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Milne-Mercer type inequalities for h-convex functions

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Abstract. In this study, Milne-Mercer type inequalities for h-convex functions involving the generalized fractional integral operators are established. In addition, new results are presented that generalized various inequalities known in the literature.

1. Introduction

The analysis of fractional calculation is a generalization of classical analysis, and it progressed quickly with the fascinating concept of convexity. Its numerous applications in functional analysis, stochastic theory, and optimization theory have made it a highly desirable research topic. The author of [12] introduces a novel class of functions called h-convex functions.

Definition 1.1. *Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, if f is non-negative and for all $x, y \in I$, $\lambda \in (0, 1)$ we have*

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y). \tag{1}$$

If the inequality (1) is reversed, then f is said to be h-concave.

By setting

1. $h(\lambda) = \lambda$, Definition 1.1 reduces to convex function [9].
2. $h(\lambda) = 1$, Definition 1.1 reduces to P-functions [5, 10].
3. $h(\lambda) = \lambda^s$, Definition 1.1 reduces to s-convex functions [4].
4. $h(\lambda) = \dfrac{1}{n}\sum_{k=1}^{n} \lambda^{\frac{1}{k}}$, Definition 1.1 reduces to polynomial n-fractional convex functions [6].

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In [8], the famous Jensen-Mercer inequality was presented as follows: If f is a convex function on $[a, b]$, then

$$f\left(a+b-\sum_{j=1}^{n}\lambda_j z_j\right)\leq f(a)+f(b)-\sum_{j=1}^{n}\lambda_j f(z_j),$$

for each $z_j \in [a, b]$ and $\lambda_j \in [0, 1]$ $(j = \overline{1;n})$ with $\sum\limits_{j=1}^{n} \lambda_j = 1$.

In [1], the authors presented the following interesting result (Lemma 4.1).

Lemma 1.2. *Let f be an h-convex function. Then for every $z \in [a, b]$, there exists $\lambda \in [0, 1]$ such that*

$$f(a+b-z)\leq[h(\lambda)+h(1-\lambda)]\,[f(a)+f(b)]-f(z). \tag{2}$$

The left- and right-sided generalized fractional integral operators of order $\alpha > 0$ and $\rho \in (0, 1]$ are expressed as follows [7]:

$$^{\rho}\Im^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\frac{(x-a)^{\rho}-(t-a)^{\rho}}{\rho}\right)^{\alpha-1}(t-a)^{\rho-1}f(t)dt,\quad x>a,$$

$$^{\rho}\Im^{\alpha}_{b^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-t)^{\rho-1}f(t)dt,\quad x<b.$$

For $\rho = 1$, the previous operators are reduced to Riemann-Liouville fractional operators with order $\alpha > 0$ as follows:

$$\Im^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\quad x>a,$$

$$^{\rho}\Im^{\alpha}_{b^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\quad x<b.$$

The beta-Euler function $\beta(.,.)$ is defined for any $p, q > 0$ as follows:

$$\beta(q,p)=\int_0^1(1-y)^{q-1}\,y^{p-1}\,dy.$$

In 2022, Djenaoui and Meftah presented a Milne inequality for convex functions with Riemann integral as follows [3, Corollary 2.4.].

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{5(b-a)}{24}\left(|f'(a)|+|f'(b)|\right). \tag{3}$$

In 2023, Budak et al. established a fractional Milne type inequality for convex functions using Riemann-Liouville integral operators [2, Theorem 1.].

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im^{\alpha}_{\frac{a+b}{2}^+}f(b)+\Im^{\alpha}_{\frac{a+b}{2}^-}f(a)\right]\right|\\&\leq\frac{4\alpha+1}{12(\alpha+1)}(b-a)\left(|f'(a)|+|f'(b)|\right).\end{aligned} \tag{4}$$

Based on earlier research, we developed an additional version of Milne-Mercer inequality for h-convex functions using the generalized fractional integral operators.

2. Milne-Mercer type inequalities

Lemma 2.1. *Let $\alpha > 0$, $\rho \in (0,1]$ and $x, y \in [a,b]$ where $x < y$. If $f : [x,y] \to \mathbb{R}$ is a differentiable mapping such that $f' \in L_1([x,y])$, then the following identity holds.*

$$\begin{aligned}
&\frac{1}{3}\left[2f(a+b-y) - f\left(a+b-\frac{x+y}{2}\right) + 2f(a+b-x)\right] \\
&-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y) + {}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x)\right] \\
&= \frac{y-x}{4}\int_0^1\left(\frac{4}{3} - (1-t^{\rho})^{\alpha}\right)\left\{f'\left((1-t)\left(a+b-\frac{x+y}{2}\right) + t(a+b-x)\right)\right. \\
&\left.- f'\left((1-t)\left(a+b-\frac{x+y}{2}\right) + t(a+b-y)\right)\right\}dt.
\end{aligned} \tag{5}$$

