Adjoint Appell-Euler polynomials and approximation

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Dedicated to our mentor Professor Vijay Gupta

Abstract. In the present article, linear positive operators connected with the adjoint Appell-Euler polynomials are proposed. Their moments are calculated and quantitative convergence estimates are provided using the *K*-functional and various moduli of smoothness such as first-order, exponential, and weighted moduli. Furthermore, Korovkin type theorems are established for different sets of test functions. Moreover, the associated modified operators are introduced which preserve constant functions and $e^{2\mu x}$, $\mu > 0$. Asymptotic formula and some analogous convergence results are established for these operators. Finally, we examine the convergence behaviour of both operators through numerical tables and graphical examples.

1. Introduction

Approximation theory and positive linear operators have been one of the most active fields among researchers due to their applicability in various areas such as computer-aided geometric design (CAGD) and Bézier curves [30], numerical analysis [29], integro-differential equations [20], probability and distributions [7], differential equations and boundary value problems [8, 11, 45] and fractional calculus [27]. Special functions and polynomial families are frequently used in constructing linear positive operators making them essential in approximation theory. Among these, Appell polynomials hold particular significance due to their generating function structure and their connections to various operator theory frameworks. An Appell sequence $\{q_k(x)\}_{k=0}^{\infty}$ is expressed via the exponential generating function, given as [5]:

$$A(t)e^{xt} = \sum_{k=0}^{\infty} q_k(x)\frac{t^k}{k!}, \tag{1}$$

where $A(t) = \sum_{k=0}^{\infty} q_k(0)\frac{t^k}{k!}$ is analytic and $A(0) \neq 0$.

Well-known examples such as the Bernoulli, Euler, Hermite and Genocchi polynomials fall within the Appell polynomials family, depending on the choice of $A(t)$, such as $A(t) = \frac{t}{e^t-1}$, $\frac{2}{e^t+1}$, $e^{-t^2/2}$ and $\frac{2t}{e^t+1}$,

respectively. These polynomials have various applications in numerous areas such as quantum mechanics [42], differential and integral equations [19, 25], computer science, physics, engineering, fluid dynamics, finance, medical science, number theory [23] and umbral algebra [12]. Euler numbers are crucial in number theory, topology, and the theory of modular forms. These numbers link to algebraic geometry, cryptography, and combinatorics.

The construction of linear positive operators based on Appell polynomials was initiated by Jakimovski and Leviatan [24]. Since then, numerous generalizations and extensions have been proposed within this framework. For related developments, see [4, 10, 19, 22, 36–38, 41, 43]. Natalini and Ricci (see [28]) introduced adjoint Appell polynomials by considering the reciprocal of the function $A(t)$ (in Eq. (1)). This approach provided a new family of polynomials with distinctive properties. Very recently, in [44], Yılmaz introduced Kantorovich-type operators involving adjoint Appell-Bernoulli polynomials, which were further generalized by Malik-Gupta in [26]. Generalized Phillips operators based on the Appell polynomials of class $A^{(2)}$ were given by Aksoy in [4].

The adjoint Appell-Euler polynomials (of the first kind), denoted by $\varepsilon_k(x)$, are defined in the following manner [28]:

$$\frac{e^t + 1}{2} e^{xt} = \sum_{k=0}^{\infty} \varepsilon_k(x) \frac{t^k}{k!}. \tag{2}$$

Based on (2), for $n \in \mathbb{N}$ and $x \in [0, \infty)$, we introduce the operators $A_n(f; x)$ as follows:

$$A_n(f; x) = \sum_{k=0}^{\infty} a_{n,k}(x) f\left(\frac{k}{n}\right), \tag{3}$$

where

$$a_{n,k}(x) = \frac{2}{e+1} e^{-nx} \frac{\varepsilon_k(nx)}{k!}. \tag{4}$$

In recent years, apart from constructions based on generating functions of special polynomials, many new operators have also been proposed using a combination of probabilistic and stochastic techniques [7, 17], solutions of certain partial differential equations [15, 18], convolution methods [31, 32, 40], and various composition procedures [6, 16–18].

The structure of this paper is as follows. In Section 2, we give some auxiliary lemmas related to the operators A_n. Section 3 focuses on the approximation properties of the operators A_n, where we establish estimates for the convergence using the K-functional and provide Voronovskaya-type formulae in terms of various moduli of smoothness.

There has been considerable interest in constructing linear positive operators that reproduce exponential functions. In [1], Acar et al. introduced generalized Szász–Mirakyan operators preserving the functions e^{ax} and e^{2ax}, $a > 0$. Acar-Aral-Gonska [2] presented Szász–Mirakyan operators preserving constant functions and e^{2ax} with $a > 0$. In [9], Bozkurt et al. proposed Bivariate Bernstein type operators that reproduce exponential functions on the simplex. Gupta and Aral [14] presented modified Szász–Mirakyan–Kantorovich operators preserving the constant function and e^{-x}. In [33], Özsaraç and Acar gave modified Baskakov operators preserving the functions $e^{\mu x}$ and $e^{2\mu x}$ with $\mu > 0$ (also see [34]). Motivated by these developments, Section 4 is devoted to the modified operators $\overline{A}_n$ which preserve both constant functions and $e^{2\mu x}$ ($\mu > 0$). We provide some analogous results and give asymptotic formula. Finally, in Section 5, we present graphical examples and numerical tables to analyse the effectiveness of both operators A_n and $\overline{A}_n$ in approximating functions.

2. Preliminary Lemmas

We compute the moments and central moments for the operators A_n, which will benefit in studying their convergence properties.

Lemma 2.1. *For the operators A_n, one has*

$$A_n(e^{At}; x) = \frac{e^{e^{A/n}} + 1}{e + 1} e^{nx(e^{A/n} - 1)}, \tag{5}$$

and

$$A_n(e_0; x) = 1,$$

$$A_n(e_1; x) = x + \frac{e}{n(e+1)},$$

$$A_n(e_2; x) = x^2 + \frac{x}{n} + \frac{2e}{n(e+1)}\left(x + \frac{1}{n}\right),$$

$$A_n(e_3; x) = x^3 + \frac{3x^2}{n} + \frac{x}{n^2} + \frac{e}{n(e+1)}\left(3x^2 + \frac{9x}{n} + \frac{5}{n^2}\right),$$

$$A_n(e_4; x) = x^4 + \frac{6x^3}{n} + \frac{7x^2}{n^2} + \frac{x}{n^3} + \frac{e}{n(e+1)}\left(4x^3 + \frac{24x^2}{n} + \frac{36x}{n^2} + \frac{15}{n^3}\right),$$

where $e_l(t) := t^l$ (from now on) for $l = 0, 1, \ldots$.

