The weak perturbation of matrices for the Moore-Penrose inverse in rings with involution

Liyun Wu[a], Jianlong Chen[a,*]

[a]School of Mathematics, Southeast University, Nanjing 210096, China

Abstract. Let R be a $*$-ring. For any $T \in R_{m\times n}$, $E \in R_{m\times m}$ and $F \in R_{n\times n}$, a matrix M over a $*$-ring is called a multiplicative perturbation of T if $M = ETF^*$. The initial goal of this paper is to introduce a type of multiplicative perturbation, called weak perturbation. Suppose that T, L_E and R_F have Moore-Penrose inverses $T^\dagger$, $L_E^\dagger$ and $R_F^\dagger$, respectively. Then, a multiplicative perturbation ETF^* of T is said to be a weak perturbation if $L_E^\dagger L_E T = TR_F^\dagger R_F$, where $L_E = ETT^\dagger + I_m - TT^\dagger$ and $R_F = FT^\dagger T + I_n - T^\dagger T$. It is worth noting that the definition of weak perturbation for the Moore-Penrose inverse of matrices over rings only satisfies one of the equations in the case of a complex matrix. It is shown that if M is a weak perturbation of T, then M has a Moore-Penrose inverse if and only if the Moore-Penrose inverses of $ETT^\dagger$ and $T^\dagger TF^*$ both exist. Moreover, $M^\dagger = (T^\dagger TF^*)^\dagger T^\dagger (ETT^\dagger)^\dagger$. Necessary and sufficient conditions for the existence as well as the expressions for $M^\dagger$ are derived under the condition that M is the weak perturbation of T. If M is a multiplicative perturbation of T, and $(ETT^\dagger)^\dagger$ and $(T^\dagger TF^*)^\dagger$ both exist, inspired by an alternative expression for M as $M = (ETT^\dagger) \cdot T \cdot (T^\dagger TF^*)$, we prove that $M^\dagger$ exists if and only if $S_{L,N^{-1}}^\dagger$ exists, where $S_{L,N^{-1}}^\dagger$ is the weighted Moore-Penrose inverse for certain matrices S, L and N associated to the triple (T, E, F). Moreover, $M^\dagger = (T^\dagger TF^*)^\dagger S_{L,N^{-1}}^\dagger (ETT^\dagger)^\dagger$.

1. Introduction

Let $\mathbb{C}_{m\times n}$ be the set of all $m \times n$ complex matrices. I_n and 0 denote the identity matrix and the zero matrix in $\mathbb{C}_{m\times n}$, respectively. For any $A \in \mathbb{C}_{m\times n}$, the symbol A^* denotes the conjugate transpose of A.

Recall that the Moore-Penrose inverse of $A \in \mathbb{C}_{m\times n}$, denoted by $A^\dagger$ [20], is the unique solution to the following equations

$$\text{(i) } AXA = A, \text{ (ii) } XAX = X, \text{ (iii) } (AX)^* = AX, \text{ (iv) } (XA)^* = XA.$$

Let $T \in \mathbb{C}_{m\times n}$. A matrix $M \in \mathbb{C}_{m\times n}$ is called a multiplicative perturbation of T if M can be expressed as

$$M = ETF^*, \text{ where } E \in \mathbb{C}_{m\times m} \text{ and } F \in \mathbb{C}_{n\times n}. \tag{1}$$

In recent years, many papers have focused on multiplicative perturbations of the Moore-Penrose inverse of matrices or operators [8, 10, 11, 14, 15, 17, 23]. In the present paper, we mainly consider the existence and expressions for the Moore-Penrose inverse of multiplicative perturbations of matrices over an arbitrary ∗-ring. We next introduce some results which directly inspire the present study.

In 2015, Castro-González et al. [2] presented a formula for the Moore-Penrose inverse $M^\dagger$, where M is given by (1) such that E and F are nonsingular,

$$E(I_m − TT^\dagger) = I_m − TT^\dagger \text{ and } F(I_n − T^\dagger T) = I_n − T^\dagger T. \tag{2}$$

Furthermore, they provided expressions for the Moore-Penrose inverse of block matrices involving the Schur complement. Later, Castro-González et al. [1] extended the similar formulae in (2) above to matrices over a ring and derived some characterizations of the existence for the Moore-Penrose inverse of a matrix product. When E and F are both singular, in order to establish formulae for $M^\dagger$, Zhang et al. [24] introduced an alternative expression for the multiplicative perturbation by the following two matrices:

$$L_E = ETT^\dagger + I_m − TT^\dagger \text{ and } R_F = FT^\dagger T + I_n − T^\dagger T. \tag{3}$$

In view of Equation (3), the following three facts can be obtained: (i) $ET = L_E T$ and $TF^* = TR_F^*$; (ii) $M = ETF^* = L_E TR_F^*$; (iii) Equation (2) is always true if E and F are replaced with L_E and R_F, respectively. They also introduced the following type of multiplicative perturbation to study the representations for the Moore-Penrose inverse.

Definition 1.1. [24] Let L_E and R_F be defined by (3) for $T \in \mathbb{C}_{m\times n}$, $E \in \mathbb{C}_{m\times m}$ and $F \in \mathbb{C}_{n\times n}$. The multiplicative perturbation M given by (1) is said to be a weak perturbation if

$$TT^\dagger L_E^\dagger(I_m − TT^\dagger) = 0, \; T^\dagger T R_F^\dagger(I_n − T^\dagger T) = 0, \; L_E^\dagger L_E T = TR_F^\dagger R_F. \tag{4}$$

Applying singular value decomposition (SVD) of complex matrices, formulae for $M^\dagger$ are derived in [24, Theorem 4.1] under the condition that M is the weak perturbation of T. Later, Xu et al. [21] obtained the generalized triple reverse order law for the Moore-Penrose inverse $M^\dagger$ in terms of another alternative expression for the multiplicative perturbation (1) as $M = (ETT^\dagger) \cdot T \cdot (T^\dagger TF^*)$. It is worth noting that in [21] and [24], applying singular value decomposition of complex matrices and a technical result in [24, Theorem 3.1] concerning representations for the Moore-Penrose inverse of certain 2×2 partitioned matrices, the authors derived many formulae for $M^\dagger$. However, this method is not applicable to matrices in rings.

Throughout this paper, R is a ∗-ring, that is, a ring with an involution ∗ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a + b)^* = a^* + b^*$ for any $a, b \in R$. Let $R_{m\times n}$ denote the set of all $m \times n$ matrices over R. In particular, we write R_n instead of $R_{n\times 1}$ for short. For a matrix $A = (a_{ij}) \in R_{m\times n}$, we write $A^* = (a_{ji}^*)$ and $\mathcal{R}(A) = \{Ax | x \in R_n\}$. We say that $A \in R_{m\times m}$ is Hermitian if $A^* = A$.

Recall that $A \in R_{m\times n}$ is said to have a Moore-Penrose inverse [20] if there exists some $X \in R_{n\times m}$ such that

$$\text{(i) } AXA = A, \text{ (ii) } XAX = X, \text{ (iii) } (AX)^* = AX, \text{ (iv) } (XA)^* = XA.$$

Such an X is called a Moore-Penrose inverse of A. It is unique if it exists, and is denoted by $A^\dagger$. More generally, if A and X satisfy the equations (i) and (iii), then X is called a {1, 3}-inverse of A, and is denoted by $A^{(1,3)}$. If A and X satisfy the equations (i) and (iv), then X is called a {1, 4}-inverse of A, and is denoted by $A^{(1,4)}$. As usual, we denote by $R_{m\times n}^\dagger$, $R_{m\times n}^{\{1,3\}}$ and $R_{m\times n}^{\{1,4\}}$ the sets of all Moore-Penrose invertible, {1, 3}-invertible and {1, 4}-invertible matrices in $R_{m\times n}$, respectively. It is well known that $A \in R_{m\times n}^\dagger$ if and only if $A \in R_{m\times n}^{\{1,3\}} \cap R_{m\times n}^{\{1,4\}}$. In this case, $A^\dagger = A^{(1,4)}AA^{(1,3)}$.

For any matrices in a ring, multiplicative perturbation can be defined similarly to that of (1).

