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The left non-conformable-Mercer fractional Hermite-Hadamard type inequalities using convex functions

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Abstract. This paper introduces a significant advancement in the field of fractional integral inequalities by establishing new Hermite-Hadamard (H-H) type inequalities for convex functions, specifically of the Jensen-Mercer kind, within the framework of non-conformable fractional calculus. The core of our approach leverages the support line property of convex functions to derive a foundational fractional (H-H)-Mercer inequality. From this cornerstone result, we develop new integral identities related to non-conformable fractional operators. These identities serve as powerful tools to prove a suite of associated trapezoidal and midpoint type inequalities that provide explicit bounds for the approximation error. This research is important as it deepens the link between convex analysis and fractional calculus, offering more versatile tools for analyzing the behavior of convex functions under generalized integral operators. The results are expected to have applications in various domains where fractional modeling and optimization are crucial, such as mathematical physics, engineering systems, and economic modeling, providing refined methods for approximation and error estimation.

1. Introduction

Convex function plays a notable character in the field of both theoretical and applied sciences. The study of convex functions always presents stunning and magnificent sight of the beauty in advanced mathematics. The mathematicians always put potential in this direction as a result, discover and survey a large variety of results that are beneficial and remarkable for applications. This method is effective in dealing with a wide range of problems, the majority of which may be found in both the pure and applied sciences. Convexity also has a finest effect on our daily lives through numerous applications in medicine, industry, business and art. The formulation of inequalities is one of the most important applications of the convex function. Many novel inequalities of various kind of categories related to convex function have been obtained and

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implemented to other fields of studies can be seen in [2–5]. In the literature, the (H-H) inequality is a highly familiar results. Furthermore, in many fields of science and technology, such as engineering, mathematical statistics, financial economics, and computer science, this inequality has been employed to solve a variety of problems. The definitions and outcomes listed below are considered necessary for our research.

Definition 1.1. *[6] A function $\Psi : [\kappa_1, \kappa_2] \to \mathbb{R}$ is convex, if*

$$\Psi(\lambda\zeta_1 + (1-\lambda)\zeta_2) \leq \lambda\Psi(\zeta_1) + (1-\lambda)\Psi(\zeta_2), \tag{1}$$

for all $\zeta_1, \zeta_2 \in [\kappa_1, \kappa_2]$ and $\lambda \in [0, 1]$.

Ψ is considered a strongly convex function if the inequality in (1) is squeezed for $\zeta_1 \neq \zeta_2$, and a concave function if $-\Psi$ is convex. Convex functions are covered by a number of significant inequalities, including the Jensen, Jensen-Mercer, (H-H), and support line inequalities. One of the most well-known inequalities is the standard Jensen's inequality, which is listed below [7, 8].:

Definition 1.2. *If $\Psi : [\kappa_1, \kappa_2] \to \mathbb{R}$ is convex, then*

$$\Psi\left(\sum_{i=1}^{n} \omega_i \zeta_i\right) \leq \sum_{i=1}^{n} \omega_i \Psi(\zeta_i),$$

for all $\zeta_i \in [\kappa_1, \kappa_2]$ and $\omega_i \in [0, 1]$ $(i = 1, 2, 3, ..., n)$ with $\sum_{i=1}^{n} \omega_i = 1$.

Mercer introduced the Jensen-Mercer inequality, a variation of Jensen's inequality, in [9].

Theorem 1.3. *If $\Psi : [\kappa_1, \kappa_2] \to \mathbb{R}$ is convex, then*

$$\Psi\left(\kappa_1 + \kappa_2 - \sum_{i=1}^{n} \omega_i \zeta_i\right) \leq \Psi(\kappa_1) + \Psi(\kappa_2) - \sum_{i=1}^{n} \omega_i \Psi(\zeta_i), \tag{2}$$

for all $\zeta_i \in [\kappa_1, \kappa_2]$ and $\omega_i \in [0, 1]$ $(i = 1, 2, 3, ..., n)$ with $\sum_{i=1}^{n} \omega_i = 1$.

There is at least one line on or below the function's graph for a convex function.

Definition 1.4. *[10] A function $\Psi : I \to \mathbb{R}$ has a support at $x_o \in I$, if*

$$\Psi(x_o) + c(\mathrm{u} - x_o) \leq \Psi(\mathrm{u}), \tag{3}$$

for all $x_o \in I$ and for each $\mathrm{u} \in [\kappa_1, \kappa_2] \subset I$. The inequality (3) is said to be the support line inequality.

The support line inequality and convex functions are related by the following theorem.

Theorem 1.5. *[10] The function $\Psi : [\kappa_1, \kappa_2] \to \mathbb{R}$ is convex if and only if Ψ has at least one line of support at each $x_o \in [\kappa_1, \kappa_2]$.*

Because of its geometrical significance and applications, the H-H inequality is one of the most studied inequality in convex function theory. A significant amount of research has been devoted to the extensions, generalizations, improvements, and applications of the H-H inequality due to its significance. Below is the Hadamard inequality [11]:

Let $\Psi : I \to \mathbb{R}$ be a convex function, where I is an interval and $\kappa_1, \kappa_2 \in I$ such

$$\Psi\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \Psi(\mathrm{u}) d\mathrm{u} \leq \frac{\Psi(\kappa_1) + \Psi(\kappa_2)}{2}. \tag{4}$$

In the opposite direction, (4) holds if Ψ is concave. For other findings related to H-H inequality, refer to [12–20].

Fractional integral operators have been used to expand the H-H inequality. The Non-Conformable fractional operator, as defined below, is the most often used of all.

Definition 1.6. *[21] Let $\varpi \in \mathbb{R}$ and $0 < \vartheta_1 < \vartheta_2$. For each function $\Psi \in L^1[\vartheta_1, \vartheta_2]$, we define*

$$ {}_{N_3}J_{\nu}^{\varpi}\Psi(\mathfrak{u}) = \int_{\nu}^{\mathfrak{u}} \lambda^{-\varpi}\Psi(\lambda)d\lambda, \tag{5} $$

for every $\mathfrak{u}, \nu \in [\vartheta_1, \vartheta_2]$.

Definition 1.7. *[21] Let $\varpi \in \mathbb{R}$ and $\zeta_1 < \zeta_2$. For each function $\Psi \in L_{\varpi,0}[\zeta_1, \zeta_2]$ let us define the fractional integrals*

$$ {}_{N_3}J_{\zeta_1^+}^{\varpi}\Psi(\mathfrak{u}) = \int_{\zeta_1}^{\mathfrak{u}} (\mathfrak{u} - \lambda)^{-\varpi}\,\Psi(\lambda)d\lambda, \tag{6} $$

$$ {}_{N_3}J_{\zeta_2^-}^{\varpi}\Psi(\mathfrak{u}) = \int_{\mathfrak{u}}^{\zeta_2} (\lambda - \mathfrak{u})^{-\varpi}\,\Psi(\lambda)d\lambda, \tag{7} $$

for every $\mathfrak{u} \in [\zeta_1, \zeta_2]$. Here, for $\varpi = 0$ ${}_{N_3}J_{\zeta_1^+}^{\varpi}\Psi(\mathfrak{u}) = {}_{N_3}J_{\zeta_2^-}^{\varpi}\Psi(\mathfrak{u}) = \int_{\zeta_1}^{\zeta_2}\Psi(\lambda)d\lambda$.

In [22–24], the authors obtained trapezoidal and midpoint type inequalities and provided associated results using the following lemmas.

Lemma 1.8. *Let $\Psi : I^o \to \mathbb{R}$ (where I^o is the interior of I) be a differentiable function and $\zeta_1, \zeta_2 \in I$ such that $\zeta_1 < \zeta_2$, If $\Psi' \in L[\zeta_1, \zeta_2]$, then*

$$ \frac{\Psi(\zeta_1) + \Psi(\zeta_2)}{2} - \frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2}\Psi(\mathfrak{u})d\mathfrak{u} = \frac{\zeta_2 - \zeta_1}{2}\int_0^1 (1 - 2\lambda)\Psi'(\lambda\zeta_1 + (1 - \lambda)\zeta_2)d\lambda. \tag{8} $$

Lemma 1.9. *Let all the assumptions of Lemma 1.8 hold, then*

$$ \begin{aligned} &\frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2}\Psi(\mathfrak{u})d\mathfrak{u} - \Psi\left(\frac{\zeta_1 + \zeta_2}{2}\right) \\ &= (\zeta_2 - \zeta_1)\left(\int_0^{\frac{1}{2}} \lambda\Psi'(\lambda\zeta_1 + (1 - \lambda)\zeta_2)d\lambda + \int_{\frac{1}{2}}^1 (\lambda - 1)\Psi'(\lambda\zeta_1 + (1 - \lambda)\zeta_2)d\lambda\right). \end{aligned} \tag{9} $$

In this article, we establish a fractional Hermite-Hadamard type inequality of the Jensen-Mercer type. We first present certain identities involving fractional integrals and, based on these identities, derive trapezoidal and midpoint type inequalities. Throughout the article, ϖ denotes a positive real number. In a well-known work [25], the authors proved Hermite-Hadamard type inequalities for Riemann-Liouville left-sided fractional integrals in the class of convex functions. A remarkable feature of generalized fractional integrals is their ability to encompass Riemann-Liouville, Katugampola, and several other well-known fractional integrals. Related results can be found in [26–33]. The fractional Hermite-Hadamard inequality, which incorporates fractional operators of various orders such as the Riemann-Liouville and Caputo operators, bridges the concepts of convexity and fractional calculus by analyzing the behavior of fractional integrals of convex functions. It establishes connections between endpoint values and the fractional integral of a convex function, thereby offering valuable insights into complex phenomena arising in physics, engineering, and related fields. In recent years, this inequality has attracted considerable attention from researchers.

Several mathematicians have investigated new inequalities, analyzed their properties, and explored potential applications [34–40], leading to a deeper understanding of the relationship between fractional calculus and convex functions [41–45]. The remainder of this paper is organized as follows: In Section 2, we present a variant of the existing result for left non-conformable fractional integrals using the support line approach; in Section 3, we establish essential identities for fractional trapezoid and midpoint type inequalities involving Jensen-Mercer convex functions, which serve as auxiliary tools to derive new inequalities for left non-conformable Jensen-Mercer convex functions, along with several corollaries; finally, Section 4 provides concluding remarks and outlines possible directions for future research.

2. Main Results

We begin this section with our first major finding, a cornerstone result that establishes direction and significance for subsequent discussions.