Proof. By the definition (3),

$$A_n(e^{At}; x) = \frac{2}{e+1} e^{-nx} \sum_{k=0}^{\infty} \frac{\epsilon_k(nx)}{k!} e^{Ak/n}$$

$$= \frac{e^{e^{A/n}} + 1}{e + 1} e^{nx(e^{A/n} - 1)},$$

which follows from (2).
The moments of A_n can now be obtained by using the following relation:

$$A_n(e_u; x) = \left[\frac{\partial^u}{\partial A^u}\left(\frac{e^{e^{A/n}} + 1}{e + 1} e^{nx(e^{A/n} - 1)}\right)\right]\Bigg|_{A=0}, \quad u = 0, 1, 2, \ldots.$$

□

Lemma 2.2. *Let $\psi_n^q(x) := A_n\left((e_1 - xe_0)^q; x\right)$ denote the central moment of q-th order for the operators A_n, where $q \in \mathbb{N} \cup \{0\}$. Then*

$$\psi_n^0(x) = 1, \qquad \psi_n^1(x) = \frac{e}{n(e+1)}, \qquad \psi_n^2(x) = \frac{1}{n}\left(x + \frac{2e}{n(e+1)}\right),$$

$$\psi_n^3(x) = \frac{x}{n^2} + \frac{e}{n^2(e+1)}\left(3x + \frac{5}{n}\right), \qquad \psi_n^4(x) = \frac{3x^2}{n^2} + \frac{x}{n^3} + \frac{e}{n^3(e+1)}\left(16x + 15\right).$$

In general,

$$\psi_n^s(x) = O\left(n^{-\lfloor \frac{s+1}{2} \rfloor}\right),$$

as $n \to \infty$, where $\lfloor a \rfloor$ denotes the integer part of a and $s = 0, 1, \ldots$.

Proof. The central moments can be evaluated either by using the linearity of the operators A_n or by the following formula:

$$\psi_n^q(x) = \left[\frac{\partial^q}{\partial A^q}\left(\frac{e^{e^{A/n}} + 1}{e + 1} e^{x\{n(e^{A/n} - 1) - A\}}\right)\right]\Bigg|_{A=0}.$$

□

3. Approximation by the operators A_n

Let $C(I)$ denote the space of real-valued uniformly continuous functions on $I \subseteq \mathbb{R}$, and let $C^j(I)$ be the space of functions that are j-times continuously differentiable on I, where $j \in \mathbb{N} \cup \{0\}$. Define $C_B(I)$ as the subspace of bounded functions in $C(I)$, and $C_B^j(I)$ as the subspace of functions in $C_B(I)$ whose derivatives up to order j also belong to $C_B(I)$.

Consider $f \in C^j(I)$. For any $l_1, l_2 \in I$:

$$f(l_1) = \sum_{q=0}^{j} \frac{f^{(q)}(l_2)}{q!} (l_1 - l_2)^q + \zeta_j(f; l_1, l_2),$$

where the remainder term is

$$\zeta_j(f; l_1, l_2) := (l_1 - l_2)^j \, \xi(l_1, l_2),$$

and $\xi(l_1, l_2)$ is some suitable function. Also (see [13]),

$$|\zeta_j(f; l_1, l_2)| \leq \frac{2|l_1 - l_2|^j}{j!} \mathcal{K}\left(f^{(j)}, \frac{|l_1 - l_2|}{2(j+1)}\right), \tag{6}$$

where $\mathcal{K}(f; \delta)$ denotes Peetre's K-functional given as [35]:

$$\mathcal{K}(f; \delta) = \inf_{g \in C^{j+1}(I)} \{\|f - g\|_\infty + \delta \|g'\|_\infty\}. \tag{7}$$

Now, we provide a quantitative convergence estimate for the operators A_n using $\mathcal{K}(f; \cdot)$.

Theorem 3.1. *Let* $f \in C_B^2[0, \infty)$, *then*

$$\left| n\{A_n(f; x) - f(x)\} - \left(\frac{e}{e+1}\right) f'(x) - \frac{1}{2}\left(x + \frac{2e}{n(e+1)}\right) f''(x) \right| \leq \left(x + \frac{2e}{n(e+1)}\right) \mathcal{K}\left(f''; \frac{nx(1+4e)+5e}{6n(2e+nx(1+e))}\right).$$

Proof. For $f \in C_B^2[0, \infty)$,

$$f\left(\frac{k}{n}\right) = \sum_{q=0}^{2} \left(\frac{k}{n} - x\right)^q \frac{f^{(q)}(x)}{q!} + \zeta_2\left(f; \frac{k}{n}, x\right),$$

where

$$\zeta_2\left(f; \frac{k}{n}, x\right) = \left(\frac{k}{n} - x\right)^2 \xi\left(\frac{k}{n}; x\right), \quad \text{with } \xi\left(\frac{k}{n}; x\right) \to 0 \text{ as } \frac{k}{n} \to x.$$

Then

$$\left| n\left\{A_n(f; x) - \sum_{q=0}^{2} \frac{\psi_n^q(x)}{q!} f^{(q)}(x)\right\} \right| \leq n \sum_{k=0}^{\infty} a_{n,k}(x) \left| \zeta_2\left(f; \frac{k}{n}, x\right) \right|,$$

where $a_{n,k}(x)$ is given in (4). In view of (6), we get

$$\left| n\left\{A_n(f; x) - \sum_{q=0}^{2} \frac{\psi_n^q(x)}{q!} f^{(q)}(x)\right\} \right| \leq n \sum_{k=0}^{\infty} a_{n,k}(x) \left(\frac{k}{n} - x\right)^2 \mathcal{K}\left(f''; \frac{\left|\frac{k}{n} - x\right|}{6}\right).$$

Let $g \in C_B^3[0, \infty)$ be fixed. Then by (7),

$$\left| n\left\{A_n(f; x) - \sum_{q=0}^{2} \frac{\psi_n^q(x)}{q!} f^{(q)}(x)\right\} \right| \leq n \sum_{k=0}^{\infty} a_{n,k}(x) \left(\frac{k}{n} - x\right)^2 \left\{\|(f - g)''\|_\infty + \frac{\left|\frac{k}{n} - x\right|}{6} \|g'''\|_\infty\right\}$$

$$= n\left\{\|(f - g)''\|_\infty \left|\psi_n^2(x)\right| + \frac{\|g'''\|_\infty}{6} \left|\psi_n^3(x)\right|\right\}.$$

Taking the infimum over $g'' \in C_B^1[0, \infty)$ and using Lemma 2.2 ,

$$\left| n\left\{A_n(f;x) - f(x)\right\} - \left(\frac{e}{e+1}\right) f'(x) - \frac{1}{2}\left(x + \frac{2e}{n(e+1)}\right) f''(x) \right| \leq \left(x + \frac{2e}{n(e+1)}\right) \mathcal{K}\left(f''; \frac{nx(1+4e) + 5e}{6n(2e + nx(1+e))}\right).$$

□

Corollary 3.2. *Under the assumptions of Theorem 3.1, we have*

$$\lim_{n\to\infty} n\left[A_n(f;x) - f(x)\right] = \frac{e}{e+1}f'(x) + \frac{x}{2}f''(x).$$

Denote by $C^*[0, \infty) = \{f : C[0, \infty) : \lim_{x\to\infty} f(x) \text{ exists and is finite }\}$. We proceed to prove a Korovkin-type theorem (see [16, 21]) using the test functions $\{1, e^{-x}, e^{-2x}\}$ and the following associated modulus of continuity:

$$\omega^*(f, \sigma) = \sup_{x,v\geq 0}\{|f(x) - f(v)|, \left|e^{-x} - e^{-v}\right| \leq \sigma \text{ and } \sigma \geq 0\},$$

defined for $f \in C^*[0, \infty)$.

