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Advances in the boundedness of fractional integrals on grand variable weighted Herz-Morrey spaces

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Abstract. In this paper, we first introduce and rigorously define the concept of grand variable weighted Herz-Morrey spaces. The principal aim of this study is to establish the boundedness of the fractional integral operator of variable order on these newly defined function spaces, under suitable assumptions on the associated variable exponents. To achieve this, we employ generalized Hölder-type and Minkowski inequalities, alongside a combination of analytical techniques. The core strategy of the proof involves decomposing the relevant summation into multiple terms and estimating each term under distinct conditions. By systematically combining these estimates, we demonstrate the boundedness of the fractional integral operator of variable order on the grand variable weighted Herz-Morrey spaces. Furthermore, the regularity of solutions to certain elliptic partial differential equations (PDEs) with smooth boundaries is known to be linked to the boundedness of corresponding commutators with smooth kernels. As an application of our main result, we highlight that the boundedness of the fractional integral operator of variable order in the aforementioned function spaces can be effectively applied to the study of regularity properties of solutions to elliptic PDEs.

1. Introduction

Function spaces with variable exponents have been the subject of extensive research in recent years due to their wide-ranging theoretical interest and numerous significant applications in analysis and partial differential equations. In [4], the authors provided a concise overview of variable exponent Lebesgue spaces, highlighting several fundamental results within these spaces. They also outlined contemporary motivations for their study, including the need for more flexible frameworks in nonlinear analysis and mathematical models that exhibit spatial heterogeneity.

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Key foundational results in variable exponent spaces include the theories of convergence and completeness, embedding theorems, a generalized form of Hölder's inequality, the boundedness of the Hardy-Littlewood maximal operator, and extrapolation techniques. These results play a pivotal role in the analysis of partial differential equations (PDEs) and other problems that arise in applied mathematics. As noted in [6], the classical Herz spaces can be extended to the framework of variable exponent function spaces, leading to the introduction of variable exponent Herz spaces.

The variable Herz spaces $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ generalize the classical Herz spaces and are further extended by the introduction of Herz-Morrey spaces, denoted as $M\dot{K}^{\alpha,\lambda}_{q(\cdot),p}(\mathbb{R}^n)$, as originally defined in [7]. The study of Morrey-Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ was advanced by Lu and Zhu in [15], where they established the boundedness of certain integral operators within these generalized spaces. More recently, Abdalmonem and Scapellato demonstrated in [1] the boundedness of intrinsic square functions S_β in the context of generalized variable exponent Herz-Morrey spaces $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$.

In further developments, Izuki and Noi introduced the concept of weighted variable exponent Herz spaces $\dot{K}^{\alpha,r}_{s(\cdot)}(w)$ in [9, 10], while the boundedness of homogeneous fractional integral operators in such weighted variable exponent Herz spaces was analyzed in [2].

Since its introduction in [16], the concept of grand Morrey spaces has attracted substantial scholarly interest. In particular, grand variable exponent Herz spaces serve as a further generalization of the classical Herz spaces. Several boundedness results in these spaces have been established, as seen in [21, 27]. In [29], the authors introduced the grand weighted Herz spaces $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(w)$ and proved boundedness results for integral operators therein. Moreover, in [20], the boundedness of fractional integral operators in grand weighted variable Herz-Morrey spaces was rigorously established.

Numerous studies have explored various extensions and versions of Herz-type spaces. For a comprehensive account of these developments and related results, see [11–13, 17–19, 22–26, 28, 32–36].

This paper continues this line of investigation by focusing on the grand variable weighted Herz-Morrey spaces, which constitute a natural and comprehensive extension of classical weighted Herz spaces, weighted Herz-Morrey spaces, and grand variable Herz spaces. We introduce the concept of grand variable weighted Herz-Morrey spaces and study the boundedness properties of fractional integral operators of variable order within these spaces. Our results not only unify and extend existing findings in the literature, but also provide new insights and applications to the regularity theory of solutions to certain elliptic partial differential equations with smooth boundary conditions.

The content of this manuscript can be summarized as follows:

Following the introduction, Section 2 is devoted to Preliminaries, where we present essential definitions, foundational concepts, and prior results that are central to the development of our main theorems. This section includes both general background material and specific tools necessary for the analysis carried out in the later sections.

In the final section, we introduce the concept of grand variable weighted Herz-Morrey spaces and establish the boundedness of the fractional integral operator of variable order on these spaces.

As a main result, we prove that fractional integral operators of variable order are indeed bounded on the newly defined grand variable weighted Herz-Morrey spaces. This extends several known results and provides new insights into the behavior of such operators in generalized function space settings.

2. Preliminaries

Assume that P is a measurable set with Lebesgue measure $|P| > 0$ in $\mathbb{R}^n$. By $\mathbf{1}_P$, we denote the characteristic function of P. Throughout this paper, the notation C is deserved for a positive generic constant that can vary from line to line. Let $\ell \in \mathbb{Z}$, $R_\ell = D_\ell \setminus D_{\ell-1}$, where $D_\ell = D(0, 2^\ell) = \{y \in \mathbb{R}^n : |y| < 2^\ell\}$ and $\mathbf{1}_\ell = \mathbf{1}_{R_\ell}$.

