to establish new inequalities and explore analytical properties that may encourage further research in this area. The remainder of the article is organized as follows: In Section 2, we derive several inequalities concerning the q-numerical radius for both individual operators and operator matrices using the Buzano inequality. These results are then compared with corresponding inequalities for the classical numerical radius to highlight their significance and potential advantages. In order to make progress our research work, we need the following lemmas to prove our results. The very first lemma represents a consequence of the results obtained in [8, Proposition 3.1], as well as (8).

Lemma 1.1. *Let $T, S \in \mathcal{L}(\mathcal{H}), q \in \bar{\mathbb{D}}$ and $\lambda \in \mathbb{C}$. Then we have following properties:*

(i) $w_q(\lambda T) = |\lambda| w_q(T)$.
(ii) $w_q(T + S) \le w_q(T) + w_q(S)$.
(iii) $w_q(U^*TU) = w_q(T)$, *where $U \in \mathcal{B}(\mathcal{H})$ is an unitary operator.*
(iv) $w_{\lambda q}(T) = w_q(T)$ *for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.*

Lemma 1.2. [30, Theorem 1.4] *If $T \in \mathcal{L}(\mathcal{H})$ and $q \in \overline{\mathbb{D}}$, then*

$$\frac{|q|}{2}\|T\| \le w_q(T) \le \|T\|.$$

It is well known that, for $T_1, T_2 \in \mathcal{L}(H)$, $w\left(\begin{bmatrix} T_1 & O \\ O & T_2 \end{bmatrix}\right) = \max\{w(T_1), w(T_2)\}$, but the same result does not hold for q-numerical radius of the operator matrix [21]. In the forthcoming lemmas, we have collected some results for q-numerical radius of 2×2 and $n \times n$ operator matrices.

Lemma 1.3. [30, Lemma 5.2] *Let $T_1, T_2 \in \mathcal{L}(\mathcal{H}), q \in \bar{\mathbb{D}}$ and $\theta \in \mathbb{R}$. Then*

(i) $w_q\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) = w_q\left(\begin{bmatrix} O & T_2 \\ T_1 & O \end{bmatrix}\right)$.

(ii) $w_q\left(\begin{bmatrix} O & T_1 \\ e^{i\theta}T_2 & O \end{bmatrix}\right) = w_q\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right)$ *for any* $\theta \in \mathbb{R}$.

(iii) $w_q\left(\begin{bmatrix} T_1 & O \\ O & T_2 \end{bmatrix}\right) = w_q\left(\begin{bmatrix} T_2 & O \\ O & T_1 \end{bmatrix}\right)$.

The classical numerical radius version of pinching type inequalities has been established in [1]. Using unitarily equivalent condition for the q-numerical radius, one can deduce the following result.

Lemma 1.4. [23] *Let* $T_1, T_2, T_3, T_4 \in \mathcal{L}(\mathcal{H})$. *Then*

(i) $w_q\left(\begin{bmatrix} T_1 & O \\ O & T_4 \end{bmatrix}\right) \leq w_q\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right)$.

(ii) $w_q\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq w_q\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right)$.

The following result is from [30, Theorem 1.5]

Lemma 1.5. [30, Theorem 1.5] *Let* $(\mathcal{H}_n)_{n\in\mathbf{N}}$ *be a sequence of Hilbert spaces and let* $T_n \in \mathcal{L}(\mathcal{H}_n)$ *for all* $n \in \mathbf{N}$. *If* $q \in \bar{\mathbb{D}} \setminus \{0\}$. *Then*

$$\sup_{n\in\mathbf{N}} w_q(T_n) \leq w_q\left(\bigoplus_{n=1}^{+\infty} T_n\right) \leq \frac{|q| + 2\sqrt{1-|q|^2}}{|q|} \sup_{n\in\mathbf{N}} w_q(T_n).$$

As a special case of the above result we have the following.