Theorem 3.3. *Let $f \in C^*[0, \infty)$. For the operators A_n ,*

$$\sup_{x\geq 0} |A_n(f;x) - f(x)| \leq 2\omega^*\left(f, \sqrt{\mathcal{H}_n}\right),$$

where $\mathcal{H}_n = \frac{(4e + \frac{2}{e} + 2)}{n(1+e)} + \frac{(6e + \frac{3}{e^2} + \frac{4}{e} + 4)}{n^2(1+e)} + O\left(\frac{1}{n}\right)^3 \to 0$, as $n \to \infty$.

Proof. For $x, v \geq 0$ and $f \in C^*[0, \infty)$ (see [21]),

$$|f(v) - f(x)| \leq \left(1 + \left(\frac{e^{-v} - e^{-x}}{\sigma}\right)^2\right)\omega^*(f, \sigma).$$

By applying the operators A_n,

$$|A_n(f;x) - f(x)| \leq \left(1 + A_n((e^{-v} - e^{-x})^2; x)\frac{1}{\sigma^2}\right)\omega^*(f, \sigma).$$

Now,

$$A_n((e^{-v} - e^{-x})^2; x) = A_n(e^{-2v}; x) + e^{-2x}A_n(1; x) - 2e^{-x}A_n(e^{-v}; x)$$
$$= \left[\frac{e^{-2x}(2ex + 2x - 2e)}{n(1+e)} + \frac{e^{-2x}(24e - 8x - 32ex + 12x^2 + 12ex^2)}{6n^2(1+e)} + O\left(\frac{1}{n}\right)^3\right]$$
$$- \left[2e^{-x}\left(\frac{e^{-x}(ex + x - 2e)}{2n(1+e)}\right) + \frac{e^{-x}(24e - 4x - 16ex + 3x^2 + 3ex^2)}{24n^2(1+e)} + O\left(\frac{1}{n}\right)^3\right].$$

Taking the supremum over $x \geq 0$ and utilizing $\sup_{x\in[0,\infty)} x^p e^{-qx} = \left(\frac{p}{qe}\right)^p$, we obtain

$$\sup_{x\geq 0} |A_n((e^{-v} - e^{-x})^2; x)| \leq \frac{(4e + \frac{2}{e} + 2)}{n(1+e)} + \frac{(6e + \frac{3}{e^2} + \frac{4}{e} + 4)}{n^2(1+e)} + O\left(\frac{1}{n}\right)^3$$
$$=: \mathcal{H}_n.$$

Therefore,

$$\sup_{x\geq 0} |A_n(f;x) - f(x)| \leq \left(1 + \frac{\mathcal{H}_n}{\sigma^2}\right)\omega^*(f, \sigma).$$

Choosing $\sigma = \sqrt{\mathcal{H}_n}$ completes the proof. □

Likewise, for the next estimation we use the test functions $\left\{1, \frac{x}{1+x}, \frac{x^2}{(1+x)^2}\right\}$ and the following modulus of smoothness:

$$\widehat{\omega}(f, \sigma) = \sup_{x,t \geq 0} \left\{ |f(x) - f(t)|, \left| \frac{x}{1+x} - \frac{t}{1+t} \right| \leq \sigma \text{ and } \sigma \geq 0 \right\}.$$

Theorem 3.4. *Let* $f \in C^*[0, \infty)$, *then for the operators* A_n,

$$\sup_{x \geq 0} |A_n(f;x) - f(x)| \leq 2\,\widehat{\omega}\left(f, \sqrt{\frac{1}{4n} + \frac{2e}{n^2(e+1)}}\right).$$

Proof. For $x, \theta \geq 0$ and $f \in C^*[0, \infty)$ (see [21]),

$$|f(\theta) - f(x)| \leq \left(1 + \frac{(\theta - x)^2}{\sigma^2(1+x)^2}\right)\widehat{\omega}(f, \sigma).$$

Then from Lemma 2.2,

$$|A_n(f;x) - f(x)| \leq \left(1 + \frac{1}{\sigma^2(1+x)^2}\left(\frac{x}{n} + \frac{2e}{(e+1)n^2}\right)\right)\widehat{\omega}(f, \sigma).$$

Taking the supremum over $x \in [0, \infty)$,

$$\sup_{x \geq 0} |A_n(f;x) - f(x)| \leq \left(1 + \frac{1}{\sigma^2}\left(\frac{1}{4n} + \frac{2e}{(e+1)n^2}\right)\right)\widehat{\omega}(f, \sigma).$$

Choosing $\sigma = \sqrt{\frac{1}{4n} + \frac{2e}{(e+1)n^2}}$ gives the desired result. $\quad\square$

Define the Lipschitz class

$$\mathcal{L}_{\alpha,A} = \left\{g : \omega_1^*(g; \delta, A) \leq M\delta^\alpha,\ \delta < 1\right\}, \quad 0 < \alpha \leq 1,\ M > 0,$$

where

$$\omega_1^*(g; \delta, A) = \sup_{\substack{0 \leq u < \infty \\ h \leq \delta}} |g(u + h) - g(u)|\, e^{-Au}$$

is the exponential modulus of continuity [39]. Also consider the space

$$C_A[0, \infty) = \left\{g \in C[0, \infty) : \|g\|_A := \sup_{u \geq 0} |g(u)|e^{-Au} < \infty\right\}.$$

We give another quantitative Voronovskaya type Theorem (see [17, 39]) using $\omega_1^*(f''; \cdot, A)$.

Theorem 3.5. *For* $A_n : C_A[0, \infty) \to C[0, \infty)$,

$$A_n\left((v - x)^2 e^{Av}; x\right) \leq \psi_n^2(x)\, \mathcal{S}_A(x),$$

where $A > 0$ *and* $x \in [0, \infty)$. *Moreover, if* $f \in C_A[0, \infty) \cap C^2[0, \infty)$ *and* $f'' \in \mathcal{L}_{\alpha,A}$, *then*

$$\left| A_n(f;x) - f(x) - \left(\frac{e}{(1+e)n}\right)f'(x) - \left(\frac{x}{n} + \frac{2e}{(1+e)n^2}\right)f''(x) \right|$$

$$\leq \left(\frac{x}{n} + \frac{2e}{(1+e)n^2}\right)\left[e^{2Ax} + \frac{\mathcal{S}_A(x)}{2} + \frac{\sqrt{\mathcal{S}_{2A}(x)}}{2}\right]\omega_1^*\left(f''; \sqrt{\frac{e(15 + nx(17 + 3nx)) + nx(1 + 3nx)}{n^2(2e + (1+e)nx)}}; A\right),$$

where $\mathcal{S}_A(x) := e^{Ax}\left(2e^{(1+e^{\frac{1}{2}})} + 4A(e+1)x^2\right)$.