Theorem 2.1. *Let $\Psi : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a convex function and $\zeta_1, \zeta_2 \in [\kappa_1, \kappa_2]$ such that $\zeta_1 < \zeta_2$, then*

$$\Psi\left(\kappa_1 + \kappa_2 - \frac{(1-\varpi)\zeta_1 + \zeta_2}{2-\varpi}\right) \leq \Psi(\kappa_1) + \Psi(\kappa_2) - \frac{(1-\varpi)}{(\zeta_2 - \zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{\zeta_1^+}\Psi(\zeta_2)$$
$$\leq \Psi(\kappa_1) + \Psi(\kappa_2) - \Psi\left(\frac{(1-\varpi)\zeta_1 + \zeta_2}{2-\varpi}\right) \tag{10}$$

and

$$\Psi\left(\kappa_1 + \kappa_2 - \frac{(1-\varpi)\zeta_1 + \zeta_2}{2-\varpi}\right) \leq \frac{(1-\varpi)}{(\zeta_2 - \zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1 + \kappa_2 - \zeta_2)$$
$$\leq \frac{(1-\varpi)\Psi(\kappa_1 + \kappa_2 - \zeta_1) + \Psi(\kappa_1 + \kappa_2 - \zeta_2)}{2-\varpi}$$
$$\leq \Psi(\kappa_1) + \Psi(\kappa_2) - \frac{(1-\varpi)\Psi(\zeta_1) + \Psi(\zeta_2)}{2-\varpi}. \tag{11}$$

Proof. Since Ψ is convex, it has a support line at each point $x_o \in [\kappa_1, \kappa_2]$, that is

$$\Psi(x_o) + c(\mathrm{u} - x_o) \leq \Psi(\mathrm{u}), \tag{12}$$

for each $\mathrm{u} \in [\kappa_1, \kappa_2]$. Substituting $x_o = \kappa_1 + \kappa_2 - \frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}$ and $\mathrm{u} = \kappa_1 + \kappa_2 - \lambda\zeta_1 - (1-\lambda)\zeta_2$, where $\lambda \in [0, 1]$, in inequality (12), we get

$$\Psi\left(\kappa_1 + \kappa_2 - \frac{(1-\varpi)\zeta_1 + \zeta_2}{2-\varpi}\right) + c\left[-\lambda\zeta_1 - (1-\lambda)\zeta_2 + \frac{(1-\varpi)\zeta_1 + \zeta_2}{2-\varpi}\right]$$
$$\leq \Psi(\kappa_1 + \kappa_2 - \lambda\zeta_1 - (1-\lambda)\zeta_2). \tag{13}$$

Multiplying (13) with $(1-\varpi)\lambda^{-\varpi}$ and integrating w.r.t "λ", we get

$$\Psi\left(\kappa_1 + \kappa_2 - \frac{(1-\varpi)\zeta_1 + \zeta_2}{2-\varpi}\right) + c\left[-\frac{(1-\varpi)\zeta_1 + \zeta_2}{2-\varpi} + \frac{(1-\varpi)\zeta_1 + \zeta_2}{2-\varpi}\right]$$
$$\leq (1-\varpi)\int_0^1 \lambda^{-\varpi}\Psi(\kappa_1 + \kappa_2 - \lambda\zeta_1 - (1-\lambda)\zeta_2)d\lambda$$

$$\Psi\left(\kappa_1 + \kappa_2 - \frac{(1-\varpi)\zeta_1 + \zeta_2}{2-\varpi}\right) \leq (1-\varpi)\int_0^1 \lambda^{-\varpi}\Psi(\kappa_1 + \kappa_2 - \lambda\zeta_1 - (1-\lambda)\zeta_2)d\lambda. \tag{14}$$

Using Mercer's inequality, we get

$$\Psi\left(\kappa_1 + \kappa_2 - \frac{(1-\varpi)\zeta_1 + \zeta_2}{2-\varpi}\right)$$
$$\leq (1-\varpi)\int_0^1 \lambda^{-\varpi}(\Psi(\kappa_1) + \Psi(\kappa_2) - (\lambda\Psi(\zeta_1) + (1-\lambda)\Psi(\zeta_2)))d\lambda$$
$$= \Psi(\kappa_1) + \Psi(\kappa_2) - (1-\varpi)\int_0^1 \lambda^{-\varpi}(\lambda\Psi(\zeta_1) + (1-\lambda)\Psi(\zeta_2))d\lambda. \tag{15}$$

Since Ψ is convex, we have $-(\lambda\Psi(\zeta_1)+(1-\lambda)\Psi(\zeta_2)) \leq -\Psi(\lambda\zeta_1+(1-\lambda)\zeta_2)$ and (15) becomes

$$\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right) \leq \Psi(\kappa_1)+\Psi(\kappa_2)-(1-\varpi)\int_0^1 \lambda^{-\varpi}\Psi(\lambda\zeta_1+(1-\lambda)\zeta_2)d\lambda. \tag{16}$$

Substituting $\lambda\zeta_1+(1-\lambda)\zeta_2 = w$ in (16), we obtain

$$\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right) \leq \Psi(\kappa_1)+\Psi(\kappa_2)-\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}\int_{\zeta_1}^{\zeta_2}(\zeta_2-w)^{-\varpi}\Psi(w)dw$$

$$\Rightarrow \Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right) \leq \Psi(\kappa_1)+\Psi(\kappa_2)-\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{\zeta_1^+}\Psi(\zeta_2). \tag{17}$$

Now we prove the second inequality (10). Putting $x_o = \frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}$ and $\mathfrak{u} = \lambda\zeta_1+(1-\lambda)\zeta_2$ in (12), we get

$$\Psi\left(\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)+c\left[\lambda\zeta_1+(1-\lambda)\zeta_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right] \leq \Psi(\lambda\zeta_1+(1-\lambda)\zeta_2).$$

Multiplying the above inequality with $(1-\varpi)\lambda^{-\varpi}$ and integrating over $[0,1]$, we obtain

$$\Psi\left(\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right) \leq (1-\varpi)\int_0^1 \lambda^{-\varpi}\Psi(\lambda\zeta_1+(1-\lambda)\zeta_2)d\lambda.$$

Putting $\lambda\zeta_1+(1-\lambda)\zeta_2 = w$, we obtain

$$\Psi\left(\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right) \leq \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}\int_{\zeta_1}^{\zeta_2}(\zeta_2-w)^{-\varpi}\Psi(w)dw$$

$$\Rightarrow \Psi\left(\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right) \leq \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{\zeta_1^+}\Psi(\zeta_2)$$

$$\Rightarrow -\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{\zeta_1^+}\Psi(\zeta_2) \leq -\Psi\left(\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right). \tag{18}$$

Adding $\Psi(\kappa_1)+\Psi(\kappa_2)$ on both sides of (18), we get

$$\Psi(\kappa_1)+\Psi(\kappa_2)-\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{\zeta_1^+}\Psi(\zeta_2) \leq \Psi(\kappa_1)+\Psi(\kappa_2)-\Psi\left(\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right), \tag{19}$$

and on combining (17) and (19), we obtain (10).
Now we prove the inequalities (11). Let $\mathfrak{u} = \kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2$
$\Rightarrow \lambda = \frac{\mathfrak{u}-\kappa_1-\kappa_2+\zeta_2}{\zeta_2-\zeta_1}$ and (14) becomes

$$\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right) \leq (1-\varpi)\int_{\kappa_1+\kappa_2-\zeta_2}^{\kappa_1+\kappa_2-\zeta_1}\left(\frac{\mathfrak{u}-\kappa_1-\kappa_2+\zeta_2}{\zeta_2-\zeta_1}\right)^{-\varpi}\Psi(\mathfrak{u})d\mathfrak{u}$$

$$= \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}\int_{\kappa_1+\kappa_2-\zeta_2}^{\kappa_1+\kappa_2-\zeta_1}(\mathfrak{u}-(\kappa_1+\kappa_2-\zeta_2))^{-\varpi}\Psi(\mathfrak{u})d\mathfrak{u}$$

$$\Rightarrow \Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right) \leq \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2). \tag{20}$$

We now proceed to prove the remaining two inequalities in (11). Since Ψ is convex, we have

$$\Psi(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)$$

$$
\begin{aligned}
&= \Psi(\lambda(\kappa_1 + \kappa_2 - \zeta_1) + (1 - \lambda)(\kappa_1 + \kappa_2 - \zeta_2)) \\
&\leq \lambda\Psi(\kappa_1 + \kappa_2 - \zeta_1) + (1 - \lambda)\Psi(\kappa_1 + \kappa_2 - \zeta_2) \\
&\leq \Psi(\kappa_1) + \Psi(\kappa_2) - \lambda\Psi(\zeta_1) - (1 - \lambda)\Psi(\zeta_2).
\end{aligned}
$$

Multiplying with $(1 - \varpi)\lambda^{-\varpi}$ and integrating, we get

$$
\begin{aligned}
&(1 - \varpi)\int_0^1 \lambda^{-\varpi}\Psi(\kappa_1 + \kappa_2 - \lambda\zeta_1 - (1 - \lambda)\zeta_2)d\lambda \\
&\leq (1 - \varpi)\Psi(\kappa_1 + \kappa_2 - \zeta_1)\int_0^1 \lambda^{1-\varpi}d\lambda + (1 - \varpi)\Psi(\kappa_1 + \kappa_2 - \zeta_2)\int_0^1 (\lambda^{-\varpi} - \lambda^{1-\varpi})d\lambda \\
&\leq \Psi(\kappa_1) + \Psi(\kappa_2) - (1 - \varpi)\Psi(\zeta_1)\int_0^1 \lambda^{1-\varpi}d\lambda - (1 - \varpi)\Psi(\zeta_2)\int_0^1 (\lambda^{-\varpi} - \lambda^{1-\varpi})d\lambda \\
\Rightarrow &(1 - \varpi)\int_0^1 \lambda^{-\varpi}\Psi(\kappa_1 + \kappa_2 - \lambda\zeta_1 - (1 - \lambda)\zeta_2)d\lambda \\
&\leq \frac{(1 - \varpi)\Psi(\kappa_1 + \kappa_2 - \zeta_1) + \Psi(\kappa_1 + \kappa_2 - \zeta_2)}{2 - \varpi} \\
&\leq \Psi(\kappa_1) + \Psi(\kappa_2) - \frac{(1 - \varpi)\Psi(\zeta_1) + \Psi(\zeta_2)}{2 - \varpi}. \qquad (21)
\end{aligned}
$$