Definition 1.2. A matrix $M \in R_{m\times n}$ is called a multiplicative perturbation of $T \in R_{m\times n}$ if M can be expressed as

$$M = ETF^*, \text{ where } E \in R_{m\times m} \text{ and } F \in R_{n\times n}. \tag{5}$$

The purpose of this paper is to extend weak perturbation for the Moore-Penrose inverse from complex matrices to matrices over arbitrary $*$-ring, and to investigate characterizations and expressions for the Moore-Penrose inverse of multiplicative perturbations of matrices, generalizing some corresponding results in [21, 24]. Different from the definition of weak perturbation in complex matrices, the weak perturbation for the Moore-Penrose inverse of matrices in $*$-ring only satisfies the third equation in (4). Let $T \in R^\dagger_{m \times n}$, $L_E = ETT^\dagger + I_m - TT^\dagger$ and $R_F = FT^\dagger T + I_n - T^\dagger T$. If $T^\dagger$, $L^\dagger_E$ and $R^\dagger_F$ exist, then, a multiplicative perturbation M is called a weak perturbation of T if $L^\dagger_E L_E T = TR^\dagger_F R_F$. Based on this definition, we can also generalize some corresponding results in complex matrices. Inspired by an alternative expression for M as $M = (ETT^\dagger) \cdot T \cdot (T^\dagger TF^*)$, we want to investigate whether there exists a matrix X such that the generalized triple reverse order law holds for $M^\dagger = (T^\dagger TF^*)^\dagger X (ETT^\dagger)^\dagger$. We show in Theorem 2.7 that if M is a weak perturbation of T, then $M^\dagger$ exists if and only if $(ETT^\dagger)^\dagger$ and $(T^\dagger TF^*)^\dagger$ both exist. Moreover, $M^\dagger = (T^\dagger TF^*)^\dagger T^\dagger L^\dagger_E L_E TT^\dagger (ETT^\dagger)^\dagger$. Inspired by another alternative expression for M as $M = L_E TR^*_F$, more existence criteria and representations for $M^\dagger$ are derived under the condition that M is the weak perturbation of T. In particular, if M is a weak perturbation of T, we prove in Theorem 2.11 that if $M^\dagger$ exists, then $M^\dagger = (T^\dagger TF^*)^\dagger T^\dagger (ETT^\dagger)^\dagger$ and $(ETT^\dagger)^\dagger ET = TF^*(T^\dagger TF^*)^\dagger$. In addition, some counterexamples are given to show that the assumption that M is the weak perturbation of T cannot be removed in a general $*$-ring. Finally, we establish the relationship between $M^\dagger$ and $S^\dagger_{L,N^{-1}}$, where $S^\dagger_{L,N^{-1}}$ is the weighted Moore-Penrose inverse for certain matrices S, L and N associated to the triple (T, E, F) given by (5). We establish in Theorem 3.4 that if M is a multiplicative perturbation of T, then $M^\dagger$ exists if and only if $S^\dagger_{L,N^{-1}}$ exists provided that $(ETT^\dagger)^\dagger$ and $(T^\dagger TF^*)^\dagger$ both exist. Moreover, $M^\dagger = (T^\dagger TF^*)^\dagger S^\dagger_{L,N^{-1}} (ETT^\dagger)^\dagger$ and $S^\dagger_{L,N^{-1}} = (T^\dagger TF^*)M^\dagger(ETT^\dagger)$.

2. Weak perturbation and reverse order law

In this section, we extend the notion of weak perturbation for the Moore-Penrose inverse of complex matrices to matrices over rings.

In what follows, suppose $T \in R^\dagger_{m \times n}$, $E \in R_{m \times m}$ and $F \in R_{n \times n}$. Define

$$L_E = ETT^\dagger + I_m - TT^\dagger \text{ and } R_F = FT^\dagger T + I_n - T^\dagger T. \tag{6}$$

Lemma 2.1. *Let $T \in R^\dagger_{m \times n}$, $E \in R_{m \times m}$ and $F \in R_{n \times n}$. We have the following results*:

  (i) $ET = L_E T$ and $TF^* = TR^*_F$.

  (ii) $TT^\dagger E^* = TT^\dagger L^*_E$ and $FT^\dagger T = R_F T^\dagger T$.

Proof. (i). Note that $L_E = ETT^\dagger + I_m - TT^\dagger$ and $R_F = FT^\dagger T + I_n - T^\dagger T$, then by direct computation, we have $L_E T = (ETT^\dagger + I_m - TT^\dagger)T = ET$ and $TR^*_F = T(FT^\dagger T + I_n - T^\dagger T)^* = TF^*$.

  (ii). By (i), we get $TT^\dagger E^* = (ETT^\dagger)^* = (L_E TT^\dagger)^* = TT^\dagger L^*_E$ and $FT^\dagger T = (T^\dagger TF^*)^* = (T^\dagger TR^*_F)^* = R_F T^\dagger T$. $\square$

We remark that an alternative expression for the multiplicative perturbation (5) as $M = ETF^* = L_E TR^*_F$ by Lemma 2.1.

Compared with the weak perturbation of complex matrices, we now introduce the definition of weak perturbation for the Moore-Penrose inverse of matrices over rings, which only satisfies the third equation in (4). Based on this definition, it is surprising to find that we can also generalize some corresponding results in complex matrices.

Definition 2.2. *Let $T \in R_{m \times n}$, $E \in R_{m \times m}$ and $F \in R_{n \times n}$, and let L_E and R_F be defined by (6) such that $T \in R^\dagger_{m \times n}$, $L_E \in R^\dagger_{m \times m}$ and $R_F \in R^\dagger_{n \times n}$. A multiplicative perturbation ETF^* of T is said to be a weak perturbation if*

$$L^\dagger_E L_E T = TR^\dagger_F R_F. \tag{7}$$

Let $T \in R_{m \times n}^{\dagger}$ and $M = ETF^*$ be the multiplicative perturbation of T given by (5). Another alternative expression for M can be given as

$$M = (ETT^{\dagger}) \cdot T \cdot (T^{\dagger}TF^*).$$

Inspired by the reverse order law for matrix products [4–7, 13, 18, 19], we next consider whether there exists a matrix X such that

$$M^{\dagger} = (T^{\dagger}TF^*)^{\dagger}X(ETT^{\dagger})^{\dagger}. \tag{8}$$

If the equality (8) holds, then the equality (8) is called the generalized triple reverse order law for the Moore-Penrose inverse of M.

We first give some auxiliary lemmas and results to investigate the generalized triple reverse order law for $M^{\dagger}$.

Lemma 2.3. [12, P.201] *Let $A \in R_{m \times n}$. We have the following results*:

(i) *A is $\{1, 3\}$-invertible with $\{1, 3\}$-inverse X if and only if $X^*A^*A = A$;*

(ii) *A is $\{1, 4\}$-invertible with $\{1, 4\}$-inverse Y if and only if $AA^*Y^* = A$.*

Lemma 2.4. ([25, Theorem 2.19 and 2.20], [20]) *Let $A \in R_{m \times n}$. The following results are equivalent*:

(i) *$A \in R_{m \times n}^{\dagger}$;*

(ii) *$A \in R_{m \times m}AA^*A$;*

(iii) *$A \in AA^*AR_{n \times n}$;*

(iv) *$A \in AA^*R_{m \times n} \cap R_{m \times n}A^*A$.*

In this case,

$$A^{\dagger} = A^*AX^2A^* = A^*Y^2AA^* = SAT,$$

where $A = AA^*AX = YAA^*A = T^*A^*A = AA^*S^*$.

Combining Lemmas 2.3 and 2.4, if $A^{\dagger}$ exists, then we can obtain another formula for $A^{\dagger} = A^{(1,4)}AA^{(1,3)} = (AX)^*A(YA)^* = X^*A^*AA^*Y^*$, where $A = AA^*AX = YAA^*A$ by Lemma 2.4 above.

Lemma 2.5. *Let $T \in R_{m \times n}$, $E \in R_{m \times m}$ and $F \in R_{n \times n}$ be such that $T \in R_{m \times n}^{\dagger}$.*

(i) *$ETT^{\dagger} \in R_{m \times m}^{\{1,3\}}$ if and only if $ET \in R_{m \times n}^{\{1,3\}}$.*

(ii) *$T^{\dagger}TF^* \in R_{n \times n}^{\{1,4\}}$ if and only if $TF^* \in R_{m \times n}^{\{1,4\}}$.*

Proof. We only prove the statement (i) as the statement (ii) can be proved similarly.

We first assume $ETT^{\dagger} \in R_{m \times m}^{\{1,3\}}$ and $X \in ETT^{\dagger}\{1, 3\}$. By Lemma 2.3, we have $ETT^{\dagger} = X^*(ETT^{\dagger})^*ETT^{\dagger}$. Post-multiplying $ETT^{\dagger} = X^*(ETT^{\dagger})^*ETT^{\dagger}$ by T gives

$$ET = X^*(ETT^{\dagger})^*ET = (T^{\dagger}X)^*(ET)^*ET.$$

Therefore, $ET \in R_{m \times n}^{\{1,3\}}$ and $T^{\dagger}X \in ET\{1, 3\}$ by Lemma 2.3(i).

Conversely, assume that $ET \in R_{m \times n}^{\{1,3\}}$ and $Y \in ET\{1, 3\}$. It follows from Lemma 2.3 that $ET = Y^*(ET)^*ET$. Post-multiplying $ET = Y^*(ET)^*ET$ by $T^{\dagger}$ implies

$$ETT^{\dagger} = Y^*(ET)^*ETT^{\dagger} = Y^*(ETT^{\dagger}T)^*ETT^{\dagger} = (TY)^*(ETT^{\dagger})^*ETT^{\dagger}.$$

Therefore, $ETT^{\dagger} \in R_{m \times m}^{\{1,3\}}$ and $TY \in ETT^{\dagger}\{1, 3\}$ by Lemma 2.3(i). $\square$

The following result indicates that $(ETT^\dagger)^{(1,4)}$ and $(T^\dagger TF^*)^{(1,3)}$ exist, even without the condition that $(ETF^*)^\dagger$ exists.