Proof. Differentiating Eq. (5) both sides with respect to A,

$$A_n\left(ve^{Av};x\right) = \frac{x\left(e^{e^{A/n}}+1\right)ye^{\frac{A}{n}}}{1+e} + \frac{ye^{\frac{A}{n}+e^{A/n}}}{(1+e)n},$$

where $y := e^{nx(e^{A/n}-1)}$.

Next,

$$A_n\left(v^2 e^{Av};x\right) = \frac{ye^{\frac{A}{n}}}{(1+e)n^2}\left[n^2 x^2 e^{A/n} + e^{\frac{A}{n}+e^{A/n}}(1+xn)^2 + e^{e^{A/n}}(xn+1) + xn\right].$$

By elementary computations, we get

$$A_n\left((v-x)^2 e^{Av};x\right) = \frac{\left(\frac{e^{A/n}+1}{n}\right)ye^{\frac{A}{n}+e^{A/n}}}{(1+e)n} + \frac{2xye^{\frac{2A}{n}+e^{A/n}}}{(1+e)n} + \frac{x\left(e^{e^{A/n}}+1\right)ye^{\frac{A}{n}}\left(xe^{A/n}+\frac{1}{n}\right)}{1+e} + \frac{x^2\left((e^{e^{\frac{A}{n}}}+1)y\right)}{e+1}$$
$$- 2x\left(\frac{x\left(e^{e^{A/n}}+1\right)ye^{\frac{A}{n}}}{1+e} + \frac{ye^{\frac{A}{n}+e^{A/n}}}{(1+e)n}\right).$$

Suppose that $2A < n < 2Ax$. This yields:

$$ye^{\frac{2A}{n}+e^{\frac{A}{n}}} \le e^{Ax+1+e^{\frac{1}{2}}},$$

$$\frac{2x}{(1+e)n} + \frac{1}{n^2(1+e)} \le \frac{2x}{n} + \frac{4e}{n^2(1+e)},$$

$$\left(\frac{1+e^{e^{\frac{A}{n}}}}{1+e}\right)\frac{x}{n} + \frac{e^{\frac{A}{n}}}{n^2(1+e)} \le \frac{2x}{n} + \frac{4e}{n^2(1+e)}.$$

Moreover,

$$\frac{x^2(e^{e^{\frac{A}{n}}}+1)y}{e+1} + \frac{x\left(e^{e^{A/n}}+1\right)ye^{\frac{A}{n}}\left(xe^{A/n}\right)}{1+e} - 2x\left(\frac{x\left(e^{e^{A/n}}+1\right)ye^{\frac{A}{n}}}{1+e}\right) \le \left(\frac{x}{n} + \frac{2e}{(1+e)n^2}\right)\left(4A(e+1)x^2 e^{Ax}\right).$$

It follows that

$$A_n\left((v-x)^2 e^{Av};x\right) \le \psi_n^2(x)\cdot 2e^{Ax+1+e^{\frac{1}{2}}} + \psi_n^2(x)\left(4A(e+1)x^2 e^{Ax}\right)$$
$$\le \psi_n^2(x)\cdot e^{Ax}\left(2e^{(1+e^{\frac{1}{2}})} + 4A(e+1)x^2\right). \tag{8}$$

For $x, v \in [0,\infty)$ and $f \in C_A[0,\infty)$,

$$f(v) = f(x) + (v-x)f'(x) + \frac{(v-x)^2}{2}f''(x) + R_2(f;v,x),$$

where

$$R_2(f;v,x) = \frac{(v-x)^2}{2}\left(f''(\eta) - f''(x)\right) \tag{9}$$

and η is located between x and v.

By operating A_n,

$$\left|A_n\left(f;x\right) - f(x) - \psi_n^1(x)f'(x) - \frac{1}{2}\psi_n^2(x)f''(x)\right| \le A_n\left(\left|R_2(f;v,x)\right|;x\right).$$

Also, there holds the following estimate [39]

$$\left|R_2(f;v,x)\right| \le \frac{1}{2}\left(e^{2Ax} + e^{Av}\right)\left(1 + \frac{|v-x|}{\delta}\right)|v-x|^2\,\omega_1^*(f'';\delta,A).$$

This, together with Lemma 2.2, yields

$$\left|A_n\,(f;x) - f(x) - \frac{e}{(1+e)n}f'(x) - \left(x + \frac{2e}{(1+e)n}\right)\frac{f''(x)}{n}\right|$$

$$\le \frac{1}{2}\left[A_n\left(\left(e^{2Ax} + e^{Av}\right)\left(|v-x|^2 + \frac{|v-x|^3}{\delta}\right);x\right)\right]\omega_1^*(f'';\delta,A). \tag{10}$$

Furthermore, applying the Cauchy–Schwarz inequality along with (8) yields

$$A_n\left(|v-x|^3e^{Av};x\right) \le \sqrt{S_{2A}(x)\psi_n^2(x)} \cdot \sqrt{\psi_n^4(x)}. \tag{11}$$

By using (8) and (11) in (10) and choosing $\delta := \sqrt{\dfrac{\psi_n^4(x)}{\psi_n^2(x)}}$, we get the required result. □

Remark 3.6. *For $x \ge 0$,*

$$\frac{\psi_n^4(x)}{\psi_n^2(x)} = \frac{e\,(15 + nx(17 + 3nx)) + nx(1 + 3nx)}{n^2(2e + (1+e)nx)} \to 0 \ as \ n \to \infty.$$

Thus, the Theorem 3.5 is validated.

For $v \in [0, \infty)$, consider the space

$$\widehat{C}[0,\infty) := \{f : [0,\infty) \to \mathbb{R} \ : \ f \text{ is continuous and } |f(v)| \le \Gamma_f\,(1 + v^2)\},$$

where $\Gamma_f > 0$ is some constant dependent on f and the corresponding weighted modulus of continuity [3], for $f \in \widehat{C}[0,\infty)$, is given as:

$$\Omega(f;\gamma) = \sup_{0 \le h < \gamma,\, v \in [0,\infty)} \frac{|f(v + h) - f(v)|}{(1 + h^2)(1 + v^2)}.$$

Consider a subspace $\mathcal{W} \subseteq C[0,\infty)$ that includes all polynomials. For $f \in C^2[0,\infty)$, we can consider the same remainder as mentioned in (9) and it satisfies [3]:

$$|R_2(f;v,x)| \le 2\left(1 + \delta^2\right)^2 (1 + x^2)\,\Omega(f'';\delta)\left(1 + \frac{(v-x)^4}{\delta^4}\right). \tag{12}$$

Furthermore, let $\widehat{C}_2[0,\infty)$ be the class of functions $g \in \widehat{C}[0,\infty)$ such that $\lim\limits_{y\to\infty}\dfrac{g(y)}{1 + y^2}$ exists and is finite. Next we establish a quantitative form of the Voronovskaya-type theorem in the weighted space.