After a change of variables, (21) can be rewritten as

$$
\begin{aligned}
&(1 - \varpi)\int_{\kappa_1+\kappa_2-\zeta_2}^{\kappa_1+\kappa_2-\zeta_1} \left(\frac{u - \kappa_1 - \kappa_2 + \zeta_2}{\zeta_2 - \zeta_1}\right)^{-\varpi}\Psi(u)du \\
&\leq \frac{(1 - \varpi)\Psi(\kappa_1 + \kappa_2 - \zeta_1) + \Psi(\kappa_1 + \kappa_2 - \zeta_2)}{2 - \varpi} \\
&\leq \Psi(\kappa_1) + \Psi(\kappa_2) - \frac{(1 - \varpi)\Psi(\zeta_1) + \Psi(\zeta_2)}{2 - \varpi}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \frac{(1 - \varpi)}{(\zeta_2 - \zeta_1)^{1-\varpi}}\int_{\kappa_1+\kappa_2-\zeta_2}^{\kappa_1+\kappa_2-\zeta_1} (u - (\kappa_1 + \kappa_2 - \zeta_2)^{-\varpi}\Psi(u)du \\
&\leq \frac{(1 - \varpi)\Psi(\kappa_1 + \kappa_2 - \zeta_1) + \Psi(\kappa_1 + \kappa_2 - \zeta_2)}{2 - \varpi} \\
&\leq \Psi(\kappa_1) + \Psi(\kappa_2) - \frac{(1 - \varpi)\Psi(\zeta_1) + \Psi(\zeta_2)}{2 - \varpi}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \frac{(1 - \varpi)}{(\zeta_2 - \zeta_1)^{1-\varpi}}\, {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1 + \kappa_2 - \zeta_2) \\
&\leq \frac{(1 - \varpi)\Psi(\kappa_1 + \kappa_2 - \zeta_1) + \Psi(\kappa_1 + \kappa_2 - \zeta_2)}{2 - \varpi} \\
&\leq \Psi(\kappa_1) + \Psi(\kappa_2) - \frac{(1 - \varpi)\Psi(\zeta_1) + \Psi(\zeta_2)}{2 - \varpi}, \qquad (22)
\end{aligned}
$$

Combining (22) and (20), we arrive at (11). □

Corollary 2.2. *Substituting $\zeta_1 = \kappa_1$ and $\zeta_2 = \kappa_2$ in (11), we obtain*

$$
\Psi\left(\frac{\kappa_1 + (1 - \varpi)\kappa_2}{2 - \varpi}\right) \leq \frac{(1 - \varpi)}{(\kappa_2 - \kappa_1)^{1-\varpi}}\int_{\kappa_1}^{\kappa_2} (u - \kappa_1)^{-\varpi}\Psi(u)du
$$

$$
\begin{aligned}
&\le \frac{(1-\varpi)\Psi(\kappa_2)+\Psi(\kappa_1)}{2-\varpi}\\
&\le \frac{\Psi(\kappa_1)+(1-\varpi)\Psi(\kappa_2)}{2-\varpi}.
\end{aligned}
$$

Remark 2.3. *If we put $\varpi = 0$ in Theorem 2.1, we obtain*

$$
\begin{aligned}
&\Psi\left(\kappa_1+\kappa_2-\frac{\zeta_1+\zeta_2}{2}\right) \le \Psi(\kappa_1)+\Psi(\kappa_2)-\int_0^1 \Psi(\lambda\zeta_1+(1-\lambda)\zeta_2)d\lambda\\
&\le \Psi(\kappa_1)+\Psi(\kappa_2)-\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right)
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
&\Psi\left(\kappa_1+\kappa_2-\frac{\zeta_1+\zeta_2}{2}\right) \le \frac{1}{(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2} \Psi(\mathrm{u})d\mathrm{u}\\
&\le \frac{\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2}\\
&\le \Psi(\kappa_1)+\Psi(\kappa_2)-\frac{\Psi(\zeta_1)+\Psi(\zeta_2)}{2}.
\end{aligned}
\tag{24}
$$

The inequalities (23) *and* (24) *were established in [46].*

Remark 2.4. *Substituting $\varpi = 0$, $\zeta_1 = \kappa_1$ and $\zeta_2 = \kappa_2$ in* (11), *we obtain*

$$
\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \Psi(\mathrm{u})d\mathrm{u} \le \frac{\Psi(\kappa_1)+\Psi(\kappa_2)}{2},
$$

which appeared in [11].

3. (H-H)-type inequality via Jensen-Mercer Inequality

Throughout this section, we consider a differentiable function $\Psi : [\kappa_1, \kappa_2] \to \mathbb{R}$. To establish bounds for the difference in the Hermite-Hadamard inequality, we first present the following Lemmas.

Lemma 3.1. *Let $\zeta_1, \zeta_2 \in [\kappa_1, \kappa_2]$ such that $\zeta_1 < \zeta_2$ and let $\Psi' \in L[\kappa_1, \kappa_2]$, then*

$$
\begin{aligned}
&\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}\, {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\\
&= \frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_0^1 ((2-\varpi)\lambda^{1-\varpi}-1)\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda.
\end{aligned}
\tag{25}
$$

Proof. Using the technique of integration by parts, we have

$$
\begin{aligned}
&\frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_0^1 ((2-\varpi)\lambda^{1-\varpi}-1)\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda\\
&= (\zeta_2-\zeta_1)\int_0^1 \lambda^{1-\varpi}\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda\\
&- \frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_0^1 \Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda
\end{aligned}
$$

$$= \lambda^{1-\varpi}\Psi(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)|_0^1 - (1-\varpi)\int_0^1 \lambda^{-\varpi}\Psi(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda$$
$$- \frac{1}{2-\varpi}\Psi(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)|_0^1$$
$$= \frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - (1-\varpi)\int_0^1 \lambda^{-\varpi}\Psi(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda.$$

By changing of variable in above equation, we obtain

$$\frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_0^1((2-\varpi)\lambda^{1-\varpi}-1)\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda$$
$$= \frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi}$$
$$- \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}\int_{\kappa_1+\kappa_2-\zeta_2}^{\kappa_1+\kappa_2-\zeta_1}(\mathrm{u}-(\kappa_1+\kappa_2-\zeta_2))^{-\varpi}\Psi(\mathrm{u})d\mathrm{u}$$
$$= \frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi}$$
$$- \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2).$$

This completes the proof. □

Remark 3.2. *Substituting $\varpi = 0$, $\kappa_1 = \zeta_1$ and $\kappa_2 = \zeta_2$ in* (25), *we get*

$$\frac{\Psi(\zeta_1)+\Psi(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\Psi(\mathrm{u})d\mathrm{u} = \frac{\zeta_2-\zeta_1}{2}\int_0^1(1-2\lambda)\Psi'(\lambda\zeta_1+(1-\lambda)\zeta_2)d\lambda,$$

which appeared in [22].

Lemma 3.3. *Let $\zeta_1, \zeta_2 \in [\kappa_1, \kappa_2]$, such that $\zeta_1 < \zeta_2$ and let $\Psi' \in L[\kappa_1, \kappa_2]$, then*

$$\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2) - \Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)$$
$$= (\zeta_2-\zeta_1)\int_{\frac{1-\varpi}{2-\varpi}}^1(1-\lambda^{1-\varpi})\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda$$
$$- (\zeta_2-\zeta_1)\int_0^{\frac{1-\varpi}{2-\varpi}}\lambda^{1-\varpi}\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda. \tag{26}$$

Proof. Using the technique of integration by parts, we have

$$(\zeta_2-\zeta_1)\int_{\frac{1-\varpi}{2-\varpi}}^1(1-\lambda^{1-\varpi})\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda$$
$$- (\zeta_2-\zeta_1)\int_0^{\frac{1-\varpi}{2-\varpi}}\lambda^{1-\varpi}\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda$$
$$= (\zeta_2-\zeta_1)\int_{\frac{1-\varpi}{2-\varpi}}^1\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda$$
$$- (\zeta_2-\zeta_1)\int_0^1\lambda^{1-\varpi}\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda$$

$$= \Psi(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)|^{1}_{\frac{1-\varpi}{2-\varpi}} - \lambda^{1-\varpi}\Psi(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)|^{1}_{0}$$
$$+ (1-\varpi)\int_0^1 \lambda^{-\varpi}\Psi(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda$$
$$= -\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right) + (1-\varpi)\int_0^1 \lambda^{-\varpi}\Psi(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda.$$

By changing of variable in above equation, we obtain

$$= \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2) - \Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right). \tag{27}$$

This completes the proof. □

Remark 3.4. *If we put $\varpi = 0$, $\zeta_1 = \kappa_1$ and $\zeta_2 = \kappa_2$ in* (26), *we obtain*

$$\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\Psi(u)du - \Psi\left(\frac{\zeta_1+\zeta_2}{2}\right)$$
$$= (\zeta_2-\zeta_1)\left(\int_0^{\frac{1}{2}} \lambda\Psi'(\lambda\zeta_1+(1-\lambda)\zeta_2)d\lambda + \int_{\frac{1}{2}}^1 (\lambda-1)\Psi'(\lambda\zeta_1+(1-\lambda)\zeta_2)d\lambda\right). \tag{28}$$

which appeared in [22].

