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The boundedness of rough generalized commutators with Lipschitz functions on homogeneous variable exponent Herz type spaces

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Abstract. With the development of science, many nonlinear problems have emerged. At this time, the classical function space has certain restrictions. For example, it has lost its effectiveness for nonlinear problems under nonstandard growth conditions. In the process of studying such nonlinear problems, scholars are paying more and more attention to the transition from classical function space to variable exponent function space. Also, there is a big difference between variable exponent space and classical function space, mainly because variable exponent function space has lost translation invariance. This difference leads to many properties that hold in classical space no longer hold in variable exponent space. It is important to emphasize that variable exponent function spaces are a fundamental building block in harmonic analysis. In recent years, there has been a growing interest in the study of function spaces equipped with variable exponents, leading to the development of a new framework known as variable exponent analysis. These spaces provide a powerful tool for analyzing functions with variable growth or decay rates and have found applications in various areas of mathematics, including partial differential equations, harmonic analysis and image processing. One can better understand the heterogeneity and complexity inherent in many real world phenomena by taking into consideration the theory of variable exponent function spaces. Thus, by using certain properties of Lipschitz functions and variable exponents, in this article, we establish the boundedness of a class of rough generalized commutators with Lipschitz functions on homogeneous variable exponent Herz and Herz-Morrey spaces.

1. Introduction and Main Results

Assume that $\Omega \in L^s(S^{n-1})\,(s>1)$, Ω is homogeneous of degree zero on $\mathbb{R}^n$ with zero mean value on S^{n-1}, S^{n-1} denotes the unit sphere on $\mathbb{R}^n$, m is a positive integer, $A(x)$ is a function defined on $\mathbb{R}^n$ with m-th order derivatives on $L_{loc}(\mathbb{R}^n)$ and

$$L_{loc}(\mathbb{R}^n) = \left\{ f : \int_K \left| f(x) \right| dx < \infty;\ \text{for all compact subset } K \subset \mathbb{R}^n \right\}.$$

In analysis, theory and applications, the generalized commutators are popular operators extensively studied over the past hundred years and subjected to many generalizations in various settings. There are many

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others, but we will limit ourselves to these two, for these are the main focus of our objective. Thus, in this paper, we investigate the following generalized commutators

$$I_{\Omega,\phi}^{A,m}f(x)=\int\limits_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\phi+m-1}}R_m\left(A;x,y\right)f(y)dy \qquad 0<\phi<n$$

and

$$M_{\Omega,\phi}^{A,m}f(x)=\sup_{r>0}\frac{1}{r^{n-\phi+m-1}}\int\limits_{|x-y|<r}\left|\Omega(x-y)R_m\left(A;x,y\right)f(y)\right|dy \qquad 0<\phi<n,$$

where $m\in\mathbb{N}$, and $R_m\left(A;x,y\right)$ denotes the m-th remainder of Taylor series of A at x about y, more precisely,

$$R_m\left(A;x,y\right)=A\left(x\right)-\sum_{\left|\gamma\right|<m}\frac{1}{\gamma!}D^{\gamma}A\left(y\right)\left(x-y\right)^{\gamma},$$

where $D^{\gamma}A\in L^{r}\left(\mathbb{R}^n\right)$ $(1<r\leq\infty)$, $D^{\gamma}A\in BMO\left(\mathbb{R}^n\right)$ or $D^{\gamma}A\in\dot{\Lambda}_{\beta}\left(\mathbb{R}^n\right)$ for $\left|\gamma\right|=m-1$ $(m\geq 1)$.

When $m=1$ above, it is obvious to see that $R_1\left(A;x,y\right)=A\left(x\right)-A\left(y\right)$. In this case, $I_{\Omega,\phi}^{A,1}=I_{\Omega,\phi}^{A}$ and $M_{\Omega,\phi}^{A,1}=M_{\Omega,\phi}^{A}$ are just commutator operators,

$$\begin{aligned} I_{\Omega,\phi}^{A}f\left(x\right) &= A\left(x\right)I_{\Omega,\phi}f\left(x\right)-I_{\Omega,\phi}\left(Af\right)\left(x\right)\\ &= \int\limits_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\phi}}\left(A\left(x\right)-A\left(y\right)\right)f(y)dy \qquad 0<\phi<n\end{aligned}$$

and

$$\begin{aligned} M_{\Omega,\phi}^{A}f\left(x\right) &= A\left(x\right)M_{\Omega,\phi}f\left(x\right)-M_{\Omega,\phi}\left(Af\right)\left(x\right)\\ &= \sup_{r>0}\frac{1}{r^{n-\phi}}\int\limits_{|x-y|<r}\left|\Omega\left(x-y\right)\right|\left|A\left(x\right)-A\left(y\right)\right|\left|f(y)\right|dy \qquad 0<\phi<n.\end{aligned}$$

Here, $I_{\Omega,\phi}^{A,m}$ and $M_{\Omega,\phi}^{A,m}$ are trivial generalizations of the above commutators, respectively.

It is well known that the multilinear operators have been widely studied by many authors. (For example, see [2, 4, 12] etc.) In 2013, Wu and Lan [12] proved that $I_{\Omega,\phi}^{A,m}$ and $M_{\Omega,\phi}^{A,m}$ are bounded from $L^{p(\cdot)}$ to $L^{q(\cdot)}$ for $D^{\gamma}A\in\dot{\Lambda}_{\beta}\left(\mathbb{R}^n\right)$. However, it is worth pointing out that so far Lipschitz boundedness for $I_{\Omega,\phi}^{A,m}$ and $M_{\Omega,\phi}^{A,m}$ on homogeneous variable exponent Herz type spaces has not been proved for $D^{\gamma}A\in\dot{\Lambda}_{\beta}\left(\mathbb{R}^n\right)$.

In this sense, we recall the definition of homogenous Lipschitz space $\dot{\Lambda}_{\beta}\left(\mathbb{R}^n\right)$ as follows:

Definition 1.1. (***Homogenous Lipschitz space***) *Let* $0<\beta\leq 1$. *The homogeneous Lipschitz space* $\dot{\Lambda}_{\beta}$ *is defined by*

$$\dot{\Lambda}_{\beta}\left(\mathbb{R}^n\right)=\left\{f:\left\|f\right\|_{\dot{\Lambda}_{\beta}}=\sup_{x,h\in\mathbb{R},h\neq 0}\frac{\left|\Delta_h^{[\beta]+1}f\left(x\right)\right|}{|h|^{\beta}}<\infty\right\},$$

where $\Delta_h^1 f\left(x\right)=f\left(x+h\right)-f\left(x\right)$, $\Delta_h^{k+1}f\left(x\right)=\Delta_h^k f\left(x+h\right)-\Delta_h^k f\left(x\right)$, $k\geq 1$.

