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The generalized Hermite polynomials: Application to the heat equation

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Abstract. We consider the family of Hermite Polynomials (HP) and we introduce the γ-order Generalized Hermite Polynomials (GHP) $H_{r,\gamma}(w), r \in \mathbb{N}, \gamma = \gamma(m) = \frac{m}{m-1}, m > 1, m \in \mathbb{N}$. Briefly, we prove the following points: (i) the Differential of the γ-order Generalized Normal distribution is related to $H_{r,\gamma}(w)$, (ii) the Generalized form of the Heat Equation (GHE) is an application of the new extension $H_{r,\gamma}(w)$, (iii) due to our definitions with $\gamma = 2$ we are reduced to the classical case, (iv) the introduced GHE covers an existing extension. The provided examples and figures clarify the introduced extensions.

1. Introduction

The target of this paper is to provide an extension of the Heat Equation (HE), a Generalized Heat Equation (GHE), based on a Statistical line of thought, that is, due to an extension of the Normal distribution. The definition of a new class of Generalized Hermite Polynomials (GHP) supports our target. The background is the γ-order Generalized Normal distribution (γGN), reviewed in Introduction, see also [15]. The γGN offered the creation of the Generalized Heat Equation, which is reduced to the classical HE, when $\gamma = 2$, see [11]. Therefore the most popular differential equation, strongly related with the Probability Theory has been generalized, see Section 2, through an extended form of the Normal distribution, the γGN.

One of the merits of the γ-order Generalized Normal distribution, introduced in [16], is that it emerged from the Logarithmic Sobolev Inequality (LSI), [15]. There is an extra shape parameter γ so the parameter vector is $\theta = (\boldsymbol{\mu}, \boldsymbol{\Sigma}; \gamma)$ with $\boldsymbol{\mu}$ the mean and $\boldsymbol{\Sigma}$ the variance covariance matrix. The density function $\phi_\gamma(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ is given by, [18],

$$\phi_\gamma(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = C \exp\left\{-\frac{\gamma-1}{\gamma}[Q(\boldsymbol{x})]^{\frac{\gamma}{2(\gamma-1)}}\right\}, \quad \boldsymbol{x} \in \mathbb{R}^p, \tag{1}$$

which is in the form of the usual multivariate Normal distribution with

$$Q(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}), \tag{2}$$

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exactly as in the multivariate case and

$$C = C(p, \gamma, \mathbf{\Sigma}) = \frac{(\frac{\gamma-1}{\gamma})^{p\frac{\gamma-1}{\gamma}}}{\pi^{p/2}|\mathbf{\Sigma}|^{1/2}} \frac{\Gamma(\frac{p}{2}+1)}{\Gamma(p\frac{\gamma-1}{\gamma}+1)}, \tag{3}$$

the normalizing constant. Briefly, concerning the notation we use:

- $\boldsymbol{\mu}$ is the position (mean) vector $\in \mathbb{R}^p$,
- $\mathbf{\Sigma} \in \mathbb{R}^{p\times p}$ is the positive definite scale parameter matrix,
- $\gamma \in \mathbb{R} - [0, 1]$ is the extra defined shape parameter.

We say that the random variable (rv) X follows the γ-order Generalized Normal distribution and write $X \sim N_\gamma(\boldsymbol{\mu}, \mathbf{\Sigma})$. This distribution, as pointed out in [16], is a Kotz-type distribution, while for an application to Environmental Economics, see [8], applications to Finance, see [12], and applications to Information theory see [17].

The effect of the extra shape parameter γ is discussed in [15] and when

- $\gamma \uparrow 0$ then $\phi_\gamma(x)$ coincides with the Dirac distribution,
- $\gamma \downarrow 1$ then $\phi_\gamma(x)$ coincides with the Uniform distribution,
- $\gamma = 2$ then $\phi_\gamma(x)$ coincides with Normal distribution,
- $\gamma \to \pm\infty$ then $\phi_\gamma(x)$ coincides with Laplace distribution.

That is, this "family" of distributions is enriched with well-known distributions.

Moreover, the cumulative distribution function (cdf) of ϕ_γ, say Φ_γ, for $z = \frac{x-\mu}{\sigma}$ is obtained and equals to, for $p = 1$,

$$\Phi_\gamma(z) = \frac{1}{2} + \frac{\sqrt{\pi}\mathrm{sgn}(z)}{2\Gamma(\frac{\gamma-1}{\gamma})\Gamma(\frac{\gamma}{\gamma-1})} \mathrm{Erf}_{\frac{\gamma}{\gamma-1}} \left\{ \left(\frac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}} |z| \right\}, \tag{4}$$

with the generalized error function defined as, see [1],

$$\mathrm{Erf}_a(x) = \frac{\Gamma(a+1)}{\sqrt{\pi}} \int_0^x e^{-t^a} dt. \tag{5}$$

Furthermore, the covariance matrix of $N_\gamma(\boldsymbol{\mu}, \mathbf{\Sigma})$ can be evaluated as

$$Cov(\boldsymbol{X}; \gamma) = \frac{\Gamma((p+2)\frac{\gamma-1}{\gamma})}{\Gamma(p\frac{\gamma-1}{\gamma})} \left(\frac{\gamma}{\gamma-1}\right)^{2\frac{\gamma-1}{\gamma}} (rank\mathbf{\Sigma})^{-1}\mathbf{\Sigma}. \tag{6}$$

The Laplace transform of $N_\gamma(\mu, \sigma^2)$ has been evaluated in [14], and with $\gamma = 2$ the Laplace transform of the classical Normal is obtained.

For the univariate case $p = 1$, relation (1) becomes

$$\phi_\gamma(x; \mu, \sigma^2) = \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{\gamma-1}{\gamma}+1)} \left(\frac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}} \frac{1}{\sqrt{\pi\sigma^2}} \exp\left\{ -\frac{\gamma-1}{\gamma} \left(\frac{|x-\mu|}{\sigma}\right)^{\frac{\gamma}{\gamma-1}} \right\}. \tag{7}$$

Notice that $\phi_2 = \phi$, the pdf of the well-known Normal distribution. For the simulation problem of the γGN see [13].

The most well-known continuous Markov stochastic process $\{X(t); t \in [0, \infty)\}$, [24], is the Brownian motion, [23], [9, Chapter 2], also known as Wiener process, [4, Chapter 4]. The basic framework is that for the stochastic process, as above, $X(t)$ is considered the x component of a particle, always as a function of time. Let at the time t_0, $X(t_0) = x_0$ and let the conditional probability density of $X(t + t_0)$ given $X(t_0) = x_0$ to be $p(x, t|x_0)$. We assume that for "small t" $X(t + t_0) \approx X(t_0)$ i.e. $\lim_{t\to 0} p(x, t|x_0) = 0$, see for details ([22], [7]).