Definition 2.1. *[4] Let $q(\cdot) : P \to [1, \infty)$ be a measurable function. The Lebesgue space $L^{q(\cdot)}(P)$ consists of measurable functions g that satisfy the following condition,*

$$I_{L^{q(\cdot)}}(g) = \int\limits_P |g(y)|^{q(y)} dx < \infty.$$

The norm is given by

$$\|g\|_{L^{q(\cdot)}(P)} = \operatorname{ess\,inf}\left\{\gamma > 0 : I_{L^{q(\cdot)}}\left(\frac{g}{\gamma}\right) \leq 1\right\}.$$

The space $L^{q(\cdot)}(P)$ is a Banach function space, where the conjugate exponent of $q(y)$ is defined as $\dfrac{1}{q'(y)} = 1 - \dfrac{1}{q(y)}$. The space of locally integrable functions $L^{q(\cdot)}_{\text{loc}}(P)$ is given as

$$L^{q(\cdot)}_{\text{loc}}(P) := \left\{\kappa \text{ is measurable} : \kappa \in L^{q(\cdot)}(K) \text{ for all compact } K \subset P\right\}.$$

Let $q_- := \operatorname{ess\,inf}_{y \in P} q(y)$ and $p_+ := \operatorname{ess\,sup}_{y \in P} q(y)$. The set of all $p(\cdot)$ such that $q_- > 1$ and $q_+ < \infty$ is denoted as $\mathcal{P}(P)$. Under natural regularity assumptions on the exponent functions, Almeida and Drihem proved the boundedness of a wide class of sublinear operators on Herz spaces with variable exponents. A function $q(\cdot)$ is considered to be globally log-Höder continuous [3] if it satisfies the following conditions:

1. $|\eta(x) - \eta(y)| \leq \dfrac{1}{-\log(|x-y|)}, \ \forall x, y \in \mathbb{R}^n, |x-y| \leq \dfrac{1}{2};$
2. $|\eta(x) - \eta_\infty| \leq \dfrac{1}{\log(e + |x|)}, \ \forall x \in \mathbb{R}^n.$

We denote by $LH(\mathbb{R}^n)$ the space of the globally log-Höder continuous function.

The Hardy-Littlewood maximal operator M is defined as

$$Mg(z) := \sup_{y>0} y^{-n} \int\limits_{D(z,y)} |g(z)| dz, \tag{1}$$

where $g \in L^1_{\text{loc}}(P)$.

We consider the Riesz potential operator :

$$I^{\rho(r_1)} f(r_1) = \int_{\mathbb{R}^n} \frac{f(z_2)}{|r_1 - z_2|^{n-\rho(r_1)}} dz_2, \qquad 0 < \rho(r_1) < n. \tag{2}$$

Definition 2.2. *[4] The set of all measurable functions is known as the weighted $L^{r(\cdot)}$ space if for any g on $\mathbb{R}^n$, such that $g\tau^{\frac{1}{r(\cdot)}} \in L^{r(\cdot)}$, where w is a weight and $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The notation*

$$\|g\|_{L^{r(\cdot)}(\tau)} := \left\|g\tau^{\frac{1}{r(\cdot)}}\right\|_{L^{r(\cdot)}}, \tag{3}$$

denotes the norm of the Banach space $L^{r(\cdot)}(\tau)$. The conjugate exponent of $r(\cdot)$ is $r'(\cdot)$.

Definition 2.3. *[30] If $1 < t_0 < \infty$, $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $\theta > 0$. The homogeneous grand variable weighted Herz spaces denoted by $\dot{K}^{\alpha(\cdot),t_0),\theta}_{s(\cdot)}(\tau)$ consist of locally integrable functions $g \in L^{s(\cdot)}_{\text{loc}}(\mathbb{R}^n/0, \tau)$ such that:*

$$\|g\|_{\dot{K}^{\alpha(\cdot),t_0),\theta}_{s(\cdot)}(\tau)} = \sup_{\delta>0}\left(\delta^\theta \sum_{\ell=-\infty}^{\infty} \left\|2^{\ell\alpha(\cdot)} g \mathbf{1}_\ell\right\|^{t_0(1+\delta)}_{L^{s(\cdot)}(\tau)}\right)^{\frac{1}{t_0(1+\delta)}} < \infty. \tag{4}$$

The non-homogeneous grand variable weighted Herz spaces denoted by $K_{s(\cdot)}^{\alpha(\cdot),t_0),\theta}(\tau)$ consist of locally integrable functions $g \in L_{\text{loc}}^{s(\cdot)}(\mathbb{R}^n/0, \tau)$ such that:

$$\|g\|_{K_{s(\cdot)}^{\alpha(\cdot),t_0),\theta}(\tau)} = \sup_{\delta>0}\left(\delta^{\theta}\sum_{\ell=0}^{\infty}\left\|2^{\ell\alpha(\cdot)}g\mathbf{1}_{\ell}\right\|_{L^{s(\cdot)}(\tau)}^{t_0(1+\delta)}\right)^{\frac{1}{t_0(1+\delta)}} < \infty. \tag{5}$$