Lemma 1.6. [30, Corollary 5.1] *Let* $T, S \in \mathcal{L}(\mathcal{H})$ *and* $q \in (0, 1]$. *Then*

$$\max\{w_q(T), w_q(S)\} \leq w_q\left(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}\right) \leq \frac{q + 2\sqrt{1-q^2}}{q} \max\{w_q(T), w_q(S)\}.$$

The next result deals with the power inequality for q-numerical radius of an operator.

Lemma 1.7. *Let* $T \in \mathcal{L}(\mathcal{H})$ *and* $q \in \overline{\mathbb{D}}$. *Then*

$$|q|^{n-1} w_q(T^n) \leq w_q^n(T) \text{ for all } n \in \mathbb{N}. \tag{9}$$

Lemma 1.8. [22] *Let* $0 \leq q \leq 1$ *and* $T \in \mathcal{M}_2(\mathbb{C})$. *Then* T *is unitarily similar to* $e^{it}\begin{bmatrix} \gamma & a \\ b & \gamma \end{bmatrix}$ *for some* $0 \leq t \leq 2\pi$ *and* $0 \leq b \leq a$. *Also,*

$$W_q(T) = e^{it}\left\{\gamma q + r\left((c + pd)\cos(s) + i(d + pc)\sin(s)\right) : 0 \leq r \leq 1, 0 \leq s \leq 2\pi\right\},$$

with $c = \frac{a+b}{2}$, $d = \frac{a-b}{2}$ *and* $p = \sqrt{1-q^2}$.

2. Bounds for q-numerical radius

In this section, we obtain some bounds for q-numerical radii of operators using Buzano inequality. Further, we have investigated the q-numerical radius inequalities for commutators of positive operators. To prove our results, the following key lemmas are essential for our purpose.

Lemma 2.1. [11, Lemma 2.1] *Let* $a, b \in \mathcal{H}$ *and* $t \in \mathbb{R}$, *then*

$$\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2 \leq \|a\|^2\|b - ta\|^2. \tag{10}$$

Lemma 2.2. [7] *Let $a, b, c \in \mathcal{H}$ and $\|c\| = 1$, then*

$$\|a\|\|b\| \geq |\langle a, b\rangle - \langle a, c\rangle\langle c, b\rangle| + |\langle a, c\rangle\langle c, b\rangle| \geq |\langle a, b\rangle|. \tag{11}$$

By the first inequality in (11), *one can deduce*

$$\|a\|\|b\| + |\langle a, b\rangle| \geq 2|\langle a, c\rangle\langle c, b\rangle|. \tag{12}$$

This inequality is widely known as Buzano inequality.

Theorem 2.3 describes a relation between q-numerical radius and the classical numerical radius of an operator.

Theorem 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ and $q \in (0, 1]$. Then*

$$w_q^2(T) \leq w(T^2) + \frac{1}{|q|^2} \inf_{|\lambda| \neq 0} \|\lambda T \pm T^*\|^2. \tag{13}$$

Proof. Let $b = T^*x, a = \lambda Tx$ and $t = 1$, where $x \in \mathcal{H}, \|x\| = 1$ in (10), we have

$$\begin{aligned} |\lambda|^2\|Tx\|^2\|T^*x\|^2 &\leq |\langle \lambda Tx, T^*x\rangle|^2 + \|\lambda Tx\|^2\|T^*x - \lambda Tx\|^2 \\ &= |\langle \lambda T^2x, x\rangle|^2 + |\lambda|^2\|Tx\|^2\|T^*x - \lambda Tx\|^2 \\ &= |\lambda|^2|\langle T^2x, x\rangle|^2 + |\lambda|^2\|Tx\|^2\|\lambda Tx - T^*x\|^2. \end{aligned}$$

Taking supremum over $x \in \mathcal{H},\ \|x\| = 1$, we get

$$\|T\|^2\|T^*\|^2 \leq w^2(T^2) + \|T\|^2\|\lambda T - T^*\|^2. \tag{14}$$

On the other hand put $a = Ty,\ b = T^*y,\ c = x$, in (12), we have

$$\|Ty\|\|T^*y\| + |\langle Ty, T^*y\rangle| \geq 2|\langle Ty, x\rangle\langle x, T^*y\rangle|.$$