The behavior of this stochastic process has been explained in an excellent work, [6], where it was proved that

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} \tag{8}$$

where $D = 2RT/Nf$ is the so called diffusion coefficient, [5], with R being the gas constant, T is the temperature, N is the Avogadro number and f is the coefficient of friction. After this result, the diffusion equation (8) attracted a special interest, see also Discussion. From an Analysis point of view as a partial differential equation, [26], among others, from physical point of view, known as Heat Equation - modeling the proportion of the amount of heat divided by the "amount" (precisely the mass) of the material, with a proportionality factor, which under a proper scale can be $D = 1/2$ i.e. (8) is reduced to

$$\frac{\partial^2 p}{\partial x^2} = 2\frac{\partial p}{\partial t}. \tag{9}$$

Eventually Probability theory is also involved as we can easily verify that the unique solution of (9), under the boundary conditions

(a) $\lim_{t\to 0} p(x, t|x_0) = 0, \quad x \neq x_0$

(b) $p(x, t|x_0)$ is a density function in x, thus $p(x, t|x_0) \geq 0$ and $\int_{-\infty}^{\infty} p(x, t|x_0)dx = 1$.

is:

$$p(x, t|x_0) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{1}{2t}(x - x_0)^2\right\} \tag{10}$$

i.e. if, without loss of generality, we assume that $x_0 = 0$, $p(x, t|x_0)$ coincides with the (distribution function of the) standard Normal distribution

$$\phi(x) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{1}{2}\left(\frac{x}{\sqrt{t}}\right)^2\right\} := \phi(x; 0, t). \tag{11}$$

We work with a general form of (11), that is (7), for $\mu = 0$ and $\sigma^2 = \sigma_t^2 = t$ as

$$\phi_\gamma(x; 0, t) = \frac{\lambda_\gamma}{\sqrt{\pi t}} \exp\left\{-\frac{\gamma - 1}{\gamma}\left(\frac{|x|}{\sqrt{t}}\right)^{\frac{\gamma}{\gamma-1}}\right\} \tag{12}$$

where the constant term, depending on γ, is

$$\lambda_\gamma = \frac{\Gamma(\frac{1}{2} + 1)}{\Gamma(\frac{\gamma-1}{\gamma} + 1)}\left(\frac{\gamma - 1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}}. \tag{13}$$

The stochastic process in (12), see [7], [10] among others, follows a γ-order Generalized Normal distribution.

Therefore we have considered a Brownian motion $\{X(t); t \geq 0\}$ and have assumed that every increment follows a γ-order Normal distribution with mean 0 and variance σ_t^2 with σ fixed. Letting also $X(0) = 0$ as usually, and assuming $X(t)$ is continuous at $t = 0$ we may also assume w.l.o.g. that $\sigma = 1$, or that the

Brownian motion is standard, i.e. $N_\gamma(0,t)$ is considered. This stochastic process is applied to generalize the Heat Equation (HE) [11], as we present below, while in Section 3.1 a new extension is presented due to the introduced in Section 2.1 Generalized Hermite Polynomials.

For solutions of the HE see [25] among others, while the generalization of the HE due to the γGN is presented at the following theorem. For other mathematically oriented generalizations of the HE, see [19] where the fractional heat equation is treated.

Theorem 1.1 ([11]). *The γ-order generalized form of the Heat Equation is*

$$\frac{\partial^2 \phi_\gamma}{\partial x^2} = K \frac{\partial \phi_\gamma}{\partial t}, \tag{14}$$

where

$$K = K(x;t,\gamma) = \frac{t^{-\frac{\gamma}{\gamma-1}} |x|^{\frac{2}{\gamma-1}} - \frac{1}{\gamma-1} t^{-\frac{\gamma}{2(\gamma-1)}} |x|^{\frac{2-\gamma}{\gamma-1}}}{\frac{1}{2}(-t^{-1} + t^{-\frac{3\gamma-2}{2(\gamma-1)}} |x|^{\frac{\gamma}{\gamma-1}})} = \frac{N(x;t,\gamma)}{D(x;t,\gamma)} \tag{15}$$

and ϕ_γ from (12), namely $\phi_\gamma = \phi_\gamma(x;\mu,\sigma^2) = \phi_\gamma(x;\mu,t)$.

Traditionally in Statistics, we consider $t = 1$, see [10].

The following plots of γGN and $K(x;t,\gamma)$ try to clarify the described above situation. In Figure 1 the first column presents the functions ϕ_γ while the second column presents the corresponding values of the $K(x;t,\gamma)$. Notice that at the value $\gamma = 2$ (the red solid line) it is calculated that $K = 2$ while for the other members of the family of γGN the corresponding values of $K(x;t,\gamma)$, which is a nonconstant function of $|x|/\sqrt{t}$, appear to have easily calculated functional forms.

The physical interpretation of the above results is beyond the target of this paper. In the following Section 2 the Hermite Polynomials (HP) are discussed and extended and eventually, as it is proved, related to a new generalized form of the presented above HE.

2. The γ-order Generalized Hermite Polynomials

Recall that the Hermite Polynomials are related to the Normal distribution. In this section we introduce a generalized γ-order form of Hermite Polynomials linked with the γGN, see for example (29). This generalization provides evidence that an extended form of the Generalized Heat Equation can be produced, see Section 3.1. The Hermite Polynomials are discussed below as an introduction to the Generalized Hermite Polynomials.

The Hermite Polynomials, defined by Laplace in 1811, and named on 1864 with a paper of Hermite referring to these polynomials, see [3, Chapter 5], are expressed as follows, where for the case $k = 1$ see [3, Theorem 5.2] and for the case $k = 1/2$ see [1, Section 22.5.18]

$${}_kH_n(x) = (-1)^n \exp\{kx^2\} \frac{d^n}{dx^n}[\exp\{-kx^2\}]. \tag{16}$$

The main two families of HP are:

- ${}_{1/2}H_n(x)$ are referred as the probabilistic HP [$k = 1/2$]
- ${}_1H_n(x)$ are referred as the physicist's HP [$k = 1$]

The above families of HP are symmetric and orthogonal wrt an appropriate measure, see also [21, Chapter 4], as stated in the following Proposition, extended below, Corollary 3.5.