Proof. First, the following integral calculus is required. Let $x, y \in [a,b]$ where $x < y$ and putting $\tau = t(a+b-y) + (1-t)(a+b-\frac{x+y}{2})$, we get

$$\begin{aligned}
&\int_0^1 (1-t^{\rho})^{\alpha-1}t^{\rho-1}f\left(t(a+b-y) + (1-t)\left(a+b-\frac{x+y}{2}\right)\right)dt \\
&= \left(\frac{2}{y-x}\right)^{\rho\alpha}\int_{a+b-y}^{a+b-\frac{x+y}{2}}\left[\left(\left(a+b-\frac{x+y}{2}\right) - (a+b-y)\right)^{\rho} - \left(\left(a+b-\frac{x+y}{2}\right) - \tau\right)^{\rho}\right]^{\alpha-1} \\
&\times\left(\left(a+b-\tfrac{x+y}{2}\right) - \tau\right)^{\rho-1}f(\tau)d\tau \\
&= \left(\frac{2}{y-x}\right)^{\rho\alpha}\rho^{\alpha-1}\Gamma(\alpha)\ {}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y).
\end{aligned}$$

Taking $\tau = (1-t)(a+b-\frac{x+y}{2}) + t(a+b-x)$, we get

$$\begin{aligned}
&\int_0^1 (1-t^{\rho})^{\alpha-1}t^{\rho-1}f\left((1-t)(a+b-\frac{x+y}{2}) + t(a+b-x)\right)dt \\
&= \left(\frac{2}{y-x}\right)^{\rho\alpha}\int_{a+b-\frac{x+y}{2}}^{a+b-x}\left[\left((a+b-x) - \left(a+b-\frac{x+y}{2}\right)\right)^{\rho} - \left(\tau - \left(a+b-\frac{x+y}{2}\right)\right)^{\rho}\right]^{\alpha-1} \\
&\times\left(\tau - \left(a+b-\tfrac{x+y}{2}\right)\right)^{\rho-1}f(\tau)d\tau \\
&= \left(\frac{2}{y-x}\right)^{\rho\alpha}\rho^{\alpha-1}\Gamma(\alpha)\ {}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x).
\end{aligned}$$

By utilizing the integration by parts method, we can obtain the following expression:

$$
\begin{aligned}
I_1 &= \int_0^1 \left((1-t^\rho)^\alpha - \frac{4}{3}t\right) f'\left(t(a+b-y) + (1-t)\left(a+b-\frac{x+y}{2}\right)\right)dt \\
&= -\left(\frac{2}{y-x}\right)\left((1-t^\rho)^\alpha - \frac{4}{3}t\right) f\left(t(a+b-y) + (1-t)\left(a+b-\frac{x+y}{2}\right)\right)\Big|_0^1 \\
&-\left(\frac{2\,\alpha\rho}{y-x}\right)\int_0^1 (1-t^\rho)^{\alpha-1}\, t^{\rho-1} f\left(t(a+b-y) + (1-t)\left(a+b-\frac{x+y}{2}\right)\right)dt \\
&= \left(\frac{2}{y-x}\right)\left[\frac{4}{3}f(a+b-y) - \frac{1}{3}f\left(a+b-\frac{x+y}{2}\right)\right] \\
&-\left(\frac{2}{y-x}\right)^{\rho\alpha+1} \rho^\alpha\Gamma(\alpha+1)\ {}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^-} f(a+b-y).
\end{aligned}
$$

Similarly to the following integral,

$$
\begin{aligned}
I_2 &= \int_0^1 \left((1-t^\rho)^\alpha - \frac{4}{3}t\right) f'\left((1-t)\left(a+b-\frac{x+y}{2}\right) + t(a+b-x)\right)dt \\
&= \left(\frac{2}{y-x}\right)\left((1-t^\rho)^\alpha - \frac{4}{3}t\right) f\left((1-t)\left(a+b-\frac{x+y}{2}\right) + t(a+b-x)\right)\Big|_0^1 \\
&+\left(\frac{2\,\alpha\rho}{y-x}\right)\int_0^1 (1-t^\rho)^{\alpha-1}\, t^{\rho-1} f\left((1-t)\left(a+b-\frac{x+y}{2}\right) + t(a+b-x)\right)dt \\
&= -\left(\frac{2}{y-x}\right)\left[\frac{4}{3}f(a+b-x) - \frac{1}{3}f\left(a+b-\frac{x+y}{2}\right)\right] \\
&+\left(\frac{2}{y-x}\right)^{\rho\alpha+1} \rho^\alpha\Gamma(\alpha+1)\ {}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^+} f(a+b-x).
\end{aligned}
$$

As a result, the following equality is valid:

$$
\begin{aligned}
\frac{y-x}{4}(I_1 - I_2) = \ & \frac{1}{3}\left[2\,f(a+b-y) - f\left(a+b-\frac{x+y}{2}\right) + 2\,f(a+b-x)\right] \\
& -\frac{2^{\rho\alpha-1}\,\rho^\alpha\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^-} f(a+b-y) + {}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^+} f(a+b-x)\right].
\end{aligned}
$$

So the proof is complete. □

We present the first result for Milne-Mercer inequality with the generalized fractional integral operators.