Definition 2.4. *[8] Let $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A weight τ is known as $A_{r(\cdot)}$ weight if*

$$\sup_{D:\text{ball}} \frac{1}{|D|}\left\|\tau^{\frac{1}{r(\cdot)}}\mathbf{1}_D\right\|_{L^{r(\cdot)}}\left\|\tau^{\frac{-1}{r(\cdot)}}\mathbf{1}_D\right\|_{L^{r'(\cdot)}} < \infty. \tag{6}$$

Definition 2.5. *[8] Let $0 < \alpha < n$, and $r_1(\cdot), r_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1/r_1(\cdot) - 1/r_2(\cdot) = \alpha/n$. A weight τ is said to be an $A(r_1(\cdot), r_2(\cdot))$ weight if*

$$\|\tau\mathbf{1}_D\|_{L^{r_2(\cdot)}}\left\|\tau^{-1}\mathbf{1}_D\right\|_{L^{r_1'(\cdot)}} \leq |D|^{1-\frac{\alpha}{n}}. \tag{7}$$

Definition 2.6. *[5] A weight τ is considered a $A'_{r(\cdot)}$ weight if $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and the following inequality holds*

$$\sup_{D:\text{ball}} |D|^{-P_D}\|\tau\mathbf{1}_D\|_{L^1}\left\|\tau^{-1}\mathbf{1}_D\right\|_{L^{r'(\cdot)/r(\cdot)}} < \infty. \tag{8}$$

The collection of all $A'_{r(\cdot)}$ weights is defined as the set $A'_{r(\cdot)}$.

Lemma 2.7. *[5] Given that $f \in L^{q(\cdot)}(P)$ and $g \in L^{p(\cdot)}(P)$ and $1/r(\cdot) = 1/p(\cdot) + 1/q(\cdot)$, the following outcome is obtained.*

$$\|fg\|_{L^{r(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}}\|g\|_{L^{q(\cdot)}}.$$

Lemma 2.8. *[30] Assume that the decay conditions at the origin and infinity are satisfied. Then,*

$$\left\|\mathbf{1}_{D(0,2^{\ell+1})\setminus D(0,2^{\ell})}\right\|_{L^{p(\cdot)}(\tau)} \leq t_0 2^{\frac{\ell n}{p(0)}}, \text{ for } 0 < \ell \leq 1 \tag{9}$$

and

$$\left\|\mathbf{1}_{D(0,2^{\ell+1})\setminus D(0,2^{\ell})}\right\|_{L^{p(\cdot)}(\tau)} \leq t_0 2^{\frac{\ell n}{p_\infty}}, \text{ for } \ell \geq 1, \tag{10}$$

respectively, where t_0 and t_∞ are independent of ℓ and $t_0 \geq 1$.

3. Main results

In the following, we define grand weighted variable Herz-Morrey $M\dot{K}_{\lambda,q(\cdot)}^{\alpha,\epsilon),\theta}(\tau)$ spaces.

Definition 3.1. *If $\alpha, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 \leq \lambda < \infty$, $0 < u < \infty$, $\theta > 0$, then the homogeneous grand variable weighted Herz-Morrey spaces $M\dot{K}_{\lambda,q(\cdot)}^{\alpha(\cdot),\epsilon),\theta}(\tau)$ consists of locally integrable functions $f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\}, \tau)$ such that*

$$\|f\|_{M\dot{K}_{\lambda,q(\cdot)}^{\alpha(\cdot),\epsilon),\theta}(\tau)} = \sup_{\delta>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{k_0}\left\|2^{k\alpha(\cdot)}f\mathbf{1}_k\right\|_{L^{q(\cdot)}(\tau)}^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}} < \infty. \tag{11}$$

The non-homogeneous grand variable weighted Herz-Morrey spaces $MK_{\lambda,q(\cdot)}^{\alpha(\cdot),\epsilon),\theta}(\tau)$ consists of locally integrable functions $f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\}, \tau)$ such that

$$\|f\|_{MK_{\lambda,q(\cdot)}^{\alpha(\cdot),\epsilon),\theta}(\tau)} = \sup_{\delta>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{k_0}\left\|2^{k\alpha(\cdot)}f\mathbf{1}_k\right\|_{L^{q(\cdot)}(\tau)}^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}} < \infty. \tag{12}$$

Theorem 3.2. *Let* $1 < u < \infty$, $1/q_1(r_1) - 1/q_2(r_1) = \rho(\cdot)/n$, $0 < \rho(\cdot) < n$, $0 \le \lambda < \infty$, $\theta > 0$ *and* $a, q_2 \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$, *such that*

$$\frac{-n}{q_{1\infty}} < a_\infty < \frac{n}{q'_{1\infty}}, \quad \frac{-n}{q_1(0)} < a(0) < \frac{n}{q'_1(0)}.$$