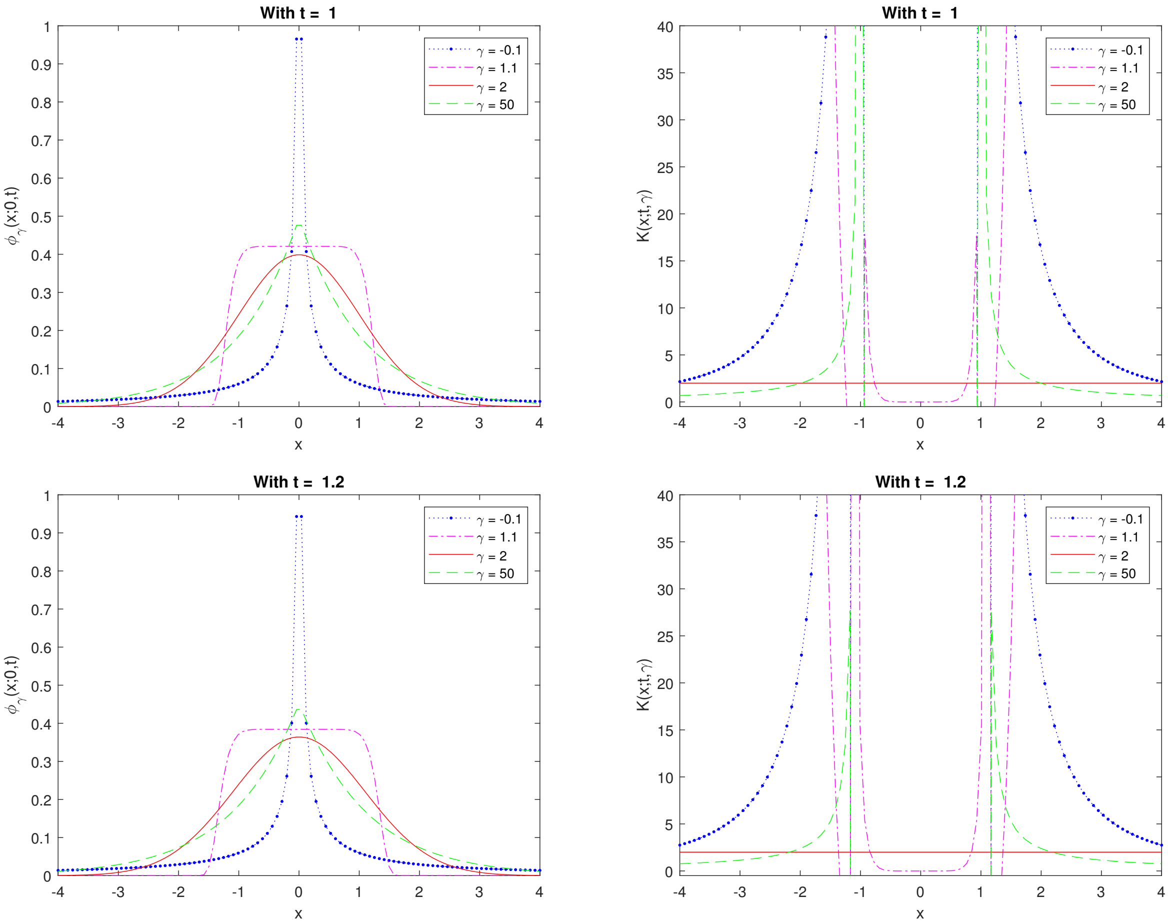

Figure 1: Plots of the univariate $\phi_\gamma(x; 0, t)$ and $K(x; t, \gamma)$ for different t, γ.

Proposition 2.1. *Consider the Hermite polynomials as in (16). Then they are either even (when $n = 2k, k \in \mathbb{N}$) or odd functions (when $n = 2k + 1, k \in \mathbb{N}$),*

$$_kH_n(-x) = (-1)^n {_kH_n}(x) \tag{17}$$

and they are orthogonal wrt to the weight measure $w(x) = e^{-kx^2}$

$$\int_{-\infty}^{\infty} {_kH_q}(x) {_kH_n}(x) e^{-kx^2} dx = \begin{cases} \sqrt{2\pi} n! \delta_{n,q} & k = 1/2 \\ \sqrt{\pi} 2^n n! \delta_{n,q} & k = 1 \end{cases}, \tag{18}$$

with $\delta_{n,q}$ being the Kronecker delta, that is $\delta_{n,q} = 1$ when $n = q$ and zero otherwise.

For a survey on Hermite polynomials we refer to [20] and references therein.

2.1. *Extensions*

Theorem 2.2 supports the development of the introduced line of thought about the γ-order GHP. In particular, the following theorem provides the partial derivatives of $\phi_\gamma(x, t)$ with respect to x of degree r for any γ.

Theorem 2.2 ([15]). *The r-th partial derivative with respect to x of $\phi_\gamma(x,t)$ satisfies*

$$\begin{aligned}&(-1)^r\frac{\partial^r}{\partial x^r}\phi_\gamma(x,t)=\phi_\gamma(x,t)\Big(|x|^{r(\frac{\gamma}{\gamma-1}-1)}t^{-r\frac{\gamma}{2(\gamma-1)}}\\&+c_{r,1}(\gamma)|x|^{r(\frac{\gamma}{\gamma-1}-1)-\frac{\gamma}{\gamma-1}}t^{-(r-1)\frac{\gamma}{2(\gamma-1)}}+c_{r,2}(\gamma)|x|^{r(\frac{\gamma}{\gamma-1}-1)-2\frac{\gamma}{\gamma-1}}t^{-(r-2)\frac{\gamma}{2(\gamma-1)}}\\&+\cdots+c_{r,r-1}(\gamma)|x|^{r(\frac{\gamma}{\gamma-1}-1)-(r-1)\frac{\gamma}{\gamma-1}}t^{-\frac{\gamma}{2(\gamma-1)}}\Big)(\mathrm{sgn}(x))^r.\end{aligned}\tag{19}$$

Notice the interesting characteristic that the coefficients $c_{r,j}$, being functions of γ, satisfy for $j=2,\dots,r-2$ an iterative scheme, a typical characteristic for the known families of HP. Here it is,

$$c_{r,j}(\gamma)=c_{r-1,j}(\gamma)-(\gamma-1)^{-1}c_{r-1,j-1}(\gamma)(r-1-(j-1)\gamma),\tag{20}$$

with

$$\begin{aligned}c_{r,0}&=1,\quad c_{r,r}=0\\c_{r,1}(\gamma)&=-\frac{\frac{\gamma}{\gamma-1}-1}{2}r(r-1)\\c_{r,r-1}(\gamma)&=(-1)^{r-1}\left(\frac{\gamma}{\gamma-1}-1\right)\left(\frac{\gamma}{\gamma-1}-2\right)\cdots\left(\frac{\gamma}{\gamma-1}-r+1\right).\end{aligned}\tag{21}$$

Note that for $r=1$ (19) implies that

$$\frac{\partial}{\partial x}\phi_\gamma(x,t)=-\phi_\gamma(x,t)|x|^{\frac{1}{\gamma-1}}t^{-\frac{\gamma}{2(\gamma-1)}}\mathrm{sgn}(x)=\phi_\gamma(x,t)B_1(x;t,\gamma)\tag{22}$$

while for $r=2$ in (19)

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}\phi_\gamma(x,t)=\phi_\gamma(x,t)\left(|x|^{2(\frac{\gamma}{\gamma-1}-1)}t^{-\frac{\gamma}{\gamma-1}}+c_{2,1}(\gamma)|x|^{2(\frac{\gamma}{\gamma-1}-1)-\frac{\gamma}{\gamma-1}}t^{-\frac{\gamma}{2(\gamma-1)}}\right)(\mathrm{sgn}(x))^2\\=\;&\phi_\gamma(x,t)\left(|x|^{\frac{2}{\gamma-1}}t^{-\frac{\gamma}{\gamma-1}}-\left(\frac{\gamma}{\gamma-1}-1\right)|x|^{\frac{\gamma}{\gamma-1}-2}t^{-\frac{\gamma}{2(\gamma-1)}}\right)=\phi_\gamma(x,t)B_2(x;t,\gamma).\end{aligned}\tag{23}$$

Theorem 2.2 gives us the appropriate theoretical background to introduce the following definition of γ-order GHP.