Equivalently, it can be written as

$$2|\langle Ty, x\rangle\langle x, T^*y\rangle| \leq |\langle T^2y, y\rangle| + \|Ty\|\|T^*y\|.$$

So, we get

$$2|\langle T^*x, y\rangle\langle Tx, y\rangle| \leq |\langle T^2y, y\rangle| + \|Ty\|\|T^*y\|,$$

which implies

$$2w_q(T^*)w_q(T) \leq |\langle T^2y, y\rangle| + \|Ty\|\|T^*y\|.$$

Since, $w_q(T) = w_q(T^*)$ and $0 < q \leq 1$, we have

$$2w_q^2(T) \leq w(T^2) + \|T\|\|T^*\|.$$

Using (14), we have

$$(2w_q^2(T) - w(T^2))^2 \leq w^2(T^2) + \|T^2\|\|\lambda T - T^*\|^2,$$

which is same as

$$4w_q^4(T) + w^2(T^2) - 4w_q^2(T)w(T^2) \leq w^2(T^2) + \|T^2\|\|\lambda T - T^*\|^2.$$

This implies that

$$4w_q^2(T)\Big(w_q^2(T) - w(T^2)\Big) \leq \frac{4}{|q|^2} w_q^2(T)\|\lambda T - T^*\|^2 \text{ (by Lemma 1.2)}$$

which implies

$$w_q^2(T) - w(T^2) \le \frac{1}{|q|^2}\|\lambda T - T^*\|^2.$$

Taking infimum over $\lambda \in \mathbb{C} \setminus \{0\}$, gives

$$w_q^2(T) - w(T^2) \le \frac{1}{|q|^2}\inf_{|\lambda|\neq 0}\|\lambda T - T^*\|^2. \tag{15}$$

Now, replacing T by iT in (15), one can get

$$w_q^2(T) - w(T^2) \le \frac{1}{|q|^2}\inf_{|\lambda|\neq 0}\|\lambda T + T^*\|^2. \tag{16}$$

The result follows from (15) and (16). □

Remark 2.4. *Taking $q \to 1$ in Theorem 2.3, we have*

$$w^2(T) - w(T^2) \le \inf_{|\lambda|\neq 0}\|\lambda T \pm T^*\|^2,$$

which is already established in [12, Theorem 2.1].

Theorem 2.5 provides an upper bound for q-numerical radius of a 2×2 off-diagonal operator matrix.

Theorem 2.5. *Let $T, S \in \mathcal{L}(\mathcal{H})$ and $q \in (0, 1]$. Then*

$$w_q\left(\begin{bmatrix} O & T \\ S & O \end{bmatrix}\right) \le \sqrt{\max(w(TS), w(ST)) + \frac{1}{|q|^2}\|T \pm S^*\|^2}.$$

Proof. Using $\lambda = 1$ in (13), we have

$$\begin{aligned} w_q^2\left(\begin{bmatrix} O & T \\ S & O \end{bmatrix}\right) &\le w\left(\begin{bmatrix} O & T \\ S & O \end{bmatrix}^2\right) + \frac{1}{|q|^2}\left\|\begin{bmatrix} O & T \\ S & O \end{bmatrix} \pm \begin{bmatrix} O & S^* \\ T^* & O \end{bmatrix}\right\|^2 \\ &= w\left(\begin{bmatrix} TS & O \\ O & ST \end{bmatrix}\right) + \frac{1}{|q|^2}\left\|\begin{bmatrix} O & T \pm S^* \\ S \pm T^* & O \end{bmatrix}\right\|^2. \end{aligned}$$

Since $\|T \pm S^*\| = \|S \pm T^*\|$ and using equalities (6) and (7), we have

$$w_q^2\left(\begin{bmatrix} O & T \\ S & O \end{bmatrix}\right) \le \max(w(TS), w(ST)) + \frac{1}{|q|^2}\|T \pm S^*\|^2.$$

□

Remark 2.6. *When $q \to 1$ in Theorem 2.5, we have*

$$w^2\left(\begin{bmatrix} O & T \\ S & O \end{bmatrix}\right) \le \max(w(TS), w(ST)) + \|T \pm S^*\|^2,$$

which is obtained by earlier authors in [12, Theorem 2.2].