Lemma 2.6. *Let $T \in R_{m\times n}$, $E \in R_{m\times m}$ and $F \in R_{n\times n}$. If $M = ETF^*$ is a weak perturbation of T, then $(ETT^\dagger)^{(1,4)}$ and $(T^\dagger TF^*)^{(1,3)}$ exist.*

Proof. As $M = ETF^*$ is a weak perturbation of T, then, by Lemma 2.1 and Definition 2.2, we have

$$
\begin{aligned}
M = L_E T R_F^* \;&=\; L_E(TT^\dagger T)\cdot(R_F R_F^\dagger R_F)^* = L_E TT^\dagger(TR_F^\dagger R_F)R_F^* \\
&\overset{(7)}{=}\; L_E TT^\dagger L_E^\dagger L_E TR_F^* = L_E(TT^\dagger L_E^*)(L_E^\dagger)^* TR_F^* \\
&=\; ETT^\dagger E^*(L_E^\dagger)^* TR_F^* = ETT^\dagger(ETT^\dagger)^*(L_E^\dagger)^* TR_F^*.
\end{aligned}
$$

Post-multiplying $M = ETT^\dagger(ETT^\dagger)^*(L_E^\dagger)^* TR_F^*$ by $(R_F^\dagger)^* T^\dagger$ gives

$$
\begin{aligned}
L_E TR_F^*(R_F^\dagger)^* T^\dagger &= L_E(TR_F^\dagger R_F)T^\dagger \overset{(7)}{=} (L_E L_E^\dagger L_E)TT^\dagger = L_E TT^\dagger = ETT^\dagger \\
&= ETT^\dagger(ETT^\dagger)^*(L_E^\dagger)^* TR_F^*(R_F^\dagger)^* T^\dagger = ETT^\dagger(ETT^\dagger)^*(L_E^\dagger)^*(TR_F^\dagger R_F)T^\dagger \\
&\overset{(7)}{=} ETT^\dagger(ETT^\dagger)^*[(L_E^\dagger)^* L_E^\dagger L_E]TT^\dagger = ETT^\dagger(ETT^\dagger)^*(L_E^\dagger)^* TT^\dagger.
\end{aligned}
$$

i.e., $ETT^\dagger \in R_{m\times m}^{\{1,4\}}$ and $TT^\dagger L_E^\dagger \in ETT^\dagger\{1,4\}$ by Lemma 2.3(ii).

Similarly, since

$$
\begin{aligned}
M = L_E T R_F^* \;&=\; (L_E L_E^\dagger L_E)T(T^\dagger TF^*) = L_E(L_E^\dagger L_E T)(T^\dagger TF^*) \\
&\overset{(7)}{=}\; L_E TR_F^\dagger R_F(T^\dagger TF^*) = L_E TR_F^\dagger(T^\dagger TF^*)^*(T^\dagger TF^*).
\end{aligned}
$$

Pre-multiplying $M = L_E TR_F^\dagger(T^\dagger TF^*)^*(T^\dagger TF^*)$ by $T^\dagger L_E^\dagger$ gives

$$
\begin{aligned}
T^\dagger L_E^\dagger L_E TR_F^* &\overset{(7)}{=} T^\dagger T(R_F^\dagger R_F R_F^*) = T^\dagger TR_F^* = T^\dagger TF^* \\
&= T^\dagger(L_E^\dagger L_E T)R_F^\dagger(T^\dagger TF^*)^*(T^\dagger TF^*) \overset{(7)}{=} T^\dagger T(R_F^\dagger R_F R_F^\dagger)(T^\dagger TF^*)^*(T^\dagger TF^*) \\
&= T^\dagger TR_F^\dagger(T^\dagger TF^*)^*(T^\dagger TF^*).
\end{aligned}
$$

i.e., $T^\dagger TF^* \in R_{n\times n}^{\{1,3\}}$ and $(R_F^\dagger)^* T^\dagger T \in T^\dagger TF^*\{1,3\}$ by Lemma 2.3(i). $\quad\square$

The following theorem gives the necessary and sufficient conditions for the existence of $M^\dagger$ under the hypothesis that M is a weak perturbation of T. In addition, we presents the first formula for $M^\dagger$ revealing the generalized triple reverse order law, where $X = T^\dagger TR_F^\dagger R_F T^\dagger = T^\dagger L_E^\dagger L_E TT^\dagger$ in the equality (8).

Theorem 2.7. *Let $T \in R_{m\times n}$, $E \in R_{m\times m}$ and $F \in R_{n\times n}$ be such that $M = ETF^*$ is a weak perturbation of T. Then the following conditions are equivalent*:
 (i) $M \in R_{m\times n}^\dagger$.
 (ii) $ETT^\dagger \in R_{m\times m}^{\{1,3\}}$ and $T^\dagger TF^* \in R_{n\times n}^{\{1,4\}}$.
 (iii) $ETT^\dagger \in R_{m\times m}^\dagger$ and $T^\dagger TF^* \in R_{n\times n}^\dagger$.
 In this case,

$$
\begin{aligned}
M^\dagger \;&=\; (T^\dagger TF^*)^{(1,4)} T^\dagger TR_F^\dagger R_F T^\dagger(ETT^\dagger)^{(1,3)} \\
&=\; (T^\dagger TF^*)^{(1,4)} T^\dagger L_E^\dagger L_E TT^\dagger(ETT^\dagger)^{(1,3)} \\
&=\; (T^\dagger TF^*)^\dagger T^\dagger TR_F^\dagger R_F T^\dagger(ETT^\dagger)^\dagger \\
&=\; (T^\dagger TF^*)^\dagger T^\dagger L_E^\dagger L_E TT^\dagger(ETT^\dagger)^\dagger.
\end{aligned}
$$

Proof. Note that M is a weak perturbation of T, it follows from Equation (7) that $L_E^\dagger L_E T = TR_F^\dagger R_F$.

(i)$\Rightarrow$ (iii). By Lemma 2.6, it suffices to verify that both $(ETT^\dagger)^{(1,3)}$ and $(T^\dagger TF^*)^{(1,4)}$ exist.

We first prove that $(ETT^\dagger)^{(1,3)}$ exists. If $M = L_E TR_F^* \in R_{m\times n}^\dagger$, then there exists $X_1 \in R_{n\times m}$ such that $L_E TR_F^* = X_1^*(L_E TR_F^*)^* L_E TR_F^*$ by Lemma 2.3(i). Post-multiplying $L_E TR_F^* = X_1^*(L_E TR_F^*)^* L_E TR_F^*$ by $(R_F^\dagger)^*$ gives

$$L_E TR_F^*(R_F^\dagger)^* = L_E(TR_F^\dagger R_F) \stackrel{(7)}{=} L_E L_E^\dagger L_E T = L_E T = ET$$
$$= X_1^*(L_E TR_F^*)^* ET = X_1^* F(ET)^* ET.$$

Then, we have $ET = X_1^* F(ET)^* ET$, i.e., $ET \in R_{m\times n}^{\{1,3\}}$ and $F^* X_1 \in ET\{1,3\}$ by Lemma 2.3. Hence, $ETT^\dagger \in R_{m\times m}^{\{1,3\}}$ and $TF^* X_1 \in ETT^\dagger\{1,3\}$ by Lemma 2.5.

Similarly, we can also prove that $(T^\dagger TF^*)^{(1,4)}$ exists. Note also that $L_E TR_F^* \in R_{m\times n}^\dagger$, then there exists $Y_1 \in R_{n\times n}$ such that $L_E TR_F^* = L_E TR_F^*(L_E TR_F^*)^* Y_1^*$ by Lemma 2.3. Pre-multiplying $L_E TR_F^* = L_E TR_F^*(L_E TR_F^*)^* Y_1^*$ by $L_E^\dagger$ implies

$$L_E^\dagger L_E TR_F^* \stackrel{(7)}{=} TR_F^\dagger R_F R_F^* = TR_F^* = TF^*$$
$$= TF^*(L_E TR_F^*)^* Y_1^* = TF^*(TF^*)^*(Y_1 E)^*.$$

Then, we get that $TF^* = TF^*(TF^*)^*(Y_1 E)^*$, i.e., $TF^* \in R_{m\times n}^{\{1,4\}}$ and $Y_1 E \in TF^*\{1,4\}$ by Lemma 2.3. Hence, $T^\dagger TF^* \in R_{n\times n}^{\{1,4\}}$ and $Y_1 ET \in T^\dagger TF^*\{1,4\}$ by Lemma 2.5.

(iii)$\Rightarrow$ (ii). It is clear.

(ii)$\Rightarrow$ (i). Suppose that $(ETT^\dagger)^{(1,3)}$ exists, and let $X_2 \in R_{m\times m}$ be a $\{1,3\}$-inverse of $ETT^\dagger$. Then, by Lemma 2.3, we have $ETT^\dagger = X_2^*(ETT^\dagger)^* ETT^\dagger$, which gives

$$M = (ETT^\dagger)TF^* = X_2^*(ETT^\dagger)^*(ETT^\dagger)TF^*$$
$$= X_2^*(T^\dagger)^*(ET)^*(ETF^*) = (T^\dagger X_2)^*[L_E(L_E^\dagger L_E T)]^*(ETF^*)$$
$$\stackrel{(7)}{=} (T^\dagger X_2)^*[L_E(TR_F^\dagger R_F)]^*(ETF^*) = (T^\dagger X_2)^* R_F^\dagger(L_E TR_F^*)^*(ETF^*)$$
$$= [(R_F^\dagger)^* T^\dagger X_2]^* M^* M.$$

Thus, $M \in R_{m\times n}^{\{1,3\}}$ and $(R_F^\dagger)^* T^\dagger X_2 \in M\{1,3\}$ by Lemma 2.3(i).