Definition 2.3. *The γ-order Generalized Hermite Polynomials $H_{r,\gamma}$ with particular values of γ as in (24) and degree $n=r(m-1), r=0,1,2,\dots$*

$$\gamma=\gamma(m)=\frac{m}{m-1},\quad m>1, m\in\mathbb{N},\tag{24}$$

are defined as

$$\begin{aligned}H_{r,\gamma(m)}(w)&=w^{-r}\Big(|w|^{rm}+c_{r,1}(\gamma(m))|w|^{(r-1)m}+\ldots+c_{r,r-1}(\gamma(m))|w|^m\Big)\\&=w^{-r}\sum_{j=0}^{j-1}c_{r,j}(\gamma(m))|w|^{(r-j)m}.\end{aligned}\tag{25}$$

For $r=0$ according to (19), (20) and (21) it is $H_{0,\gamma}(w)=1$ for all γ of the form (24). Concerning the coefficients in (25) it holds that

$$\begin{aligned}c_{r,1}(\gamma(m))&=-\frac{(m-1)}{2}r(r-1)\\c_{r,r-1}(\gamma(m))&=(-1)^{r-1}(m-1)(m-2)\cdots(m-r+1)\\c_{r,j}(\gamma(m))&=c_{r-1,j}(\gamma(m))-c_{r-1,j-1}(\gamma(m))\left((r-1)(m-1)-(j-1)m\right),\end{aligned}\tag{26}$$

for $j=2,\dots,r-2$. See also Table 1 for the values of the coefficients $c_{r,j}$ of the γ-order GHP for selected values of r and m.

Example 2.4. *Through (26) the polynomial formula of different γ-order GHP is obtained, which can be used on desired applications. For $r = 1$ note that $c_{1,0}(\gamma(m)) = 1$ and using the identity $|w| = w\mathrm{sgn}(w)$, we find that*

$$\begin{aligned} H_{1,\gamma(m)}(w) &= w^{-1}c_{1,0}(\gamma(m))|w|^m \\ &= |w|^m w^{-1} = w^{m-1}[\mathrm{sgn}(w)]^m. \end{aligned} \tag{27}$$

For $m = 2, 3, 4$ and 5, $r = 1$, and using again the sign function instead of the absolute value we get that

$$H_{1,\gamma(2)}(w) = w, H_{1,\gamma(3)}(w) = w^2\mathrm{sgn}(w), H_{1,\gamma(4)}(w) = w^3, H_{1,\gamma(5)}(w) = w^4\mathrm{sgn}(w).$$

For $r = 2$ note that by (21) it is $c_{2,1}(\gamma(m)) = -(m-1)$ therefore

$$\begin{aligned} H_{2,\gamma(m)}(w) &= w^{-2}(|w|^{2m} + c_{2,1}(\gamma(m))|w|^m) \\ &= w^{2m-2} + (1-m)|w|^m w^{-2} \\ &= w^{2m-2} + (1-m)w^{m-2}[\mathrm{sgn}(w)]^m. \end{aligned} \tag{28}$$

For $m = 2, 3, 4$ and 5, $r = 2$ as well as for other values of r the results are collected in Table 1.

Table 1: The generalized Hermite polynomials $H_{r,\gamma(m)}$ for different values of r and m.

$r \backslash m$	2	3	4	5
0	1	1	1	1
1	w	$w^2\mathrm{sgn}(w)$	w^3	$w^4\mathrm{sgn}(w)$
2	$w^2 - 1$	$w^4 - 2w\mathrm{sgn}(w)$	$w^6 - 3w^2$	$w^8 - 4w^3\mathrm{sgn}(w)$
3	$w^3 - 3w$	$w^6\mathrm{sgn}(w) - 6w^3 + 2\mathrm{sgn}(w)$	$w^9 - 9w^5 + 6w$	$w^{12} - 12w^7 + 12w^2\mathrm{sgn}(w)$
4	$w^4 - 6w^2 + 3$	$w^8 - 12w^5\mathrm{sgn}(w) + 20w^2$	$w^{12} - 18w^8 + 51w^4 - 6$	$w^{16} - 24w^{11}\mathrm{sgn}(w) + 96w^6 - 24w\mathrm{sgn}(w)$

Note that for $m = 2$ which by (24) gives $\gamma(2) = 2$ and corresponds to the classical Normal distribution, the GHP (27), (28), become, see also Table 1,

$H_{0,2}(w) = 1$ $\quad H_{1,2}(w) = w$ $\quad H_{2,2}(w) = w^2 - 1$,
$H_{3,2}(w) = w^3 - 3w$, $\quad H_{4,2}(w) = w^4 - 6w^2 + 3$.

Recall (16) and notice that the above coincide with the first five probabilistic HP, ${}_{1/2}H_0(w)$, ${}_{1/2}H_1(w)$, ${}_{1/2}H_2(w)$, ${}_{1/2}H_3(w)$ and ${}_{1/2}H_4(w)$ respectively.

3. The solution of the GHE due to the γ-order GHP

Recall that the HP are related to the partial derivatives of the Normal distribution. The way we introduced the γ-order GHP $H_{r,\gamma(m)}$ provides similar results and justifies the γ-notation: the γ-order GHP are related to the γGN distribution, $\phi_\gamma(x,t)$. Notice the differential equation (19) in Theorem 2.2 and Definition 2.3. Then the following example clarifies the preceding discussion.

Proposition 3.1. *The γ-order GHP are related with the differential equation (19) as follows*

$$(-1)^r \frac{\partial^r}{\partial x^r}\phi_\gamma(x,t) = \phi_\gamma(x,t)t^{-r/2}H_{r,\gamma}(\frac{x}{\sqrt{t}}). \tag{29}$$

For $\gamma = 2$ relation (29) is reduced to

$$(-1)^r \frac{\partial^r}{\partial x^r}\phi_2(x,t) = \phi_2(x,t)t^{-\frac{r}{2}}H_{r,2}(\frac{x}{\sqrt{t}}), \tag{30}$$

with

$$H_{r,2}(\frac{x}{\sqrt{t}}) = (\frac{x}{\sqrt{t}})^r + a_{r,1}(\frac{x}{\sqrt{t}})^{r-2} + \cdots + a_{r,r-1}(\frac{x}{\sqrt{t}}),$$

while the coefficients $a_{r,j}$ of the Hermite polynomial $H_r(x) = H_{r,2}(x)$ are now

$$a_{r,1} = -\frac{r(r-1)}{2}, \; a_{r,2} = \frac{r(r-1)(r-2)(r-3)}{2^2 2!}, \ldots$$

Consider (29) and (12) with $t = 1$. Then the γ-order GHP are expressed as

$$\begin{aligned} H_{r,\gamma}(x) &= (-1)^r \frac{1}{\phi_\gamma(x,1)} \frac{\partial^r}{\partial x^r} \phi_\gamma(x,1) \\ &= (-1)^r \frac{\exp\left\{\frac{\gamma-1}{\gamma}|x|^{\frac{\gamma}{\gamma-1}}\right\}}{\frac{\lambda_\gamma}{\sqrt{\pi}}} \frac{\partial^r}{\partial x^r} \frac{\lambda_\gamma}{\sqrt{\pi}} \exp\left\{-\frac{\gamma-1}{\gamma}|x|^{\frac{\gamma}{\gamma-1}}\right\} \end{aligned} \tag{31}$$

Applying the values of $\gamma = \frac{m}{m-1}$ as in (24) in Definition 2.3 and applying the differential operator $D^r(x) = \frac{\partial^r}{\partial x^r}$, (31) becomes after some algebra

$$H_{r,\gamma}(x) = (-1)^r \exp\left\{\frac{1}{m}|x|^m\right\} D^r\left(\exp\left\{-\frac{1}{m}|x|^m\right\}\right), \quad \gamma = \gamma(m), m > 1. \tag{32}$$

These introductory notes provide the appropriate material to prove the orthogonality property wrt the weight measure $e^{-\frac{1}{m}|x|^m}$ in Theorem 3.2 below.