Lemma 2.7. [30, Theorem 5.5] *Let $T, S \in \mathcal{L}(\mathcal{H})$ and $q \in (0, 1]$. Then*

$$\begin{aligned} \frac{1}{2}\max\{w_q(T+S), w_q(T-S)\} &\le w_q\left(\begin{bmatrix} O & T \\ S & O \end{bmatrix}\right) \\ &\le \frac{q + 2\sqrt{1-q^2}}{2q}\{w_q(T+S) + w_q(T-S)\}. \end{aligned}$$

Example 2.8. *Consider* $T = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, *and* $q \in (0, 1]$. *Then, Theorem 2.5 gives* $w_q\left(\begin{bmatrix} O & T \\ S & O \end{bmatrix}\right) \leq \sqrt{6 + \frac{1}{q^2}}$, *whereas the right hand inequality of Lemma 2.7 gives* $w_q\left(\begin{bmatrix} O & T \\ S & O \end{bmatrix}\right) \leq 3(q + 2\sqrt{1 - q^2})$. *One can check that the bound obtained in Theorem 2.5 is better than the upper bound obtained in Lemma 2.7. We have calculated the bounds for different values of q in the following table.*

Expression	**Value for** $q = 0.5$	**Value for** $q = 1$	**Results**
$\sqrt{6 + \frac{1}{q^2}}$	3.16227	2.64575	Theorem 2.5
$3(q + 2\sqrt{1 - q^2})$	6.69615	3	Lemma 2.7

Lemma 2.9. [30, Lemma 5.3] *If* $T, S \in \mathcal{L}(\mathcal{H})$, $q \in (0, 1]$. *Then*

$$\max\{w_q(T + S), w_q(T - S)\} \leq w_q\left(\begin{bmatrix} T & S \\ S & T \end{bmatrix}\right) \leq \frac{q + 2\sqrt{1 - q^2}}{q} \max\{w_q(T + S), w_q(T - S)\}. \tag{17}$$

In particular,

$$w_q(T) \leq w_q\left(\begin{bmatrix} O & T \\ T & O \end{bmatrix}\right) \leq \frac{q + 2\sqrt{1 - q^2}}{q} w_q(T). \tag{18}$$

In the forthcoming result, we present an inequality related to q-numerical radius for the product and sum of operators in $\mathcal{L}(\mathcal{H})$.

Theorem 2.10. *If* $P, Q, R \in \mathcal{L}(\mathcal{H})$ *and* $q \in (0, 1]$. *Then*

$$w_q(PR \pm R^*Q) \leq 2\sqrt{\max(w(PRR^*Q), w(R^*QPR)) + \frac{1}{|q|^2}\|PR \pm Q^*R\|^2}. \tag{19}$$

Proof. Using (18) and Lemma 1.3, we have

$$\begin{aligned} w_q(PR + R^*Q) &\leq w_q\left(\begin{bmatrix} O & PR + R^*Q \\ PR + R^*Q & O \end{bmatrix}\right) \\ &= w_q\left(\begin{bmatrix} O & R^*Q \\ PR & O \end{bmatrix} + \begin{bmatrix} O & PR \\ R^*Q & O \end{bmatrix}\right) \\ &\leq w_q\left(\begin{bmatrix} O & R^*Q \\ PR & O \end{bmatrix}\right) + w_q\left(\begin{bmatrix} O & PR \\ R^*Q & O \end{bmatrix}\right) \\ &= w_q\left(\begin{bmatrix} O & PR \\ R^*Q & O \end{bmatrix}\right) + w_q\left(\begin{bmatrix} O & PR \\ R^*Q & O \end{bmatrix}\right) \\ &= 2w_q\left(\begin{bmatrix} O & PR \\ R^*Q & O \end{bmatrix}\right). \end{aligned}$$