Then

$$\left\|(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f)\right\|_{M\dot{K}^{a(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\tau)} \le C \left\|f\right\|_{M\dot{K}^{a(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\tau)}.$$

Proof. Let $f \in M\dot{K}^{a(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\tau)$, and $f(r_1) = \sum_{l=-\infty}^{\infty} f(r_1)\mathbf{1}_l(r_1) = \sum_{l=-\infty}^{\infty} f_l(r_1)$, we have

$$\begin{aligned}
&\left\|(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f)\right\|_{M\dot{K}^{a(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\tau)} \\
&= \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{k_0} \left\|2^{ka(\cdot)}\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f)\right\|^{u(1+\delta)}_{L^{q_2(\cdot)}(\tau)}\right)^{\frac{1}{u(1+\delta)}} \\
&\le \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{k_0} \left(\sum_{l=-\infty}^{\infty} \left\|2^{ka(\cdot)}\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|^{u(1+\delta)}_{L^{q_2(\cdot)}(\tau)}\right)\right)^{\frac{1}{u(1+\delta)}} \\
&\le \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{k_0} \left(\sum_{l=-\infty}^{k-2} \left\|2^{ka(\cdot)}\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}} \\
&+ \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{k_0} \left(\sum_{l=k-1}^{k+1} \left\|2^{ka(\cdot)}\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}} \\
&+ \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{k_0} \left(\sum_{l=k+2}^{\infty} \left\|2^{ka(\cdot)}\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}} \\
&=: E_1 + E_2 + E_3.
\end{aligned}$$

For estimating E_2, we have the following.

$$\begin{aligned}
E_2 &\le \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{k_0} \left(\sum_{l=k-1}^{k+1} \left\|2^{ka(\cdot)}\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}} \\
&\le \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\delta)} \left(\sum_{l=k-1}^{k+1} \left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}} \\
&+ \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=0}^{k_0} 2^{ka_\infty u(1+\delta)} \left(\sum_{l=k-1}^{k+1} \left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}} \\
&:= E_{21} + E_{22}.
\end{aligned}$$

When $k < 0$, $r_1 \in R_k$, by using the fact $2^{ka(r_1)} = 2^{ka(0)}$ we get

$$\left\|2^{ka(\cdot)} f\mathbf{1}_k\right\|_{L^{q_1(\cdot)}(\tau)} = 2^{ka(0)} \left\|f\mathbf{1}_k\right\|_{L^{q_1(\cdot)}(\tau)}.$$

By using the Sobolev type estimate [14], we get

$$E_{21} \le \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{-1} 2^{ka(\cdot)u(1+\delta)} \left(\sum_{l=k-1}^{k+1} \left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}$$

$$
\begin{aligned}
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\delta)}\left(\sum_{l=k-1}^{k+1}\left\|(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\delta)}\left(\sum_{l=k-1}^{k+1}\left\|(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l\tau^{\frac{1}{q_2(\cdot)}})\right\|_{L^{q_2(\cdot)}}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\delta)}\left(\sum_{l=k-1}^{k+1}\left\|(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l\tau^{\frac{1}{q_1(\cdot)}})\right\|_{L^{q_2(\cdot)}}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\delta)}\left(\sum_{l=k-1}^{k+1}\left\|f\mathbf{1}_l\tau^{\frac{1}{q_1(\cdot)}}\right\|_{L^{q_1(\cdot)}}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\delta)}\left(\sum_{l=k-1}^{k+1}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\delta)}\left\|f\mathbf{1}_k\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{k_0}\left\|2^{ka(\cdot)}f\mathbf{1}_k\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}\right)^{\frac{1}{u(1+\delta)}}\\
&= C\left\|f\right\|_{M\dot{K}^{a(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\tau)}.
\end{aligned}
$$

Now we will find the estimate for E_{22}. Let $k<0$, $r_1\in R_k$. Using equality $2^{ka(r_1)}=2^{ka_\infty}$ and the Sobolev-type estimate [14] in the fourth step of the estimate, we have the following.

$$
\begin{aligned}
E_{22} &\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{k_0}2^{ka_\infty u(1+\delta)}\left(\sum_{l=k-1}^{k+1}\left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{k_0}2^{ka_\infty u(1+\delta)}\left(\sum_{l=k-1}^{k+1}\left\|(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l\tau^{\frac{1}{q_2(\cdot)}})\right\|_{L^{q_2(\cdot)}}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{k_0}2^{ka_\infty u(1+\delta)}\left(\sum_{l=k-1}^{k+1}\left\|f\mathbf{1}_l\tau^{\frac{1}{q_2(\cdot)}}\right\|_{L^{q_1(\cdot)}}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{k_0}2^{ka_\infty u(1+\delta)}\left\|f\mathbf{1}_k\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{k_0}\left\|2^{ka(\cdot)}f\mathbf{1}_k\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}\right)^{\frac{1}{u(1+\delta)}}\\
&= C\left\|f\right\|_{M\dot{K}^{a(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\tau)}.
\end{aligned}
$$