Theorem 3.2. *Consider the γ-order GHP as in Definition 2.3. Then they are orthogonal wrt the weight measure $w_{\gamma(m)} = e^{-\frac{1}{m}|x|^m}$, that is*

$$\int_{-\infty}^{\infty} e^{-\frac{1}{m}|x|^m} H_{r,\gamma}(x) H_{s,\gamma}(x) dx = [s(m-1)]! 2 \cdot m^{\frac{1}{m}-1} \Gamma\left(\frac{1}{m}\right) \delta_{r,s(m-1)}, \tag{33}$$

where $m > 1, m \in \mathbb{N}$, with $\delta_{r,s(m-1)} = 1$ when $r = s(m-1)$ and zero otherwise.

Proof. Denote by I the integral on the lhs of (33). By representation (32) and for the case $r > s(m-1)$ it is

$$\begin{aligned} I &= \int_{-\infty}^{\infty} e^{-\frac{1}{m}|x|^m} H_{r,\gamma}(x) H_{s,\gamma}(x) dx \\ &= \int_{-\infty}^{\infty} e^{-\frac{1}{m}|x|^m} (-1)^r e^{-\frac{1}{m}|x|^m} D^r\left(\exp\left\{-\frac{1}{m}|x|^m\right\}\right) H_{s,\gamma}(x) dx \\ &= (-1)^r \int_{-\infty}^{\infty} H_{s,\gamma}(x) D^r\left(\exp\left\{-\frac{1}{m}|x|^m\right\}\right) dx. \end{aligned} \tag{34}$$

Integration by parts r times results to the r-th derivative of $H_{s,\gamma}(x)$ since all boundary terms vanish because of the super exponential decay of $e^{-\frac{1}{m}|x|^m}$ at infinity beating any polynomial. Indeed the integral (34) becomes

$$I = (-1)^r \Big[H_{s,\gamma}(x) D^{r-1}\left(\exp\left\{-\frac{1}{m}|x|^m\right\}\right)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} D\left(H_{s,\gamma}(x)\right) D^{r-1}\left(\exp\left\{-\frac{1}{m}|x|^m\right\}\right) dx \Big] \tag{35}$$

$$\begin{aligned} &= (-1)^{r+1} \int_{-\infty}^{\infty} D(H_{s,\gamma}(x)) D^{r-1}\left(\exp\left\{-\frac{1}{m}|x|^m\right\}\right) dx \\ &= \ldots = (-1)^{2r} \int_{-\infty}^{\infty} D^r(H_{s,\gamma}(x)) \exp\left\{-\frac{1}{m}|x|^m\right\} dx = 0 \end{aligned} \tag{36}$$

by noting that the degree of $H_{s,\gamma}$ is $s(m-1)$ and $r > s(m-1)$. When $r = s(m-1)$ and recalling that the leading coefficient in (25) is 1 then (36) becomes

$$\begin{aligned} I &= \int_{-\infty}^{\infty} D^{s(m-1)}(H_{s,\gamma}(x)) \exp\left\{-\frac{1}{m}|x|^m\right\} dx \\ &= [s(m-1)]! \frac{\sqrt{\pi}}{\lambda_{\gamma(m)}} \int_{-\infty}^{\infty} \frac{\lambda_{\gamma(m)}}{\sqrt{\pi}} \exp\left\{-\frac{1}{m}|x|^m\right\} dx, \end{aligned} \tag{37}$$

where in (37) by technicality the quantity $\frac{\lambda_{\gamma(m)}}{\sqrt{\pi}}$ multiplied by its reciprocal equals 1. Using the fact that $\Gamma(x+1) = x\Gamma(x)$ and the value $\Gamma(1/2) = \sqrt{\pi}$ the quantity $\lambda_{\gamma(m)}$ for $\gamma(m) = \frac{m}{m-1}$ in (13) equals

$$\lambda_{\gamma(m)} = \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{1}{m}+1)}\left(\frac{1}{m}\right)^{\frac{1}{m}} = \frac{\sqrt{\pi}}{2\Gamma(\frac{1}{m})}\left(\frac{1}{m}\right)^{\frac{1}{m}-1}. \tag{38}$$

Moreover the integral on the rhs of (37) is equal to 1 since it is the integral of the γGN over all possible values of x and thus plugging (38) into (37) yields

$$I = [s(m-1)]!\frac{\sqrt{\pi}}{\frac{\sqrt{\pi}}{2\Gamma(\frac{1}{m})}\left(\frac{1}{m}\right)^{\frac{1}{m}-1}} = [s(m-1)]!2\cdot m^{\frac{1}{m}-1}\Gamma(\frac{1}{m}). \tag{39}$$

(36) and (39) imply (33). □

Corollary 3.3. *Consider the* 2*-order GHP as in Definition 2.3. Then they are orthogonal wrt the weight measure* $w_2(x) = e^{-\frac{1}{2}|x|^2}$*, that is*

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}|x|^2} H_{r,2}(x)H_{s,2}(x)dx = s!\sqrt{2\pi}\delta_{r,s}. \tag{40}$$

Proof. Application of Theorem 3.2 with $\gamma = \gamma(m) = 2$ and thus $m = 2$ implies that

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\frac{1}{2}|x|^2} H_{r,2}(x)H_{s,2}(x)dx &= s!2\cdot 2^{\frac{1}{2}-1}\Gamma\left(\frac{1}{2}\right)\delta_{r,s} \\ &= s!\sqrt{2}\sqrt{\pi}\delta_{r,s} = s!\sqrt{2\pi}\delta_{r,s}.\end{aligned}$$

Thus (18) in Proposition 2.1 has been generalized by (40). □

Concerning the symmetry of the γ-order GHP, it still holds as in the classical case due to the following stated and proved proposition.

Proposition 3.4. *Consider the γ-order GHP as in (25). Then, they are either even (when $n = 2k, k \in \mathbb{N}$) or odd functions (when $n = 2k+1, k \in \mathbb{N}$),*

$$H_{r,\gamma(m)}(-x) = (-1)^r H_{r,\gamma(m)}(x) \tag{41}$$

where $m > 1, m \in \mathbb{N}$.