Obviously, if $\beta>1$, *then* $\dot{\Lambda}_{\beta}\left(\mathbb{R}^n\right)$ *only includes constant. So we restrict* $0<\beta\leq 1$ *(see [10] for details)*

Let us now give some necessary definitions and notations. Throughout this work, Q will denote a cube on $\mathbb{R}^n$ with edges parallel to the axes. We will denote the cube with center x_0 and edge length r by $Q = Q(x_0, r)$. Given a cube Q and $\delta > 0$, we will denote the cube with center Q and edge length δ times the edge length of Q by δQ. For a cube Q, we use the notation

$$f_Q = \frac{1}{|Q|}\int_Q f,$$

where f_Q is the center of Q.

Throughout this work, the constant $C > 0$ may vary from step to another and do depentent on parameters involved. The expression $f \lesssim g$ means $f \leqslant Cg$ and $f \approx g$ implies that $f \lesssim g \lesssim f$. Also, for simplicity, we denote $L^{p(\cdot)}(\mathbb{R}^n)$ by $L^{p(\cdot)}$ and similarly $B(x, r)$ by B.

Some scholars found that as long as it is proved that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, many conclusions in the corresponding classical harmonic analysis and function space theory can be established in the corresponding variable exponent function space. In this context, we denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all functions $p(\cdot)$ which are measurable and satisfy: $1 \leq p_- := \operatorname{essinf}_{x\in\mathbb{R}^n} p(x)$ and $p_+ := \operatorname{esssup}_{x\in\mathbb{R}^n} p(x) < \infty$.

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$\left\|f\right\|_{L^{p(\cdot)}} := \inf\left\{\eta > 0 : \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx \leq 1\right\} < \infty.$$

Let $f \in L_{loc}(\mathbb{R}^n)$. The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by

$$\mathcal{M}f(x) := \sup_{r>0} r^{-n}\int_B |f(y)|\,dy, \qquad \forall x \in \mathbb{R}^n,$$

where and follows $B = \left\{y \in \mathbb{R}^n : |x - y| < r\right\}$ is the open ball centered at x with radius r. $\mathcal{B}(\mathbb{R}^n)$ is the collection of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ that satisfy the boundedness of $\mathcal{M}$ on $L^{p(\cdot)}(\mathbb{R}^n)$ as $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ in [1].

For all $x, y \in \mathbb{R}^n$ and $C > 0$, if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies the requirement given below

$$|p(y) - p(x)| \leq \frac{-C}{\ln(|x-y|)}, \qquad \text{if } |x - y| \leq \frac{1}{2} \tag{1}$$

$$|p(y) - p(x)| \leq \frac{C}{\ln(e + |x|)}, \qquad \text{if } |x| \leq |y|, \tag{2}$$

then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ in [9].

As we all know, Hölder's inequality is a very important tool in studying the boundedness of operators. Of course, similar inequalities are also needed in variable exponent function space, so there is a generalized Hölder's inequality.

For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, the integral form of Hölder's inequality in the context of variable exponent spaces takes the form

$$\int_{\mathbb{R}^n} |f(x)\,g(x)|\,dx \leq r_p \left\|f\right\|_{L^{p(\cdot)}} \left\|g\right\|_{L^{p'(\cdot)}}, \tag{3}$$

where the constant r_p is given by

$$r_p = 1 + \frac{1}{p_-} - \frac{1}{p_+},$$

see Theorem 2.1 in [7].

On the other hand, Nakai and Sawano [8] defined another variable exponent $\tilde{q}(\cdot)$ by

$$\frac{1}{q}+\frac{1}{\tilde{q}(\cdot)}=\frac{1}{p(\cdot)}.$$

Then, we have

$$\left\|f.g\right\|_{L^{p(\cdot)}}\lesssim\left\|f\right\|_{L^{\tilde{q}(\cdot)}}\left\|g\right\|_{L^{q}} \tag{4}$$

for $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $(p)_+<q$ and for all measurable functions f and g.

Assume that $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and satisfies (1) and (2). Then so does $p'(\cdot)$. Generally, we can observe that $p(\cdot), p'(\cdot)\in\mathcal{B}(\mathbb{R}^n)$ from [9]. Thus, by virtue of Lemma 1 in [6], we can consider constants $\delta_1\in\left(0,\frac{1}{(p)_+}\right)$ and $\delta_2\in\left(0,\frac{1}{(p')_+}\right)$ such that

$$\frac{\|\chi_S\|_{L^{p(\cdot)}}}{\|\chi_B\|_{L^{p(\cdot)}}}\leq C\left(\frac{|S|}{|B|}\right)^{\delta_1},\frac{\|\chi_S\|_{L^{p'(\cdot)}}}{\|\chi_B\|_{L^{p'(\cdot)}}}\leq C\left(\frac{|S|}{|B|}\right)^{\delta_2} \tag{5}$$

for $S\subset B$.

When $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, then

$$\|\chi_B\|_{L^{p(\cdot)}}\|\chi_B\|_{L^{p'(\cdot)}}\leq C|B| \tag{6}$$

was proved in [6].

Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfying conditions (1) and (2), then

$$\left\|\chi_Q\right\|_{L^{p(\cdot)}}\approx\begin{cases}|Q|^{\frac{1}{p(x)}}, \text{ if } |Q|\leq 2^n \text{ and } x\in Q,\\ |Q|^{\frac{1}{p(\infty)}}, \text{ if } |Q|>1,\end{cases} \tag{7}$$

for all cubes (balls) $Q\subset\mathbb{R}^n$, where $p(\infty)=\lim_{|x|\to\infty}p(x)$ (see [3]).

Let $l\in\mathbb{Z}$, $B_l:=\{x\in\mathbb{R}^n:|x|\leq 2^l\}$, $\Delta_l:=B_l\setminus B_{l-1}$, $\chi_l:=\chi_{\Delta_l}$. For any $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, we define

$$\tilde{\chi}_m:=\begin{cases}\chi_{\Delta_m} & , \quad m\geq 1\\ \chi_{B_0} & , \quad m=0\end{cases}.$$

By virtue of (5), we obtain

$$\frac{\|\chi_l\|_{L^{p(\cdot)}}}{\left\|\chi_{B_l}\right\|_{L^{p(\cdot)}}}\leq C\left(\frac{|\Delta_l|}{|B_l|}\right)^{\delta_1}\Longrightarrow\|\chi_l\|_{L^{p(\cdot)}}\lesssim\|\chi_{B_l}\|_{L^{p(\cdot)}}. \tag{8}$$

Definition 1.2. *Let $\alpha\in\mathbb{R}$, $0<q\leq\infty$ and $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. Then, the homogeneous variable exponent Herz space $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is defined by*

$$\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n):=\left\{f\in L^{p(\cdot)}_{loc}(\mathbb{R}^n\setminus\{0\}):\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}<\infty\right\},$$

where

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}:=\left(\sum_{l=-\infty}^{\infty}\|2^{l\alpha}f\chi_l\|^q_{L^{p(\cdot)}}\right)^{\frac{1}{q}}$$

with the usual modifications made when $q=\infty$.

The first main result we wanted to find in this work is as follows.