Theorem 3.7. *Let $f \in \mathcal{W}$, $f'' \in \widehat{C}_2[0,\infty)$. Then $A_n : \mathcal{W} \to C[0,\infty)$ satisfies*

$$\left|A_n(f;x) - f(x) - \frac{e}{n(1+e)}f'(x) - \frac{1}{2}\left(\frac{x}{n} + \frac{2e}{n^2(1+e)}\right)f''(x)\right|$$

$$\le 16(1 + x^2)\,\Omega\left(f'';(\tau_n(x))^{1/4}\right)\left(\frac{x}{n} + \frac{2e}{n^2(1+e)}\right),$$

where

$$\tau_n(x) = \frac{nx\left(15n^2x^2 + 25nx + 1\right) + e\left(15n^3x^3 + 175n^2x^2 + 362nx + 203\right)}{n^4(nx + e(nx + 2))}. \tag{13}$$

Proof. For $f \in \mathcal{W}$, applying A_n to its Taylor expansion yields

$$A_n(f;x) - f(x) - \psi_n^1(x)f'(x) - \frac{1}{2!}\psi_n^2(x)f''(x) = A_n((v-x)^2 R_2(f;v,x);x).$$

Using Lemma 2.2 and equation (12), we get

$$\left| A_n(f;x) - f(x) - \frac{e}{n(1+e)}f'(x) - \frac{1}{2}\left(\frac{x}{n} + \frac{2e}{n^2(1+e)}\right)f''(x) \right|$$

$$\leq 2(1+\delta^2)^2(1+x^2)\Omega\left(f'';\delta\right)\left[\left(\frac{x}{n} + \frac{2e}{(1+e)n^2}\right)\right.$$

$$\left. + \frac{1}{\delta^4}\left(\frac{203e}{(1+e)n^6} + \frac{(1+362e)x}{(1+e)n^5} + \frac{25(1+7e)x^2}{(1+e)n^4} + \frac{15x^3}{n^3}\right)\right]$$

$$\leq 8(1+x^2)\Omega\left(f'';\delta\right)\left(\frac{x}{n} + \frac{2e}{(1+e)n^2}\right)\left(1 + \frac{1}{\delta^4}\tau_n(x)\right),$$

which follows by taking $\delta < 1$. Moreover, choosing $\delta = (\tau_n(x))^{1/4}$ gives the desired assertion, where $\tau_n(x)$ is given in (13). $\square$

Remark 3.8. *By simple computations,*

$$\lim_{n\to\infty}\left(\frac{\psi_n^6(x)}{\psi_n^2(x)}\right) = 0,$$

which validates the Theorem 3.7.

4. Modified operators preserving $e^{2\mu x}$

Motivated by earlier works on linear positive operators that preserve exponential functions such as $e^{2ax}, a > 0$ (see [2, 9, 33]) and e^{-x} (see [14]), we introduce modified adjoint Appell-Euler operators, preserving $e^{2\mu x}$ ($\mu > 0$), as follows:

$$\overline{A}_n(f;x) = A_n(f;a_n(x))$$

$$= \left(\frac{2}{e+1}\right)e^{-na_n(x)}\sum_{k=0}^{\infty}\frac{\varepsilon_k(na_n(x))}{k!}f\left(\frac{k}{n}\right), \tag{14}$$

where, in view of (5), $a_n(x)$ satisfies the following:

$$e^{2\mu x} = \left(\frac{e^{e^{\frac{2\mu}{n}}} + 1}{e+1}\right)e^{na_n(x)\left(e^{\frac{2\mu}{n}} - 1\right)},$$

which gives

$$a_n(x) = \frac{2\mu x - \log\left(\frac{e^{e^{\frac{2\mu}{n}}}+1}{e+1}\right)}{n\left(e^{\frac{2\mu}{n}} - 1\right)}, \quad x \in \left[\frac{\log\left(\frac{e^{e^{\frac{2\mu}{n}}}+1}{e+1}\right)}{2\mu}, \infty\right). \tag{15}$$

Hence, the operators in (14) have the form:

$$\overline{A}_n(f;x) = \left(\frac{2}{e+1}\right)e^{-\frac{2\mu x - \log\left(\frac{e^{e^{\frac{2\mu}{n}}}+1}{e+1}\right)}{e^{\frac{2\mu}{n}}-1}}\sum_{k=0}^{\infty}\frac{\varepsilon_k\left(\frac{2\mu x - \log\left(\frac{e^{e^{\frac{2\mu}{n}}}+1}{e+1}\right)}{e^{\frac{2\mu}{n}}-1}\right)}{k!}f\left(\frac{k}{n}\right). \tag{16}$$

Lemma 4.1. *The following holds for the operators* $\overline{A}_n$:

$$\overline{A}_n(e^{At}; x) = \left(\frac{e^{e^{\frac{A}{n}}} + 1}{e + 1} \right) e^{na_n(x)\left(e^{\frac{A}{n}} - 1\right)}.$$

Proof. We have

$$\overline{A}_n(e^{At}; x) = \left(\frac{2}{e + 1} \right) e^{-na_n(x)} \sum_{k=0}^{\infty} \frac{\varepsilon_k na_n(x)}{k!} e^{\frac{Ak}{n}} = \left(\frac{e^{e^{\frac{A}{n}}} + 1}{e + 1} \right) e^{na_n(x)\left(e^{\frac{A}{n}} - 1\right)}.$$

Interestingly, the operators $\overline{A}_n$ preserve constant functions, since $\overline{A}_n(e_0; x) = 1$. $\square$

Lemma 4.2. *The central moments* $\overline{A}_n(\psi_x^r(t), x) := \overline{A}_n((t - x)^r, x)$ *are mentioned for* $r = 0, 1, 2, 3$ *as:*

$$\overline{A}_n(\psi_x^0(t); x) = 1,$$

$$\overline{A}_n(\psi_x^1(t); x) = a_n(x) + \frac{e}{(e + 1)n} - x,$$

$$\overline{A}_n(\psi_x^2(t); x) = (a_n(x) - x)^2 + \frac{a_n(x)}{n} + \frac{2e}{n(e + 1)} \left(a_n(x) - x + \frac{1}{n} \right),$$

$$\overline{A}_n(\psi_x^3(t); x) = (a_n(x) - x)^3 + \frac{a_n(x)}{n^2} - \frac{3xa_n(x)}{n} + \frac{3a_n^2(x)}{n} + \frac{e}{n(e + 1)} \left(\frac{9a_n(x)}{n} - \frac{6x}{n} \right.$$

$$\left. + 3a_n^2(x) + 3x^2 - 6xa_n(x) + \frac{5}{n^2} \right).$$

Lemma 4.3. *For the operators* $\overline{A}_n$, *we have*

$$\overline{A}_n(e^{\mu t}; x) = \frac{1 + e^{e^{\frac{\mu}{n}}}}{1 + e} e^{\frac{\left(e^{\frac{\mu}{n}} - 1 \right)\left(2x\mu - \log\left(\frac{1 + e^{e^{\frac{2\mu}{n}}}}{1 + e} \right) \right)}{e^{\frac{2\mu}{n}} - 1}},$$

$$\overline{A}_n(e^{2\mu t}; x) = e^{2\mu x},$$

$$\overline{A}_n(e^{3\mu t}; x) = \frac{1 + e^{e^{\frac{3\mu}{n}}}}{1 + e} e^{\frac{\left(e^{\frac{3\mu}{n}} - 1 \right)\left(2x\mu - \log\left(\frac{1 + e^{e^{\frac{2\mu}{n}}}}{1 + e} \right) \right)}{e^{\frac{2\mu}{n}} - 1}},$$

$$\overline{A}_n(e^{4\mu t}; x) = \frac{1 + e^{e^{\frac{4\mu}{n}}}}{1 + e} e^{\frac{\left(e^{\frac{4\mu}{n}} - 1 \right)\left(2x\mu - \log\left(\frac{1 + e^{e^{\frac{2\mu}{n}}}}{1 + e} \right) \right)}{e^{\frac{2\mu}{n}} - 1}}.$$