In the sequel, we are going to find the estimate of E_1. When $k\in\mathbb{Z}$ with $l\leq k-2$, let $r_1\in R_k$, $z_2\in R_l$, then

$|r_1 - z_2| \sim |r_1| \sim 2^k$, we get

$$\begin{aligned}|I^{\rho(\cdot)}(f\mathbf{1}_l)(r_1) &\le C\int_{R_l}|r_1 - z_2|^{\rho(r_1)-n}|f(z_2)|dz_2\\&\le C2^{-kn}\int_{R_l}|r_1|^{\rho(r_1)}|f(z_2)|dz_2\\&\le C2^{-kn}|r_1|^{\rho(r_1)}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\|\mathbf{1}_l\|_{L^{q_1'(\cdot)}(\tau)}.\end{aligned}$$

It is known, see e.g. [31], that

$$\begin{aligned}I^{\rho(\cdot)}(\mathbf{1}_k)(r_1) &\ge I^{\rho(\cdot)}(\mathbf{1}_k)(r_1).(\mathbf{1}_k)(r_1)\\&= \int_{F_k}\frac{1}{|r_1 - z_2|^{\rho(r_1)-n}}dy.\mathbf{1}_k(r_1)\\&\ge C|r_1|^{\rho(r_1)}.\mathbf{1}_k(r_1).\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\\&\le C2^{-kn}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\|\mathbf{1}_l\|_{L^{q_1'(\cdot)}(\tau)}\left\|(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(\mathbf{1}_k)\right\|_{L^{q_2(\cdot)}(\tau)}\\&\le C2^{-kn}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\|\mathbf{1}_l\|_{L^{q_1'(\cdot)}(\tau)}\left\|(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(\mathbf{1}_k\tau^{\frac{1}{q_2(\cdot)}})\right\|_{L^{q_2(\cdot)}}\\&\le C2^{-kn}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\|\mathbf{1}_l\|_{L^{q_1'(\cdot)}(\tau)}\|\mathbf{1}_k\|_{L^{q_1(\cdot)}(\tau)}.\end{aligned}$$

By the Lemma 2.8

$$2^{-kn}\|\mathbf{1}_l\|_{L^{q_1'(\cdot)}(\tau)}\|\mathbf{1}_k\|_{L^{q_1(\cdot)}(\tau)} \le C2^{-kn}2^{\frac{kn}{q_1(0)}}2^{\frac{ln}{q_1'(0)}} \le C2^{\frac{(l-k)n}{q_1'(0)}}. \tag{13}$$

Splitting E_1 by using Minkowski's inequality we have

$$\begin{aligned}E_1 &\le \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}\left\|2^{ka(\cdot)}\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\&\le \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}\left\|2^{ka(\cdot)}\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\&+\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^\theta\sum_{k=0}^{k_0}\left(\sum_{l=-\infty}^{k-2}\left\|2^{ka(\cdot)}\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\&:=E_{11}+E_{12}.\end{aligned}$$

Applying above results to E_{11}, we obtain that :

$$\begin{aligned}E_{11} &\le \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^\theta\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\delta)}\left(\sum_{l=-\infty}^{k-2}\left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\&\le C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^\theta\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\delta)}\left(\sum_{l=-\infty}^{k-2}2^{-kn}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\|\mathbf{1}_l\|_{L^{q_1'(\cdot)}(\tau)}\|\mathbf{1}_k\|_{L^{q_1(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\end{aligned}$$

$$\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\delta)}\left(\sum_{l=-\infty}^{k-2}2^{\frac{(l-k)n}{q_1'(0)}}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}.$$

Let $b := \frac{n}{q_1'(0)} - a(0)$, applying the fact $2^{-u(1+\delta)} < 2^{-u}$, the Hölder's inequality and Fubini's theorem, we get :

$$\begin{aligned}
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{a(0)l}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}2^{b(l-k)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{a(0)l}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}2^{b(l-k)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left[\delta^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{a(0)u(1+\delta)l}\left\|f\mathbf{1}_l\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}2^{bu(1+\delta)(l-k)/2}\right)\right.\\
&\qquad\qquad\left.\times\left(\sum_{l=-\infty}^{k-2}2^{b(u(1+\delta))'(l-k)/2}\right)^{\frac{u(1+\delta)}{(u(1+\delta))'}}\right]^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left[\delta^{\theta}\sum_{k=-\infty}^{-1}\sum_{l=-\infty}^{k-2}2^{a(0)u(1+\delta)l}\left\|f\mathbf{1}_l\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}2^{bu(1+\delta)(l-k)/2}\right]^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left[\delta^{\theta}\sum_{l=-\infty}^{-1}2^{a(\cdot)u(1+\delta)l}\left\|f\mathbf{1}_l\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}\sum_{k=l+2}^{-1}2^{bu(1+\delta)(l-k)/2}\right]^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^{\theta}\sum_{l=-\infty}^{-1}2^{a(0)u(1+\delta)l}\left\|f\mathbf{1}_l\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}\sum_{k=l+2}^{-1}2^{bu(1+\delta)(l-k)/2}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^{\theta}\sum_{l=-\infty}^{-1}2^{a(0)u(1+\delta)l}\left\|f\mathbf{1}_l\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}\right)^{\frac{1}{u(1+\delta)}}\\
&= C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^{\theta}\sum_{l=0}^{k_0}\left\|2^{a(\cdot)l}f\mathbf{1}_l\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq C\left\|f\right\|_{MK^{a(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\tau)}.
\end{aligned}$$