Theorem 2.2. *Assume that the assumptions of Lemma 5 hold. If $|f'|$ is a h-convex mapping on $[a,b]$, then the* *following Milne-Mercer type inequality for the generalized fractional integral operators holds*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]\\
&-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x)\right]\Bigg|\\
&\leq\frac{y-x}{4}\left[\left|f'(a+b-y)\right|+\left|f'(a+b-x)\right|\right]\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)\left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\right]dt \qquad (6)\\
&\leq\frac{y-x}{4}\left\{2\left[h(\lambda)+h(1-\lambda)\right]\left[\left|f'(a)\right|+\left|f'(b)\right|\right]-\left[\left|f'(x)\right|+\left|f'(y)\right|\right]\right\}\\
&\times\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)\left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\right]dt.
\end{aligned}$$

Proof. Using the absolute value of identity (5) and the h-convexity of the function $|f'|$, we deduce

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]\\
&-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x)\right]\Bigg|\\
&\leq\frac{y-x}{4}\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)\\
&\times\left[\left|f'\left((1-t)\left(a+b-\frac{x+y}{2}\right)+t(a+b-x)\right)\right|+\left|f'\left((1-t)\left(a+b-\frac{x+y}{2}\right)+t(a+b-y)\right)\right|\right]dt\\
&\leq\frac{y-x}{4}\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)\left(2h(1-t)\left|f'\left(a+b-\frac{x+y}{2}\right)\right|+h(t)\left[\left|f'(a+b-y)\right|+\left|f'(a+b-x)\right|\right]\right)dt.
\end{aligned}$$

Utilizing h-convexity of $|f'|$, we get

$$\begin{aligned}
\left|f'\left(a+b-\frac{x+y}{2}\right)\right| &=\left|f'\left(\frac{a+b-y}{2}+\frac{a+b-x}{2}\right)\right|\\
&\leq h\left(\frac{1}{2}\right)\left[\left|f'(a+b-y)\right|+\left|f'(a+b-x)\right|\right],
\end{aligned}$$

therefore

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2f(a+b+y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]\\
&-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x)\right]\Bigg|\\
&\leq\frac{y-x}{4}\left[\left|f'(a+b-y)\right|+\left|f'(a+b-x)\right|\right]\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)\left(2h(1-t)h\left(\frac{1}{2}\right)+h(t)\right)dt.
\end{aligned}$$

This accomplishes the first inequality in (6). Applying (2) yields to the second inequality in (6). □

To prove all of the results, we need to use the following lemma:

Lemma 2.3. *Let* $t \in (0,1), \quad s \in (0,1]$*. The following inequalities hold:*

$$(1+s)^{\frac{1}{s}} \geq 2\,, \tag{7}$$

$$t^s + (1-t)^s \leq 2^{1-s}\,, \tag{8}$$

and

$$t^s + 2^{1-s}(1-t)^s \leq \left(\frac{1+2^{1-s}}{2^s}\right). \tag{9}$$

Proof. • Let $\Phi(x) = \ln(x+1)$, where $\Phi(0) = 0$, $\phi(1) = \ln 2$ and $A(1, \ln 2)$. On the interval $[0,1]$, the graph of the function Φ appears over the line (O, A). It therefore provides us

$$\text{for all } s \in (0,1], \quad \ln(s+1) \geq s \ln 2 \Leftrightarrow (1+s)^{\frac{1}{s}} \geq 2.$$

• We use absurdity to demonstrate inequality (8), suppose that exist $s \in (0,1]$ verified

$$t^s + (1-t)^s > \left(\frac{1}{2}\right)^{s-1},$$

we obtain

$$\int_0^1 [t^s + (1-t)^s]\, dt > \left(\frac{1}{2}\right)^{s-1},$$

then

$$2\left(\frac{1}{s+1}\right) > \left(\frac{1}{2}\right)^{s-1} \Leftrightarrow s+1 < 2^s,$$

therefore

$$(1+s)^{\frac{1}{s}} < 2 \text{ which is absurd with } (7)\,.$$

• For $t \in (0,1), \quad s \in (0,1]$, by using $\mu = 1-t$, we get

$$J = t^s + 2^{1-s}(1-t)^s = (1-t)^s + 2^{1-s} t^s,$$

then

$$2J = (1+2^{1-s})\,[t^s + (1-t)^s].$$

Applying (8) yields

$$J \leq (1+2^{1-s})\, 2^{-s}.$$

□

Next, consider some particular cases of Theorem 2.2 with h-convexity involving the generalized fractional integral operators.

1. Given $h(t) = t^s$ with $s \in (0, 1]$ in Theorem 2.2 and apply (9), we get

$$\begin{aligned}
I_1 &= \int_0^1 \left(\frac{4}{3} - (1-t^\rho)^\alpha\right)\left[h(t) + 2h\left(\frac{1}{2}\right)h(1-t)\right]dt \\
&= \int_0^1 \left(\frac{4}{3} - (1-t^\rho)^\alpha\right)\left[t^s + 2^{1-s}(1-t)^s\right]dt \\
&\leq \left(\frac{1+2^{1-s}}{2^s}\right)\int_0^1 \left(\frac{4}{3} - (1-t^\rho)^\alpha\right)dt \\
&= \left(\frac{1+2^{1-s}}{2^s}\right)\left(\frac{4}{3} - \frac{1}{\rho}\beta\left(\alpha+1, \frac{1}{\rho}\right)\right),
\end{aligned}$$

where we used

$$\int_0^1 (1-t^\rho)^\alpha \, dt = \frac{1}{\rho}\int_0^1 (1-t)^\alpha \, t^{\frac{1}{\rho}-1}\, dt = \frac{1}{\rho}\beta\left(\alpha+1, \frac{1}{\rho}\right).$$

By using (8), we get

$$h(\lambda) + h(1-\lambda) \leq 2^{1-s}.$$

Yields the next Corollary.