Theorem 3.5. *Assume the conditions of Lemma 3.1 hold. Let $|\Psi'|$ be a convex function defined on $[\kappa_1, \kappa_2]$ and let $\zeta_1, \zeta_2 \in [\kappa_1, \kappa_2]$, such that $\zeta_1 < \zeta_2$, then*

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}[\mathcal{M}_1(\varpi)|\Psi'(\kappa_1)| + \mathcal{M}_1(\varpi)|\Psi'(\kappa_2)| - \mathcal{M}_2(\varpi)|\Psi'(\zeta_1)| - \mathcal{M}_3(\varpi)|\Psi'(\zeta_2)|], \tag{29}$$

where

$$\mathcal{M}_1(\varpi) = \frac{2(1-\varpi)}{(2-\varpi)^{\frac{2-\varpi}{1-\varpi}}},$$
$$\mathcal{M}_2(\varpi) = \frac{(1-\varpi)\left((2-\varpi)^{\frac{2}{1-\varpi}}+2\right)}{2(3-\varpi)(2-\varpi)^{\frac{2}{1-\varpi}}}$$

and

$$\mathcal{M}_3(\varpi) = \frac{(1-\varpi)\left(4(3-\varpi)(2-\varpi)^{\frac{1}{1-\varpi}-1} - (2-\varpi)^{\frac{2}{1-\varpi}} - 2\right)}{2(3-\varpi)(2-\varpi)^{\frac{2}{1-\varpi}}}.$$

Proof. From Lemma 3.1, we have

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_0^1 |((2-\varpi)\lambda^{1-\varpi}-1)||\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)|d\lambda. \tag{30}$$

Given that $|\Psi'|$ is convex, applying Jensen-Mercer inequality, we obtain

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$

$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_0^1 |((2-\varpi)\lambda^{1-\varpi}-1)|(|\Psi'(\kappa_1)|+|\Psi'(\kappa_2)|-\lambda|\Psi'(\zeta_1)|-(1-\lambda)|\Psi'(\zeta_2)|)d\lambda$$
$$= \frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_0^{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}} (1-(2-\varpi)\lambda^{1-\varpi})(|\Psi'(\kappa_1)|+|\Psi'(\kappa_2)|-\lambda|\Psi'(\zeta_1)|-(1-\lambda)|\Psi'(\zeta_2)|)d\lambda$$
$$+ \frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}^1 ((2-\varpi)\lambda^{1-\varpi}-1)(|\Psi'(\kappa_1)|+|\Psi'(\kappa_2)|-\lambda|\Psi'(\zeta_1)|-(1-\lambda)|\Psi'(\zeta_2)|)d\lambda$$

which is equivalent to

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi}-\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}([\mathcal{L}_1(\varpi)+\mu_1(\varpi)]|\Psi'(\kappa_1)|+[\mathcal{L}_1(\varpi)+\mu_1(\varpi)]|\Psi'(\kappa_2)|$$
$$-[\mathcal{L}_2(\varpi)+\mu_2(\varpi)]|\Psi'(\zeta_1)|+[\mathcal{L}_3(\varpi)+\mu_3(\varpi)]|\Psi'(\zeta_2)|), \tag{31}$$

where

$$\mathcal{L}_1(\varpi)=\int_0^{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}(1-(2-\varpi)\lambda^{1-\varpi})d\lambda=\frac{1-\varpi}{(2-\varpi)^{\frac{2-\varpi}{1-\varpi}}},$$
$$\mathcal{L}_2(\varpi)=\int_0^{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}(1-(2-\varpi)\lambda^{1-\varpi})\lambda d\lambda=\frac{(1-\varpi)}{2(3-\varpi)(2-\varpi)^{\frac{2}{1-\varpi}}},$$
$$\mathcal{L}_3(\varpi)=\int_0^{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}(1-(2-\varpi)\lambda^{1-\varpi})(1-\lambda)d\lambda=\frac{(1-\varpi)\left(2(3-\varpi)(2-\varpi)^{\frac{1}{1-\varpi}-1}-1\right)}{2(3-\varpi)(2-\varpi)^{\frac{2}{1-\varpi}}},$$
$$\mu_1(\varpi)=\int_{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}^1((2-\varpi)\lambda^{1-\varpi}-1)d\lambda=\frac{1-\varpi}{(2-\varpi)^{\frac{2-\varpi}{1-\varpi}}},$$
$$\mu_2(\varpi)=\int_{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}^1((2-\varpi)\lambda^{1-\varpi}-1)\lambda d\lambda=\frac{(1-\varpi)\left((2-\varpi)^{\frac{2}{1-\varpi}}+1\right)}{2(3-\varpi)(2-\varpi)^{\frac{2}{1-\varpi}}},$$

and

$$\mu_3(\varpi)=\int_{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}^1((2-\varpi)\lambda^{1-\varpi}-1)(1-\lambda)d\lambda$$
$$=\frac{(1-\varpi)\left(2(3-\varpi)(2-\varpi)^{\frac{1}{1-\varpi}-1}-(2-\varpi)^{\frac{2}{1-\varpi}}-1\right)}{2(3-\varpi)(2-\varpi)^{\frac{2}{1-\varpi}}}.$$

Substituting these values in (31), we get (29). □

Remark 3.6. *If we put $\varpi=0$, $\zeta_1=\kappa_1$ and $\zeta_2=\kappa_2$ in* (29)*, we get the inequality given in [[22], Theorem* (2.2)*].*

Remark 3.7. *If we put $\varpi=0$ in* (29)*, we get the following inequality*

$$\left|\frac{\Psi(\kappa_1+\kappa_2-\zeta_2)}{2}-\frac{1}{(\zeta_2-\zeta_1)}\int_{(\kappa_1+\kappa_2-\zeta_1)}^{(\kappa_1+\kappa_2-\zeta_2)}\Psi(u)du\right|$$
$$\leq\frac{(\zeta_2-\zeta_1)}{4}[|\Psi'(\kappa_1)|+|\Psi'(\kappa_2)|-\frac{1}{2}(|\Psi'(\zeta_1)|-\frac{1}{2}|\Psi'(\zeta_2)|].$$

Remark 3.8. *If we put $\zeta_1 = \kappa_1$ and $\zeta_2 = \kappa_2$ in* (29), *we get the following inequality*

$$\left|\frac{(1-\varpi)\Psi(\kappa_2)+\Psi(\kappa_1)}{2-\varpi} - \frac{(1-\varpi)}{(\kappa_2-\kappa_1)^{1-\varpi}}\int_{\kappa_2}^{\kappa_1}(\mathfrak{u}-\kappa_1)^{-\varpi}\Psi(\mathfrak{u})d\mathfrak{u}\right|$$
$$\leq \frac{(\kappa_2-\kappa_1)}{2-\varpi}\left[\left(\mathcal{M}_1(\varpi)-\mathcal{M}_2(\varpi)\right)|\Psi'(\kappa_1)| + \left(\mathcal{M}_1(\varpi)-\mathcal{M}_3(\varpi)\right)|\Psi'(\kappa_2)|\right].$$

Corollary 3.9. *Taking $|\Psi'| \leq K$, in Theorem 3.5, then*

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq K\frac{(\zeta_2-\zeta_1)}{2-\varpi}[2\mathcal{M}_1(\varpi)-\mathcal{M}_2(\varpi)-\mathcal{M}_3(\varpi)].$$

Theorem 3.10. *Assume the conditions of Lemma 3.1 hold. Let $|\Psi'|^{\mathfrak{q}}$ be a convex function for $\mathfrak{q} \geq 1$ and let $\zeta_1, \zeta_2 \in [\kappa_1, \kappa_2]$, such that $\zeta_1 < \zeta_2$, then*

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}\left(\frac{2(1-\varpi)}{(2-\varpi)^{\frac{2-\varpi}{1-\varpi}}}\right)^{1-\frac{1}{\mathfrak{q}}}$$
$$\times\left(\mathcal{M}_1(\varpi)|\Psi'(\kappa_1)|^{\mathfrak{q}} + \mathcal{M}_1(\varpi)|\Psi'(\kappa_2)|^{\mathfrak{q}} - \mathcal{M}_2(\varpi)|\Psi'(\zeta_1)|^{\mathfrak{q}} - \mathcal{M}_3(\varpi)|\Psi'(\zeta_2)|^{\mathfrak{q}}\right)^{\frac{1}{\mathfrak{q}}}, \tag{32}$$

where $\mathcal{M}_1(\varpi)$, $\mathcal{M}_2(\varpi)$ and $\mathcal{M}_3(\varpi)$ are provided in Theorem 3.5.

Proof. Lemma 3.1 gives us

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_0^1 |((2-\varpi)\lambda^{1-\varpi}-1)||\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)|d\lambda. \tag{33}$$

Given that $|\Psi'|^q$ is convex, applying power mean and Jensen-Mercer inequality, we obtain

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}\left(\int_0^1 |(2-\varpi)\lambda^{1-\varpi}-1|d\lambda\right)^{1-\frac{1}{\mathfrak{q}}}$$
$$\left(\int_0^1 |((2-\varpi)\lambda^{1-\varpi}-1)||\Psi'(\kappa_1)+\Psi'(\kappa_2)-\lambda\Psi'(\zeta_1)-(1-\lambda)\Psi'(\zeta_2)|^{\mathfrak{q}}d\lambda\right)^{\frac{1}{\mathfrak{q}}}.$$

This implies that

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}\left(\int_0^{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}(1-(2-\varpi)\lambda^{1-\varpi})d\lambda + \int_{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}^1((2-\varpi)\lambda^{1-\varpi}-1)\right)^{1-\frac{1}{\mathfrak{q}}}$$
$$\times\left(\int_0^1 |((2-\varpi)\lambda^{1-\varpi}-1)||\Psi'(\kappa_1)|^{\mathfrak{q}}+|\Psi'(\kappa_2)|^{\mathfrak{q}}-\lambda|\Psi'(\zeta_1)|^{\mathfrak{q}}-(1-\lambda)|\Psi'(\zeta_2)|^{\mathfrak{q}}d\lambda\right)^{\frac{1}{\mathfrak{q}}}$$

$$= \frac{(\zeta_2 - \zeta_1)}{2-\varpi}\left(\frac{2(1-\varpi)}{(2-\varpi)^{\frac{2-\varpi}{1-\varpi}}}\right)^{1-\frac{1}{q}}$$
$$\times \Big(\int_0^{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}} (1-(2-\varpi)\lambda^{1-\varpi})(|\Psi'(\kappa_1)|^q + |\Psi'(\kappa_2)|^q - \lambda|\Psi'(\zeta_1)|^q - (1-\lambda)|\Psi'(\zeta_2)|^q)d\lambda$$
$$+ \int_{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}^1 ((2-\varpi)\lambda^{1-\varpi} - 1)(|\Psi'(\kappa_1)|^q + |\Psi'(\kappa_2)|^q - \lambda|\Psi'(\zeta_1)|^q - (1-\lambda)|\Psi'(\zeta_2)|^q d\lambda)\Big)^{\frac{1}{q}},$$

which is equivalent to

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1) + \Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}\, {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2 - \zeta_1)}{2-\varpi}\left(\frac{2(1-\varpi)}{(2-\varpi)^{\frac{2-\varpi}{1-\varpi}}}\right)^{1-\frac{1}{q}}$$
$$\times ([\mathcal{L}_1(\varpi) + \mu_1(\varpi)]|\Psi'(\kappa_1)|^q + [\mathcal{L}_1(\varpi) + \mu_1(\varpi)]|\Psi'(\kappa_2)|^q$$
$$- [\mathcal{L}_2(\varpi) + \mu_2(\varpi)]|\Psi'(\zeta_1)|^q + [\mathcal{L}_3(\varpi) + \mu_3(\varpi)]|\Psi'(\zeta_2)|^q)^{\frac{1}{q}}. \tag{34}$$