So, we have

$$w_q(PR + R^*Q) \leq 2w_q\left(\begin{bmatrix} O & PR \\ R^*Q & O \end{bmatrix}\right). \tag{20}$$

By Theorem 2.5, we have

$$w_q(PR + R^*Q) \leq 2\sqrt{\max(w(PRR^*Q), w(R^*QPR)) + \frac{1}{|q|^2}\|PR \pm Q^*R\|^2}.$$

Replacing R by iR in the above inequality, we have

$$w_q(PR - R^*Q) \leq 2\sqrt{\max(w(PRR^*Q), w(R^*QPR)) + \frac{1}{|q|^2}\|PR \pm Q^*R\|^2}.$$

From the above two inequalities, we get the required result. □

Corollary 2.11. *Let $P, Q, U \in \mathcal{L}(\mathcal{H})$, $q \in (0,1]$ and U be a unitary operator, then*

$$w_q(PU \pm U^*Q) \leq 2\sqrt{\max(w(PQ), w(QP)) + \frac{1}{|q|^2}\|P \pm Q^*\|^2}.$$

In particular,

$$w_q(PU \pm U^*P) \leq 2\sqrt{w(P^2) + \frac{1}{|q|^2}\|P \pm P^*\|^2}. \tag{21}$$

Proof. Put $R = U$ in (19), we have

$$w_q(PU \pm U^*Q) \leq 2\sqrt{\max(w(PUU^*Q), w(U^*QPU)) + \frac{1}{|q|^2}\|PU \pm Q^*U\|^2}. \tag{22}$$

Since U is unitary and $w(U^*QPU) = w(QP)$, now the required result follows from (22). □

Corollary 2.12. *Let $P, Q, R, S \in \mathcal{L}(\mathcal{H})$, $q \in (0,1]$ and $\mathbb{T} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, then*

$$\max\{w_q(P), w_q(S)\} \leq \sqrt{w(\mathbb{T}^2) \pm \frac{1}{|q|^2}\|\mathbb{T} \pm \mathbb{T}^*\|^2}.$$

Proof. By using Lemma 1.6

$$\max\{w_q(P), w_q(S)\} \leq w_q\left(\begin{bmatrix} P & O \\ O & S \end{bmatrix}\right).$$

Since S is arbitrary, we have $\max\{w_q(P), w_q(S)\} \leq w_q\left(\begin{bmatrix} P & O \\ O & -S \end{bmatrix}\right)$. Let $\mathbb{U} = \begin{bmatrix} I & O \\ O & -I \end{bmatrix}$ be a unitary operator on $\mathcal{H} \oplus \mathcal{H}$. Then,

$$\begin{aligned}
\mathbb{T}\mathbb{U} + \mathbb{U}^*\mathbb{T} &= \begin{bmatrix} P & Q \\ R & S \end{bmatrix}\begin{bmatrix} I & O \\ O & -I \end{bmatrix} + \begin{bmatrix} I & O \\ O & -I \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} \\
&= \begin{bmatrix} P & -Q \\ R & -S \end{bmatrix} + \begin{bmatrix} P & Q \\ -R & -S \end{bmatrix} \\
&= \begin{bmatrix} 2P & O \\ O & -2S \end{bmatrix} \\
&= 2\begin{bmatrix} P & O \\ O & -S \end{bmatrix}.
\end{aligned}$$

Thus using (21), we have

$$\begin{aligned}
\max\{w_q(P), w_q(S)\} &\leq w_q(\mathbb{T}\mathbb{U} + \mathbb{U}^*\mathbb{T}) \\
&\leq \sqrt{w(\mathbb{T}^2) \pm \frac{1}{|q|^2}\|\mathbb{T} \pm \mathbb{T}^*\|^2}.
\end{aligned}$$

□

Remark 2.13. *When $q = 1$ in Corollary 2.12, we have*

$$\max\{w(P), w(S)\} \leq \sqrt{w(\mathbb{T}^2) \pm \|\mathbb{T} \pm \mathbb{T}^*\|^2}.$$

This is obtained by earlier authors in [12, Corollary 2.3.2].