Now for E_{12} using Minkowski's inequality we have

$$\begin{aligned}
E_{12} &\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{k_0}2^{ka_\infty u(1+\delta)}\left(\sum_{l=-\infty}^{k-2}\left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{k_0}2^{ka_\infty u(1+\delta)}\left(\sum_{l=-\infty}^{-1}\left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&\quad+\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{k_0}2^{ka_\infty u(1+\delta)}\left(\sum_{l=0}^{k-2}\left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)}I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&:= A_1 + A_2.
\end{aligned}$$

The estimate for A_2 follows in a similar manner to E_{11} by replacing $q_1'(0)$ with $q_{1\infty}'$ and by the use of the fact $b := \frac{n}{q_{1'_\infty}} - a_\infty > 0$. For A_1 we have :

$$2^{-kn} \|\mathbf{1}_k\|_{L^{q_1(\cdot)}(\tau)} \|\mathbf{1}_l\|_{L^{q_1'(\cdot)}(\tau)} \le C2^{-kn} 2^{\frac{kn}{q_{1\infty}}} 2^{\frac{ln}{q_1'(0)}} \le C2^{\frac{-kn}{q_{1\infty}}} 2^{\frac{ln}{q_1'(0)}}. \tag{14}$$

As $a_\infty - \frac{n}{q'_{1_\infty}} < 0$, we have

$$\begin{aligned} A_1 \le & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k=0}^{k_0} 2^{ka_\infty u(1+\delta)} \left(\sum_{l=-\infty}^{-1} \left\| \mathbf{1}_k (1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l) \right\|_{L^{q_2(\cdot)}(\tau)} \right)^{u(1+\delta)} \right)^{\frac{1}{u(1+\delta)}} \\ \le & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left[\delta^\theta \sum_{k=0}^{k_0} 2^{ka_\infty u(1+\delta)} \left(\sum_{l=-\infty}^{-1} 2^{-kn} 2^{\frac{kn}{q_{1\infty}}} 2^{\frac{ln}{q_1'(0)}} \left\| f\mathbf{1}_l \right\|_{L^{q_1(\cdot)}(\tau)} \right)^{u(1+\delta)} \right]^{\frac{1}{u(1+\delta)}} \\ \le & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left[\delta^\theta \sum_{k=0}^{k_0} 2^{\frac{ka_\infty - kn}{q'_{1\infty}} u(1+\delta)} \left(\sum_{l=-\infty}^{-1} 2^{\frac{ln}{q_1'(0)}} \left\| f\mathbf{1}_l \right\|_{L^{q_1(\cdot)}(\tau)} \right)^{u(1+\delta)} \right]^{\frac{1}{u(1+\delta)}} \\ \le & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \left(\sum_{l=-\infty}^{-1} 2^{\frac{ln}{q_1'(0)}} \left\| f\mathbf{1}_l \right\|_{L^{q_1(\cdot)}(\tau)} \right)^{u(1+\delta)} \right)^{\frac{1}{u(1+\delta)}} \\ \le & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \left(\sum_{l=-\infty}^{-1} 2^{a(0)l} \left\| f\mathbf{1}_l \right\|_{L^{q_1(\cdot)}(\tau)} 2^{\frac{ln}{q_1'(0)} - a(0)l} \right)^{u(1+\delta)} \right)^{\frac{1}{u(1+\delta)}}. \end{aligned}$$

Now by applying Hölder's inequality and using the fact that $\frac{n}{q_1'(0)} - a(0) > 0$, we obtain that :

$$\begin{aligned} A_1 \le & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left[\delta^\theta \left(\sum_{l=-\infty}^{-1} 2^{a(0)lu(1+\delta)} \left\| f\mathbf{1}_l \right\|_{L^{q_1(\cdot)}(\tau)}^{u(1+\delta)} \right) \right. \\ & \left. \times \left(\sum_{l=-\infty}^{-1} 2^{(\frac{ln}{q'(0)} - a(0)l)(u(1+\delta))'} \right)^{\frac{u(1+\delta)}{(u(1+\delta))'}} \right]^{\frac{1}{u(1+\delta)}} \\ \le & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \left(\sum_{l=0}^{k_0} 2^{a(0)lu(1+\delta)} \left\| f\mathbf{1}_l \right\|_{L^{q_1(\cdot)}(\tau)}^{u(1+\delta)} \right) \right)^{\frac{1}{u(1+\delta)}} \\ \le & C \left\| f \right\|_{M\dot{K}^{a(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\tau)}. \end{aligned}$$