Note that $(T^\dagger TF^*)^{(1,4)}$ exists. Again, by Lemma 2.3, there exists $Y_2 \in R_{n\times n}$ such that $T^\dagger TF^* = (T^\dagger TF^*)(T^\dagger TF^*)^* Y_2^*$. It follows that

$$M = ET(T^\dagger TF^*) = ET(T^\dagger TF^*)(T^\dagger TF^*)^* Y_2^*$$
$$= (ETF^*)[T(R_F R_F^\dagger R_F)^*]^*(Y_2 T^\dagger)^* = (ETF^*)[(TR_F^\dagger R_F)R_F^*]^*(Y_2 T^\dagger)^*$$
$$\stackrel{(7)}{=} (ETF^*)[L_E^\dagger(L_E TR_F^*)]^*(Y_2 T^\dagger)^* = MM^*(Y_2 T^\dagger L_E^\dagger)^*,$$

which implies $M \in R_{m\times n}^{\{1,4\}}$ and $Y_2 T^\dagger L_E^\dagger \in M\{1,4\}$ by Lemma 2.3(ii).

Therefore, $M \in R^\dagger$ and

$$M^\dagger = M^{(1,4)} MM^{(1,3)} = Y_2 T^\dagger L_E^\dagger(ETF^*)(R_F^\dagger)^* T^\dagger X_2$$
$$= Y_2 T^\dagger(L_E^\dagger L_E T)R_F^*(R_F^\dagger)^* T^\dagger X_2 \stackrel{(7)}{=} Y_2 T^\dagger TR_F^\dagger R_F R_F^\dagger R_F T^\dagger X_2$$
$$= Y_2 T^\dagger TR_F^\dagger R_F T^\dagger X_2 \stackrel{(7)}{=} Y_2 T^\dagger L_E^\dagger L_E TT^\dagger X_2$$
$$= (T^\dagger TF^*)^\dagger T^\dagger TR_F^\dagger R_F T^\dagger(ETT^\dagger)^\dagger \stackrel{(7)}{=} (T^\dagger TF^*)^\dagger T^\dagger L_E^\dagger L_E TT^\dagger(ETT^\dagger)^\dagger.$$

This completes the proof. $\square$

Motivated by this fact in Theorem 2.7, one may suspect that if M is a multiplicative perturbation of T, we also get that $M \in R^{\dagger}_{m \times n}$ if and only if $(ETT^{\dagger})^{\dagger}$ and $(T^{\dagger}TF^*)^{\dagger}$ both exist. But the following example eliminates the possibility.

Example 2.8. Let $R = M_2(\mathbb{Z})$ be the ring of all 2×2 integer matrices and take the involution $*$ as the transpose.

(1) Suppose $T = I$, $E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. Clearly, $M = ETF^* = 0 \in R^{\dagger}$. However, $ETT^{\dagger} = E \notin R^{\dagger}$ and $T^{\dagger}TF^* = F^* \notin R^{\dagger}$.

(2) Suppose $T = I$, $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is clear that E is invertible, $(ETT^{\dagger})^{\dagger} = E^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $(T^{\dagger}TF^*)^{\dagger} = F$. However, $M = ETF^* = EF = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \notin R^{\dagger}$.

For any $A \in R_{m \times n}$, it is known from [3, Theorem 2.2.7] that if $A \in R^{\dagger}_{m \times n}$, then $A^* \in R^{\dagger}_{n \times m}$ and $A^*A \in R^{\dagger}_{n \times n}$. Moreover, $(A^*)^{\dagger} = (A^{\dagger})^*$ and $(A^*A)^{\dagger} = A^{\dagger}(A^*)^{\dagger}$. Then we can obtain another formula for the Moore-Penrose inverse of M in the following result.

Theorem 2.9. *Let E, T, F be as in Theorem 2.7 such that $M = ETF^*$ is a weak perturbation of T. If $M^{\dagger}$ exists, then*

$$
\begin{aligned}
M^{\dagger} &= (R^{\dagger}_F)^*(I_n + Z)^* T^{\dagger} T R^{\dagger}_F R_F T^{\dagger}(I_m + Y)L^{\dagger}_E \\
&= (R^{\dagger}_F)^*(I_n + Z)^* T^{\dagger} L^{\dagger}_E L_E T T^{\dagger}(I_m + Y)L^{\dagger}_E,
\end{aligned}
$$

where

$$
Y = (TT^{\dagger}E^*ETT^{\dagger})^{\dagger} \cdot TT^{\dagger}E^*(I_m - TT^{\dagger}),
$$

$$
Z = (T^{\dagger}TF^*FT^{\dagger}T)^{\dagger} \cdot T^{\dagger}TF^*(I_n - T^{\dagger}T).
$$

Proof. If $M = ETF^*$ is a weak perturbation of T, it follows from Theorem 2.7 that $M^{\dagger}$ exists if and only if both $(ETT^{\dagger})^{\dagger}$ and $(T^{\dagger}TF^*)^{\dagger}$ exist.

We next give the formula of $M^{\dagger}$. Since $ETT^{\dagger} \in R^{\dagger}_{m \times m}$ and $T^{\dagger}TF^* \in R^{\dagger}_{n \times n}$, one can get $TT^{\dagger}E^*ETT^{\dagger} \in R^{\dagger}_{m \times m}$ and $T^{\dagger}TF^*FT^{\dagger}T \in R^{\dagger}_{n \times n}$. Furthermore, we have $(TT^{\dagger}E^*ETT^{\dagger})^{\dagger} = (ETT^{\dagger})^{\dagger}(TT^{\dagger}E^*)^{\dagger}$ and $(T^{\dagger}TF^*FT^{\dagger}T)^{\dagger} = (FT^{\dagger}T)^{\dagger}(T^{\dagger}TF^*)^{\dagger}$.

Note that

$$
\begin{aligned}
(ETT^{\dagger})(I_m + Y)L^{\dagger}_E &= (ETT^{\dagger})[I + (ETT^{\dagger})^{\dagger}(TT^{\dagger}E^*)^{\dagger}TT^{\dagger}E^*(I_m - TT^{\dagger})]L^{\dagger}_E \\
&\overset{(6)}{=} (ETT^{\dagger})[I + (ETT^{\dagger})^{\dagger}(L_E - ETT^{\dagger})]L^{\dagger}_E \\
&= (ETT^{\dagger})(ETT^{\dagger})^{\dagger}L_E L^{\dagger}_E = (TT^{\dagger}E^*)^{\dagger}TT^{\dagger}(L^*_E L_E L^{\dagger}_E) \\
&= (TT^{\dagger}E^*)^{\dagger}(TT^{\dagger}L^*_E) = (ETT^{\dagger})(ETT^{\dagger})^{\dagger}
\end{aligned}
$$

and

$$
\begin{aligned}
(R^{\dagger}_F)^*(I_n + Z)^*(T^{\dagger}TF^*) &= (R^{\dagger}_F)^*[I_n + (FT^{\dagger}T)^{\dagger}(T^{\dagger}TF^*)^{\dagger}T^{\dagger}TF^*(I_n - T^{\dagger}T)]^*(T^{\dagger}TF^*) \\
&= (R^{\dagger}_F)^*[I_n + (FT^{\dagger}T)^{\dagger}(I_n - T^{\dagger}T)]^*(T^{\dagger}TF^*) \\
&\overset{(6)}{=} (R^{\dagger}_F)^*[I_n + (R^*_F - T^{\dagger}TF^*)(T^{\dagger}TF^*)^{\dagger}](T^{\dagger}TF^*) \\
&= (R^{\dagger}_F)^*R^*_F(T^{\dagger}TF^*)^{\dagger}(T^{\dagger}TF^*) \\
&= (R_F R^{\dagger}_F R_F)T^{\dagger}T(FT^{\dagger}T)^{\dagger} = (FT^{\dagger}T)(FT^{\dagger}T)^{\dagger},
\end{aligned}
$$

then, by Theorem 2.7, we have

$$
\begin{aligned}
M^\dagger &= (T^\dagger TF^*)^\dagger T^\dagger TR_F^\dagger R_F T^\dagger (ETT^\dagger)^\dagger = [(T^\dagger TF^*)^\dagger (T^\dagger TF^*)](R_F^\dagger)^* T^\dagger (ETT^\dagger)^\dagger \\
&= (R_F^\dagger)^* (I_n + Z)^* [T^\dagger TR_F^*(R_F^\dagger)^* T^\dagger](ETT^\dagger)^\dagger = (R_F^\dagger)^* (I_n + Z)^* [T^\dagger TR_F^\dagger R_F T^\dagger](ETT^\dagger)^\dagger \\
&\overset{(7)}{=} (R_F^\dagger)^* (I_n + Z)^* T^\dagger L_E^\dagger [(ETT^\dagger)(ETT^\dagger)^\dagger] = (R_F^\dagger)^* (I_n + Z)^* T^\dagger L_E^\dagger L_E TT^\dagger (I_m + Y) L_E^\dagger \\
&\overset{(7)}{=} (R_F^\dagger)^* (I_n + Z)^* T^\dagger TR_F^\dagger R_F T^\dagger (I_m + Y) L_E^\dagger.
\end{aligned}
$$

The proof is completed. $\square$

Remark 2.10. Now we can provide another proof for Theorem 2.9 as follows. Through the proof of Theorem 2.9 above, we find that $(I_m + Y)L_E^\dagger \in ETT^\dagger\{1,3\}$ and $(R_F^\dagger)^*(I_n + Z)^* \in T^\dagger TF^*\{1,4\}$. Indeed, we have $(ETT^\dagger)(I_m + Y)L_E^\dagger = (ETT^\dagger)(ETT^\dagger)^\dagger$ and $(R_F^\dagger)^*(I_n + Z)^*(T^\dagger TF^*) = (FT^\dagger T)(FT^\dagger T)^\dagger$ by Theorem 2.9. Hence, $(ETT^\dagger)(I_m + Y)L_E^\dagger(ETT^\dagger) = ETT^\dagger$ and $(T^\dagger TF^*)(R_F^\dagger)^*(I_n + Z)^*(T^\dagger TF^*) = T^\dagger TF^*$. Therefore, by Theorem 2.7, we have $M^\dagger = (T^\dagger TF^*)^{(1,4)} T^\dagger TR_F^\dagger R_F T^\dagger (ETT^\dagger)^{(1,3)} = (R_F^\dagger)^*(I_n + Z)^* T^\dagger TR_F^\dagger R_F T^\dagger (I_m + Y) L_E^\dagger$.