Theorem 4.4. *Let* $f \in C^*[0, \infty)$ *and its second derivative exists at some point* $x \in [0, \infty)$. *Then*

$$\lim_{n \to \infty} n \left(\overline{A}_n(f; x) - f(x) \right) = -\mu x f'(x) + \frac{x}{2} f''(x).$$

Proof. Consider the substitution $u = e^{\mu t}$ and define the auxiliary function $g(u) := f\left(\frac{\ln u}{\mu} \right)$ so that $f(t) = g(e^{\mu t})$. Using Taylor's theorem for $g(u)$ around $u = e^{\mu x}$, one has

$$f(t) = g(e^{\mu t}) = g(e^{\mu x}) + g'(e^{\mu x})(e^{\mu t} - e^{\mu x}) + \frac{1}{2} g''(e^{\mu x})(e^{\mu t} - e^{\mu x})^2 + h_x(t)(e^{\mu t} - e^{\mu x})^2,$$

where $h_x(t) \to 0$ as $t \to x$.
Now

$$\overline{A}_n(f; x) = f(x) + g'(e^{\mu x}) \cdot \overline{A}_n(\varphi_1(t); x) + \frac{1}{2} g''(e^{\mu x}) \cdot \overline{A}_n(\varphi_2(t); x) + \overline{A}_n(h_x(t) \cdot \varphi_2(t); x),$$

where $\varphi_r(t) := (e^{\mu t} - e^{\mu x})^r$ for $r \in \mathbb{N}$.

Also,

$$g'(e^{\mu x}) = \frac{1}{\mu e^{\mu x}} f'(x), \quad g''(e^{\mu x}) = e^{-2\mu x}\left(\frac{1}{\mu^2} f''(x) - \frac{1}{\mu} f'(x)\right).$$

Substituting these into the expansion, we get

$$\overline{A}_n(f;x) - f(x) = \frac{1}{\mu e^{\mu x}} f'(x) \cdot \overline{A}_n(\varphi_1(t);x) + \frac{1}{2} e^{-2\mu x}\left(\frac{1}{\mu^2} f''(x) - \frac{1}{\mu} f'(x)\right) \cdot \overline{A}_n(\varphi_2(t);x) + \overline{A}_n(h_x(t) \cdot \varphi_2(t);x).$$

Since

$$\lim_{n\to\infty} n\overline{A}_n(\varphi_1(t);x) = -\frac{1}{2}\mu^2 x e^{\mu x} \text{ and } \lim_{n\to\infty} n\overline{A}_n(\varphi_2(t);x) = \mu^2 x e^{2\mu x}.$$

Also, by Cauchy-Schwarz inequality,

$$n|\overline{A}_n(h_x(t)\varphi_2(t);x)| \le \left[\overline{A}_n(h_x^2(t);x)\right]^{\frac{1}{2}} \left[n^2\overline{A}_n(\varphi_4(t);x)\right]^{\frac{1}{2}}$$

Moreover,

$$\lim_{n\to\infty} \overline{A}_n(h_x^2(t);x) = 0.$$

The above, together with Lemma 4.3, implies that

$$\lim_{n\to\infty} n|\overline{A}_n(h_x(t)\varphi_2(t);x)| = 0.$$

Thus,

$$\lim_{n\to\infty} n\left(\overline{A}_n(f;x) - f(x)\right) = \frac{x}{2} f''(x) - \mu x f'(x).$$

□

Let us denote

$$C_\mu[0,\infty) := \{f : [0,\infty) \to \mathbb{R} \ : \ f \text{ is continuous and } |f(x)| \le Ke^{\mu x}\},$$

$$C_\mu^s[0,\infty) := \{f \in C_\mu[0,\infty) \ : \ \lim_{x\to\infty} e^{-\mu x}|f(x)| = s\},$$

where $\mu > 0$ is fixed and s, K are some positive constants. Also,

$$\|f\|_\mu := \sup_{x\in[0,\infty)} e^{-\mu x}|f(x)|.$$

For any $f \in C_\mu^s[0,\infty)$, consider the weighted modulus of continuity defined as [1, 34]:

$$\breve{\omega}(f;\delta) := \sup\left\{\frac{f(y) - f(x)}{e^{\mu y} + e^{\mu x}}, \ x, y \ge 0 \text{ and } |y - x| \le \delta\right\}.$$

For some basic properties of $\breve{\omega}(f;\cdot)$, we refer the readers to [1].

Theorem 4.5. *Let* $f \in C_\mu^s[0,\infty)$, *then*

$$\|\overline{A}_n(f) - f\|_{3\mu} \le 2\left(\frac{\left(1 + e^{e^{\frac{\mu}{n}}}\right)}{1 + e}\left(\frac{1 + e}{1 + e^{2\mu/n}}\right)^{\frac{e^{\mu/n}-1}{e^{2\mu/n}-1}} + 3\right)\breve{\omega}\left(f;\mu^{-1}\sqrt{2 - 2\frac{\left(1 + e^{e^{\frac{\mu}{n}}}\right)}{1 + e}\left(\frac{1 + e}{1 + e^{2\mu/n}}\right)^{\frac{e^{\mu/n}-1}{e^{2\mu/n}-1}}}\right).$$

Proof. By elementary calculations, one has for $x \in [0, \infty)$ and $f \in C_\mu^s[0, \infty)$,

$$|f(y) - f(x)| \le 2 \left(e^{\mu y} + e^{\mu x} \right) \left(1 + \frac{|e^{\mu y} - e^{\mu x}|}{\mu \delta} \right) \breve{\omega}(f; \delta).$$

By applying the operators $\overline{A}_n$ on the above and using Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left| \overline{A}_n(f; x) - f(x) \right| \\
&\le \overline{A}_n \left(|f(y) - f(x)|; x \right) \\
&\le 2 \, \overline{A}_n \left(\left(e^{\mu y} + e^{\mu x} \right) \left(1 + \frac{|e^{\mu y} - e^{\mu x}|}{\mu \delta} \right); x \right) \breve{\omega}(f; \delta) \\
&\le 2 \left(\overline{A}_n(e^{\mu y}; x) + \frac{1}{\mu \delta} \sqrt{e^{2\mu x}} \sqrt{\overline{A}_n(\psi_x^2(y); x)} + e^{\mu x} + \frac{e^{\mu x}}{\mu \delta} \sqrt{\overline{A}_n(\psi_x^2(y); x)} \right) \breve{\omega}(f; \delta).
\end{aligned}$$