Theorem 1.3. *Let* $0 < \phi < n$ *and* Ω *be homogeneous of degree zero with* $\Omega \in L^s(S^{n-1})\,(s > 1)$. *Let also* $0 < q_1 \le q_2 < \infty$, $0 < \beta < 1$ *and* $\alpha \in \mathbb{R}$ *such that* $\phi + \beta + n\delta_2 < \alpha < n\delta_1 - \left(\phi + \beta + \frac{n-1}{s}\right)$ *with* δ_1, $\delta_2 \in (0,1)$ *satisfying (5). Assume that* $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfy (1) and (2) and define* $p_1(\cdot)$ *and* $p_2(\cdot)$ *by* $\frac{1}{p_2(\cdot)} = \frac{1}{p_1(\cdot)} - \frac{\beta+\phi}{n}$. *If* $D^\gamma A \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ $\left(|\gamma| = m-1, m \ge 2\right)$ *and* $\left(p_1'\right)_+ < s$, *then the following inequalities hold:*

$$\left\|I^{A,m}_{\Omega,\phi} f\right\|_{\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} \lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|_{\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}, \tag{9}$$

$$\left\|M^{A,m}_{\Omega,\phi} f\right\|_{\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} \lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|_{\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}. \tag{10}$$

When $m = 1$ in Theorem 1.3, we have the following:

Corollary 1.4. *Under the conditions of Theorem 1.3, the following boundedness estimates hold:*

$$\left\|I^{A}_{\Omega,\phi} f\right\|_{\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} \lesssim \|A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|_{\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)},$$

$$\left\|M^{A}_{\Omega,\phi} f\right\|_{\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} \lesssim \|A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|_{\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Definition 1.5. *Let* $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* $0 \le \lambda < \infty$. *Then, the homogeneous variable exponent Herz-Morrey space* $M\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ *is defined by*

$$M\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n) := \left\{f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{l=-\infty}^{L} \|2^{l\alpha} f\chi_l\|^q_{L^{p(\cdot)}}\right)^{\frac{1}{q}}$$

with the usual modifications made when $q = \infty$. *Obviously, when* $\lambda = 0$, $M\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$.

Finally, the other main result we wanted to find in this article is as follows.

Theorem 1.6. *Let* $0 \le \lambda < \infty$ *such that* $\phi + \beta + n\delta_2 + \lambda < \alpha < n\delta_1 + \lambda - \left(\phi + \beta + \frac{n-1}{s}\right)$ *with* δ_1, $\delta_2 \in (0,1)$ *satisfying (5) and under stipulations in Theorem 1.3, the generalized commutators* $I^{A,m}_{\Omega,\phi}$ *and* $M^{A,m}_{\Omega,\phi}$ *satisfy*

$$\left\|I^{A,m}_{\Omega,\phi} f\right\|_{M\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} \lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}, \tag{11}$$

$$\left\|M^{A,m}_{\Omega,\phi} f\right\|_{M\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} \lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}. \tag{12}$$

For $m = 1$ in Theorem 1.6, we get

Corollary 1.7. *Let* $0 \le \lambda < \infty$ *such that* $\phi + \beta + n\delta_2 + \lambda < \alpha < n\delta_1 + \lambda - \left(\phi + \beta + \frac{n-1}{s}\right)$ *with* δ_1, $\delta_2 \in (0,1)$ *satisfying (5) and under the conditions of Theorem 1.3, the following fundamental inequalities hold:*

$$\left\|I^{A}_{\Omega,\phi} f\right\|_{M\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} \lesssim \|A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)},$$

$$\left\|M^{A}_{\Omega,\phi} f\right\|_{M\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} \lesssim \|A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

2. Main Lemmas

Before proving Theorems 1.3 and 1.6, following lemmas are needed. These lemmas will be helpful in proving main results.

Lemma 2.1. *(see [2]) Let $A(x)$ be a function defined on $\mathbb{R}^n$ with m-th order derivatives on $L^q_{loc}(\mathbb{R}^n)$ for $|\gamma| = m$ and some $q > n$. Then,*

$$\left|R_m(A;x,y)\right| \le C\left|x-y\right|^m \sum_{|\gamma|=m}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\gamma}A(z)|^q\,dz\right)^{\frac{1}{q}},$$

where $\tilde{Q}$ is the cube centered at x and having diameter $5\sqrt{n}\left|x-y\right|$.

Lemma 2.2. *(see [10]) For $0<\beta<1$, $1\le q<\infty$, we have*

$$\begin{aligned}\|f\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} &= \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q\left|f-f_Q\right| \\ &\approx \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\left(\frac{1}{|Q|}\int_Q\left|f-f_Q\right|^q dx\right)^{\frac{1}{q}}.\end{aligned}$$

Lemma 2.3. *(see [10]) Let $Q^*\subset Q$. If $f\in\dot{\Lambda}_{\beta}(\mathbb{R}^n)$ $(0<\beta<1)$, then*

$$\left|f_{Q^*}-f_Q\right| \le C\left\|f\right\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}|Q|^{\frac{\beta}{n}}.$$

In the proof of our main results we need to obtain an local estimate of $\left|R_m(A;x,y)\right|$, similar to the Lemma 2.1. To do this, applying the above results, we obtain the following:

Lemma 2.4. *Let $0<\beta<1$, $A(x)$ be a function defined on $\mathbb{R}^n$ with m-th order derivatives on $L^q_{loc}(\mathbb{R}^n)$ for $|\gamma| = m$ and some $q > n$. Then,*

$$\left|R_m(A;x,y)\right| \lesssim \sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}\left|x-y\right|^{m-1+\beta}. \tag{13}$$

Proof. For any $x\in\mathbb{R}^n$, let $Q(x,y)=Q$ be the cube centered at x and having diameter $4\sqrt{n}\left|x-y\right|$. For fixed $x\in\mathbb{R}^n$, let

$$A_Q(y) = A(y) - \sum_{|\gamma|=m-1}\frac{1}{\gamma!}(D^{\gamma}A)_Q\,y^{\gamma},$$

where $(D^{\gamma}A)_Q$ is the average of $D^{\gamma}A$ on Q. From the definitions of $R_m(A;x,y)$ and A_Q, it is easy to see that $R_m(A;x,y) = R_m\left(A_Q;x,y\right)$ and

$$R_m\left(A_Q;x,y\right) = R_{m-1}\left(A_Q;x,y\right) - \sum_{|\gamma|=m-1}\frac{1}{\gamma!}D^{\gamma}A_Q(x)\left|x-y\right|^{\gamma}.$$

Then,

$$\left|R_m(A;x,y)\right| \lesssim \left|R_{m-1}\left(A_Q;x,y\right)\right| + \sum_{|\gamma|=m-1}\frac{1}{\gamma!}\left|D^{\gamma}A_Q(x)\right|\left|x-y\right|^{m-1}. \tag{14}$$