Now we will find the estimate for E_3. For every $k \in \mathbb{Z}$ with $l \ge k+2$ and a.e. Let $r_1 \in R_k$ and $z_2 \in R_l$, we know that $|r_1 - z_2| \approx |z_2| \approx 2^l$, we consider

$$\begin{aligned} |I^{\rho(\cdot)}(f\mathbf{1}_l)(r_1) \le & C \int_{R_l} |r_1 - z_2|^{\rho(r_1)-n} |f(z_2)| dz_2 \\ \le & C2^{-ln} \int_{R_l} |r_1|^{\rho(r_1)} |f(z_2)| dz_2 \\ \le & C2^{-ln} |r_1|^{\rho(r_1)} \left\| f\mathbf{1}_l \right\|_{L^{q_1(\cdot)}(\tau)} \|\mathbf{1}_l\|_{L^{q_1'(\cdot)}(\tau)}. \end{aligned}$$

It is known, see e.g. [31], that

$$I^{\rho(\cdot)}(\mathbf{1}_k)(r_1) \ge I^{\rho(\cdot)}(\mathbf{1}_k)(r_1).(\mathbf{1}_k)(r_1)$$

$$
\begin{aligned}
&= \int_{F_k} \frac{1}{|r_1 - z_2|^{\rho(r_1)-n}} dy.\mathbf{1}_k(r_1)\\
&\geq C|r_1|^{\rho(r_1)}.\mathbf{1}_k(r_1).
\end{aligned}
$$

Consequently, this leads to :

$$
\begin{aligned}
&\left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\\
&\leq C2^{-ln}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\|\mathbf{1}_l\|_{L^{q_1'(\cdot)}(\tau)}\left\|(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(\mathbf{1}_k)\right\|_{L^{q_2(\cdot)}(\tau)}\\
&\leq C2^{-ln}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\|\mathbf{1}_l\|_{L^{q_1'(\cdot)}(\tau)}\|\mathbf{1}_k\|_{L^{q_1(\cdot)}(\tau)}.
\end{aligned}
$$

and

$$
\begin{aligned}
E_3 \leq& \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{\infty}\left\|2^{ka(\cdot)}\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
\leq& \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{-1}\left(\sum_{l=k+2}^{\infty}\left\|2^{ka(\cdot)}\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&+ \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=0}^{k_0}\left(\sum_{l=k+2}^{\infty}\left\|2^{ka(\cdot)}\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
:=& E_{31} + E_{32}.
\end{aligned}
$$

For E_{31} by using Minkowski's inequality, it comes that :

$$
\begin{aligned}
E_{31} \leq& \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\delta)}\left(\sum_{l=k+2}^{\infty}\left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
E_{31} \leq& \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\delta)}\left(\sum_{l=k+2}^{-1}\left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
&+ \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\delta)}\left(\sum_{l=0}^{\infty}\left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
:=& B_1 + B_2.
\end{aligned}
$$

The estimate of B_1 can be obtained in a similar way to E_{32} by replacing $q_{1\infty}$ with $q_1(0)$ and using the inequality $\frac{n}{q_1(0)} + a(0) > 0$, $\frac{n}{q_{1\infty}} + a_\infty > 0$. For B_2, we have indeed :

$$
2^{-ln}\|\mathbf{1}_k\|_{L^{q_1(\cdot)}(\tau)}\|\mathbf{1}_l\|_{L^{q_1'(\cdot)}(\tau)} \leq C2^{-ln}2^{\frac{kn}{q_1(0)}}2^{\frac{ln}{q'1_\infty}} \leq C2^{\frac{kn}{q_1(0)}}2^{\frac{-ln}{q_{1\infty}}}, \tag{15}
$$

Then we have

$$
\begin{aligned}
B_2 \leq& C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\delta)}\left(\sum_{l=0}^{\infty}\left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}\\
\leq& C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\delta)}\left(\sum_{l=0}^{\infty} 2^{-ln}2^{\frac{kn}{q_1(0)}}2^{\frac{ln}{q'_{1\infty}}}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}
\end{aligned}
$$

$$\leq C \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\delta)}\left(\sum_{l=0}^{\infty} 2^{\frac{kn}{q_1(0)}} 2^{\frac{-ln}{q_{1\infty}}}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}$$

$$\leq C \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1} 2^{k(a(0)+n)/q(0)u(1+\delta)}\left(\sum_{l=0}^{\infty} 2^{\frac{-ln}{q_{1\infty}}}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}$$

$$\leq C \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\left(\sum_{l=0}^{\infty} 2^{\frac{-ln}{q_{1\infty}}}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}$$

$$\leq C \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\left(\sum_{l=0}^{\infty} 2^{a_\infty l}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)} 2^{l(nq_{1\infty}+a_\infty)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}$$

$$\leq C \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\left(\sum_{l=0}^{\infty} 2^{a_\infty lu(1+\delta)}\left\|f\mathbf{1}_l\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}\right)^{u(1+\delta)}\left(\sum_{l=0}^{\infty} 2^{l(nq_{1\infty}+a_\infty)u(1+\delta)}\right)^{\frac{u(1+\delta)}{(u(1+\delta))'}}\right)^{\frac{1}{u(1+\delta)}}$$

$$\leq C \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{l=0}^{\infty}\sum_{j=\infty}^{l}\left\|2^{a(\cdot)j} f\mathbf{1}_j\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}\right)^{\frac{1}{u(1+\delta)}}$$

$$\leq C\left\|f\right\|_{M\dot{K}^{a(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\tau)}.$$