By substituting the values of $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, μ_1, μ_2, and μ_3 from Theorem 3.5 into (34), we obtain (32). □

Remark 3.11. *If we put $\varpi = 0$, $\zeta_1 = \kappa_1$ and $\zeta_2 = \kappa_2$ in Theorem 3.10, we get the inequality given in [24, Theorem 1], .*

Remark 3.12. *If we put $\varpi = 0$ in* (32)*, we get the following inequality*

$$\left|\frac{\Psi(\kappa_1+\kappa_2-\zeta_2)}{2} - \frac{1}{(\zeta_2-\zeta_1)}\int_{(\kappa_1+\kappa_2-\zeta_1)}^{(\kappa_1+\kappa_2-\zeta_2)} \Psi(u)du\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(|\Psi'(\kappa_1)|^q + |\Psi'(\kappa_2)|^q - \frac{1}{2}|\Psi'(\zeta_1)|^q - \frac{1}{2}|\Psi'(\zeta_2)|^q\right)^{\frac{1}{q}}.$$

Remark 3.13. *If we put $\zeta_1 = \kappa_1$ and $\zeta_2 = \kappa_2$ in* (32)*, we get the following inequality*

$$\left|\frac{(1-\varpi)\Psi(\kappa_2) + \Psi(\kappa_1)}{2-\varpi} - \frac{(1-\varpi)}{(\kappa_2-\kappa_1)^{1-\varpi}}\int_{\kappa_2}^{\kappa_1}(u-\kappa_1)^{-\varpi}\Psi(u)du\right|$$
$$\leq \frac{(\kappa_2-\kappa_1)}{2-\varpi}\left(\frac{2(1-\varpi)}{(2-\varpi)^{\frac{2-\varpi}{1-\varpi}}}\right)^{1-\frac{1}{q}}\left((\mathcal{M}_1(\varpi) - \mathcal{M}_2(\varpi))|\Psi'(\kappa_1)|^q + (\mathcal{M}_1(\varpi) - \mathcal{M}_3(\varpi))|\Psi'(\kappa_2)|^q\right)^{\frac{1}{q}}.$$

Corollary 3.14. *Taking $|\Psi'| \leq K$, in Theorem 3.10, then*

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1) + \Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}\, {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq K\frac{(\zeta_2-\zeta_1)}{2-\varpi}\left(\frac{(1-\varpi)}{(2-\varpi)^{\frac{2-\varpi}{1-\varpi}}}\right)^{1-\frac{1}{q}}(2\mathcal{M}_1(\varpi) - \mathcal{M}_2(\varpi) - \mathcal{M}_3(\varpi))^{\frac{1}{q}}.$$

Theorem 3.15. *Assume the conditions of Lemma 3.1 hold. Let $|\Psi'|^q$ be a convex function for $q > 1$ and let $\zeta_1, \zeta_2 \in [\kappa_1, \kappa_2]$ such that $\zeta_1 < \zeta_2$. Then*

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1) + \Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}\, {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$

$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}(\mathcal{L}_4(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}\left(|\Psi'(\kappa_1)|^{\mathfrak{q}}+|\Psi'(\kappa_2)|^{\mathfrak{q}}-\frac{|\Psi'(\zeta_1)|^{\mathfrak{q}}+|\Psi'(\zeta_2)|^{\mathfrak{q}}}{2}\right)^{\frac{1}{\mathfrak{q}}}, \tag{35}$$

where

$$\mathcal{L}_4(\varpi,\mathfrak{p})=\int_0^1|(2-\varpi)\lambda^{\varpi}-1|^{\mathfrak{p}}d\lambda,$$

with $\frac{1}{\mathfrak{p}}+\frac{1}{\mathfrak{q}}=1$.

Proof. From Lemma 3.1, we have

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi}-\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_0^1|((2-\varpi)\lambda^{1-\varpi}-1)||\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)|d\lambda.$$

Applying Hölder's inequality, we get

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi}-\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}\left(\int_0^1|((2-\varpi)\lambda^{1-\varpi}-1)|^{\mathfrak{p}}d\lambda\right)^{\frac{1}{\mathfrak{p}}}\left(\int_0^1|\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)|^{\mathfrak{q}}d\lambda\right)^{\frac{1}{\mathfrak{q}}}$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}(\mathcal{L}_4(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}\left(\int_0^1(|\Psi'(\kappa_1)|^{\mathfrak{q}}+|\Psi'(\kappa_2)|^{\mathfrak{q}}-\lambda|\Psi'(\zeta_1)|^{\mathfrak{q}}-(1-\lambda)|\Psi'(\zeta_2)|^{\mathfrak{q}})d\lambda\right)^{\frac{1}{\mathfrak{q}}}$$
$$=\frac{(\zeta_2-\zeta_1)}{2-\varpi}(\mathcal{L}_4(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}\left(|\Psi'(\kappa_1)|^{\mathfrak{q}}+|\Psi'(\kappa_2)|^{\mathfrak{q}}-\frac{|\Psi'(\zeta_1)|^{\mathfrak{q}}+|\Psi'(\zeta_2)|^{\mathfrak{q}}}{2}\right)^{\frac{1}{\mathfrak{q}}}.$$

□

Remark 3.16. *Substituting* $\varpi=0$, $\zeta_1=\kappa_1$ *and* $\zeta_2=\kappa_2$ *in Theorem 3.15, we obtain Theorem* 2.3, *which appeared in [22].*

Remark 3.17. *If we put* $\zeta_1=\kappa_1$ *and* $\zeta_2=\kappa_2$ *in* (35), *we get the following inequality*

$$\left|\frac{(1-\varpi)\Psi(\kappa_2)+\Psi(\kappa_1)}{2-\varpi}-\frac{(1-\varpi)}{(\kappa_2-\kappa_1)^{1-\varpi}}\int_{\kappa_2}^{\kappa_1}(\mathfrak{u}-\kappa_1)^{-\varpi}\Psi(\mathfrak{u})d\mathfrak{u}\right|$$
$$\leq \frac{(\kappa_2-\kappa_1)}{2-\varpi}(\mathcal{L}_4(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}\left(\frac{|\Psi'(\kappa_1)|^{\mathfrak{q}}+|\Psi'(\kappa_2)|^{\mathfrak{q}}}{2}\right)^{\frac{1}{\mathfrak{q}}}.$$

Remark 3.18. *If we put* $\varpi=0$ *in* (35), *we get the following inequality*

$$\left|\frac{(1-\varpi)\Psi(\kappa_2)+\Psi(\kappa_1)}{2-\varpi}-\frac{(1-\varpi)}{(\kappa_2-\kappa_1)^{1-\varpi}}\int_{\kappa_2}^{\kappa_1}\Psi(\mathfrak{u})d\mathfrak{u}\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2}\left(|\Psi'(\kappa_1)|^{\mathfrak{q}}+|\Psi'(\kappa_2)|^{\mathfrak{q}}-\frac{|\Psi'(\zeta_1)|^{\mathfrak{q}}+|\Psi'(\zeta_2)|^{\mathfrak{q}}}{2}\right)^{\frac{1}{\mathfrak{q}}}.$$

Corollary 3.19. *Taking* $|\Psi'|\leq K$, *in Theorem 3.15, then*

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi}-\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq K\frac{(\zeta_2-\zeta_1)}{2-\varpi}(\mathcal{L}_4(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}.$$

Theorem 3.20. *Assume the conditions of Lemma 3.1 hold. Let $|\Psi'|$ be a convex function and let $\zeta_1, \zeta_2 \in [\kappa_1, \kappa_2]$, such that $\zeta_1 < \zeta_2$, then*

$$\left|\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)-\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)\right|$$
$$\leq (\zeta_2-\zeta_1)(\mathcal{M}_5(\varpi)|\Psi'(\zeta_1)|+\mathcal{M}_6(\varpi)|\Psi'(\zeta_2)|), \tag{36}$$

where

$$\mathcal{M}_5(\varpi)=\frac{-(1-\varpi)}{2(3-\varpi)(2-\varpi)^2}$$

and

$$\mathcal{M}_6(\varpi)=\frac{(1-\varpi)}{2(3-\varpi)(2-\varpi)^2}.$$

Proof. Utilising Lemma 3.3, we have

$$\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)-\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)$$
$$=(\zeta_2-\zeta_1)\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda$$
$$-(\zeta_2-\zeta_1)\int_0^{\frac{1-\varpi}{2-\varpi}}\lambda^{1-\varpi}\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda. \tag{37}$$

Given that $|\Psi'|$ is convex, applying Jensen-Mercer inequality, we obtain

$$\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)-\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)$$
$$\leq(\zeta_2-\zeta_1)\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})(|\Psi'(\kappa_1)|+|\Psi'(\kappa_2)|-\lambda|\Psi'(\zeta_1)|-(1-\lambda)|\Psi'(\zeta_2)|)d\lambda$$
$$-(\zeta_2-\zeta_1)\int_0^{\frac{1-\varpi}{2-\varpi}}\lambda^{1-\varpi}(|\Psi'(\kappa_1)|+|\Psi'(\kappa_2)|-\lambda|\Psi'(\zeta_1)|-(1-\lambda)|\Psi'(\zeta_2)|)d\lambda,$$

which implies that

$$\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)-\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)$$
$$\leq(\zeta_2-\zeta_1)((\mu_5(\varpi)-\mathcal{L}_5(\varpi))|\Psi'(\zeta_1)|+(\mu_6(\varpi)-\mathcal{L}_6(\varpi))|\Psi'(\zeta_2)|), \tag{38}$$

where

$$\mathcal{L}_5(\varpi)=\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})\lambda d\lambda=\frac{(1-\varpi)(2-\varpi)^{1-\varpi}+2(1-\varpi)^{3-\varpi}}{2(3-\varpi)(2-\varpi)^{3-\varpi}},$$
$$\mathcal{L}_6(\varpi)=\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})(1-\lambda)\lambda d\lambda=\frac{4(1-\varpi)^{2-\varpi}-(1-\varpi)(2-\varpi)^{1-\varpi}}{2(3-\varpi)(2-\varpi)^{3-\varpi}},$$
$$\mu_5(\varpi)=\int_0^{\frac{1-\varpi}{2-\varpi}}\lambda^{2-\varpi}d\lambda=\frac{(1-\varpi)^{3-\varpi}}{(3-\varpi)(2-\varpi)^{3-\varpi}}$$

and

$$\mu_6(\varpi) = \int_0^{\frac{1-\varpi}{2-\varpi}} \lambda^{1-\varpi}(1-\lambda)d\lambda = \frac{2(1-\varpi)^{2-\varpi}}{(3-\varpi)(2-\varpi)^{3-\varpi}},$$

substituting these values in (38), we get (36). □

Remark 3.21. *If we put $\zeta_1 = \kappa_1$ and $\zeta_2 = \kappa_2$ in* (36)*, we get the following inequality*

$$\left|\frac{(1-\varpi)}{(\kappa_2-\kappa_1)^{1-\varpi}}\int_{\kappa_2}^{\kappa_1}(\mathfrak{u}-\kappa_1)^{-\varpi}\Psi(\mathfrak{u})d\mathfrak{u} - \Psi\left(\frac{\kappa_1+(1-\varpi)\kappa_2}{2-\varpi}\right)\right|$$
$$\leq (\kappa_2-\kappa_1)(\mathcal{M}_5(\varpi)|\Psi'(\kappa_1)| + \mathcal{M}_6(\varpi)|\Psi'(\kappa_2)|),$$