Theorem 2.14 provides an upper bound for the q-numerical radius of commutator of operators.

Theorem 2.14. *Let $T, P \in \mathcal{L}(\mathcal{H})$ such that P is a projection and $q \in (0, 1]$. Then*

$$w_q(TP - PT) \leq \sqrt{w(T^2) + \frac{1}{|q|^2}\|T \pm T^*\|^2}. \tag{23}$$

Proof. Let us use the decomposition $\mathcal{H} = ran(P) \oplus ker(P)$ and using (5) we can represent P as the form $P = \begin{bmatrix} I_1 & O \\ O & O \end{bmatrix}$, where I_1 is the identity operator on $ran(P)$. With respect to this decomposition, T can be written as $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$. Then

$$\begin{aligned} PT - TP &= \begin{bmatrix} I_1 & O \\ O & O \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} - \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} I_1 & O \\ O & O \end{bmatrix} \\ &= \begin{bmatrix} T_{11} & T_{12} \\ O & O \end{bmatrix} - \begin{bmatrix} T_{11} & O \\ T_{21} & O \end{bmatrix} \\ &= \begin{bmatrix} O & T_{12} \\ -T_{21} & O \end{bmatrix}. \end{aligned}$$

Suppose I_2 is the identity operator on $ker(P)$ and if $U = \begin{bmatrix} I_1 & O \\ O & -I_2 \end{bmatrix}$, then U is unitary and

$$\begin{aligned} U^*T - TU &= \begin{bmatrix} I_1 & O \\ O & -I_2 \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} - \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} I_1 & O \\ O & -I_2 \end{bmatrix} \\ &= \begin{bmatrix} T_{11} & T_{12} \\ -T_{21} & -T_{22} \end{bmatrix} - \begin{bmatrix} T_{11} & -T_{12} \\ T_{21} & -T_{22} \end{bmatrix} \\ &= 2\begin{bmatrix} O & T_{12} \\ -T_{21} & O \end{bmatrix}. \end{aligned}$$

So,

$$\frac{1}{2}(U^*T - TU) = \begin{bmatrix} O & T_{12} \\ -T_{21} & O \end{bmatrix}.$$

Similarly,

$$TU - U^*T = \begin{bmatrix} T_{11} & -T_{12} \\ T_{21} & -T_{22} \end{bmatrix} - \begin{bmatrix} T_{11} & T_{12} \\ -T_{21} & -T_{22} \end{bmatrix} = 2\begin{bmatrix} O & -T_{12} \\ T_{21} & O \end{bmatrix}.$$

So, we have $\frac{1}{2}(TU - U^*T) = \begin{bmatrix} O & -T_{12} \\ T_{21} & O \end{bmatrix}$ and $TP - PT = \begin{bmatrix} T_{11} & O \\ T_{21} & O \end{bmatrix} - \begin{bmatrix} T_{11} & T_{12} \\ O & O \end{bmatrix}$.

This implies

$$TP - PT = \begin{bmatrix} O & -T_{12} \\ T_{21} & O \end{bmatrix}.$$

So,

$$w_q(TP - PT) = w_q\left(\begin{bmatrix} O & -T_{12} \\ T_{21} & O \end{bmatrix}\right) = \frac{1}{2}w_q(TU - U^*T).$$

Now, using (21), we have $w_q(TP - PT) \leq \sqrt{w(T^2) + \frac{1}{|q|^2}\|T \pm T^*\|^2}$. □