Indeed: By (25) the following can be proved

$$\begin{aligned}H_{r,\gamma(m)}(-x) &= (-x)^{-r}\Big(|-x|^{rm} + c_{r,1}(\gamma(m))|-x|^{(r-1)m} + \ldots + c_{r,r-1}(\gamma(m))|-x|^{m}\Big) \\ &= (-1)^r x^{-r}\Big(|x|^{rm} + c_{r,1}(\gamma(m))|x|^{(r-1)m} + \ldots + c_{r,r-1}(\gamma(m))|x|^{m}\Big) \\ &= (-1)^r H_{r,\gamma(m)}(x),\end{aligned}$$

thus their symmetry, see (17), is retained.

Corollary 3.5. *For the γ-order GHP as in (25) there is a symmetry property as is (41) and an orthogonality relation as in (33).*

Example 3.6. *Consider the γ-order GHP, $H_{2,\gamma(2)}$ and $H_{3,\gamma(2)}$, recall Table 1 and the weight function $w_{\gamma(2)} = e^{-\frac{1}{2}|x|^2}$. Then*

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}|x|^2} H_{2,\gamma(2)}(x)H_{3,\gamma(2)}(x)dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}|x|^2}(x^2-1)(x^3-3x)dx = 0,$$

since the integration of an odd function in a symmetric domain is zero, while by integration by parts it follows that

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{-\frac{1}{2}|x|^2}(H_{2,\gamma(2)}(x))^2 dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}|x|^2}(x^2-1)^2 dx \\ = & \int_{-\infty}^{\infty} 3x^2 e^{-\frac{1}{2}|x|^2} dx - \int_{-\infty}^{\infty} 2x^2 e^{-\frac{1}{2}|x|^2} dx + \int_{-\infty}^{\infty} e^{-\frac{1}{2}|x|^2} dx \\ = & \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}|x|^2} dx + \int_{-\infty}^{\infty} e^{-\frac{1}{2}|x|^2} dx \\ = & \ 2\int_{-\infty}^{\infty} e^{-\frac{1}{2}|x|^2} dx = 2\sqrt{2\pi} \end{aligned}$$

which correspond to the result of Corollary 3.3. Now, for the γ-order GHP, $H_{1,\gamma(4)}, H_{2,\gamma(4)}$ and $H_{3,\gamma(4)}$, recall Table 1 and the weight function $w_{\gamma(4)} = e^{-\frac{1}{4}|x|^4}$ it holds that

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{-\frac{1}{4}|x|^4} H_{2,\gamma(4)}(x) H_{3,\gamma(4)}(x) dx = \int_{-\infty}^{\infty} e^{-\frac{1}{4}|x|^4}(x^6-3x^2)(x^9-9x^5+6x)dx \\ = & \int_{-\infty}^{\infty} e^{-\frac{1}{4}|x|^4}(x^{15}-12x^{11}+33x^7-18x^3)dx = 0, \end{aligned}$$

since the integration of an odd function in a symmetric domain is zero, while by integration by parts and (38) it follows that

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{-\frac{1}{4}|x|^4}(H_{1,\gamma(4)}(x))(H_{3,\gamma(4)}(x))dx = \int_{-\infty}^{\infty} e^{-\frac{1}{4}|x|^4} x^3(x^9-9x^5+6x)dx \\ = & -\int_{-\infty}^{\infty} \left(e^{-\frac{1}{4}|x|^4}\right)' (x^9-9x^5+6x)dx \\ = & \int_{-\infty}^{\infty} (9x^8-45x^4) e^{-\frac{1}{4}|x|^4} dx + 6\int_{-\infty}^{\infty} x^4 e^{-\frac{1}{4}|x|^4} dx \\ = & -\int_{-\infty}^{\infty} \left(e^{-\frac{1}{4}|x|^4}\right)' (9x^5-45x)dx + 6\frac{\sqrt{\pi}}{\lambda_{\gamma(4)}} \int_{-\infty}^{\infty} \frac{\lambda_{\gamma(4)}}{\sqrt{\pi}} e^{-\frac{1}{4}|x|^4} dx \\ = & \int_{-\infty}^{\infty} e^{-\frac{1}{4}|x|^4}(45x^4-45)dx + 6\frac{\sqrt{\pi}\Gamma(\frac{1}{4}+1)}{\Gamma(\frac{1}{2}+1)\left(\frac{1}{4}\right)^{\frac{1}{4}}} \\ = & -\int_{-\infty}^{\infty} 45x\left(e^{-\frac{1}{4}|x|^4}\right)' dx - 45\int_{-\infty}^{\infty} e^{-\frac{1}{4}|x|^4} dx + 6\frac{\sqrt{\pi}\frac{1}{4}\Gamma(\frac{1}{4})}{\frac{1}{2}\Gamma(\frac{1}{2})} 4^{\frac{1}{4}} \\ = & \ 3!2\cdot 4^{\frac{1}{4}-1}\Gamma(\frac{1}{4}) \end{aligned}$$

which corresponds to the new results stated in Theorem 3.2, see (33).

3.1. *The γ-order GHP and the Heat Equation*

Equation (29) provides the appropriate food for thought to work in Theorem 3.7 offering a theoretical extension of the already generalized form of the Heat Equation as in Theorem 1.1. The generalized form of the γ-order GHE can be now achieved due to the following theorem which is a combination of (15) and (29) and we omit the proof. It is emphasized that the coefficient K_r in (43) below is an extended form of K in (14).

Theorem 3.7. *The partial derivatives with respect to x of order r and the first order partial derivative with respect to t of the γGN distribution are linked as*

$$(-1)^r \frac{\partial^r}{\partial x^r}\phi_\gamma(x,t) = K_r \frac{\partial \phi_\gamma}{\partial t}, \tag{42}$$

where

$$K_r = K_r(x,t,\gamma) = \frac{t^{-r/2}H_{r,\gamma}(\frac{x}{\sqrt{t}})}{\frac{1}{2}\left(-\frac{1}{t} + \frac{|x|^{\frac{\gamma}{\gamma-1}}}{t^{\frac{3\gamma-2}{2(\gamma-1)}}}\right)} =: \frac{N_r(x;t,\gamma)}{D(x;t,\gamma)}, \tag{43}$$

where the definition of $N_r(x;t,\gamma)$ *is obvious.*

Notice that the coefficient function K_r in (43) above under particular values of r and γ coincides with the coefficient function K in (15) and in that sense it is a generalization of it, see Figure 4.

Example 3.8. *For* $t = 1, \gamma = 2$, *the Normal distribution case, (43) provides the value*

$$K_r(x;1,2) = \frac{H_{r,2}(x)}{\frac{1}{2}(-1 + |x|^2)},$$

while the coefficient $K_r(x;t,\gamma)$ *in (15) for* $t = 1, r = 2$ *and* $\gamma = 2$ *equals*

$$K_2(x;1,2) \quad = \quad \frac{|x|^2 - 1}{\frac{1}{2}(-1 + |x|^2)} = 2,$$

therefore by combing the above relations it holds that

$$K_r(x;1,2) = K_2(x;1,2)\frac{H_{r,2}(x)}{|x|^2 - 1}. \tag{44}$$

By the above evaluations for $K_r(x;1,\gamma)$ *and* $K(x;1,\gamma)$ *it becomes apparent how the generalized coefficient affects in the numerator the coefficient K.*

Notice that with $r = 2$ the γ-order GHE (15) is once again obtained, while with $r = 2$ and $\gamma = 2$ the classical HE is obtained. The γ-order notation, as it has been introduced, provides in all cases the classical situation for $\gamma = 2$. In that sense (43) is an extended form of (15). In particular, the numerator of (15) has been extended while the denominator remains the same. The application of the γ-order GHP provided a theoretical extension to the Generalized form of the HE.