Corollary 2.4. *Assume α, ρ and f are defined according to Theorem 2.2. If $|f'|$ is a s-convex function on $[a, b]$, then*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a+b-y) - f\left(a+b-\frac{x+y}{2}\right) + 2f(a+b-x)\right]\right. \\
&\left.-\frac{2^{\rho\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^\rho\mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y) + {}^\rho\mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x)\right]\right| \\
&\leq \frac{y-x}{4}\left(\frac{1+2^{1-s}}{2^s}\right)\left(\frac{4}{3} - \frac{1}{\rho}\beta\left(\alpha+1, \frac{1}{\rho}\right)\right)\left[\left|f'(a+b-y)\right| + \left|f'(a+b-x)\right|\right] \\
&\leq \frac{y-x}{4}\left(\frac{1+2^{1-s}}{2^s}\right)\left(\frac{4}{3} - \frac{1}{\rho}\beta\left(\alpha+1, \frac{1}{\rho}\right)\right)\left\{2^{2-s}[\left|f'(a)\right| + \left|f'(b)\right|] - \left[\left|f'(x)\right| + \left|f'(y)\right|\right]\right\}.
\end{aligned} \tag{10}$$

Remark 2.5.
- *Corollary 2.4 is a generalization of Theorem 3 in [11], simply by setting $s = 1$, $x = a$ and $y = b$.*
- *Taking $\rho = 1$ in inequality (10), we get Milne-Mercer inequality via Riemann-Liouville operators for s-convex function.*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a+b-y) - f\left(a+b-\frac{x+y}{2}\right) + 2f(a+b-x)\right]\right. \\
&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y) + \mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x)\right]\right| \\
&\leq \frac{y-x}{4}\left(\frac{1+2^{1-s}}{2^s}\right)\left(\frac{4\alpha+1}{3(\alpha+1)}\right)\left[\left|f'(a+b-y)\right| + \left|f'(a+b-x)\right|\right] \\
&\leq \frac{y-x}{4}\left(\frac{1+2^{1-s}}{2^s}\right)\left(\frac{4\alpha+1}{3(\alpha+1)}\right)\left\{2^{2-s}[\left|f'(a)\right| + \left|f'(b)\right|] - \left[\left|f'(x)\right| + \left|f'(y)\right|\right]\right\}.
\end{aligned} \tag{11}$$

The above inequality (11) is a generalization (4).

- *Putting $\rho = 1$ and $\alpha = 1$ in inequality (10), we get Milne-Mercer inequality via Riemann integral for s-convex function.*

$$\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right|$$
$$\leq \frac{5(y-x)}{24}\left(\frac{1+2^{1-s}}{2^s}\right)\left[\left|f'(a+b-y)\right|+\left|f'(a+b-x)\right|\right] \tag{12}$$
$$\leq \frac{5(y-x)}{24}\left(\frac{1+2^{1-s}}{2^s}\right)\left\{2^{2-s}[\left|f'(a)\right|+\left|f'(b)\right|]-\left[\left|f'(x)\right|+\left|f'(y)\right|\right]\right\}.$$

The above inequality (12) is a generalization (3).

2. Setting $h(\lambda) = 1$ in Theorem 2.2 gives the following new result about the class P-function. Consider $s \to 0^+$ in the inequalities (10), (11) and (12).

Corollary 2.6. *Assume α and f are defined according to Theorem 2.2. If $|f'|$ is a P-function on $[a, b]$, then*

$$\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^\rho\mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y)+{}^\rho\mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x)\right]\right|$$
$$\leq \frac{y-x}{4}\left(4-\frac{3}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)\right)\left[\left|f'(a+b-y)\right|+\left|f'(a+b-x)\right|\right] \tag{13}$$
$$\leq \frac{y-x}{4}\left(4-\frac{3}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)\right)\left\{4[\left|f'(a)\right|+\left|f'(b)\right|]-\left[\left|f'(x)\right|+\left|f'(y)\right|\right]\right\}.$$

Remark 2.7. *Taking $\rho = 1$, we derive the following Milne-Mercer inequality via Riemann-Liouville operators, where f is a P-function.*

$$\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y)+\mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x)\right]\right|$$
$$\leq \frac{y-x}{4}\left(\frac{4\alpha+1}{\alpha+1}\right)\left[\left|f'(a+b-y)\right|+\left|f'(a+b-x)\right|\right] \tag{14}$$
$$\leq \frac{y-x}{4}\left(\frac{4\alpha+1}{\alpha+1}\right)\left\{4[\left|f'(a)\right|+\left|f'(b)\right|]-\left[\left|f'(x)\right|+\left|f'(y)\right|\right]\right\}.$$

Remark 2.8. *By setting $\rho = 1$ and $\alpha = 1$, we use the Riemann integral to derive the following Milne-Mercer* *inequality for the class P function.*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right| \\
&\leq \frac{5(y-x)}{8}\left[\left|f'(a+b-y)\right|+\left|f'(a+b-x)\right|\right] \qquad (15)\\
&\leq \frac{5(y-x)}{8}\left\{4[\left|f'(a)\right|+\left|f'(b)\right|]-\left[\left|f'(x)\right|+\left|f'(y)\right|\right]\right\}.
\end{aligned}$$

3. Setting $h(\lambda) = \frac{1}{n}\sum_{k=1}^{n}\lambda^{\frac{1}{k}}$. Given that $\lambda^{\frac{1}{k}} = \lambda^{s}$ where $s = \frac{1}{k}$ and the properties of the integral with the sum, applying (10) yields the following new result for the class n-fractional polynomial convex function.