If M is a multiplicative perturbation of T given by (1), in [21, Theorem 3.3], Xu et. al proved that if $(ETT^\dagger)^\dagger ET = TF^*(T^\dagger TF^*)^\dagger$, then $M^\dagger = (T^\dagger TF^*)^\dagger T^\dagger (ETT^\dagger)^\dagger$. This result also applies to matrices in the ring.

The following theorem, under the hypothesis that M is a weak perturbation of T, we find that the equation $(ETT^\dagger)^\dagger ET = TF^*(T^\dagger TF^*)^\dagger$ can be derived from the Moore-Penrose invertibility of M. Furthermore, we give the second formula for $M^\dagger$ revealing the generalized triple reverse order law, where $X = T^\dagger$ in the equality (8). This extends the result in [21, Theorem 3.3 and Corollary 3.6].

Theorem 2.11. *Let $T \in R_{m \times n}$, $E \in R_{m \times m}$ and $F \in R_{n \times n}$ be such that $M = ETF^*$ is a weak perturbation of T. If $M^\dagger$ exists, then $M^\dagger = (T^\dagger TF^*)^\dagger T^\dagger (ETT^\dagger)^\dagger$ and $(ETT^\dagger)^\dagger ET = TF^*(T^\dagger TF^*)^\dagger$.*

Proof. As M is a weak perturbation of T and $M^\dagger$ exists, then $(ETT^\dagger)^\dagger$ and $(T^\dagger TF^*)^\dagger$ both exist and $L_E^\dagger L_E T = TR_F^\dagger R_F$ by Theorem 2.7 and Definition 2.2.

Let $X = (T^\dagger TF^*)^\dagger T^\dagger (ETT^\dagger)^\dagger = (T^\dagger TR_F^*)^\dagger T^\dagger (L_E TT^\dagger)^\dagger$. We first show that X is the Moore-Penrose inverse of M. Noting that

$$
\begin{aligned}
MX &= (ETF^*) \cdot (T^\dagger TR_F^*)^\dagger T^\dagger (L_E TT^\dagger)^\dagger = (L_E TT^\dagger)T(T^\dagger TR_F^*) \cdot (T^\dagger TR_F^*)^\dagger T^\dagger (L_E TT^\dagger)^\dagger \\
&= L_E \cdot (L_E^\dagger L_E T) \cdot T^\dagger T \cdot [(T^\dagger TR_F^*)(T^\dagger TR_F^*)^\dagger] \cdot T^\dagger (L_E TT^\dagger)^\dagger \\
&\overset{(7)}{=} L_E TR_F^\dagger \cdot [(R_F T^\dagger T)(R_F T^\dagger T)^\dagger (R_F T^\dagger T)] \cdot T^\dagger (L_E TT^\dagger)^\dagger \\
&= L_E \cdot (TR_F^\dagger R_F) \cdot T^\dagger TT^\dagger \cdot (L_E TT^\dagger)^\dagger \overset{(7)}{=} L_E L_E^\dagger L_E TT^\dagger (L_E TT^\dagger)^\dagger \\
&= (L_E TT^\dagger)(L_E TT^\dagger)^\dagger,
\end{aligned}
$$

this yields $MX = (MX)^*$.

Since $TR_F^* = T(R_F R_F^\dagger R_F)^* = (TR_F^\dagger R_F)R_F^* \overset{(7)}{=} (L_E^\dagger L_E)TR_F^* = L_E^*(L_E^\dagger)^* TR_F^*$, on can get that

$$
\begin{aligned}
XM &= (T^\dagger TF^*)^\dagger T^\dagger \cdot [(ETT^\dagger)^\dagger (L_E TT^\dagger)] \cdot T(T^\dagger TR_F^*) \\
&= (T^\dagger TF^*)^\dagger T^\dagger (TT^\dagger L_E^*)(TT^\dagger L_E^*)^\dagger TT^\dagger [L_E^*(L_E^\dagger)^* TR_F^*] \\
&= (T^\dagger TF^*)^\dagger T^\dagger [(TT^\dagger L_E^*)(TT^\dagger L_E^*)^\dagger (TT^\dagger L_E^*)](L_E^\dagger)^* TR_F^* \\
&= (T^\dagger TF^*)^\dagger T^\dagger TT^\dagger [L_E^*(L_E^\dagger)^* TR_F^*] = (T^\dagger TF^*)^\dagger T^\dagger TR_F^* \\
&= (FT^\dagger T)(FT^\dagger T)^\dagger = (XM)^*.
\end{aligned}
$$

Finally, $MXM = (L_E TT^\dagger)(L_E TT^\dagger)^\dagger \cdot (L_E TT^\dagger)TF^* = M$ and $XMX = (FT^\dagger T)(FT^\dagger T)^\dagger \cdot (T^\dagger TF^*)^\dagger T^\dagger (ETT^\dagger)^\dagger = X$. So, X is the Moore-Penrose inverse of M.

Next, we prove $(ETT^\dagger)^\dagger ET = TF^*(T^\dagger TF^*)^\dagger$. As $M \in R_{m\times n}^\dagger$, then there exist $Y' \in R_{n\times n}$, $Z' \in R_{m\times m}$ such that $M = MM^*MY' = Z'MM^*M$ by Lemma 2.4. It follows from Lemma 2.1 that

$$
\begin{aligned}
(ETT^\dagger)^\dagger ET &= (ETT^\dagger)^\dagger (L_E L_E^\dagger L_E)T = (ETT^\dagger)^\dagger L_E(L_E^\dagger L_E T) \\
&\overset{(7)}{=} (ETT^\dagger)^\dagger (L_E TR_F^*)(R_F^\dagger)^* = (ETT^\dagger)^\dagger MM^*MY'(R_F^\dagger)^* \\
&= (ETT^\dagger)^\dagger [(ETT^\dagger)T(R_F R_F^\dagger R_F)^*]M^*MY'(R_F^\dagger)^* \\
&= (ETT^\dagger)^\dagger [(ETT^\dagger)(TT^\dagger TR_F^\dagger R_F)R_F^*]M^*MY'(R_F^\dagger)^* \\
&\overset{(7)}{=} (ETT^\dagger)^\dagger (ETT^\dagger)TT^\dagger L_E^\dagger L_E TR_F^* M^*MY'(R_F^\dagger)^* \\
&= [(L_E TT^\dagger)^\dagger (L_E TT^\dagger)(TT^\dagger L_E^*)](L_E^\dagger)^* TR_F^* M^*MY'(R_F^\dagger)^* \\
&= (TT^\dagger L_E^*)(L_E^\dagger)^* TR_F^* M^*MY'(R_F^\dagger)^* \\
&= TT^\dagger L_E^\dagger (MM^*MY')(R_F^\dagger)^* = TT^\dagger L_E^\dagger (L_E TR_F^*)(R_F^\dagger)^* \\
&= TT^\dagger (L_E^\dagger L_E T)R_F^\dagger R_F \overset{(7)}{=} TT^\dagger TR_F^\dagger R_F R_F^\dagger R_F \\
&= TR_F^\dagger R_F
\end{aligned}
$$

and

$$
\begin{aligned}
TF^*(T^\dagger TF^*)^\dagger &= T(R_F R_F^\dagger R_F)^*(T^\dagger TF^*)^\dagger = (TR_F^\dagger R_F)R_F^*(T^\dagger TF^*)^\dagger \\
&\overset{(7)}{=} L_E^\dagger (L_E TR_F^*)(T^\dagger TF^*)^\dagger = L_E^\dagger (Z'MM^*M)(T^\dagger TF^*)^\dagger \\
&= L_E^\dagger Z'MM^*[(L_E L_E^\dagger L_E)T(T^\dagger TF^*)](T^\dagger TF^*)^\dagger \\
&= L_E^\dagger Z'MM^*L_E[(L_E^\dagger L_E T)T^\dagger T](T^\dagger TF^*)(T^\dagger TF^*)^\dagger \\
&\overset{(7)}{=} L_E^\dagger Z'MM^*L_E TR_F^\dagger [(R_F T^\dagger T)(T^\dagger TR_F^*)(T^\dagger TR_F^*)^\dagger] \\
&= L_E^\dagger Z'MM^*L_E TR_F^\dagger (R_F T^\dagger T) = L_E^\dagger (Z'MM^*M)(R_F^\dagger)^* T^\dagger T \\
&= L_E^\dagger (L_E TR_F^*)(R_F^\dagger)^* T^\dagger T = L_E^\dagger L_E (TR_F^\dagger R_F)T^\dagger T \\
&\overset{(7)}{=} L_E^\dagger L_E L_E^\dagger L_E TT^\dagger T = L_E^\dagger L_E T.
\end{aligned}
$$

Since $L_E^\dagger L_E T = TR_F^\dagger R_F$ by Equation (7), we have $(ETT^\dagger)^\dagger ET = TF^*(T^\dagger TF^*)^\dagger$.