Remark 3.22. *If we put $\varpi = 0$ in* (36)*, we get the following inequality*

$$\left|\frac{1}{(\zeta_2-\zeta_1)}\int_{(\kappa_1+\kappa_2-\zeta_1)}^{(\kappa_1+\kappa_2-\zeta_2)}\Psi(\mathfrak{u})d\mathfrak{u} - \Psi\left(\kappa_1+\kappa_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{24}(|\Psi'(\zeta_2)| - |\Psi'(\zeta_1)|).$$

Corollary 3.23. *Taking $|\Psi'| \leq K$, in Theorem 3.20, then*

$$\left|\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} - \Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)\right|$$
$$\leq K(\zeta_2-\zeta_1)(\mathcal{M}_5(\varpi)+\mathcal{M}_6(\varpi)).$$

Theorem 3.24. *Assume the condition of Lemma, let $|\Psi'^{\mathfrak{q}}$ be a convex function for $\mathfrak{q} \geq 1$ and let $\zeta_1, \zeta_2 \in [\kappa_1, \kappa_2]$, such that $\zeta_1 < \zeta_2$, then*

$$\left|\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2) - \Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)\right|$$
$$\leq (\zeta_2-\zeta_1)(\mathcal{L}_8(\varpi))^{1-\frac{1}{\mathfrak{q}}}$$
$$\times\left\{(\mathcal{L}_8(\varpi)|\Psi'(\kappa_1)|^{\mathfrak{q}} + \mathcal{L}_8(\varpi)|\Psi'(\kappa_2)|^{\mathfrak{q}} - \mathcal{L}_5(\varpi)|\Psi'(\zeta_1)|^{\mathfrak{q}} - \mathcal{L}_6(\varpi)|\Psi'(\zeta_2)|^{\mathfrak{q}})^{\frac{1}{\mathfrak{q}}}\right.$$
$$\left.-(\mathcal{L}_8(\varpi)|\Psi'(\kappa_1)|^{\mathfrak{q}} + \mathcal{L}_8(\varpi)|\Psi'(\kappa_2)|^{\mathfrak{q}} - \mu_5(\varpi)|\Psi'(\zeta_1)|^{\mathfrak{q}} - \mu_6(\varpi)|\Psi'(\zeta_2)|^{\mathfrak{q}})^{\frac{1}{\mathfrak{q}}}\right\}, \tag{39}$$

where

$$\mathcal{L}_8(\varpi) = \int_{\frac{1-\varpi}{2-\varpi}}^1 (1-\lambda^{1-\varpi})d\lambda = \int_0^{\frac{1-\varpi}{2-\varpi}} \lambda^{1-\varpi} = \frac{(1-\varpi)^{2-\varpi}}{(2-\varpi)^{3-\varpi}},$$

and $\mathcal{L}_5(\varpi)$, $\mathcal{L}_6(\varpi)$, $\mu_5(\varpi)$ and $\mu_6(\varpi)$ are given in the proof of Theorem 3.20.

Proof. Utilising Lemma 3.3, we have

$$\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2) - \Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)$$
$$\leq (\zeta_2-\zeta_1)\int_{\frac{1-\varpi}{2-\varpi}}^1 (1-\lambda^{1-\varpi})\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda$$
$$-(\zeta_2-\zeta_1)\int_0^{\frac{1-\varpi}{2-\varpi}} \lambda^{1-\varpi}\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda. \tag{40}$$

Using Power mean inequality in (40), we obtain

$$\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2) - \Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)$$
$$\leq (\zeta_2-\zeta_1)\left(\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})d\lambda\right)^{1-\frac{1}{q}}\left(\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})|\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)|^q d\lambda\right)^{\frac{1}{q}}$$
$$-(\zeta_2-\zeta_1)\left(\int_{0}^{\frac{1-\varpi}{2-\varpi}}\lambda^{1-\varpi}d\lambda\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{1-\varpi}{2-\varpi}}\lambda^{1-\varpi}|\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)|^q d\lambda\right)^{\frac{1}{q}}.$$

Given that $|\Psi'|^q$is convex, applying Jensen-Mercer inequality, we obtain

$$\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2) - \Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)$$
$$\leq (\zeta_2-\zeta_1)\left(\frac{(1-\varpi)^{2-\varpi}}{(2-\varpi)^{3-\varpi}}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})(|\Psi'(\kappa_1)|^q+|\Psi'(\kappa_2)|^q-\lambda|\Psi'(\zeta_1)|^q-(1-\lambda)|\Psi'(\zeta_2)|^q)d\lambda\right)^{\frac{1}{q}}\right.$$
$$\left.-\left(\int_{0}^{\frac{1-\varpi}{2-\varpi}}\lambda^{1-\varpi}(|\Psi'(\kappa_1)|^q+|\Psi'(\kappa_2)|^q-\lambda|\Psi'(\zeta_1)|^q-(1-\lambda)|\Psi'(\zeta_2)|^q)d\lambda\right)^{\frac{1}{q}}\right\}$$
$$=(\zeta_2-\zeta_1)(\mathcal{L}_8(\varpi))^{1-\frac{1}{q}}$$
$$\times\left\{(\mathcal{L}_8(\varpi)|\Psi'(\kappa_1)|^q+\mathcal{L}_8(\varpi)|\Psi'(\kappa_2)|^q-\mathcal{L}_5(\varpi)|\Psi'(\zeta_1)|^q-\mathcal{L}_6(\varpi)|\Psi'(\zeta_2)|^q)^{\frac{1}{q}}\right.$$
$$\left.-(\mathcal{L}_8(\varpi)|\Psi'(\kappa_1)|^q+\mathcal{L}_8(\varpi)|\Psi'(\kappa_2)|^q-\mu_5(\varpi)|\Psi'(\zeta_1)|^q-\mu_6(\varpi)|\Psi'(\zeta_2)|^q)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

Remark 3.25. *Substituting $\varpi=0$, $\zeta_1=\kappa_1$ and $\zeta_2=\kappa_2$ in Theorem 3.24, we obtain the following mid point type inequality.*

$$\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\Psi(u)du-\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{8}\left(\left(\frac{|\Psi'(\kappa_1)|^q+2|\Psi'(\kappa_2)|^q}{3}\right)^{\frac{1}{q}}-\left(\frac{2|\Psi'(\kappa_1)|^q+|\Psi'(\kappa_2)|^q}{3}\right)^{\frac{1}{q}}\right),$$

which appeared in [47, Remark 8].

Remark 3.26. *If we put $\zeta_1=\kappa_1$ and $\zeta_2=\kappa_2$ in* (39)*, we get the following inequality*

$$\left|\frac{(1-\varpi)}{(\kappa_2-\kappa_1)^{1-\varpi}}\int_{\kappa_2}^{\kappa_1}(u-\kappa_1)^{-\varpi}\Psi(u)du-\Psi\left(\frac{\kappa_1+(1-\varpi)\kappa_2}{2-\varpi}\right)\right|$$
$$\leq(\kappa_2-\kappa_1)(\mathcal{L}_8(\varpi))^{1-\frac{1}{q}}\left\{\left((\mathcal{L}_8(\varpi)-\mathcal{L}_5(\varpi))|\Psi'(\kappa_1)|^q+(\mathcal{L}_8(\varpi)-\mathcal{L}_6(\varpi))|\Psi'(\kappa_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.-\left((\mathcal{L}_8(\varpi)-\mu_5(\varpi))|\Psi'(\kappa_1)|^q+(\mathcal{L}_8(\varpi)-\mu_6(\varpi))|\Psi'(\kappa_2)|^q\right)^{\frac{1}{q}}\right\}.$$

Remark 3.27. *If we put* $\varpi = 0$ *in* (39), *we get the following inequality*

$$\left|\frac{1}{(\zeta_2-\zeta_1)}\int_{(\kappa_1+\kappa_2-\zeta_1)}^{(\kappa_1+\kappa_2-\zeta_2)}\Psi(\mathfrak{u})d\mathfrak{u}-\Psi\left(\kappa_1+\kappa_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|$$
$$\leq(\zeta_2-\zeta_1)\left(\frac{1}{8}\right)^{1-\frac{1}{\mathfrak{q}}}\left\{\left(\frac{1}{8}|\Psi'(\kappa_1)|^{\mathfrak{q}}+\frac{1}{8}|\Psi'(\kappa_2)|^{\mathfrak{q}}-\frac{1}{12}|\Psi'(\zeta_1)|^{\mathfrak{q}}-\frac{1}{24}|\Psi'(\zeta_2)|^{\mathfrak{q}}\right)^{\frac{1}{\mathfrak{q}}}\right.$$
$$\left.-\left(\frac{1}{8}|\Psi'(\kappa_1)|^{\mathfrak{q}}+\frac{1}{8}|\Psi'(\kappa_2)|^{\mathfrak{q}}-\frac{1}{24}|\Psi'(\zeta_1)|^{\mathfrak{q}}-\frac{1}{12}|\Psi'(\zeta_2)|^{\mathfrak{q}}\right)^{\frac{1}{\mathfrak{q}}}\right\}.$$