Corollary 2.9. *Assume α and f are defined according to Theorem 2.2. If $|f'|$ is a n-fractional polynomial convex mapping on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]\right.\\
&\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x)\right]\right|\\
&\leq \frac{y-x}{4n}\sum_{k=1}^{n}\left(\frac{1+2^{1-\frac{1}{k}}}{2^{\frac{1}{k}}}\right)\left(\frac{4}{3}-\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)\right)\left[\left|f'(a+b-y)\right|+\left|f'(a+b-x)\right|\right] \qquad (16)\\
&\leq \frac{y-x}{4n}\sum_{k=1}^{n}\left(\frac{1+2^{1-\frac{1}{k}}}{2^{\frac{1}{k}}}\right)\left(\frac{4}{3}-\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)\right)\left\{\frac{1}{n}\sum_{k=1}^{n}\left(2^{2-\frac{1}{k}}\right)[\left|f'(a)\right|+\left|f'(b)\right|]-\left[\left|f'(x)\right|+\left|f'(y)\right|\right]\right\}.
\end{aligned}$$

Remark 2.10.

- *Corollary 2.9 is a second generalization of Theorem 3 in [11], simply by setting $n = 1$, $x = a$ and $y = b$.*
- *Taking $\rho = 1$ in inequality (16), we get Milne-Mercer inequality via Riemann-Liouville operators for s-convex function.*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]\right.\\
&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)+\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x)\right]\right|\\
&\leq \frac{y-x}{4n}\sum_{k=1}^{n}\left(\frac{1+2^{1-\frac{1}{k}}}{2^{\frac{1}{k}}}\right)\left(\frac{4\alpha+1}{3(\alpha+1)}\right)\left[\left|f'(a+b-y)\right|+\left|f'(a+b-x)\right|\right] \qquad (17)\\
&\leq \frac{y-x}{4n}\sum_{k=1}^{n}\left(\frac{1+2^{1-\frac{1}{k}}}{2^{\frac{1}{k}}}\right)\left(\frac{4\alpha+1}{3(\alpha+1)}\right)\left\{\frac{1}{n}\sum_{k=1}^{n}\left(2^{2-\frac{1}{k}}\right)[\left|f'(a)\right|+\left|f'(b)\right|]-\left[\left|f'(x)\right|+\left|f'(y)\right|\right]\right\}.
\end{aligned}$$

The above inequality (17) is the second generalization of inequality (4).

- *Putting $\rho = 1$ and $\alpha = 1$ in inequality (16), we get Milne-Mercer inequality via Riemann integral for s-convex function.*

$$\left|\frac{1}{3}\left[2f(a+b-y) - f\left(a+b-\frac{x+y}{2}\right) + 2f(a+b-x)\right] - \frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(t)dt\right|$$
$$\leq \frac{5(b-a)}{24n}\sum_{k=1}^{n}\left(\frac{1+2^{1-\frac{1}{k}}}{2^{\frac{1}{k}}}\right)\left[\left|f'(a+b-y)\right| + \left|f'(a+b-x)\right|\right] \tag{18}$$
$$\leq \frac{5(b-a)}{24n}\sum_{k=1}^{n}\left(\frac{1+2^{1-\frac{1}{k}}}{2^{\frac{1}{k}}}\right)\left\{\frac{1}{n}\sum_{k=1}^{n}\left(2^{2-\frac{1}{k}}\right)[\left|f'(a)\right| + \left|f'(b)\right|] - \left[\left|f'(x)\right| + \left|f'(y)\right|\right]\right\}.$$

The above inequality (18) is the second generalization of inequality (3).

Now, we present the second result.

Theorem 2.11. *Let $p > 1$, $\frac{1}{p'} + \frac{1}{p} = 1$ and assume that α, ρ, f are defined as in Lemma 2.1. If $\left|f'\right|^p$ is a h-convex mapping on $[a,b]$, the following Milne-Mercer type inequality holds*

$$\left|\frac{1}{3}\left[2f(a+b-y) - f\left(a+b-\frac{x+y}{2}\right) + 2f(a+b-x)\right]\right.$$
$$\left. -\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y) + {}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x)\right]\right|$$
$$\leq \frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\right]dt\right)^{\frac{1}{p}}$$
$$\times\left[\left|f'(a+b-y)\right|^p + \left|f'(a+b-x)\right|^p\right]^{\frac{1}{p}} \tag{19}$$
$$\leq \frac{b-a}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\right]dt\right)^{\frac{1}{p}}$$
$$\times\left\{2\left[h(\lambda)+h(1-\lambda)\right]\left[\left|f'(a)\right|^p + \left|f'(b)\right|^p\right] - \left[\left|f'(x)\right|^p + \left|f'(y)\right|^p\right]\right\}^{\frac{1}{p}}.$$

Proof. Using the absolute value of identity (5), we get

$$\left|\frac{1}{3}\left[2f(a+b-y) - f\left(a+b-\frac{x+y}{2}\right) + 2f(a+b-x)\right]\right.$$
$$\left. -\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y) + {}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x)\right]\right|$$
$$\leq \frac{y-x}{4}\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)\left|f'\left((1-t)\left(a+b-\frac{x+y}{2}\right)+t\,(a+b-x)\right)\right|dt$$
$$+\frac{y-x}{4}\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)\left|f'\left((1-t)\left(a+b-\frac{x+y}{2}\right)+t\,(a+b-y)\right)\right|dt.$$