Remark 2.15. *When $q \to 1$ in Theorem 2.14, we get [12, Theorem 2.4].*

Theorem 2.16. *Let $T, X \in \mathcal{L}(\mathcal{H})$ such that T is positive and $q \in (0,1]$. Then*

$$w_q(TX - XT) \leq \|T\| \sqrt{w(X^2) + \frac{1}{|q|^2}\|X \pm X^*\|^2}.$$

Proof. Since $T\sqrt{T-T^2} = \sqrt{T-T^2}\,T$, so $P = \begin{bmatrix} T & \sqrt{T-T^2} \\ \sqrt{T-T^2} & I-T \end{bmatrix}$ is a projection on $\mathcal{H} \oplus \mathcal{H}$.
Let $S = \begin{bmatrix} X & O \\ O & O \end{bmatrix}$, then

$$\begin{aligned} PS - SP &= \begin{bmatrix} T & \sqrt{T-T^2} \\ \sqrt{T-T^2} & I-T \end{bmatrix}\begin{bmatrix} X & O \\ O & O \end{bmatrix} - \begin{bmatrix} X & O \\ O & O \end{bmatrix}\begin{bmatrix} T & \sqrt{T-T^2} \\ \sqrt{T-T^2} & I-T \end{bmatrix} \\ &= \begin{bmatrix} TX & O \\ \sqrt{T-T^2}X & O \end{bmatrix} - \begin{bmatrix} XT & X\sqrt{T-T^2} \\ O & O \end{bmatrix} \\ &= \begin{bmatrix} TX - XT & -X\sqrt{T-T^2} \\ \sqrt{T-T^2}X & O \end{bmatrix}. \end{aligned}$$

Now, using (23),

$$w_q(SP - PS) \leq \sqrt{w(S^2) + \frac{1}{|q|^2}\|S \pm S^*\|^2}.$$

Let $Q = \begin{bmatrix} I & O \\ O & O \end{bmatrix}$, then

$$\begin{aligned} Q(PS - SP)Q^* &= \begin{bmatrix} I & O \\ O & O \end{bmatrix}\begin{bmatrix} TX - XT & -X\sqrt{T-T^2} \\ \sqrt{T-T^2}X & O \end{bmatrix}\begin{bmatrix} I & O \\ O & O \end{bmatrix} \\ &= \begin{bmatrix} I & O \\ O & O \end{bmatrix}\begin{bmatrix} TX - XT & O \\ \sqrt{T-T^2}X & O \end{bmatrix} \\ &= \begin{bmatrix} TX - XT & O \\ O & O \end{bmatrix}. \end{aligned}$$

Using Lemma 1.6 we have,

$$w_q(TX - XT) \leq w_q\left(\begin{bmatrix} TX - XT & O \\ O & O \end{bmatrix}\right) = w_q(Q(PS - SP)Q^*).$$

Since $w_q((Q(PS - SP)Q^*) \leq w_q(PS - SP)$. So,

$$w_q(TX - XT) \leq w_q\left(\begin{bmatrix} TX - XT & O \\ O & O \end{bmatrix}\right) = w_q(Q(PS - SP)Q^*) \leq w_q(PS - SP).$$

This implies,

$$\begin{aligned} w_q(TX - XT) &\leq w_q(PS - SP) \\ &\leq \sqrt{w(S^2) + \frac{1}{|q|^2}\|S \pm S^*\|^2} \quad \text{by (23)} \\ &= \sqrt{w(X^2) + \frac{1}{|q|^2}\|X + X^*\|^2} \quad \text{by (6) and (7)}. \end{aligned}$$

Now, since T is a positive operator, $w_q\left(\frac{T}{\|T\|}X - X\frac{T}{\|T\|}\right) \leq \sqrt{w(X^2) + \frac{1}{|q|^2}\|X \pm X^*\|^2}$. Hence,

$$w_q(TX - XT) \leq \|T\| \sqrt{w(X^2) + \frac{1}{|q|^2}\|X \pm X^*\|^2}.$$

□

Remark 2.17. *When $q \to 1$ in Theorem 2.16, we get [12, Theorem 2.5].*

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