Example 3.9. *The function* $N(x,t;\gamma)$ *in (15), is presented in Figure 2 as an application of the introduced and defined in (25)* γ*-order GHP. The denominator* $D(x,t;\gamma)$ *in (15), remains the same in (43) and is illustrated in Figure 3. Both functions* $N(\cdot,\cdot)$ *and* $D(\cdot,\cdot)$ *have a minimum, at* (x_*,t_*)*, which has been evaluated numerically. In particular, see also Figure 2,*

$$\min_{x\in[-3,3],t\in[0.8,1.2]} N(x;t;1.5) = -1.4882$$

at the symmetric values $x_{*(1)} = -0.7080$ *and* $x_{*(2)} = 0.7080$ *with* $t_* = 0.8$. *For* $\gamma = 2$, *similarly*

$$\min_{x\in[-3,3],t\in[0.8,1.2]} N(x;t,2) = -1.25$$

at $x_* = 0, t_* = 0.8$ *for the considered values of* γ *in Figure 2. For the denominator it is*

$$\min_{x\in[-3,3],t\in[0.8,1.2]} D(x;t,\gamma) = -0.625$$

when $x_* = 0, t_* = 0.8$ *for both values of* γ*, see Figure 3. Notice that values of t close to 1 correspond to the ratio value* $N/D = 2$*, so the crucial value* $K = 2$ *is still valid for* $t = 1 \pm 0.2$.

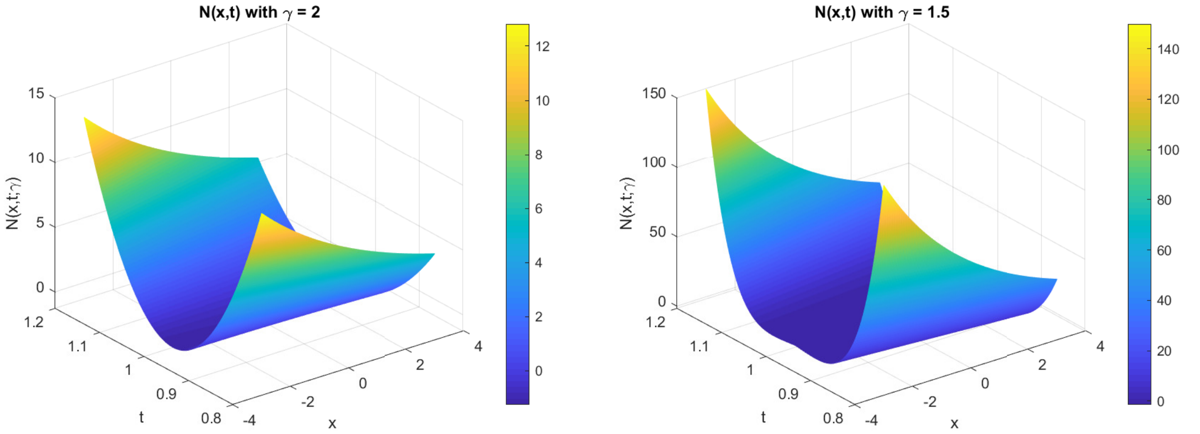

Figure 2: The polynomial $N(x;t,\gamma) = t^{-1/2}H_{2,\gamma}(\frac{x}{\sqrt{t}})$ in (15) and (29) for different values of x and t.

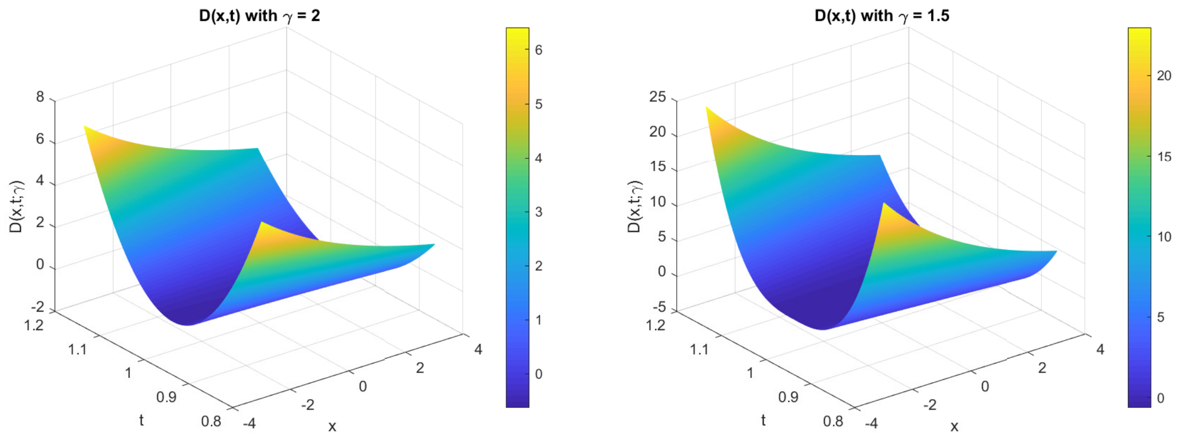

Figure 3: The coefficient function $D(x;t,\gamma)$ in (15) for different values of x and t and γ.

In Figure 4 the function $K(x;t,\gamma)$ in (15) is presented. It is easy to see that the coefficient function K in the lhs of Figure 4 is constant and equal to 2, while in the rhs the coefficient function "attempts" to behave as in the lhs for the particular values discussed above.

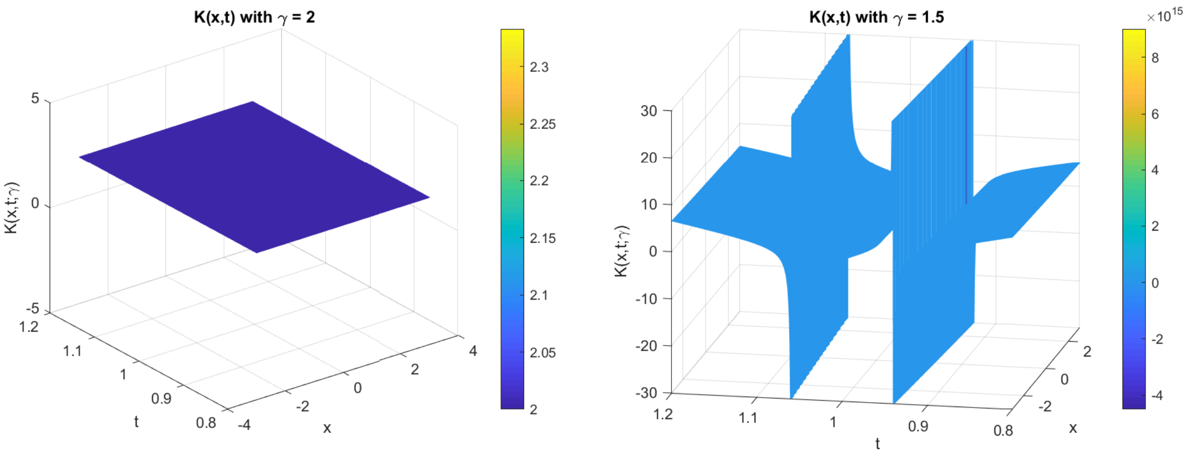

Figure 4: The coefficient function $K(x; t, \gamma)$ in (15) for different values of x and t and γ.