Since $A^{\frac{1}{p}}+B^{\frac{1}{p}}=2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$, by using Hölder inequality we deduce

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]\right.\\
&\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x)\right]\right|\\
&\le\frac{b-a}{4}\left(\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left|f'\left((1-t)\left(a+b-\frac{x+y}{2}\right)+t\,(a+b-x)\right)\right|^{p}dt\right)^{\frac{1}{p}}\\
&+\frac{y-x}{4}\left(\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left|f'\left((1-t)\left(a+b-\frac{x+y}{2}\right)+t\,(a+b-y)\right)\right|^{p}dt\right)^{\frac{1}{p}}\\
&\le\frac{y-x}{4}\left(\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}2^{1-\frac{1}{p}}\left\{\int_0^1\left|f'\left((1-t)\left(a+b-\frac{x+y}{2}\right)+t\,(a+b-x)\right)\right|^{p}dt\right.\\
&\left.+\int_0^1\left|f'\left((1-t)\left(a+b-\frac{x+y}{2}\right)+t\,(a+b-y)\right)\right|^{p}dt\right\}^{\frac{1}{p}}.
\end{aligned}$$

Given that $|f'|^p$ is a h-convex function, we result

$$\begin{aligned}
\left|f'\left(a+b-\frac{x+y}{2}\right)\right|^{p}&=\left|f'\left(\frac{a+b-y}{2}+\frac{a+b-x}{2}\right)\right|^{p}\\
&\le h\left(\frac{1}{2}\right)\left[\left|f'(a+b-y)\right|^{p}+\left|f'(a+b-x)\right|^{p}\right],
\end{aligned}$$

hence

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]\right.\\
&\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x)\right]\right|\\
&\le\frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left[\int_0^1\left(h(1-t)\left|f'\left(a+b-\frac{x+y}{2}\right)\right|^{p}+h(t)\left|f'(a+b-x)\right|^{p}\right)dt\right.\\
&\left.+\int_0^1\left(h(1-t)\left|f'\left(a+b-\frac{x+y}{2}\right)\right|^{p}+h(t)\left|f'(a+b-y)\right|^{p}\right)dt\right]^{\frac{1}{p}}\\
&\le\frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\right]dt\right)^{\frac{1}{p}}\\
&\times\left[\left|f'(a+b-y)\right|^{p}+\left|f'(a+b-x)\right|^{p}\right]^{\frac{1}{p}}.
\end{aligned}$$

This accomplishes the first inequality in (19). Apply inequality (2), we get

$$\left|f'(a+b-y)\right|^{p}+\left|f'(a+b-x)\right|^{p}\le2\left[h(\lambda)+h(1-\lambda)\right]\left[\left|f'(a)\right|^{p}+\left|f'(b)\right|^{p}\right]-\left[\left|f'(x)\right|^{p}+\left|f'(y)\right|^{p}\right],$$

which yields to the second inequality in (19). □

We propose certain particular Milne-Mercer inequalities for h-convex functions.

1. Given $h(\lambda) = \lambda^s$ with $s \in (0, 1]$ in Theorem 2.11 and using (9), we obtain

$$\begin{aligned} I_1 &= \int_0^1 \left[h(t) + 2h\left(\frac{1}{2}\right)h(1-t)\right]dt \\ &= \int_0^1 \left[t^s + 2^{1-s}(1-t)^s\right]dt \\ &\leq \left(\frac{1+2^{1-s}}{2^s}\right). \end{aligned}$$

Corollary 2.12. *Assume α, ρ and f are defined according to Theorem 2.11. If $|f'|^p$ is a s-convex function on $[a, b]$, then*

$$\begin{aligned} &\Bigg|\frac{1}{3}\left[2f(a+b-y) - f\left(a+b-\frac{x+y}{2}\right) + 2f(a+b-x)\right] \\ &-\frac{2^{\rho\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^\rho\mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y) + {}^\rho\mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x)\right]\Bigg| \\ &\leq \frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^\rho)^\alpha\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\frac{1+2^{1-s}}{2^s}\right)^{\frac{1}{p}}\left[\left|f'(a+b-y)\right|^p + \left|f'(a+b-x)\right|^p\right]^{\frac{1}{p}} \\ &\leq \frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^\rho)^\alpha\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\frac{1+2^{1-s}}{2^s}\right)^{\frac{1}{p}}\left\{2^{2-s}\left[\left|f'(a)\right|^p + \left|f'(b)\right|^p\right] - \left[\left|f'(x)\right|^p + \left|f'(y)\right|^p\right]\right\}^{\frac{1}{p}}. \end{aligned} \tag{20}$$

Remark 2.13.
- *With $s = 1$, $x = a$ and $y = b$, Corollary 2.12 improves Theorem 4 from [11].*
- *Taking $\rho = 1$ in inequality (20), we get Milne-Mercer inequality via Riemann-Liouville operators for s-convex function.*

$$\begin{aligned} &\Bigg|\frac{1}{3}\left[2f(a+b-y) - f\left(a+b-\frac{x+y}{2}\right) + 2f(a+b-x)\right] \\ &-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y) + \mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x)\right]\Bigg| \\ &\leq \frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t)^\alpha\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\frac{1+2^{1-s}}{2^s}\right)^{\frac{1}{p}}\left[\left|f'(a+b-y)\right|^p + \left|f'(a+b-x)\right|^p\right]^{\frac{1}{p}} \\ &\leq \frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t)^\alpha\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\frac{1+2^{1-s}}{2^s}\right)^{\frac{1}{p}}\left\{2^{2-s}\left[\left|f'(a)\right|^p + \left|f'(b)\right|^p\right] - \left[\left|f'(x)\right|^p + \left|f'(y)\right|^p\right]\right\}^{\frac{1}{p}}. \end{aligned} \tag{21}$$