Applying Lemmas 2.1, 2.2 and 2.3, for any y, we get

$$\begin{aligned}
\left|R_{m-1}\left(A_Q;x,y\right)\right| &\lesssim \left|x-y\right|^{m-1}\sum_{|\gamma|=m-1}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}\left|D^{\gamma}A_Q\left(z\right)\right|^q dy\right)^{\frac{1}{q}} \\
&= \left|x-y\right|^{m-1}\sum_{|\gamma|=m-1}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}\left|D^{\gamma}A\left(z\right)-(D^{\gamma}A)_Q\right|^q dy\right)^{\frac{1}{q}} \\
&\lesssim \left|x-y\right|^{m-1}\left[\begin{array}{c}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}\left|D^{\gamma}A\left(z\right)-(D^{\gamma}A)_{\tilde{Q}}\right|^q dy\right)^{\frac{1}{q}} \\ +\left|(D^{\gamma}A)_{\tilde{Q}}-(D^{\gamma}A)_{5Q}\right|+\left|(D^{\gamma}A)_{5Q}-(D^{\gamma}A)_Q\right|\end{array}\right] \\
&\lesssim \left|x-y\right|^{m+\beta-1}\|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}, \qquad (15)
\end{aligned}$$

where $\tilde{Q}$ is the cube centered at x and having diameter $5\sqrt{n}\left|x-y\right|$ and $\tilde{Q}\subset 5Q$.

On the other hand, applying Lemma 2.3 again, we have

$$\left|D^{\gamma}A_Q\left(x\right)\right| \lesssim \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}|Q|^{\frac{\beta}{n}} \lesssim \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}\left|x-y\right|^{\beta}. \qquad (16)$$

Thus, combining inequalities (14)-(16), we obtain (13). □

Lemma 2.5. *Let* $\Omega\left(x\right)$, Δ_k, Δ_z $(k,z\in\mathbb{Z})$ *be as stated above and we have*

(i) *if* $x\in\Delta_k$, $k\leq z-3$, *then*

$$\left(\int_{\Delta_z}\left|\Omega(x-y)\right|^s dy\right)^{\frac{1}{s}} \lesssim 2^{\frac{zn}{s}}\|\Omega\|_{L^s(S^{n-1})},$$

(ii) *if* $x\in\Delta_k$, $k\geq z+3$, *then*

$$\left(\int_{\Delta_z}\left|\Omega(x-y)\right|^s dy\right)^{\frac{1}{s}} \lesssim 2^{\frac{(z-k+kn)}{s}}\|\Omega\|_{L^s(S^{n-1})}.$$

Proof. When $x\in\Delta_k$, $y\in\Delta_z$, $|k-z|\geq 3$, then we have $|x|\approx 2^k$, $\left|y\right|\approx 2^z$, $\left|x-y\right|\approx\max\left\{2^z,2^k\right\}$ and

$$\left(\int_{\Delta_z}\left|\Omega(x-y)\right|^s dy\right)^{\frac{1}{s}} \leq \left(\int_{x+B_z}\left|\Omega(x-y)\right|^s dy\right)^{\frac{1}{s}}.$$

Thus, direct calculation yields (i) and (ii) (see [11] for more details). □

3. Theorem Proofs

Proof of Theorem 1.3.

Proof. Let $f \in \dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)$. Taking $f_z := f\chi_z$ for each $z \in \mathbb{Z}$, we write

$$f(x) = \sum_{z=-\infty}^{\infty} f(x)\chi_z(x) = \sum_{z=-\infty}^{\infty} f_z(x).$$

Owing to $0 < \frac{q_1}{q_2} \leq 1$, then the Jensen inequality is:

$$\left(\sum_{j=-\infty}^{\infty} \left|c_j\right|\right)^{\frac{q_1}{q_2}} \leq \sum_{j=-\infty}^{\infty} \left|c_j\right|^{\frac{q_1}{q_2}}, c_j \in \mathbb{R}, j \in \mathbb{Z}. \tag{17}$$

By (17), we have

$$\begin{aligned}
\left\|I^{A,m}_{\Omega,\phi} f\right\|^{q_1}_{\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} &= \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha q_2} \|\left(I^{A,m}_{\Omega,\phi} f\right)\chi_k\|^{q_2}_{L^{p_2(\cdot)}}\right)^{\frac{q_1}{q_2}} \\
&\lesssim \sum_{k=-\infty}^{\infty} 2^{k\alpha q_1} \|\left(I^{A,m}_{\Omega,\phi} f\right)\chi_k\|^{q_1}_{L^{p_2(\cdot)}} \\
&\lesssim \sum_{k=-\infty}^{\infty} 2^{k\alpha q_1} \left(\sum_{z=-\infty}^{k-3} \|\left(I^{A,m}_{\Omega,\phi} f_z\right)\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1} \\
&\quad + \sum_{k=-\infty}^{\infty} 2^{k\alpha q_1} \left(\sum_{z=k-2}^{k+2} \|\left(I^{A,m}_{\Omega,\phi} f_z\right)\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1} \\
&\quad + \sum_{k=-\infty}^{\infty} 2^{k\alpha q_1} \left(\sum_{z=k+3}^{\infty} \|\left(I^{A,m}_{\Omega,\phi} f_z\right)\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1} \\
&=: X + Y + Z.
\end{aligned} \tag{18}$$

We first estimate X.

Let $z \in \mathbb{Z}$, $B_z := \{x \in \mathbb{R}^n : |x| \leq 2^z\}$, $\Delta_z := B_z \setminus B_{z-1}$. For any $k, z \in \mathbb{Z}$ and $k \geq z+3$, using (3) and (13), we have

$$\begin{aligned}
\left|\left(I^{A,m}_{\Omega,\phi} f_z\right)\chi_k\right| &\lesssim \int_{\Delta_z} \frac{\left|\Omega(x-y)\right|\left|f_z(y)\right|}{|x-y|^{n-\phi+m-1}} \left|R_m(A;x,y)\right| dy \cdot \chi_k \\
&\lesssim 2^{k(\phi+\beta-n)} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left|\Omega(x-y)\right|\left|f_z(y)\right| dy \cdot \chi_k \\
&\lesssim 2^{k(\phi+\beta-n)} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f_z\right\|_{L^{p_1(\cdot)}} \left\|\Omega(x-y)\chi_z\right\|_{L^{p_1'(\cdot)}} \cdot \chi_k
\end{aligned}$$

Since $\left(p_1'\right)_+ < s$ and $\frac{1}{s} + \frac{1}{\tilde{p}_1'(\cdot)} = \frac{1}{p_1'(\cdot)}$, then using (4), (8) and Lemma 2.5 give the following

$$\begin{aligned}
\left|\left(I^{A,m}_{\Omega,\phi} f_z\right)\chi_k\right| &\lesssim 2^{k(\phi+\beta-n)} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f_z\right\|_{L^{p_1(\cdot)}} \left\|\Omega(x-y)\chi_z\right\|_{L^s} \|\chi_z\|_{L^{\tilde{p}_1'(\cdot)}} \cdot \chi_k \\
&\lesssim 2^{k(\phi+\beta-n)} 2^{\frac{(z-k+kn)}{s}} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f_z\right\|_{L^{p_1(\cdot)}} \left\|\chi_{B_z}\right\|_{L^{\tilde{p}_1'(\cdot)}} \cdot \chi_k
\end{aligned}$$

and

$$\left\|\left(I_{\Omega,\phi}^{A,m} f_z\right)\chi_k\right\|_{L^{p_2(\cdot)}} \lesssim 2^{k(\phi+\beta-n)} 2^{\frac{(z-k+kn)}{s}} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f_z\right\|_{L^{p_1(\cdot)}} \left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} \|\chi_k\|_{L^{p_2(\cdot)}}. \tag{19}$$