4. Discussion

From a physical point of view the appropriate name of the Heat Equation is diffusion equation with a source term. From an Analysis point of view the HE is known as a parabolic differential equation: briefly it describes the distribution of the heat flow (into/out) of a material in a given space over time. The proportional factor is the specific heat capacity of the material. The HE is a typical example of a continuity equation and it was related to the Gaussian. Now, with the γGN, a general form of the HE is offered. The shape parameter can describe a family of well-known distributions, even "close" to what is known as Gaussian or Normal distribution and still for this family of distributions a HE is satisfied. Most of the results concerning the new generalization of the HE are numerical results. For us the calculations are attractive for values of γ in the neighborhood of 2, i.e. "near" to the Normal distribution rather for "large" values, so to be near to Laplace distribution. The discussed in Introduction γGN has been already applied in economical problems, [12], [8], and in the present paper it has been applied in physical or biological problems, mainly in HE. We believe that the field of applications may be broadened and we are working on that direction, with Physical Chemistry, [2], being within our interests.

The theoretical results presented here, are not the only ones. It is essential for the function $K(x; t, \gamma)$, to be restricted for the "physical value" $t = 1$, which is also a Statistical consideration (when it is assumed that $\sigma_t^2 = 1$). The values of the shape parameter γ, may be those the researcher is interested in, and not only $\gamma = 2$. Thus, with the values of t and γ decided by the experimentalist, the coefficient function $K(x; t, \gamma)$ is not a complicated ratio to be evaluated, but its physical interpretation is crucial in each case. In the following Example 4.1 a way of further interpretation has been attempted.

Example 4.1. *With $p = 1$, as in (12), under the Brownian motion, the increment of particle position is regarded in the assumed time interval, say τ, from position x to position $x + y_\tau$. The coordinates are chosen so that the origin (0,0) is the initial position of the particle. The increment $y = y_\tau$ is a realization of the rv Y_τ and we assume $Y_\tau \sim N_\gamma(0, \tau)$ with pdf $\phi_\gamma(y_\tau)$ as in (12). Denoting, as usually, the density function by $\rho(\cdot, \cdot)$, then $\rho(x, t + \tau)$ can be written as*

$$\rho(x, t + \tau) = \int_{-\infty}^{\infty} \rho(x - y, t)\phi_\gamma(y)dy = \rho(x, t) + \frac{\partial^2 \rho}{\partial x^2} \int_{-\infty}^{\infty} \frac{y^2}{2} \phi_\gamma(y)dy + \ldots$$

Notice that odd moments vanish due to space symmetry. Therefore,

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \frac{\partial^2\rho}{\partial x^2}\cdot\int_{-\infty}^{\infty}\frac{y^2}{2\tau}\phi_\gamma(y)dy+\dots\\ &= \frac{1}{2\tau}\left(\int_{-\infty}^{\infty}y^2\phi_\gamma(y)dy\right)\frac{\partial^2\rho}{\partial x^2}+\dots\\ &= \frac{1}{2\tau}\mathbb{E}(Y^2)\frac{\partial^2\rho}{\partial x^2}+\dots\\ &= \frac{1}{2}\gamma_0^{-2\gamma_0}\frac{\Gamma(3\gamma_0)}{\Gamma(\gamma_0)}\frac{\partial^2\rho}{\partial x^2}+\dots \qquad (45)\end{aligned}$$

The second moment $\mathbb{E}(Y^2)$ of the γGN in (45) has been calculated by the general result, [18], as

$$\mathbb{E}(Y^{2k})=\gamma_0^{-2k\gamma_0}\frac{\Gamma((2k+1)\gamma_0)}{\Gamma(\gamma_0)}\tau^k,\quad Y\sim N_\gamma(0,\tau),$$

with $k=1$. The Taylor remainder in (45) includes higher-order moments which are not considered. The coefficient in the rhs of (45) is interpreted as a γ-order mass diffusivity, MD_γ, and it is equal to

$$MD_\gamma := \frac{1}{2}\gamma_0^{-2\gamma_0}\frac{\Gamma(3\gamma_0)}{\Gamma(\gamma_0)}, \qquad (46)$$

that is

$$\frac{\partial\rho}{\partial t}=MD_\gamma\frac{\partial^2\rho}{\partial x^2}. \qquad (47)$$

Example 4.2 (Continued). *For $\gamma=2$ in (46) it is evaluated that*

$$MD_2=\frac{1}{2}\left(\frac{1}{2}\right)^{-1}\frac{\Gamma(3/2)}{\Gamma(1/2)}=\frac{1}{2},$$

which corresponds to the known mass diffusivity value. Below, in Table 2 different values for the γ-order mass diffusivity, MD_γ, are presented.

Table 2: Values of MD_γ for different values of γ.

γ	1.8	1.9	2	2.1	2.2	3
MD_γ	0.4607	0.4811	0.5	0.5176	0.5340	0.6340

Notice that from Table 2, for valued of γ in the neighborhood of 2, $N_\gamma(2)$, the corresponding constant MD_γ is in the neighborhood of 1/2, $N_{MD_\gamma}(1/2)$. In conclusion we came across to another practical extension with a strong theoretical background. We believe other fields of applications will appear.

The γ-shape nature is emphasized, coming through the LSI. One could think to use a transformation, but this would cover the results through the LSI background. That is, the γ nature of our scenario is crucial to the provided development. That certainly needs more investigation for every particular case under study.

As HP are involved to Gaussian distribution the γ-order GHP are related to the γGN, as we have proved in Section 2, which is clear to the reader that it introduces new ideas within a strong mathematical framework. The HE is one of the most popular differential equations where Mathematicians, Physicists, Statisticians and Biologists work on. In principle Physicists and Biologists apply the classical HE. In the problems we discussed we generalized the Normal with the γGN and therefore a new of class of solutions has been provided to GHE with the GHP. This needs a particular study and investigation in the fields of applications. There is no yet a general interpretation of the proposed method, no general recipe. Concluding we would like to emphasize that:

(i) The defined GHP $H_{r,\gamma}(w)$ are related to the γGN.

(ii) The GHE, can be considered as a result of the introduced and defined GHP.

(iii) When the introduced parameter $\gamma = 2$ then we are reduced to the classical case.

There is an open window to a number of research applications. There are fields of applications to cover this framework, moving to other areas than fat-tailed economical data, that we have already worked with, while the problem of simulations has been covered, [13].

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