- *Putting $\alpha = 1$ in inequality (21), we get Milne-Mercer inequality via Riemann integral for s-convex* *function.*

$$\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right|$$
$$\leq \frac{y-x}{4}\left[\frac{2}{1+p'}\left(\left(\frac{4}{3}\right)^{1+p'}-\left(\frac{1}{3}\right)^{1+p'}\right)\right]^{\frac{1}{p'}}\left(\frac{1+2^{1-s}}{2^s}\right)^{\frac{1}{p}}\left[\left|f'(a+b-y)\right|^p+\left|f'(a+b-x)\right|^p\right]^{\frac{1}{p}}$$
$$\leq \frac{y-x}{4}\left[\frac{2}{1+p'}\left(\left(\frac{4}{3}\right)^{1+p'}-\left(\frac{1}{3}\right)^{1+p'}\right)\right]^{\frac{1}{p'}}\left(\frac{1+2^{1-s}}{2^s}\right)^{\frac{1}{p}}\left\{2^{2-s}\left[\left|f'(a)\right|^p+\left|f'(b)\right|^p\right]-\left[\left|f'(x)\right|^p+\left|f'(y)\right|^p\right]\right\}^{\frac{1}{p}}. \tag{22}$$

- *Putting $s=1$ in inequality (22), we get Milne-Mercer inequality via Riemann integral for convex function.*

$$\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right|$$
$$\leq \frac{y-x}{4}\left[\frac{2}{1+p'}\left(\left(\frac{4}{3}\right)^{1+p'}-\left(\frac{1}{3}\right)^{1+p'}\right)\right]^{\frac{1}{p'}}\left[\left|f'(a+b-y)\right|^p+\left|f'(a+b-x)\right|^p\right]^{\frac{1}{p}}$$
$$\leq \frac{y-x}{4}\left[\frac{2}{1+p'}\left(\left(\frac{4}{3}\right)^{1+p'}-\left(\frac{1}{3}\right)^{1+p'}\right)\right]^{\frac{1}{p'}}\left\{2\left[\left|f'(a)\right|^p+\left|f'(b)\right|^p\right]-\left[\left|f'(x)\right|^p+\left|f'(y)\right|^p\right]\right\}^{\frac{1}{p}}.$$

2. Setting $h(\lambda)=1$ in Theorem 2.11 gives the following new result about the class P-function. Consider $s\to 0^+$ in the inequalities (20), (21) and (22).

Corollary 2.14. *Assume α, ρ and f are defined according to Theorem 2.11. If $|f'|^p$ is a P-function on $[a,b]$, then*

$$\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^\rho\mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^-}f(a+b-y)+{}^\rho\mathfrak{I}^\alpha_{\left(a+b-\frac{x+y}{2}\right)^+}f(a+b-x)\right]\right|$$
$$\leq \frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^\rho)^\alpha\right)^{p'}dt\right)^{\frac{1}{p'}}3^{\frac{1}{p}}\left[\left|f'(a+b-y)\right|^p+\left|f'(a+b-x)\right|^p\right]^{\frac{1}{p}} \tag{23}$$
$$\leq \frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^\rho)^\alpha\right)^{p'}dt\right)^{\frac{1}{p'}}3^{\frac{1}{p}}\left\{4\left[\left|f'(a)\right|^p+\left|f'(b)\right|^p\right]-\left[\left|f'(x)\right|^p+\left|f'(y)\right|^p\right]\right\}^{\frac{1}{p}}.$$

Remark 2.15. *Taking $\rho=1$, we derive the following Milne-Mercer inequality via Riemann-Liouville operators,* *where f is a P-function.*

$$\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\Im^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)+\Im^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x)\right]\right|$$
$$\leq\frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t)^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}3^{\frac{1}{p}}\left[\left|f'(a+b-y)\right|^{p}+\left|f'(a+b-x)\right|^{p}\right]^{\frac{1}{p}} \tag{24}$$
$$\leq\frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t)^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}3^{\frac{1}{p}}\left\{4\left[\left|f'(a)\right|^{p}+\left|f'(b)\right|^{p}\right]-\left[\left|f'(x)\right|^{p}+\left|f'(y)\right|^{p}\right]\right\}^{\frac{1}{p}}.$$