By using Lemma 4.3,

$$\frac{\left| \overline{A}_n(f; x) - f(x) \right|}{e^{3\mu x}} \le 2 \left(\frac{e^{\frac{\left(e^{\frac{\mu}{n}} - 1 \right) \left(2x\mu - \log \left(\frac{1 + e^{e^{\frac{2\mu}{n}}}}{1 + e} \right) \right)}{e^{\frac{2\mu}{n}} - 1}} \left(1 + e^{e^{\frac{\mu}{n}}} \right)}{(1 + e) e^{3\mu x}} + \frac{1}{e^{2\mu x}} + \frac{2}{\mu \delta} \frac{1}{\sqrt{e^{2\mu x}}} \sqrt{\frac{\overline{A}_n(\psi_x^2(y); x)}{e^{2\mu x}}} \right) \breve{\omega}(f; \delta).$$

By taking the supremum over $x \in [0, \infty)$, one gets

$$\begin{aligned}
&\sup_{x \in [0, \infty)} \frac{\left| \overline{A}_n(f; x) - f(x) \right|}{e^{3\mu x}} \\
&\le 2 \left(\frac{\left(1 + e^{e^{\frac{\mu}{n}}} \right)}{1 + e} \left(\frac{1 + e}{1 + e^{2\mu/n}} \right)^{\frac{e^{\mu/n} - 1}{e^{2\mu/n} - 1}} + 1 + \frac{2}{\mu \delta} \left(\sqrt{2 - 2 \frac{\left(1 + e^{e^{\frac{\mu}{n}}} \right)}{1 + e} \left(\frac{1 + e}{1 + e^{2\mu/n}} \right)^{\frac{e^{\mu/n} - 1}{e^{2\mu/n} - 1}}} \right) \right) \breve{\omega}(f; \delta) \\
&< \infty,
\end{aligned}$$

since

$$\lim_{n \to \infty} \left(2 - 2 \frac{\left(1 + e^{e^{\frac{\mu}{n}}} \right)}{1 + e} \left(\frac{1 + e}{1 + e^{2\mu/n}} \right)^{\frac{e^{\mu/n} - 1}{e^{2\mu/n} - 1}} \right) = 0.$$

Selecting $\delta = \mu^{-1} \sqrt{2 - 2 \frac{\left(1 + e^{e^{\frac{\mu}{n}}} \right)}{1 + e} \left(\frac{1 + e}{1 + e^{2\mu/n}} \right)^{\frac{e^{\mu/n} - 1}{e^{2\mu/n} - 1}}}$ gives the assertion. $\square$

Next, we provide some analogous Theorems for the operators $\overline{A}_n$ to those for the operators A_n without proof.

Theorem 4.6. *Let $f \in C_B^2[0, \infty)$. For every $x \in [0, \infty)$, the operators $\overline{A}_n$ satisfies the following:*

$$\lim_{n \to \infty} \overline{A}_n(f; x) = f(x).$$

Theorem 4.7. *Let $f \in C_B^2[0, \infty)$. For the operators $\overline{A}_n$,*

$$\left| n \left\{ \overline{A}_n(f;x) - f(x) - \left(a_n(x) + \frac{e}{(e+1)n} - x \right) f'(x) \right. \right.$$

$$\left. \left. - \frac{1}{2} \left((a_n(x) - x)^2 + \frac{a_n(x)}{n} + \frac{2e}{n(e+1)} \left(a_n(x) - x + \frac{1}{n} \right) \right) f''(x) \right\} \right|$$

$$\leq n \left| (a_n(x) - x)^2 + \frac{a_n(x)}{n} + \frac{2e}{n(e+1)} \left(a_n(x) - x + \frac{1}{n} \right) \right| \mathcal{K}\left(f''; |\kappa_n(x)| \right),$$

where

$$\kappa_n(x) := \frac{\overline{A}_n(\psi_x^3(t); x)}{6\overline{A}_n(\psi_x^2(t); x)} \to 0 \text{ as } n \to \infty.$$

Theorem 4.8. *Let $f \in C^*[0, \infty)$. Then for the operators $\overline{A}_n$, we have*

$$\sup_{x \geq 0} |\overline{A}_n(f;x) - f(x)| \leq 2\omega^*\left(f, \sqrt{D_n} \right),$$

where $D_n = 2 + \frac{1}{ne} + \frac{1+2e+e^2+e^3}{n^2e^2(1+e)^2} + O\left(\frac{1}{n}\right)^3$.

Proof. Along the lines of proof for the Theorem 3.3,

$$\sup_{x \geq 0} |\overline{A}_n((e^{-t} - e^{-x})^2; x)|$$

$$\leq \sup_{x \geq 0} \left| -2e^{-x} \left[e^{-x} - \frac{e^{-x}x}{2n} + \frac{e^{-x}\left(-4e + x^2 + 2ex^2 + e^2x^2\right)\left(\frac{1}{n}\right)^2}{8(1+e)^2} + O\left(\frac{1}{n}\right)^3 \right] \right|$$

$$\leq 2 + \frac{1}{ne} + \frac{1+2e+e^2+e^3}{n^2e^2(1+e)^2} + O\left(\frac{1}{n}\right)^3$$

$$= D_n.$$

Therefore,

$$\sup_{x \geq 0} |\overline{A}_n(f;x) - f(x)| \leq \left(1 + D_n \frac{1}{\delta^2} \right) \omega^*(f, \delta).$$

$\square$

5. Graphical Analysis

Lastly, we analyse the rates of convergence of the operators $A_n(f;x)$ and $\overline{A}_n(f;x)$ with the help of graphical examples.
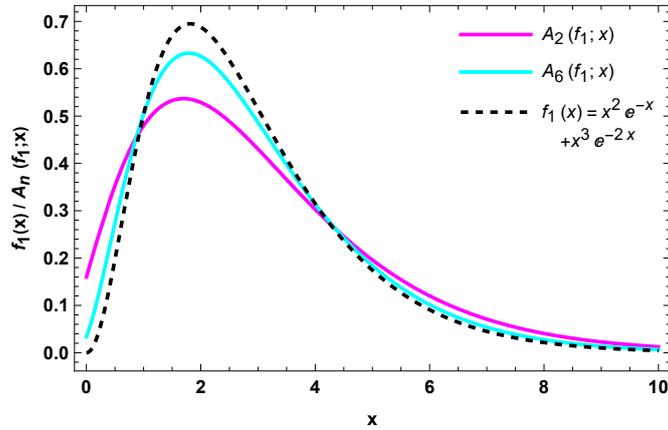Fig. 1 illustrates the approximations to the function $f_1(x) = x^3e^{-2x} + x^2e^{-x}$ by the operators A_n.

Figure 1: Graph of the function $f_1(x) = x^3 e^{-2x} + x^2 e^{-x}$ and its approximations by the operators A_n with the values $n = 2, 6$.