The proof is completed. $\square$

The following example reveals that if M is a multiplicative perturbation of T and both $(ETT^\dagger)^\dagger$ and $(T^\dagger TF^*)^\dagger$ exist, then $(T^\dagger TF^*)^\dagger T^\dagger (ETT^\dagger)^\dagger$ may not necessarily be the Moore-Penrose inverse of M.

Example 2.12. Let $R = M_2(\mathbb{C})$ be the ring of all 2×2 complex matrices and let the involution $*$ be the conjugate transpose. Take, for example, $T = I$, $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, then $M = ETF^* = E$ and $(F^*)^\dagger = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$. Hence, $M^\dagger = E$. However, $(T^\dagger TF^*)^\dagger T^\dagger (ETT^\dagger)^\dagger = (F^*)^\dagger E^\dagger = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \neq M^\dagger$.

Comparing Theorems 2.7 and 2.11, it is natural to ask that whether $T^\dagger TR_F^\dagger R_F T^\dagger$ is equal to $T^\dagger$, or $(T^\dagger TF^*)^{(1,4)} T^\dagger (ETT^\dagger)^{(1,3)}$ is the Moore-Penrose inverse of M. See the following counterexample.

Example 2.13. Let $R = M_2(\mathbb{C})$ be the ring of all 2×2 complex matrices and let the involution $*$ be the conjugate transpose. Suppose $T = I$, $E = F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $M = ETF^* = E$, $R_F = E = F^*$ and $E^\dagger = E$. However, $T^\dagger TR_F^\dagger R_F T^\dagger = E^\dagger E \neq T^\dagger = I$ and $(T^\dagger TF^*)^{(1,4)} T^\dagger (ETT^\dagger)^{(1,3)} = E^{(1,4)} E^{(1,3)}$ is not unique, i.e., $(T^\dagger TF^*)^{(1,4)} T^\dagger (ETT^\dagger)^{(1,3)}$ is not the Moore-Penrose inverse of M.

In view of Example 2.13, in general, $(T^\dagger TF^*)^{(1,4)}T^\dagger(ETT^\dagger)^{(1,3)}$ may not necessarily be the Moore-Penrose inverse of M. It is natural to ask under what conditions can it be established. A characterization is given below.

Lemma 2.14. [3, Theorem 2.3.12] *Let* $P \in R_{n \times m}$, $A \in R_{m \times n}$ *and* $Q \in R_{n \times m}$. *If there exist* $P_1, Q_1 \in R_{m \times n}$ *such that* $P_1PA = A = AQQ_1$, *then* $PAQ \in R^\dagger_{n \times m}$ *if and only if* $PA \in R^{\{1,3\}}_{n \times n}$ *and* $AQ \in R^{\{1,4\}}_{m \times m}$. *In this case,* $(PAQ)^\dagger = (AQ)^{(1,4)}A(PA)^{(1,3)}$.

In Lemma 2.14 above, set $P = L_E$ and $Q = R_F^*$, then we have the following result as a direct consequence of Lemma 2.14.

Corollary 2.15. *Let* $T \in R^\dagger_{m \times n}$, $E \in R_{m \times m}$ *and* $F \in R_{n \times n}$ *be such that* $L_E \in R^\dagger_{m \times m}$, $R_F \in R^\dagger_{n \times n}$ *and* $L_E^\dagger L_E T = T = TR_F^\dagger R_F$, *then* $M = ETF^* \in R^\dagger_{m \times n}$ *if and only if* $ET \in R^{\{1,3\}}_{m \times n}$ *and* $TF^* \in R^{\{1,4\}}_{m \times n}$. *In this case,*

$$M^\dagger = (TF^*)^{(1,4)}T(ET)^{(1,3)} = (T^\dagger TF^*)^{(1,4)}T^\dagger(ETT^\dagger)^{(1,3)}.$$

Proof. From Lemmas 2.1 and 2.14, it suffices to verify that $(TF^*)^{(1,4)}T(ET)^{(1,3)} = (T^\dagger TF^*)^{(1,4)}T^\dagger(ETT^\dagger)^{(1,3)}$. By the proof of Lemma 2.5, we have $T(ET)^{(1,3)} \in ETT^\dagger\{1,3\}$ and $(TF^*)^{(1,4)}T \in T^\dagger TF^*\{1,4\}$. Then,

$$
\begin{aligned}
(TF^*)^{(1,4)}T &= (TF^*)^{(1,4)}TT^\dagger TR_F^\dagger R_F = \{[(TF^*)^{(1,4)}T](T^\dagger TF^*)\}(R_F^\dagger)^* \\
&= (T^\dagger TF^*)^{(1,4)}(T^\dagger TF^*)(R_F^\dagger)^* = (T^\dagger TF^*)^{(1,4)}T^\dagger T
\end{aligned}
$$

and

$$
\begin{aligned}
T(ET)^{(1,3)} &= TT^\dagger T(ET)^{(1,3)} = L_E^\dagger\{(L_E TT^\dagger)[T(ET)^{(1,3)}]\} \\
&= L_E^\dagger(ETT^\dagger)(ETT^\dagger)^{(1,3)} = TT^\dagger(ETT^\dagger)^{(1,3)}.
\end{aligned}
$$

Therefore, we have

$$(TF^*)^{(1,4)}T(ET)^{(1,3)} = (T^\dagger TF^*)^{(1,4)}T^\dagger TT^\dagger(ETT^\dagger)^{(1,3)} = (T^\dagger TF^*)^{(1,4)}T^\dagger(ETT^\dagger)^{(1,3)}.$$

$\square$

3. Multiplicative perturbation and weighted generalized inverses

In this section, we discuss the relationship between the Moore-Penrose inverse of M and weighted generalized inverses, and we shall further consider the generalized triple reverse order law for the Moore-Penrose inverse of $M = (ETT^\dagger) \cdot T \cdot (T^\dagger TF^*)$, under the condition that M is the multiplicative perturbation of T given by (5).

Above all, we recall some definitions of weighted generalized inverses.

Definition 3.1. [16] Let $L \in R_{m \times m}$ and $N \in R_{n \times n}$ be two invertible Hermitian matrices. A matrix $A \in R_{m \times n}$ has a weighted Moore-Penrose inverse with weights L, N if there exists $X \in R_{n \times m}$ such that

(1) $AXA = A$, (2) $XAX = X$, (3) $(LAX)^* = LAX$, (4) $(NXA)^* = NXA$.

Such an X is called a weighted Moore-Penrose inverse of A with weights L, N (abbr. weighted Moore-Penrose inverse). It is unique if it exists, and is denoted by $A^\dagger_{L,N}$. If $L = I_m$ and $N = I_n$, $A^\dagger_{L,N}$ is called the Moore-Penrose inverse of A. More generally, any X satisfying (1) $AXA = A$ and (3) $(LAX)^* = LAX$ is called an $\{L, 1, 3\}$-inverse of A, and is denoted by $A_L^{(1,3)}$. Similarly, any X satisfying (1) $AXA = A$ and (4) $(NXA)^* = NXA$ is called an $\{N, 1, 4\}$-inverse of A, and is denoted by $A_N^{(1,4)}$. We all know that $A^\dagger_{L,N}$ exists if and only if $A_L^{(1,3)}$ and $A_N^{(1,4)}$ both exist. In this case, $A^\dagger_{L,N} = A_N^{(1,4)}AA_L^{(1,3)}$.