Remark 2.16. *By setting $\rho=1$ and $\alpha=1$, we use the Riemann integral to derive the following Milne-Mercer inequality for the class P-function.*

$$\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right|$$
$$\leq\frac{y-x}{4}\left[\frac{2}{1+p'}\left(\left(\frac{4}{3}\right)^{1+p'}-\left(\frac{1}{3}\right)^{1+p'}\right)\right]^{\frac{1}{p'}}3^{\frac{1}{p}}\left[\left|f'(a+b-y)\right|^{p}+\left|f'(a+b-x)\right|^{p}\right]^{\frac{1}{p}} \tag{25}$$
$$\leq\frac{y-x}{4}\left[\frac{2}{1+p'}\left(\left(\frac{4}{3}\right)^{1+p'}-\left(\frac{1}{3}\right)^{1+p'}\right)\right]^{\frac{1}{p'}}3^{\frac{1}{p}}\left\{4\left[\left|f'(a)\right|^{p}+\left|f'(b)\right|^{p}\right]-\left[\left|f'(x)\right|^{p}+\left|f'(y)\right|^{p}\right]\right\}^{\frac{1}{p}}.$$

3. Setting $h(\lambda)=\frac{1}{n}\sum_{k=1}^{n}\lambda^{\frac{1}{k}}$. Given that $\lambda^{\frac{1}{k}}=\lambda^{s}$ where $s=\frac{1}{k}$ and the properties of the integral with the sum, applying (20) yields the following new result for the class n-fractional polynomial convex function.

Corollary 2.17. *Assume α,ρ and f are defined according to Theorem 2.2. If $|f'|^{p}$ is a n-fractional polynomial convex mapping on $[a,b]$, then*

$$\left|\frac{1}{3}\left[2f(a+b-y)-f\left(a+b-\frac{x+y}{2}\right)+2f(a+b-x)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\Im^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)+{}^{\rho}\Im^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x)\right]\right|$$
$$\leq\frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1+2^{1-\frac{1}{k}}}{2^{\frac{1}{k}}}\right)\right)^{\frac{1}{p}}\left[\left|f'(a+b-y)\right|^{p}+\left|f'(a+b-x)\right|^{p}\right]^{\frac{1}{p}} \tag{26}$$
$$\leq\frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1+2^{1-\frac{1}{k}}}{2^{\frac{1}{k}}}\right)\right)^{\frac{1}{p}}$$
$$\times\left\{\frac{1}{n}\sum_{k=1}^{n}2^{2-\frac{1}{k}}\left[\left|f'(a)\right|^{p}+\left|f'(b)\right|^{p}\right]-\left[\left|f'(x)\right|^{p}+\left|f'(y)\right|^{p}\right]\right\}^{\frac{1}{p}}.$$

Remark 2.18.
- *For $n = 1$, $x = a$ and $y = b$, Corollary 26 improves Theorem .4 from [11].*
- *Taking $\rho = 1$ in inequality (26), we get Milne inequality via Riemann-Liouville operators for n-fractional polynomial convex function.*

$$\left|\frac{1}{3}\left[2f(a+b-y) - f\left(a+b-\frac{x+y}{2}\right) + 2f(a+b-x)\right]\right.$$
$$\left. -\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}} f(a+b-y) + \mathfrak{I}^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}} f(a+b-x)\right]\right|$$
$$\leq \frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t)^{\alpha}\right)^{p'} dt\right)^{\frac{1}{p'}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1+2^{1-\frac{1}{k}}}{2^{\frac{1}{k}}}\right)\right)^{\frac{1}{p}}\left[\left|f'(a+b-y)\right|^{p} + \left|f'(a+b-x)\right|^{p}\right]^{\frac{1}{p}} \tag{27}$$
$$\leq \frac{y-x}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t)^{\alpha}\right)^{p'} dt\right)^{\frac{1}{p'}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1+2^{1-\frac{1}{k}}}{2^{\frac{1}{k}}}\right)\right)^{\frac{1}{p}}$$
$$\times\left\{\frac{1}{n}\sum_{k=1}^{n}2^{2-\frac{1}{k}}\left[\left|f'(a)\right|^{p} + \left|f'(b)\right|^{p}\right] - \left[\left|f'(x)\right|^{p} + \left|f'(y)\right|^{p}\right]\right\}^{\frac{1}{p}}.$$

- *Putting $\rho = 1$ and $\alpha = 1$ in inequality (26), we get Milne inequality via Riemann integral for s-convex function.*

$$\left|\frac{1}{3}\left[2f(a+b-y) - f\left(a+b-\frac{x+y}{2}\right) + 2f(a+b-x)\right] - \frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(t)dt\right|$$
$$\leq \frac{y-x}{2}\left[\frac{2}{1+p'}\left(\left(\frac{4}{3}\right)^{1+p'} - \left(\frac{1}{3}\right)^{1+p'}\right)\right]^{\frac{1}{p'}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1+2^{1-\frac{1}{k}}}{2^{\frac{1}{k}}}\right)\right)^{\frac{1}{p}}\left[\left|f'(a+b-y)\right|^{p} + \left|f'(a+b-x)\right|^{p}\right]^{\frac{1}{p}}$$
$$\leq \frac{y-x}{4}\left[\frac{2}{1+p'}\left(\left(\frac{4}{3}\right)^{1+p'} - \left(\frac{1}{3}\right)^{1+p'}\right)\right]^{\frac{1}{p'}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1+2^{1-\frac{1}{k}}}{2^{\frac{1}{k}}}\right)\right)^{\frac{1}{p}}$$
$$\times\left\{\frac{1}{n}\sum_{k=1}^{n}2^{2-\frac{1}{k}}\left[\left|f'(a)\right|^{p} + \left|f'(b)\right|^{p}\right] - \left[\left|f'(x)\right|^{p} + \left|f'(y)\right|^{p}\right]\right\}^{\frac{1}{p}}. \tag{28}$$

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