Corollary 3.28. *Taking* $|\Psi'|\leq K$, *in Theorem 3.24, then*

$$\left|\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)-\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)\right|$$
$$\leq K(\zeta_2-\zeta_1)(\mathcal{L}_8(\varpi))^{1-\frac{1}{\mathfrak{q}}}\left\{(2\mathcal{L}_8(\varpi)-\mathcal{L}_5(\varpi)-\mathcal{L}_6(\varpi))^{\frac{1}{\mathfrak{q}}}-(2\mathcal{L}_8(\varpi)-\mu_5(\varpi)-\mu_6(\varpi))^{\frac{1}{\mathfrak{q}}}\right\}.$$

Theorem 3.29. *Assume the condition of Lemma, let* $|\Psi'^{\mathfrak{q}}$ *be a convex function for* $\mathfrak{q}>1$ *and let* $\zeta_1,\zeta_2\in[\kappa_1,\kappa_2]$, *such that* $\zeta_1<\zeta_2$, *then*

$$\left|\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)-\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)\right|$$
$$\leq(\zeta_2-\zeta_1)(\mathcal{L}_9(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}$$
$$\times\left(\frac{1}{2-\varpi}|\Psi'(\kappa_1)|^{\mathfrak{q}}+\frac{1}{2-\varpi}|\Psi'(\kappa_2)|^{\mathfrak{q}}-\frac{2(1-\varpi)+1}{2(2-\varpi)^2}|\Psi'(\zeta_1)|^{\mathfrak{q}}-\frac{1}{2(2-\varpi)^2}|\Psi'(\zeta_2)|^{\mathfrak{q}}\right)^{\frac{1}{\mathfrak{q}}}$$
$$-(\zeta_2-\zeta_1)(\mathcal{L}_{10}(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}$$
$$\times\left(\frac{1-\varpi}{2-\varpi}|\Psi'(\kappa_1)|^{\mathfrak{q}}+\frac{1-\varpi}{2-\varpi}|\Psi'(\kappa_2)|^{\mathfrak{q}}-\frac{(1-\varpi)^2}{2(2-\varpi)^2}|\Psi'(\zeta_1)|^{\mathfrak{q}}-\frac{(1-\varpi)(2(1-\varpi)+1)}{2(2-\varpi)^2}|\Psi'(\zeta_2)|^{\mathfrak{q}}\right)^{\frac{1}{\mathfrak{q}}},$$

where

$$\mathcal{L}_9(\varpi,\mathfrak{p})=\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})^{\mathfrak{p}}d\lambda$$

and

$$\mathcal{L}_{10}(\varpi,\mathfrak{p})=\int_0^{\frac{1-\varpi}{2-\varpi}}\lambda^{\varpi\mathfrak{p}}d\lambda.$$

Proof. Utilising Lemma 3.3, we have

$$\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)-\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)$$
$$\leq(\zeta_2-\zeta_1)\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda$$
$$-(\zeta_2-\zeta_1)\int_0^{\frac{1-\varpi}{2-\varpi}}\lambda^{1-\varpi}\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda.$$

Applying Hölder's inequality in (41), we have

$$\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)-\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)$$

$$\leq (\zeta_2-\zeta_1)\left(\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})^{\mathfrak{p}}d\lambda\right)^{\frac{1}{\mathfrak{p}}}\left(\int_{\frac{1-\varpi}{2-\varpi}}^{1}|\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)|^{\mathfrak{q}}d\lambda\right)^{\frac{1}{\mathfrak{q}}}$$
$$-(\zeta_2-\zeta_1)\left(\int_0^{\frac{1-\varpi}{2-\varpi}}\lambda^{(1-\varpi)\mathfrak{p}}d\lambda\right)^{\frac{1}{\mathfrak{p}}}\left(\int_0^{\frac{1-\varpi}{2-\varpi}}|\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)|^{\mathfrak{q}}d\lambda\right)^{\frac{1}{\mathfrak{q}}}.$$

Given that $|\Psi'|^{\mathfrak{q}}$is convex, applying Jensen-Mercer inequality, we obtain

$$\left|\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)-\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)\right|$$
$$\leq (\zeta_2-\zeta_1)(\mathcal{L}_9(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}$$
$$\times\left(\int_{\frac{1-\varpi}{2-\varpi}}^{1}(|\Psi'(\kappa_1)|^{\mathfrak{q}}+|\Psi'(\kappa_2)|^{\mathfrak{q}}-\lambda|\Psi'(\zeta_1)|^{\mathfrak{q}}-(1-\lambda)|\Psi'(\zeta_2)|^{\mathfrak{q}})\right)^{\frac{1}{\mathfrak{q}}}$$
$$-(\zeta_2-\zeta_1)(\mathcal{L}_{10}(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}$$
$$\times\left(\int_0^{\frac{1-\varpi}{2-\varpi}}(|\Psi'(\kappa_1)|^{\mathfrak{q}}+|\Psi'(\kappa_2)|^{\mathfrak{q}}-\lambda|\Psi'(\zeta_1)|^{\mathfrak{q}}-(1-\lambda)|\Psi'(\zeta_2)|^{\mathfrak{q}})\right)^{\frac{1}{\mathfrak{q}}}$$
$$=(\zeta_2-\zeta_1)(\mathcal{L}_9(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}$$
$$\times\left(\frac{1}{2-\varpi}|\Psi'(\kappa_1)|^{\mathfrak{q}}+\frac{1}{2-\varpi}|\Psi'(\kappa_2)|^{\mathfrak{q}}-\frac{(3-2\varpi)}{2(2-\varpi)^2}|\Psi'(\zeta_1)|^{\mathfrak{q}}-\frac{1}{2(2-\varpi)^2}|\Psi'(\zeta_2)|^{\mathfrak{q}}\right)^{\frac{1}{\mathfrak{q}}}$$
$$-(\zeta_2-\zeta_1)(\mathcal{L}_{10}(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}$$
$$\times\left(\frac{1-\varpi}{2-\varpi}|\Psi'(\kappa_1)|^{\mathfrak{q}}+\frac{1-\varpi}{2-\varpi}|\Psi'(\kappa_2)|^{\mathfrak{q}}-\frac{(1-\varpi)^2}{2(2-\varpi)^2}|\Psi'(\zeta_1)|^{\mathfrak{q}}-\frac{(1-\varpi)(3-2\varpi)}{2(2-\varpi)^2}|\Psi'(\zeta_2)|^{\mathfrak{q}}\right)^{\frac{1}{\mathfrak{q}}}.$$

This completes the proof. □

Remark 3.30. *Substituting $\varpi=0$, $\zeta_1=\kappa_1$ and $\zeta_2=\kappa_2$ in Theorem 3.29, we obtain*

$$\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\Psi(\mathfrak{u})d\mathfrak{u}-\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{4(\mathfrak{p}+1)}\left(\left(\frac{|\Psi'(\kappa_1)|^{\mathfrak{q}}+3|\Psi'(\kappa_2)|^{\mathfrak{q}}}{4}\right)^{\frac{1}{\mathfrak{q}}}-\left(\frac{3|\Psi'(\kappa_1)|^{\mathfrak{q}}+|\Psi'(\kappa_2)|^{\mathfrak{q}}}{4}\right)^{\frac{1}{\mathfrak{q}}}\right),$$

which appeared in [47, Remark 9].

Corollary 3.31. *Taking $|\Psi'|\leq K$, in Theorem 3.29, then*

$$\left|\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)-\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)\right|$$
$$\leq K(\zeta_2-\zeta_1)\Bigg[(\mathcal{L}_9(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}\left(\frac{2}{2-\varpi}-\frac{2(1-\varpi)+1}{2(2-\varpi)^2}-\frac{1}{2(2-\varpi)^2}\right)^{\frac{1}{\mathfrak{q}}}$$
$$-(\mathcal{L}_{10}(\varpi,\mathfrak{p}))^{\frac{1}{\mathfrak{p}}}\left(\frac{2(1-\varpi)}{2-\varpi}-\frac{(1-\varpi)^2}{2(2-\varpi)^2}-\frac{(1-\varpi)(2(1-\varpi)+1)}{2(2-\varpi)^2}\right)^{\frac{1}{\mathfrak{q}}}\Bigg].$$

Theorem 3.32. *Assume the condition of Lemma, let $\zeta_1, \zeta_2 \in [\kappa_1, \kappa_2]$ such that $\zeta_1 < \zeta_2$. If $|\Psi'|$ is concave on $[\kappa_1, \kappa_2]$, then*

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}$$
$$\times\Big(\mathcal{L}_1(\varpi)|\Psi'(\kappa_1+\kappa_2-\mathcal{L}_{11}(\varpi)\zeta_1-\mathcal{L}_{12}(\varpi)\zeta_2)| + \mu_1(\varpi)|\Psi'(\kappa_1+\kappa_2-\mathcal{L}_{13}(\varpi)\zeta_1-\mathcal{L}_{14}(\varpi)\zeta_2)|\Big), \tag{41}$$

where $\mathcal{L}_1$ and μ_1 are given in the proof of Theorem 3.5 and

$$\mathcal{L}_{11}(\varpi) = \frac{(2-\varpi)^{1-\frac{1}{1-\varpi}}}{2(3-\varpi)},$$
$$\mathcal{L}_{12}(\varpi) = \frac{(2-\varpi)^{1-\frac{1}{1-\varpi}}[2(3-\varpi)(2-\varpi)^{1-\frac{1}{1-\varpi}}-1]}{2(3-\varpi)},$$
$$\mathcal{L}_{13}(\varpi) = \frac{(2-\varpi)^{1-\frac{1}{1-\varpi}}[(2-\varpi)^{\frac{2}{1-\varpi}}+1]}{2(3-\varpi)}$$

and

$$\mathcal{L}_{14}(\varpi) = \frac{(2-\varpi)^{1-\frac{1}{1-\varpi}}[2(3-\varpi)(2-\varpi)^{\frac{1}{1-\varpi}-1}-1]}{2(3-\varpi)}.$$

Proof. Utilising Lemma 3.3, we have

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_0^1 |(2-\varpi)\lambda^{1-\varpi}-1||\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda|$$
$$= \frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_0^{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}} (1-(2-\varpi)\lambda^{1-\varpi})|\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda|$$
$$+\frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}^1 ((2-\varpi)\lambda^{1-\varpi}-1)|\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda|.$$