Lemma 3.2. [26, Propositions 2.1 and 2.2] *Let $A \in R_{m \times n}$ and let $L \in R_{m \times m}$ and $N \in R_{n \times n}$ be two invertible Hermitian matrices. We have the following results:*

(i) *A is $\{L, 1, 3\}$-invertible if and only if $A \in R_{m \times n}A^*LA$. Moreover, if $A = XA^*LA$ for some $X \in R_{m \times n}$, then X^*L is an $\{L, 1, 3\}$-inverse of A;*

(ii) *A is $\{N, 1, 4\}$-invertible if and only if $A \in AN^{-1}A^*R_{m \times n}$. Moreover, if $A = AN^{-1}A^*Y$ for some $Y \in R_{m \times n}$, then $N^{-1}Y^*$ is an $\{N, 1, 4\}$-inverse of A.*

Let $T \in R_{m \times n}^\dagger$. We know that if $(ETT^\dagger)^\dagger$ and $(T^\dagger TF^*)^\dagger$ both exist, then $(TT^\dagger E^*ETT^\dagger)^\dagger = (ETT^\dagger)^\dagger(TT^\dagger E^*)^\dagger$ and $(T^\dagger TF^*FT^\dagger T)^\dagger = (FT^\dagger T)^\dagger(T^\dagger TF^*)^\dagger$. In what follows, unless otherwise indicated, we define

$$Y = TT^\dagger E^*ETT^\dagger, \quad Z = T^\dagger TF^*FT^\dagger T, \tag{9}$$

$$L = Y + I_m - YY^\dagger, \quad N = Z + I_n - ZZ^\dagger. \tag{10}$$

Lemma 3.3. *Let $T \in R_{m \times n}^\dagger$, $E \in R_{m \times m}$ and $F \in R_{n \times n}$ and let Y, Z, L, N be as in (9) and (10). If $(ETT^\dagger)^\dagger$ and $(T^\dagger TF^*)^\dagger$ both exist, then L and N are two invertible Hermitian matrices. In this case,*

$$\begin{aligned}
L^{-1} &= (ETT^\dagger)^\dagger(TT^\dagger E^*)^\dagger + I - (ETT^\dagger)^\dagger(ETT^\dagger), \\
N^{-1} &= (FT^\dagger T)^\dagger(T^\dagger TF^*)^\dagger + I - (FT^\dagger T)^\dagger(FT^\dagger T).
\end{aligned}$$

Proof. As $ETT^\dagger \in R_{m \times m}^\dagger$ and $T^\dagger TF^* \in R_{n \times n}^\dagger$, then we have

$$\begin{aligned}
L &= Y + I_m - YY^\dagger = TT^\dagger E^*ETT^\dagger + I_m - (TT^\dagger E^*ETT^\dagger)(TT^\dagger E^*ETT^\dagger)^\dagger \\
&= TT^\dagger E^*ETT^\dagger + I_m - [(TT^\dagger E^*)(ETT^\dagger)(ETT^\dagger)^\dagger](TT^\dagger E^*)^\dagger \\
&= (TT^\dagger E^*)(ETT^\dagger) + I_m - (ETT^\dagger)^\dagger(ETT^\dagger)
\end{aligned}$$

and

$$\begin{aligned}
N &= Z + I_n - ZZ^\dagger = T^\dagger TF^*FT^\dagger T + I_n - (T^\dagger TF^*FT^\dagger T)(T^\dagger TF^*FT^\dagger T)^\dagger \\
&= T^\dagger TF^*FT^\dagger T + I_n - [(T^\dagger TF^*)(FT^\dagger T)(FT^\dagger T)^\dagger](T^\dagger TF^*)^\dagger \\
&= (T^\dagger TF^*)(FT^\dagger T) + I_n - (FT^\dagger T)^\dagger(FT^\dagger T).
\end{aligned}$$

It is clear that L, N are Hermitian. By direct computation, we have

$$L[(ETT^\dagger)^\dagger(TT^\dagger E^*)^\dagger + I - (ETT^\dagger)^\dagger(ETT^\dagger)] = (TT^\dagger E^*)(ETT^\dagger)(ETT^\dagger)^\dagger(TT^\dagger E^*)^\dagger + I - (ETT^\dagger)^\dagger(ETT^\dagger) = I,$$

it is also easy to verify that $[(ETT^\dagger)^\dagger(TT^\dagger E^*)^\dagger + I - (ETT^\dagger)^\dagger(ETT^\dagger)]L = I$, thus $L^{-1} = (ETT^\dagger)^\dagger(TT^\dagger E^*)^\dagger + I - (ETT^\dagger)^\dagger(ETT^\dagger)$.

Similarly, we can prove that $N^{-1} = (FT^\dagger T)^\dagger(T^\dagger TF^*)^\dagger + I - (FT^\dagger T)^\dagger(FT^\dagger T)$. $\square$

In what follows, we define $S = YY^\dagger TZZ^\dagger$. In [21, Theorem 3.2], Xu et. al proved that if M is a multiplicative perturbation of T given by (1), then $M^\dagger = (T^\dagger TF^*)^\dagger S_{L,N^{-1}}^\dagger(ETT^\dagger)^\dagger$.

For any matrices in a $*$-ring, we all know that weighted Moore-Penrose inverse may not necessarily exist. The following theorem gives a necessary and sufficient condition for the existence of $M^\dagger$, which establishes the relationship between $M^\dagger$ and $S_{L,N^{-1}}^\dagger$, and presents the third formula for $M^\dagger$ revealing the generalized triple reverse order law, where $X = S_{L,N^{-1}}^\dagger$ in the equality (8) and Y, Z, L, N are given by (9) and (10), under the hypothesis that M is a multiplicative perturbation of T. This extends the result in [21, Theorem 3.2].

Theorem 3.4. *Let $T \in R_{m \times n}^\dagger$, $E \in R_{m \times m}$ and $F \in R_{n \times n}$ such that $M = ETF^*$ is a multiplicative perturbation of T. If $(ETT^\dagger)^\dagger$ and $(T^\dagger TF^*)^\dagger$ both exist, then $M \in R_{m \times n}^\dagger$ if and only if $S_{L,N^{-1}}^\dagger$ exists.*

In this case,

$$M^\dagger = (T^\dagger TF^*)^\dagger S_{L,N^{-1}}^\dagger(ETT^\dagger)^\dagger \text{ and } S_{L,N^{-1}}^\dagger = (T^\dagger TF^*)M^\dagger(ETT^\dagger).$$

Proof. Suppose $ETT^\dagger \in R_{m\times m}^\dagger$ and $T^\dagger TF^* \in R_{n\times n}^\dagger$. By (9), we have

$$
\begin{aligned}
S &= YY^\dagger TZZ^\dagger \\
&= (TT^\dagger E^*)[(ETT^\dagger)(ETT^\dagger)^\dagger(TT^\dagger E^*)^\dagger]T(T^\dagger TF^*)[(FT^\dagger T)(FT^\dagger T)^\dagger(T^\dagger TF^*)^\dagger] \\
&= (ETT^\dagger)^\dagger(ETT^\dagger)T(FT^\dagger T)^\dagger(FT^\dagger T).
\end{aligned}
$$

Suppose that $S_{L,N^{-1}}^\dagger$ exists, we first show that $X = (T^\dagger TF^*)^\dagger S_{L,N^{-1}}^\dagger(ETT^\dagger)^\dagger$ is the Moore-Penrose inverse of M. Indeed,

(1) $MXM = (ETT^\dagger)SS_{L,N^{-1}}^\dagger S(T^\dagger TF^*) = (ETT^\dagger)T(T^\dagger TF^*) = M$.

(2) $XMX = (T^\dagger TF^*)^\dagger S_{L,N^{-1}}^\dagger SS_{L,N^{-1}}^\dagger(ETT^\dagger)^\dagger = (T^\dagger TF^*)^\dagger S_{L,N^{-1}}^\dagger(ETT^\dagger)^\dagger = X$.

Note that $(TT^\dagger E^*)^\dagger L = ETT^\dagger$, then we have:

(3) $MX = (ETT^\dagger)SS_{L,N^{-1}}^\dagger(ETT^\dagger)^\dagger = (TT^\dagger E^*)^\dagger(LSS_{L,N^{-1}}^\dagger)(ETT^\dagger)^\dagger = (MX)^*$.

Note also that $N^{-1} = (FT^\dagger T)^\dagger(T^\dagger TF^*)^\dagger + I - (FT^\dagger T)^\dagger(FT^\dagger T)$ by Lemma 3.3, and thus $(FT^\dagger T)N^{-1} = (T^\dagger TF^*)^\dagger$. So, we have:

(4) $XM = (T^\dagger TF^*)^\dagger S_{L,N^{-1}}^\dagger S(T^\dagger TF^*) = (FT^\dagger T)(N^{-1}S_{L,N^{-1}}^\dagger S)(T^\dagger TF^*) = (XM)^*$.

So, X is the Moore-Penrose inverse of M.

Conversely, assume that $M \in R_{m\times n}^\dagger$. To show that $S_{L,N^{-1}}^\dagger$ exists, it is sufficient to prove $S \in SNS^*R_{m\times n} \cap R_{m\times n}S^*LS$ by Lemma 3.2. Since

$$
\begin{aligned}
S &= (ETT^\dagger)^\dagger(ETT^\dagger)T(FT^\dagger T)^\dagger(FT^\dagger T) = (ETT^\dagger)^\dagger(ETT^\dagger)T(T^\dagger TF^*)(T^\dagger TF^*)^\dagger \\
&= (ETT^\dagger)^\dagger(ETF^*)(T^\dagger TF^*)^\dagger = (ETT^\dagger)^\dagger M(T^\dagger TF^*)^\dagger,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
SNS^* &= (ETT^\dagger)^\dagger M(T^\dagger TF^*)^\dagger[(T^\dagger TF^*)(FT^\dagger T) + I_n - (FT^\dagger T)^\dagger(FT^\dagger T)]S^* \\
&= (ETT^\dagger)^\dagger M(FT^\dagger T)(FT^\dagger T)^\dagger M^*(TT^\dagger E^*)^\dagger \\
&= (ETT^\dagger)^\dagger ET[(T^\dagger TF^*)(FT^\dagger T)(FT^\dagger T)^\dagger]M^*(TT^\dagger E^*)^\dagger \\
&= (ETT^\dagger)^\dagger MM^*(TT^\dagger E^*)^\dagger
\end{aligned}
$$

and

$$
\begin{aligned}
S^*LS &= (FT^\dagger T)^\dagger M^*(TT^\dagger E^*)^\dagger[(TT^\dagger E^*)(ETT^\dagger) + I_m - (ETT^\dagger)^\dagger(ETT^\dagger)]S \\
&= (FT^\dagger T)^\dagger M^*(ETT^\dagger)(ETT^\dagger)^\dagger M(T^\dagger TF^*)^\dagger \\
&= (FT^\dagger T)^\dagger M^*[(ETT^\dagger)(ETT^\dagger)^\dagger(ETT^\dagger)]TF^*(T^\dagger TF^*)^\dagger \\
&= (FT^\dagger T)^\dagger M^*M(T^\dagger TF^*)^\dagger.
\end{aligned}
$$