According to (7),

1− When $|B_z| \leq 2^n$ and $x_z \in B_z$, we have

$$\left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} \approx |B_z|^{\frac{1}{p_1'(x_z)}} \approx \left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} |B_z|^{-\frac{1}{s}}.$$

2− When $|B_z| \geq 1$, we have

$$\left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} \approx |B_z|^{\frac{1}{p_1'(\infty)}} \approx \left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} |B_z|^{-\frac{1}{s}}.$$

So, we get

$$\left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} \approx \left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} |B_z|^{-\frac{1}{s}}. \tag{20}$$

Thus, using inequality (20) in (19), we get

$$\left\|\left(I_{\Omega,\phi}^{A,m} f_z\right)\chi_k\right\|_{L^{p_2(\cdot)}} \lesssim 2^{k(\phi+\beta-n)} 2^{\frac{(k-z)(n-1)}{s}} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f_z\right\|_{L^{p_1(\cdot)}} \left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} \|\chi_k\|_{L^{p_2(\cdot)}}. \tag{21}$$

For the establishment of the boundedness of the generalized commutators, we use the fundamental properties of the Riesz potential, which is formally defined as

$$I_\phi f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\phi}} dy \qquad 0 < \phi < n.$$

Since

$$2^{\phi z}\chi_{B_z}(x) \lesssim \int_{B_z} \frac{dy}{|x-y|^{n-\phi}} \cdot \chi_{B_z}(x) \leq I_\phi\left(\chi_{B_z}\right)(x),$$

then

$$2^{z(\phi+\beta)}\chi_{B_z} \lesssim I_{\phi+\beta}\left(\chi_{B_z}\right)(x)\,\chi_{B_z} \leq I_{\phi+\beta}\left(\chi_{B_z}\right)(x) \tag{22}$$

(see [5] pp. 539–540). On the other hand, using (6) and (8), we have

$$\begin{aligned} 2^{k(\phi+\beta-n)} \left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} \left\|\chi_{B_k}\right\|_{L^{p_2(\cdot)}} &\lesssim 2^{k(\phi+\beta)} \left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} \left\|\chi_{B_k}\right\|_{L^{p_2'(\cdot)}}^{-1} \\ &\lesssim 2^{k(\phi+\beta)} \left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} \left\|\chi_{B_z}\right\|_{L^{p_2'(\cdot)}}^{-1} 2^{n\delta_1(z-k)}. \end{aligned} \tag{23}$$

Next, applying (3), (6), (22) and from $\left(L^{p_1(\cdot)}, L^{p_2(\cdot)}\right)$-boundedness of $I_{\phi+\beta}$ (see (13) in [5]), we know that

$$\begin{aligned} \left\|\chi_{B_z}\right\|_{L^{p_2'(\cdot)}}^{-1} &\lesssim 2^{-nz} \left\|\chi_{B_z}\right\|_{L^{p_2(\cdot)}} \\ &\lesssim 2^{-nz} 2^{-z(\phi+\beta)} \left\|I_{\phi+\beta}\left(\chi_{B_z}\right)\right\|_{L^{p_2(\cdot)}} \\ &\lesssim 2^{-nz} 2^{-z(\phi+\beta)} \left\|\chi_{B_z}\right\|_{L^{p_1(\cdot)}} \\ &\lesssim 2^{-z(\phi+\beta)} \left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}}^{-1}. \end{aligned} \tag{24}$$

Thus, using inequalities (23) and (24) in (21), we get

$$\left\|\left(I^{A,m}_{\Omega,\phi} f_z\right)\chi_k\right\|_{L^{p_2(\cdot)}} \lesssim 2^{(k-z)\left(\phi+\beta-n\delta_1+\frac{n-1}{s}\right)} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f_z\right\|_{L^{p_1(\cdot)}}. \tag{25}$$

Thus, by virtue of (25) and remark that $\alpha < n\delta_1 - \left(\phi+\beta+\frac{n-1}{s}\right)$,

$$\begin{aligned} X &= \sum_{k=-\infty}^{\infty} 2^{k\alpha q_1}\left(\sum_{z=-\infty}^{k-3} \|\left(I^{A,m}_{\Omega,\phi} f_z\right)\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1} \\ &\lesssim \sum_{k=-\infty}^{\infty} 2^{k\alpha q_1}\left(\sum_{z=-\infty}^{k-3} 2^{(k-z)\left(\phi+\beta-n\delta_1+\frac{n-1}{s}\right)} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f_z\right\|_{L^{p_1(\cdot)}}\right)^{q_1} \\ &\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty}\left(\sum_{z=-\infty}^{k-3} 2^{\alpha z}\left\|f_z\right\|_{L^{p_1(\cdot)}} 2^{(k-z)\left(\phi+\beta-n\delta_1+\alpha+\frac{n-1}{s}\right)}\right)^{q_1}. \end{aligned}$$

To continue estimating X, we split the problem into the following two cases:

Case 1 $(0 < q_1 \leq 1)$.

Using (17) and substituting q_1 for $\frac{q_1}{q_2}$, we obtain

$$\begin{aligned} X &\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty}\sum_{z=-\infty}^{k-3} 2^{\alpha z q_1}\left\|f_z\right\|^{q_1}_{L^{p_1(\cdot)}} 2^{(k-z)\left(\phi+\beta-n\delta_1+\alpha+\frac{n-1}{s}\right)q_1} \\ &\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|^{q_1}_{\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Case 2 $(1 < q_1 < \infty)$.