Given that $|\Psi'|$is concave, applying Jensen-Mercer inequality, we obtain

$$\left|\frac{(1-\varpi)\Psi(\kappa_1+\kappa_2-\zeta_1)+\Psi(\kappa_1+\kappa_2-\zeta_2)}{2-\varpi} - \frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_0^{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}} (1-(2-\varpi)\lambda^{1-\varpi})$$
$$\times\left|\Psi'\left(\frac{\int_0^{(2-\varpi)^{\frac{1}{1-\varpi}}}(1-(2-\varpi)\lambda^{1-\varpi})(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda}{\int_0^{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}(1-(2-\varpi)\lambda^{1-\varpi})d\lambda}\right)\right|$$
$$+\frac{(\zeta_2-\zeta_1)}{2-\varpi}\int_{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}^1 ((2-\varpi)\lambda^{1-\varpi}-1)d\lambda$$

$$\times\left|\Psi'\left(\frac{\int_{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}^{1}((2-\varpi)\lambda^{1-\varpi}-1)(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda}{\int_{\frac{1}{(2-\varpi)^{\frac{1}{1-\varpi}}}}^{1}((2-\varpi)\lambda^{1-\varpi}-1)d\lambda}\right)\right|.$$

$$=\frac{(\zeta_2-\zeta_1)}{2-\varpi}\Big[\mathcal{L}_1(\varpi)|\Psi'(\kappa_1+\kappa_2-\mathcal{L}_{11}(\varpi)\zeta_1-\mathcal{L}_{12}(\varpi)\zeta_2)|$$
$$+\mu_1(\varpi)|\Psi'(\kappa_1+\kappa_2-\mathcal{L}_{13}(\varpi)\zeta_1-\mathcal{L}_{14}(\varpi)\zeta_2)|\Big].$$

□

Remark 3.33. *If we put $\zeta_1=\kappa_1$ and $\zeta_2=\kappa_2$ in Theorem 3.32, we get the following inequality*

$$\left|\frac{(1-\varpi)}{(\kappa_2-\kappa_1)^{1-\varpi}}\int_{\kappa_2}^{\kappa_1}(u-\kappa_1)^{-\varpi}\Psi(u)du-\Psi\left(\frac{\kappa_1+(1-\varpi)\kappa_2}{2-\varpi}\right)\right|$$
$$\leq\frac{(\kappa_2-\kappa_1)}{2-\varpi}$$
$$\times\Big(\mathcal{L}_1(\varpi)|\Psi'(\kappa_1+\kappa_2-\mathcal{L}_{11}(\varpi)\kappa_1-\mathcal{L}_{12}(\varpi)\kappa_2)|+\mu_1(\varpi)|\Psi'(\kappa_1+\kappa_2-\mathcal{L}_{13}(\varpi)\kappa_1-\mathcal{L}_{14}(\varpi)\kappa_2)|\Big).$$

Remark 3.34. *If we put $\varpi=0$ in Theorem 3.32, we get the following inequality*

$$\left|\frac{1}{(\zeta_2-\zeta_1)}\int_{(\kappa_1+\kappa_2-\zeta_1)}^{(\kappa_1+\kappa_2-\zeta_2)}\Psi(u)du-\Psi\left(\kappa_1+\kappa_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|$$
$$\leq\frac{(\zeta_2-\zeta_1)}{4}\left(\left|\Psi'\left(\kappa_1+\kappa_2-\frac{1}{6}\zeta_1-\frac{5}{6}\zeta_2\right)\right|\right).$$

Corollary 3.35. *Taking $|\Psi'|\leq K$, in Theorem 3.32, then*

$$\left|\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)-\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)\right|$$
$$\leq K\frac{(\zeta_2-\zeta_1)}{2-\varpi}\Big(\mathcal{L}_1(\varpi)(2-\mathcal{L}_{11}(\varpi)-\mathcal{L}_{12}(\varpi))+\mu_1(\varpi)(2-\mathcal{L}_{13}(\varpi)-\mathcal{L}_{14}(\varpi)\Big).$$

Theorem 3.36. *Assume the condition of Lemma, let $|\Psi'|$ be a concave function and let $\zeta_1,\zeta_2\in[\kappa_1,\kappa_2]$, such that $\zeta_1<\zeta_2$, then*

$$\left|\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}}{}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2)-\Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)\right|$$
$$\leq(\zeta_2-\zeta_1)\left(\frac{(1-\varpi)^{2-\varpi}}{(2-\varpi)^{3-\varpi}}\right)$$
$$\times\Big[|\Psi'(\kappa_1+\kappa_2-\mathcal{L}_{15}(\varpi)\zeta_1-\mathcal{L}_{16}(\varpi)\zeta_2)|-|\Psi'(\kappa_1+\kappa_2-\mathcal{L}_{17}(\varpi)\zeta_1-\mathcal{L}_{18}(\varpi)\zeta_2)|\Big],\tag{42}$$

where

$$\mathcal{L}_{15}(\varpi)=\frac{(2-\varpi)^{1-\varpi}+2(1-\varpi)^{2-\varpi}}{2(1-\varpi)^{1-\varpi}(3-\varpi)},$$
$$\mathcal{L}_{16}(\varpi)=\frac{4(1-\varpi)^{1-\varpi}-(2-\varpi)^{1-\varpi}}{2(1-\varpi)^{1-\varpi}(3-\varpi)},$$
$$\mathcal{L}_{17}(\varpi)=\frac{1-\varpi}{3-\varpi}$$

and

$$\mathcal{L}_{18}(\varpi) = \frac{2}{3-\varpi}.$$

Proof. From Lemma 3.1, we have

$$\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2) - \Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)$$
$$\leq (\zeta_2-\zeta_1)\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda$$
$$-(\zeta_2-\zeta_1)\int_0^{\frac{1-\varpi}{2-\varpi}}\lambda^{1-\varpi}\Psi'(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda.$$

Given that $|\Psi'|$is concave, applying Jensen-Mercer inequality, we obtain

$$\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2) - \Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)$$
$$\leq (\zeta_2-\zeta_1)\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})d\lambda\left|\Psi'\left(\frac{\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda}{\int_{\frac{1-\varpi}{2-\varpi}}^{1}(1-\lambda^{1-\varpi})d\lambda}\right)\right|$$
$$-(\zeta_2-\zeta_1)\int_0^{\frac{1-\varpi}{2-\varpi}}\lambda^{1-\varpi}d\lambda\left|\Psi'\left(\frac{\int_{\frac{1-\varpi}{2-\varpi}}^{1}\lambda^{1-\varpi}(\kappa_1+\kappa_2-\lambda\zeta_1-(1-\lambda)\zeta_2)d\lambda}{\int_0^{\frac{1-\varpi}{2-\varpi}}\lambda^{1-\varpi}d\lambda}\right)\right|.$$
$$=(\zeta_2-\zeta_1)\left(\frac{(1-\varpi)^{2-\varpi}}{(2-\varpi)^{3-\varpi}}\right)$$
$$\times\Big[|\Psi'(\kappa_1+\kappa_2-\mathcal{L}_{15}(\varpi)\zeta_1-\mathcal{L}_{16}(\varpi)\zeta_2)|-|\Psi'(\kappa_1+\kappa_2-\mathcal{L}_{17}(\varpi)\zeta_1-\mathcal{L}_{18}(\varpi)\zeta_2)|\Big].$$

This completes the proof. □

Remark 3.37. *If we put $\zeta_1=\kappa_1$ and $\zeta_2=\kappa_2$ in Theorem 3.36, we get the following inequality*

$$\left|\frac{(1-\varpi)\Psi(\kappa_2)+\Psi(\kappa_1)}{2-\varpi}-\frac{(1-\varpi)}{(\kappa_2-\kappa_1)^{1-\varpi}}\int_{\kappa_2}^{\kappa_1}(\mathfrak{u}-\kappa_1)^{-\varpi}\Psi(\mathfrak{u})d\mathfrak{u}\right|$$
$$\leq(\kappa_2-\kappa_1)\left(\frac{(1-\varpi)^{2-\varpi}}{(2-\varpi)^{3-\varpi}}\right)$$
$$\times\Big[|\Psi'(\kappa_1+\kappa_2-\mathcal{L}_{15}(\varpi)\kappa_1-\mathcal{L}_{16}(\varpi)\kappa_2)|-|\Psi'(\kappa_1+\kappa_2-\mathcal{L}_{17}(\varpi)\kappa_1-\mathcal{L}_{18}(\varpi)\kappa_2)|\Big].$$

Remark 3.38. *If we put $\varpi=0$ in Theorem 3.36, we get the following inequality*

$$\left|\frac{(1-\varpi)\Psi(\kappa_2)+\Psi(\kappa_1)}{2-\varpi}-\frac{(1-\varpi)}{(\kappa_2-\kappa_1)^{1-\varpi}}\int_{\kappa_2}^{\kappa_1}(\mathfrak{u}-\kappa_1)^{-\varpi}\Psi(\mathfrak{u})d\mathfrak{u}\right|$$
$$\leq\frac{(\zeta_2-\zeta_1)}{8}\left[\left|\Psi'\left(\kappa_1+\kappa_2-\frac{2}{3}\zeta_1-\frac{1}{3}\zeta_2\right)\right|-\left|\Psi'\left(\kappa_1+\kappa_2-\frac{1}{3}\zeta_1-\frac{2}{3}\zeta_2\right)\right|\right].$$

Corollary 3.39. *Taking $|\Psi'|\leq K$, in Theorem 3.36, then*

$$\left|\frac{(1-\varpi)}{(\zeta_2-\zeta_1)^{1-\varpi}} {}_{N_3}J^{\varpi}_{(\kappa_1+\kappa_2-\zeta_1)^-}\Psi(\kappa_1+\kappa_2-\zeta_2) - \Psi\left(\kappa_1+\kappa_2-\frac{(1-\varpi)\zeta_1+\zeta_2}{2-\varpi}\right)\right|$$

$$\leq K(\zeta_2 - \zeta_1)\left(\frac{(1-\varpi)^{2-\varpi}}{(2-\varpi)^{3-\varpi}}\right)\Big[\mathcal{L}_{17}(\varpi) + \mathcal{L}_{18}(\varpi) - \mathcal{L}_{15}(\varpi) - \mathcal{L}_{16}(\varpi)\Big].$$

4. Conclusions

In this study, we have successfully established a fractional Hermite-Hadamard-Mercer type inequality by leveraging the support line property of convex functions, and from this foundational result, we derived new integral identities to prove associated trapezoidal and midpoint-type inequalities for left non-conformable integrals. Our results serve as a significant generalization of classical inequalities, which are naturally recovered as special cases, thereby bridging a meaningful connection between traditional and fractional convex analysis. It is expected that this work will catalyze further research into Hermite-Hadamard-type inequalities within fractional calculus, and we propose that future investigations focus on extending these findings by employing non-conformable operators to explore alternative classes of functions, such as quasi-convex or strongly convex mappings.

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