As $M \in R_{m\times n}^\dagger$, by Lemma 2.4, there exist $Y' \in R_{n\times n}$, $Z' \in R_{m\times m}$ such that $M = MM^*MY' = Z'MM^*M$. Hence, we have

$$
\begin{aligned}
S &= (ETT^\dagger)^\dagger M(T^\dagger TF^*)^\dagger = (ETT^\dagger)^\dagger MM^*MY'(T^\dagger TF^*)^\dagger \\
&= (ETT^\dagger)^\dagger MM^*[(ETT^\dagger)TF^*]Y'(T^\dagger TF^*)^\dagger \\
&= (ETT^\dagger)^\dagger MM^*[(ETT^\dagger)(ETT^\dagger)^\dagger][(ETT^\dagger)TF^*Y'](T^\dagger TF^*)^\dagger \\
&= [(ETT^\dagger)^\dagger MM^*(TT^\dagger E^*)^\dagger](TT^\dagger E^*)MY'(T^\dagger TF^*)^\dagger \\
&= SNS^*(TT^\dagger E^*)MY'(T^\dagger TF^*)^\dagger
\end{aligned}
$$

and

$$
\begin{aligned}
S &= (ETT^\dagger)^\dagger Z'MM^*M(T^\dagger TF^*)^\dagger \\
&= (ETT^\dagger)^\dagger Z'ET(T^\dagger TF^*)M^*M(T^\dagger TF^*)^\dagger \\
&= (ETT^\dagger)^\dagger[Z'ET(T^\dagger TF^*)][(T^\dagger TF^*)^\dagger(T^\dagger TF^*)]M^*M(T^\dagger TF^*)^\dagger \\
&= (ETT^\dagger)^\dagger Z'M(FT^\dagger T)[(FT^\dagger T)^\dagger M^*M(T^\dagger TF^*)^\dagger] \\
&= (ETT^\dagger)^\dagger Z'M(FT^\dagger T)S^*LS.
\end{aligned}
$$

By Lemma 3.2, we have at once that $S_L^{(1,3)}$ exists and $[(ETT^\dagger)^\dagger Z' M(FT^\dagger T)]^* L = (T^\dagger TF^*)M^*(Z')^*(ETT^\dagger) \in S\{L, 1, 3\}$, and $S_{N^{-1}}^{(1,4)}$ exists and $N[(TT^\dagger E^*)MY'(T^\dagger TF^*)^\dagger]^* = (T^\dagger TF^*)(Y')^* M^*(ETT^\dagger) \in S\{N^{-1}, 1, 4\}$. So, $S_{L,N^{-1}}^\dagger$ exists.

We next give the formula of $S_{L,N^{-1}}^\dagger$. From Lemma 2.4, $M = (ETT^\dagger)T(T^\dagger TF^*)$ and $S = (ETT^\dagger)^\dagger M(T^\dagger TF^*)^\dagger$, we get

$$
\begin{aligned}
S_{L,N^{-1}}^\dagger &= S_{N^{-1}}^{(1,4)} S S_L^{(1,3)} = N[(TT^\dagger E^*)MY'(T^\dagger TF^*)^\dagger]^* S[(ETT^\dagger)^\dagger Z' M(FT^\dagger T)]^* L \\
&= [(T^\dagger TF^*)(Y')^* M^*(ETT^\dagger)]S[(T^\dagger TF^*)M^*(Z')^*(ETT^\dagger)] \\
&= (T^\dagger TF^*)(Y')^* M^*[(ETT^\dagger)(ETT^\dagger)^\dagger M(T^\dagger TF^*)^\dagger(T^\dagger TF^*)]M^*(Z')^*(ETT^\dagger) \\
&= (T^\dagger TF^*)[(Y')^* M^* M M^*(Z')^*](ETT^\dagger) = (T^\dagger TF^*)M^\dagger(ETT^\dagger).
\end{aligned}
$$

The proof is completed. $\square$

Acknowledgments

Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] N. Castro-González, J. L. Chen, L. Wang, Further results on generalized inverses in rings with involution, Electron. J. Linear Algebra 30 (2015), 118–134.
[2] N. Castro-González, M. F. Martínez-Serrano, J. Robles, Expressions for the Moore-Penrose inverse of block matrices involving the Schur complement, Linear Algebra Appl. 471 (2015), 353–368.
[3] J.L. Chen, X.X. Zhang, Algebraic theory of generalized inverses. Science Press, Beijing, Springer, Singapore, 2024.
[4] D.S. Cvetković-Ilić, New conditions for the reverse order laws for $\{1, 3\}$ and $\{1, 4\}$-generalized inverses, Electron. J. Linear Algebra 23 (2012), 231–242.
[5] D.S. Cvetković-Ilić, M. Djikić, Various solutions to reverse order law problems, Linear multilinear Algebra 64(6) (2016), 1207–1219.
[6] D.S. Cvetković-Ilić, J. Nikolov, Reverse order laws for $\{1, 2, 3\}$-generalized inverses, Appl. Math. Comput. 234 (2014), 114–117.
[7] D.S. Cvetković-Ilić, Y.M. Wei, Algebraic Properties of Generalized Inverses. Springer, Singapore, 2017.
[8] C.Y. Deng, R.F. Liu, X.N. Wang, Expression for the multiplicative perturbation of the Moore-Penrose inverse, Linear Multilinear Algebra 66(6) (2018), 1171–1185.
[9] M.P. Drazin, Pseudo-inverses in associative rings and semigroups. Amer. Math. Monthly 65(7) (1958), 506–514.
[10] F.P. Du, J.L. Chen, Perturbation analysis for the Moore-Penrose metric generalized inverse of closed linear operators in Banach spaces, Ann. Funct. Anal. 7(2) (2016), 240–253.
[11] C.H. Fu, C.N. Song, G.R. Wang, Q.X. Xu, Norm estimations for the Moore-Penrose inverse of the weak perturbation of matrices, Linear Multilinear Algebra 70(2) (2022), 215–237.
[12] R.E. Hartwig, Block generalized inverses. Arch. Ration. Mech. Anal. 61 (1976), 197–251.
[13] R.E. Hartwig, The reverse order law revisited, Linear Algebra Appl. 76 (1986), 241–246.
[14] N. Hu, W. Luo, C.N. Song, Q.X. Xu, Norm estimations for the Moore-Penrose inverse of multiplicative perturbations of matrices. J. Math. Anal. Appl. 437(1) (2016), 498–512.
[15] B.T. Jia, Compact perturbations of Moore-Penrose invertible operators, J. Math. Anal. Appl. 523(2) (2023), 127048.
[16] K. Manjunatha Prasad, R.B. Bapat, The generalized Moore-Penrose inverse, Linear Algebra Appl. 165 (1992), 59–69.
[17] L.S. Meng, B. Zheng, New multiplicative perturbation bounds of the Moore-Penrose inverse, Linear Multilinear Algebra 63(5) (2015), 1037–1048.
[18] D. Mosić, D.S. Djordjević, Further results on the reverse order law for the Moore-Penrose inverse in rings with involution, Appl. Math. Comput. 218(4) (2011), 1478–1483.
[19] D. Mosić, D.S. Djordjević, Reverse order law in C^*-algebras. Appl. Math. Comput. 218(7) (2011), 3934–3941.
[20] R. Penrose, A generalized inverse for matrices, Proc. Camb. Phil. Soc. 51 (1955), 406–413.
[21] Q.X. Xu, C.N. Song, G.R. Wang, Multiplicative perturbations of matrices and the generalized triple reverse order law for the Moore-Penrose inverse, Linear Algebra Appl. 530 (2017), 366–383.
[22] Q.X. Xu, Y.M. Wei, Y.Y. Gu, Sharp norm-estimations for Moore-Penrose inverses of stable perturbation of Hilbert C^*-module operators. SIAM J. Numer. Anal. 47(6) (2010), 4735–4758.

[23] H. Yang, P.P. Zhang, A note on multiplicative perturbation bounds for the Moore-Penrose inverse. Linear Multilinear Algebra 62(6) (2014), 831–838.
[24] X.B. Zhang, X.X. Fang, C.N. Song, Q.X. Xu, Representations and norm estimations for the Moore-Penrose inverse of multiplicative perturbations of matrices, Linear Multilinear Algebra 65(3) (2017), 555–571.
[25] H.H. Zhu, J.L. Chen, P. Patrício, Further results on the inverse along an element in semigroups and rings, Linear Multilinear Algebra 64(3) (2015), 393–403.
[26] H.H. Zhu, Q.W. Wang, Weighted Moore-Penrose inverses and weighted core inverses in rings with involution, Chin. Ann. Math. Ser. B 42 (2021), 613–624.