Let $\frac{1}{q_1}+\frac{1}{q_1'}=1$. By (3), we get

$$\begin{aligned} X &\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty}\sum_{z=-\infty}^{k-3} 2^{\alpha z q_1}\left\|f_z\right\|^{q_1}_{L^{p_1(\cdot)}} 2^{(k-z)\left(\phi+\beta-n\delta_1+\alpha+\frac{n-1}{s}\right)\frac{q_1}{2}} \\ &\quad\times\left(\sum_{z=-\infty}^{k-3} 2^{(k-z)\left(\phi+\beta-n\delta_1+\alpha+\frac{n-1}{s}\right)\frac{q_1'}{2}}\right)^{\frac{q_1}{q_1'}} \\ &\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sum_{z=-\infty}^{\infty} 2^{\alpha z q_1}\left\|f_z\right\|^{q_1}_{L^{p_1(\cdot)}} \sum_{k=z+3}^{\infty} 2^{(k-z)\left(\phi+\beta-n\delta_1+\alpha+\frac{n-1}{s}\right)\frac{q_1}{2}} \\ &\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|^{q_1}_{\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Next, we estimate Y. When $D^\gamma A \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, from $\left(L^{p_1(\cdot)}, L^{p_2(\cdot)}\right)$-boundedness of the $I^{A,m}_{\Omega,\phi}$ (see Theorem 5 in [12]) and (17), we have

$$\begin{aligned} Y &= \sum_{k=-\infty}^{\infty} 2^{k\alpha q_1}\left(\sum_{z=k-2}^{k+2} \|\left(I^{A,m}_{\Omega,\phi} f_z\right)\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1} \\ &\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty} 2^{k\alpha q_1}\left(\sum_{z=k-2}^{k+2} \|f_z\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}^{q_1} \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha q_1} \|f\chi_k\|_{L^{p_2(\cdot)}}^{q_1}\right] \\
&= \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}^{q_1} \left\|f\right\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1}(\mathbb{R}^n)}^{q_1}.
\end{aligned}
$$

Finally, we estimate Z. For any $k, z \in \mathbb{Z}$ and $z \geq k+3$, from the process proving (21), it is easy to see that

$$
\left\|\left(I_{\Omega,\phi}^{A,m} f_z\right)\chi_k\right\|_{L^{p_2(\cdot)}} \lesssim 2^{z(\phi+\beta-n)} \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \left\|f_z\right\|_{L^{p_1(\cdot)}} \left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} \left\|\chi_{B_k}\right\|_{L^{p_2(\cdot)}}.
$$

Note that we do not go into the details of the proof process here, as the proofs are similar to each other. On the other hand, using (5) and (6), an estimate similar to (23) yields

$$
2^{z(\phi+\beta-n)} \left\|\chi_{B_z}\right\|_{L^{p_1'(\cdot)}} \left\|\chi_{B_k}\right\|_{L^{p_2(\cdot)}} \lesssim 2^{z(\phi+\beta)} \left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}} \left\|\chi_{B_k}\right\|_{L^{p_2(\cdot)}}^{-1} 2^{n\delta_2(z-k)}. \tag{26}
$$

Next, we know that

$$
2^{k(\phi+\beta)}\chi_{B_k} \lesssim I_{\phi+\beta}\left(\chi_{B_k}\right)(x)\,\chi_{B_z} \leq I_{\phi+\beta}\left(\chi_{B_k}\right)(x). \tag{27}
$$

Moreover, similar to the estimation of (24), we get

$$
\left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}}^{-1} \lesssim 2^{-nk} \left\|\chi_{B_k}\right\|_{L^{p_1'(\cdot)}} \lesssim 2^{-k(\phi+\beta)} \left\|\chi_{B_k}\right\|_{L^{p_2(\cdot)}}^{-1}. \tag{28}
$$

Hence, combining (26)-(28), we obtain

$$
\left\|\left(I_{\Omega,\phi}^{A,m} f_z\right)\chi_k\right\|_{L^{p_2(\cdot)}} \lesssim 2^{(z-k)(\phi+\beta+n\delta_2-\alpha)} \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \left\|f_z\right\|_{L^{p_1(\cdot)}}. \tag{29}
$$

Thus, by (29) and the assumption $\phi+\beta+n\delta_2<\alpha$, we conclude that

$$
\begin{aligned}
Z &= \sum_{k=-\infty}^{\infty} 2^{k\alpha q_1} \left(\sum_{z=k+3}^{\infty} \|\left(I_{\Omega,\phi}^{A,m} f_z\right)\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1} \\
&\lesssim \sum_{k=-\infty}^{\infty} 2^{k\alpha q_1} \left(\sum_{z=k+3}^{\infty} 2^{(z-k)(\phi+\beta+n\delta_2-\alpha)} \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \left\|f_z\right\|_{L^{p_1(\cdot)}}\right)^{q_1} \\
&\lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}^{q_1} \sum_{k=-\infty}^{\infty} \left(\sum_{z=k+3}^{\infty} 2^{\alpha z} \left\|f_z\right\|_{L^{p_1(\cdot)}} 2^{(z-k)(\phi+\beta+n\delta_2-\alpha)}\right)^{q_1}.
\end{aligned}
$$

To proceed, we consider Z in two cases. Indeed, if $0<q_1\leq 1$, then by replacing q_1 with $\frac{q_1}{q_2}$ in (17) to get

$$
\begin{aligned}
Z &\lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}^{q_1} \sum_{k=-\infty}^{\infty} \sum_{z=k+3}^{\infty} 2^{\alpha z q_1} \left\|f_z\right\|_{L^{p_1(\cdot)}}^{q_1} 2^{(z-k)(\phi+\beta+n\delta_2-\alpha)q_1} \\
&\lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}^{q_1} \left\|f\right\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1}(\mathbb{R}^n)}^{q_1}.
\end{aligned}
$$

Now, let $1<q_1<\infty$ and $\frac{1}{q_1}+\frac{1}{q_1'}=1$. By (3), we have

$$
Z \lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}^{q_1} \sum_{k=-\infty}^{\infty} \sum_{z=k+3}^{\infty} 2^{\alpha z q_1} \left\|f_z\right\|_{L^{p_1(\cdot)}}^{q_1} 2^{(z-k)(\phi+\beta+n\delta_2-\alpha)\frac{q_1}{2}}
$$

$$\times\left(\sum_{z=k+3}^{\infty}2^{(z-k)\left(\phi+\beta+n\delta_2-\alpha\right)\frac{q_1'}{2}}\right)^{\frac{q_1}{q_1'}}$$
$$\lesssim \sum_{|\gamma|=m-1}\|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\sum_{z=-\infty}^{\infty}2^{\alpha z q_1}\left\|f_z\right\|^{q_1}_{L^{p_1(\cdot)}}\sum_{k=z-3}^{\infty}2^{(z-k)\left(\phi+\beta+n\delta_2-\alpha\right)\frac{q_1}{2}}$$
$$\lesssim \sum_{|\gamma|=m-1}\|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\left\|f\right\|^{q_1}_{\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Thus, by introducing approximations of X, Y and Z into (18), (9) is attained.

We are now at the point of proving (10). If we remember

$$\widetilde{T}^{A}_{|\Omega|,\phi}\left(|f|\right)(x)=\int_{\mathbb{R}^n}\frac{\left|\Omega(x-y)\right|}{|x-y|^{n-\phi+m-1}}\left|R_m\left(A;x,y\right)\right|\left|f(y)\right|dy \qquad 0<\phi<n,$$

it is easy to see that the conclusions of (9) also hold for $\widetilde{T}^{A}_{|\Omega|,\phi}$. Therefore, (10) is a direct consequence of Lemma 3.2 in [4] and the above conclusions. This completes the proof of Theorem 1.3. □

Now, we prove Theorem 1.6.