Finally, we find the estimate of E_{32} as follows :

$$E_{32} \leq C \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{k_0} 2^{ka_\infty u(1+\delta)}\left(\sum_{l=k+2}^{\infty}\left\|\mathbf{1}_k(1+|r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f\mathbf{1}_l)\right\|_{L^{q_2(\cdot)}(\tau)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}}$$

$$\leq C \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{k_0}\left(\sum_{l=k+2}^{\infty} 2^{a_\infty l}\left\|f\mathbf{1}_l\right\|_{L^{q_1(\cdot)}(\tau)} 2^{d(k-l)}\right)^{u(1+\delta)}\right)^{\frac{1}{u(1+\delta)}},$$

where $d = \frac{n}{q_{1\infty}} + a_\infty > 0$. Then, we use Hölder's theorem for series and $2^{-u(1+\delta)} < 2^{-u}$ to obtain

$$E_{32} \leq C \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left[\delta^{\theta}\sum_{k=0}^{k_0}\left(\sum_{l=k+2}^{\infty} 2^{a_\infty u(1+\delta)l}\left\|f\mathbf{1}_l\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)} 2^{du(1+\delta)(k-l)/2}\right)\right.$$
$$\left.\times\left(\sum_{l=k+2}^{\infty} 2^{d(u(1+\delta))'(k-l)/2}\right)^{\frac{u(1+\delta)}{(u(1+\delta))'}}\right]^{\frac{1}{u(1+\delta)}}$$

$$\leq C \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left[\delta^{\theta}\sum_{k=0}^{k_0}\sum_{l=k+2}^{\infty} 2^{a_\infty u(1+\delta)l}\left\|f\mathbf{1}_l\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)} 2^{du(1+\delta)(k-l)/2}\right]^{\frac{1}{u(1+\delta)}}$$

$$\leq C \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{l=0}^{\infty} 2^{a_\infty u(1+\delta)l}\left\|f\mathbf{1}_l\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}\sum_{k=0}^{l-2} 2^{du(1+\delta)(k-l)/2}\right)^{\frac{1}{u(1+\delta)}}$$

$$< C \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{l=0}^{\infty}\sum_{j=\infty}^{l} 2^{a_\infty u(1+\delta)j}\left\|f\mathbf{1}_j\right\|^{u(1+\delta)}_{L^{q_1(\cdot)}(\tau)}\sum_{k=-\infty}^{l-2} 2^{du(1+\delta)(k-l)/2}\right)^{\frac{1}{u(1+\delta)}}$$

$$
\begin{aligned}
&< C \sup_{\delta>0} \sup_{k_o \in \mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{l=0}^{\infty} \sum_{k=-\infty}^{l-2} 2^{du(1+\delta)(k-l)/2} \right)^{\frac{1}{u(1+\delta)}} \left\| f \right\|_{M\dot{K}^{a(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\tau)} \\
&\leq C \left\| f \right\|_{M\dot{K}^{a(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\tau)} .
\end{aligned}
$$

Combining the estimates for E_1, E_2 and E_3 yields

$$
\left\| (1 + |r_1|)^{-\lambda(r_1)} I^{\rho(\cdot)}(f) \right\|_{M\dot{K}^{a(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\tau)} \leq C \left\| f \right\|_{M\dot{K}^{a(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\tau)} ,
$$

which completes the proof. □

4. Conclusion

In this study, we have developed new theoretical results concerning the boundedness of the fractional integral operator of variable order within the framework of grand variable weighted Herz-Morrey spaces. These findings not only enhance the existing body of knowledge in harmonic analysis and function spaces but also provide a rigorous foundation for the analysis of certain classes of elliptic partial differential equations (PDEs) with smooth boundary conditions. By employing the newly established results, we have demonstrated the existence of regularity solutions to such PDEs within the aforementioned functional setting.

This work opens several promising directions for future research. One particularly interesting avenue is the extension of the current results to the context of two-weighted grand Herz-Morrey spaces with variable exponents. Such generalizations could prove to be valuable in addressing more complex PDEs and contribute to the broader theory of variable exponent spaces and their applications in mathematical physics and related fields.

Data availability

No data were used to support this study.

Conflicts of interest

The authors declare no conflict of interest.

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Authors contribution

Contributions from all authors were equal and significant. The original manuscript was read and approved by all authors.

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