From Fig. 1, we observe that the rate of convergence for the operators $A_n(f_1; x)$ increases with an increase in the values of n. To validate this observation, we provide Table 1, which shows that the error of approximation decreases for higher values of n.

x	$E_{A_2 f_1}$	$E_{A_6 f_1}$
1	0.023141	0.000918
2.5	0.129484	0.047656
4.5	0.009317	0.006086
8.5	0.015716	0.004782

Table 1: The approximation-error of $A_n(f; x)$ to the function $f_1(x) = x^3 e^{-2x} + x^2 e^{-x}$ for different values of x and n.

Similarly, in Fig. 2, the approximations of the function $f_2(x) = x^2 \cos(x) + x \sin(x) + 15$ by the operators A_n are shown.
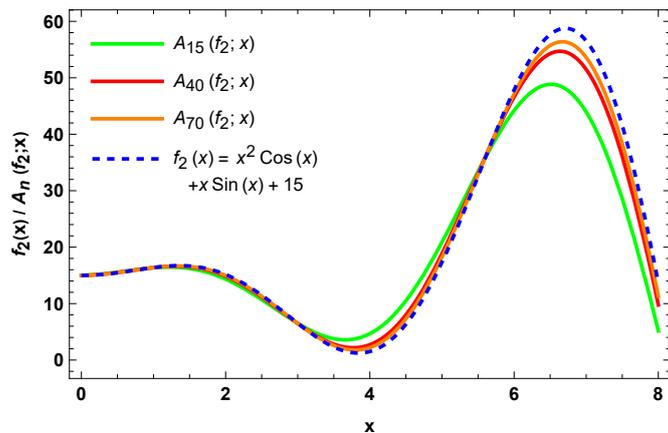

Figure 2: Graph of the function $f_2(x) = x^2 \cos(x) + x \sin(x) + 15$ and its approximations by the operators A_n with the values $n = 15, 40, 70$.

It can be seen that the operators $A_n(f_2; x)$ provide better approximation for higher values of n. Table 2 further affirms this conclusion.

x	$E_{A_{15}f_2}$	$E_{A_{40}f_2}$	$E_{A_{70}f_2}$
0.5	0.111157	0.043122	0.024799
2	0.830013	0.319071	0.183295
3.2	0.520339	0.150546	0.078981
5.5	0.855768	0.515456	0.325753

Table 2: The approximation-error of $A_n(f; x)$ to the function $f_2(x) = x^2\cos(x) + x\sin(x) + 15$ for different values of x and n.

Next, to check their preserving capability for the function $f_3(x) = e^{2x} + 2$, the approximations by the operators $\overline{A}_n$ are presented in Fig. 3.
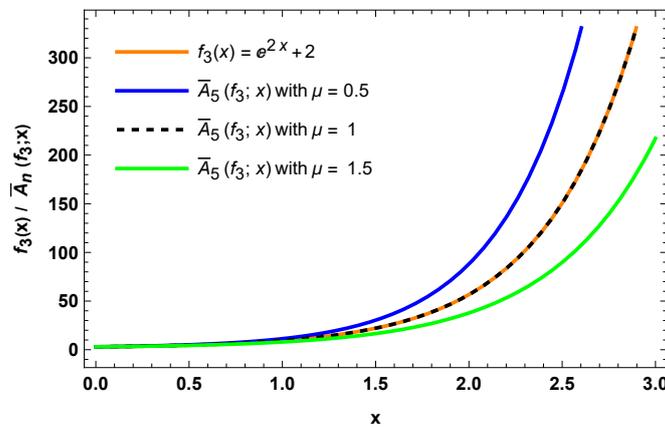

Figure 3: Graph of the function $f_3(x) = e^{2x} + 2$ and its approximations by the operators $\overline{A}_n$ with $n = 5$ and $\mu = 0.5, 1, 1.5$.

It may be noticed that for smaller values of μ (specifically close to $\mu = 1$), the operators $\overline{A}_n(f_3; x)$ provide better approximation. At $\mu = 1$, the function $f_3(x)$ and the operators $A_n(f_3; x)$ overlap together. Similar conclusion can be obtained from Table 3.

	$E_{\overline{A}_5 f_3}$		
x	$\mu = 0.5$	$\mu = 1$	$\mu = 1.5$
0.2	0.085717	0	0.077941
0.7	0.734985	0	0.586775
1.2	3.52216	0	2.51471

Table 3: The approximation-error of $\overline{A}_n(f; x)$ to the function $f_3(x) = e^{2x} + 2$ for different values of x and μ.

For particular values of $n = 20$ and $\mu = 0.05$, the approximations to the function $f_4(x) = 3\sin(4x) - e^x + x^3 + 7$ are illustrated in Fig. 4.
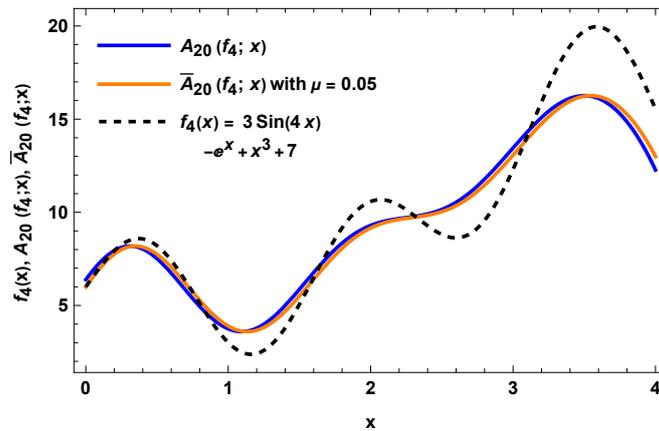
Figure 4: Graph of the function $f_4(x) = 3\sin(4x) - e^x + x^3 + 7$ and its approximations by the operators A_n and $\overline{A}_n$ with $n = 20$ and $\mu = 0.05$.

It is concluded that for specific value of μ, the operators $\overline{A}_n$ have higher rate of convergence as compared to the operators $A_n(f; x)$. Table 4 guarantees this conclusion.

x	$E_{A_{20}f_4}$	$E_{\overline{A}_{20}f_4}$
0.4	0.544655	0.43905
1.5	0.810486	0.450324
2.7	2.22101	1.96577

Table 4: The approximation-error of $A_{20}(f; x)$ and $\overline{A}_{20}(f; x)$ to the function $f_4(x) = 3\sin(4x) - e^x + x^3 + 7$ for $\mu = 0.05$ and different values of x.

6. Conclusion

We introduced linear positive operators associated with the adjoint Appell–Euler polynomials and derived their moments along with quantitative convergence estimates using the *K*-functional and various moduli of smoothness. Korovkin-type theorems were proved for test functions $\{1, e^{-x}, e^{-2x}\}$ and $\left\{1, \frac{x}{1+x}, \left(\frac{1}{1+x}\right)^2\right\}$. We also constructed modified operators preserving constants and $e^{2\mu x}$, $\mu > 0$, and studied their asymptotic behavior and convergence properties. For future investigations, it would be natural to construct and analyze similar operators associated with a broader range of special function families.

Acknowledgement

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