Proof of Theorem 1.6

Proof. Assume $f\in M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)$, $f_z := f\cdot\chi_z$ and $f=\sum\limits_{z=-\infty}^{\infty}f_z$ $(z\in\mathbb{Z})$. Then, using (17), we get

$$\begin{aligned}
\left\|I^{A,m}_{\Omega,\phi}f\right\|^{q_1}_{M\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} &= \sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left(\sum_{k=-\infty}^{L}2^{k\alpha q_2}\|\left(I^{A,m}_{\Omega,\phi}f\right)\chi_k\|^{q_2}_{L^{p_2(\cdot)}}\right)^{\frac{q_1}{q_2}}\\
&\lesssim \sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\sum_{k=-\infty}^{L}2^{k\alpha q_1}\|\left(I^{A,m}_{\Omega,\phi}f\right)\chi_k\|^{q_1}_{L^{p_2(\cdot)}}\\
&\lesssim \sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left[\sum_{k=-\infty}^{L}2^{k\alpha q_1}\left(\sum_{z=-\infty}^{k-3}\|\left(I^{A,m}_{\Omega,\phi}f_z\right)\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1}\right]\\
&\quad+\sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left[\sum_{k=-\infty}^{L}2^{k\alpha q_1}\left(\sum_{z=k-2}^{k+2}\|\left(I^{A,m}_{\Omega,\phi}f_z\right)\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1}\right]\\
&\quad+\sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left[\sum_{k=-\infty}^{L}2^{k\alpha q_1}\left(\sum_{z=k+3}^{\infty}\|\left(I^{A,m}_{\Omega,\phi}f_z\right)\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1}\right]\\
&=: X_1+Y_1+Z_1. \qquad (30)
\end{aligned}$$

First, we estimate X_1. Similar to the estimation method of X in Theorem 1.3, by virtue of (25)

$$\begin{aligned}
X_1 &= \sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left[\sum_{k=-\infty}^{L}2^{k\alpha q_1}\left(\sum_{z=-\infty}^{k-3}\|\left(I^{A,m}_{\Omega,\phi}f_z\right)\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1}\right]\\
&\lesssim \sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left[\sum_{k=-\infty}^{L}2^{k\alpha q_1}\left(\sum_{z=-\infty}^{k-3}2^{(k-z)\left(\phi+\beta-n\delta_1+\frac{n-1}{s}\right)}\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\left\|f_z\right\|_{L^{p_1(\cdot)}}\right)^{q_1}\right]\\
&\lesssim \sum_{|\gamma|=m-1}\|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left[\sum_{k=-\infty}^{L}\left(\sum_{z=-\infty}^{k-3}2^{\alpha z}\left\|f_z\right\|_{L^{p_1(\cdot)}}2^{(k-z)\left(\phi+\beta-n\delta_1+\alpha+\frac{n-1}{s}\right)}\right)^{q_1}\right].
\end{aligned}$$

On the other hand, we know that following fact:

$$\begin{aligned}\left\|f_z\right\|_{L^{p_1(\cdot)}} &= 2^{-\alpha z}\left(2^{\alpha z q_1}\left\|f\chi_z\right\|_{L^{p_1(\cdot)}}^{q_1}\right)^{\frac{1}{q_1}} \\ &\leq 2^{-\alpha z}\left(\sum_{i=-\infty}^{z}2^{\alpha i q_1}\left\|f\chi_i\right\|_{L^{p_1(\cdot)}}^{q_1}\right)^{\frac{1}{q_1}} \\ &= 2^{z(\lambda-\alpha)}\left[2^{-\lambda z}\left(\sum_{i=-\infty}^{z}2^{\alpha i q_1}\left\|f\chi_i\right\|_{L^{p_1(\cdot)}}^{q_1}\right)^{\frac{1}{q_1}}\right] \\ &\lesssim 2^{z(\lambda-\alpha)}\left\|f\right\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}^{q_1}. \end{aligned} \tag{31}$$

To continue calculating X_1, we consider the two cases $0<q_1\leq 1$ and $1<q_1<\infty$, respectively.

If $0<q_1\leq 1$ and $\alpha<n\delta_1+\lambda-\left(\phi+\beta+\frac{n-1}{s}\right)$, then using (17) and (31), we get

$$\begin{aligned} X_1 &\lesssim \sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}^{q_1}\left\|f\right\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\ &\quad\times\sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left[\sum_{k=-\infty}^{L}2^{k\lambda q_1}\sum_{z=-\infty}^{k-3}2^{(k-z)\left(\phi+\beta-n\delta_1-\lambda+\alpha+\frac{n-1}{s}\right)q_1}\right] \\ &\lesssim \sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}^{q_1}\left\|f\right\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}^{q_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left(\sum_{k=-\infty}^{L}2^{k\lambda q_1}\right) \\ &\lesssim \sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}^{q_1}\left\|f\right\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}^{q_1}. \end{aligned}$$

If $1<q_1<\infty$ and $\alpha<n\delta_1+\lambda-\left(\phi+\beta+\frac{n-1}{s}\right)$, then we use (3), (31) and obtain

$$\begin{aligned} X_1 &\lesssim \sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}^{q_1}\left\|f\right\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\ &\quad\times\sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left[\begin{array}{c}\sum\limits_{k=-\infty}^{L}2^{k\lambda q_1}\sum\limits_{z=-\infty}^{k-3}2^{(k-z)\left(\phi+\beta-n\delta_1-\lambda+\alpha+\frac{n-1}{s}\right)\frac{q_1}{2}} \\ \times\left(\sum\limits_{z=-\infty}^{k-3}2^{(k-z)\left(\phi+\beta-n\delta_1-\lambda+\alpha+\frac{n-1}{s}\right)\frac{q_1'}{2}}\right)^{\frac{q_1}{q_1'}}\end{array}\right] \\ &\lesssim \sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}^{q_1}\left\|f\right\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}^{q_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left(\sum_{k=-\infty}^{L}2^{k\lambda q_1}\right) \\ &\lesssim \sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}^{q_1}\left\|f\right\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}^{q_1}. \end{aligned}$$

Next, we estimate Y_1. If $D^\gamma A\in\dot{\Lambda}_\beta(\mathbb{R}^n)$, then from the fact that $I^{A,m}_{\Omega,\phi}$ is bounded from $L^{p_1(\cdot)}$ to $L^{p_2(\cdot)}$ (see Theorem 5 in [12]) and (17), it follows that

$$\begin{aligned} Y_1 &= \sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left[\sum_{k=-\infty}^{L}2^{k\alpha q_1}\left(\sum_{z=k-2}^{k+2}\|\left(I^{A,m}_{\Omega,\phi}f_z\right)\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1}\right] \\ &\lesssim \sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}^{q_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda q_1}\left[\sum_{k=-\infty}^{L}2^{k\alpha q_1}\left(\sum_{z=k-2}^{k+2}\|f_z\chi_k\|_{L^{p_2(\cdot)}}\right)^{q_1}\right] \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|^{q_1}_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \left[\sum_{k=-\infty}^{L} 2^{k\alpha q_1} \|f\chi_k\|^{q_1}_{L^{p_2(\cdot)}} \right] \\
&= \sum_{|\gamma|=m-1} \|D^{\gamma}A\|^{q_1}_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \left\|f\right\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Now, we estimate Z_1. By (29) and the assumption $\phi+\beta+n\delta_2<\alpha$, we know that

$$
\begin{aligned}
Z_1 &= \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \left[\sum_{k=-\infty}^{L} 2^{k\alpha q_1} \left(\sum_{z=k+3}^{\infty} \|\left(I^{A,m}_{\Omega,\phi} f_z\right)\chi_k\|_{L^{p_2(\cdot)}} \right)^{q_1} \right] \\
&\lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|^{q_1}_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \left[\sum_{k=-\infty}^{L} \left(\sum_{z=k+3}^{\infty} 2^{\alpha z} \left\|f_z\right\|_{L^{p_1(\cdot)}} 2^{(z-k)\left(\phi+\beta+n\delta_2-\alpha\right)} \right)^{q_1} \right] \\
&\lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|^{q_1}_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \left[\sum_{k=-\infty}^{L} \left(\sum_{z=k+3}^{L} 2^{\alpha z} \left\|f_z\right\|_{L^{p_1(\cdot)}} 2^{(z-k)\left(\phi+\beta+n\delta_2-\alpha\right)} \right)^{q_1} \right] \\
&\quad + \sum_{|\gamma|=m-1} \|D^{\gamma}A\|^{q_1}_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \left[\sum_{k=-\infty}^{L} \left(\sum_{z=L+1}^{\infty} 2^{\alpha z} \left\|f_z\right\|_{L^{p_1(\cdot)}} 2^{(z-k)\left(\phi+\beta+n\delta_2-\alpha\right)} \right)^{q_1} \right] \\
&=: Z_{11}+Z_{12}.
\end{aligned}
$$

To continue estimating Z_1, we look at two scenarios: $0<q_1\leq 1$ and $1<q_1<\infty$.

When $0<q_1\leq 1$ and $\phi+\beta+n\delta_2+\lambda<\alpha$, then using (17) and (31), we get

$$
\begin{aligned}
Z_1 &\lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|^{q_1}_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \left\|f\right\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \left[\sum_{k=-\infty}^{L} 2^{k\lambda q_1} \sum_{z=k+3}^{L} 2^{(z-k)\left(\phi+\beta+n\delta_2+\lambda-\alpha\right)q_1} \right] \\
&\quad + \sum_{|\gamma|=m-1} \|D^{\gamma}A\|^{q_1}_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \left[\sum_{k=-\infty}^{L} \sum_{z=L+1}^{\infty} 2^{\alpha z q_1} \left\|f_z\right\|^{q_1}_{L^{p_1(\cdot)}} 2^{(z-k)\left(\phi+\beta+n\delta_2+\lambda-\alpha\right)q_1} \right] \\
&\lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|^{q_1}_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \left\|f\right\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \left(\sum_{k=-\infty}^{L} 2^{k\lambda q_1} \right) \\
&\quad + \sum_{|\gamma|=m-1} \|D^{\gamma}A\|^{q_1}_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \\
&\quad \times \left[\sum_{k=-\infty}^{L} \sum_{z=L+1}^{\infty} 2^{z\lambda q_1} 2^{(z-k)\left(\phi+\beta+n\delta_2+\lambda-\alpha\right)q_1} 2^{-z\lambda q_1} \sum_{l=-\infty}^{z} 2^{\alpha l q_1} \left\|f_l\right\|^{q_1}_{L^{p_1(\cdot)}} \right] \\
&\lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|^{q_1}_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \left\|f\right\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

If $1<q_1<\infty$ and $\phi+\beta+n\delta_2+\lambda<\alpha$, then we use (3), (31) and obtain

$$
\begin{aligned}
Z_{11} &\lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|^{q_1}_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \left\|f\right\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)} \\
&\quad \times \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \left[\begin{array}{l} \sum\limits_{k=-\infty}^{L} 2^{k\lambda q_1} \sum\limits_{z=k+3}^{L} 2^{(z-k)\left(\phi+\beta+n\delta_2+\lambda-\alpha\right)\frac{q_1}{2}} \\ \quad \times \left(\sum\limits_{z=-\infty}^{k+3} 2^{(z-k)\left(\phi+\beta+n\delta_2+\lambda-\alpha\right)\frac{q_1'}{2}} \right)^{\frac{q_1}{q_1'}} \end{array} \right]
\end{aligned}
$$

$$\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \left(\sum_{k=-\infty}^{L} 2^{k\lambda q_1}\right)$$

$$\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Also, when $1 < q_1 < \infty$ and $\phi + \beta + n\delta_2 + \lambda < \alpha$, then we use (3), (31) and get

$$\begin{aligned} Z_{12} &\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \\ &\quad\times\left[\sum_{k=-\infty}^{L}\left(\sum_{z=L+1}^{\infty} 2^{\alpha z} \left\|f_z\right\|_{L^{p_1(\cdot)}} 2^{(z-k)\frac{(\phi+\beta+n\delta_2+\lambda-\alpha)}{2}} 2^{(z-k)\frac{(\phi+\beta+n\delta_2-\lambda-\alpha)}{2}}\right)^{q_1}\right] \\ &\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \\ &\quad\times\left[\sum_{k=-\infty}^{L}\sum_{z=L+1}^{\infty} 2^{z\lambda q_1} 2^{(z-k)\frac{(\phi+\beta+n\delta_2+\lambda-\alpha)}{2}q_1} 2^{-z\lambda q_1} \sum_{l=-\infty}^{z} 2^{\alpha l q_1} \left\|f_l\right\|^{q_1}_{L^{p_1(\cdot)}}\right] \\ &\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \\ &\quad\times\left[\sum_{k=-\infty}^{L} 2^{k\lambda q_1} \sum_{z=L+1}^{\infty} 2^{(z-k)\frac{(\phi+\beta+n\delta_2+\lambda-\alpha)}{2}q_1}\right] \\ &\lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Thus, combining the estimates of Z_{11} and Z_{12}, we find that

$$Z_1 \lesssim \sum_{|\gamma|=m-1} \|D^\gamma A\|^{q_1}_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\|f\right\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Thus, by introducing estimates of X_1, Y_1 and Z_1 into (30), (11) is obtained.

Finally, it is simple to demonstrate that (12). Indeed, we first know that

$$\widetilde{T}^A_{|\Omega|,\phi}\left(|f|\right)(x) \geq M^A_{\Omega,\phi} f(x) \tag{32}$$

for $x \in \mathbb{R}^n$ and $0 < \phi < n$ (see Lemma 3.2 in [4]). Next, from the process proving (11), the conclusions of (11) also hold for $\widetilde{T}^A_{|\Omega|,\phi}$. Thus, combining this with (32), we can immediately obtain (12), which concludes the proof of Theorem 1.6. □

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