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Simpson inequalities for generalized preinvex functions under first-order differentiability

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Abstract. In this paper, we obtain some new Simpson's inequalities for functions whose first derivative in absolute value is r-preinvex and r-preqausi-invex $(0 < r \leq 1)$. The results presented here contribute to the generalization and improvement of various earlier reported works. Some well-known results are highlighted as an immediate deduction of our main results.

1. Introduction and Literature Review

In recent years, the study of convexity and its generalizations has seen significant advancements, with numerous extensions proposed in the literature (see [20, 30] and the references therein). A notable contribution is the introduction of invex functions by Hanson [10], which offers a natural extension of classical convexity. This pioneering work has spurred extensive research on invexity in (nonlinear) optimization [30], as well as in various branches of pure and applied mathematics [4, 5, 21, 29]. Among these developments, preinvex functions have emerged as a key focus due to their fundamental properties and applications in optimization, equilibrium problems, and variational inequalities [22, 27, 30]. Moreover, Haq and Javed [28] extended the theory of preinvexity to establish new Hermite-Hadamard integral inequalities.

Consider a function $\psi : [\alpha, \beta] \to \mathbb{R}$ that is four times continuously differentiable on the interval (α, β), with

$$\left\|\psi^{(4)}\right\|_\infty = \sup\left|\psi^{(4)}(x)\right| < \infty,$$

where $\|\cdot\|_\infty$ denotes the supremum norm. Under these conditions, the classical Simpson's inequality holds [8]:

$$\left|\frac{1}{3}\left[\frac{\psi(\alpha)+\psi(\beta)}{2} + 2\psi\Big(\frac{\alpha+\beta}{2}\Big)\right] - \frac{1}{\beta-\alpha}\int_\alpha^\beta \psi(x)\,dx\right| \leq \frac{(\beta-\alpha)^4}{2880}\left\|\psi^{(4)}\right\|_\infty .$$

Simpson-type inequalities have been widely explored under different convexity assumptions. For instance, Dragomir et al. [8] obtained new Simpson-type inequalities for s-convex functions. Alomari

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et al. [2, 3] established Simpson-type inequalities for quasi-convex functions. Özdemir and Ardıç [25] derived Simpson-type inequalities for first-order differentiable preinvex and prequasi-invex functions. Sarikaya et al. [26] obtained Simpson's inequalities for s-convex functions. Noor et al. [24] studied geometrically convex functions and related Simpson-type inequalities. Luo et al. [17] extended the theory to (p, q)-differentiable functions. Kalsoom et al. [12] examined generalized strongly preinvex and quasi-preinvex functions and related Simpson-type inequalities. Chiheb et al. [7] established dual Simpson type inequalities for functions whose absolute value of the first derivatives are preinvex. Luangboon et al. [16] proved Newton-type inequalities associated with convex functions via quantum calculus. Latif et al. [15] provided weighted Simpson's type inequalities for HA-convex functions. Meftah [18] reported new integral inequalities for s-preinvex functions. İsçan [9] proved Simpson's inequalities for two times differentiable preinvex and prequasiinvex functions. Karaoğlan et al. [13] obtained Simpson integral inequalities for preinvex functions under the Atangana-Baleanu integral operator. Likewise, Almoneef et al. [1] improved Simpson's inequalities for convex functions under generalized fractional operators. Budak and Ergün [6] established new Newton-type inequalities under multiplicative conformable fractional integrals. Hezenci and Budak [11] derived factional Newton-type integral inequalities for convex and bounded functions. Additionally, Lakhdari et al. [14] derived some second-family Radau-type inequalities for convex functions and Meftah et al. [19] established some new local fractional Newton-type inequalities under generalized convexity.

In this work, we contribute to this line of research by establishing Simpson-type inequalities for generalized preinvex differentiable functions. Specifically, we derive new inequalities for first-order differentiable functions with r-preinvex and r-preqausi-invex absolute values, thereby unifying and generalizing several existing results.

The remainder of this manuscript is organized as follows:

- Section 2 provides the necessary definitions and preliminary results.
- Section 3 presents the main findings.
- Section 4 concludes a summary and discussion.

2. Preliminaries

Let C be a nonempty closed subset of $\mathbb{R}$, and let $\psi : C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ be continuous real-valued functions. We begin by recalling essential definitions central to our analysis.

Definition 2.1. *(see Ref. [21]) A set C is said to be invex with respect to ϕ if for every $a, b \in C$ and $\rho \in [0, 1]$,*

$$a + \rho\phi(b, a) \in C.$$

Geometrically, this implies the existence of a path from a to $a + \phi(b, a)$ entirely contained in C. Unlike convexity, the endpoint b need not lie on this path (see Ref. [4]). If $\phi(b, a) = b - a$, the definition reduces to classical convexity. Thus, every convex set is invex, but the converse is false (see Ref. [29] and the related works cited therein).

For simplicity, we assume $C = [\alpha, \alpha + \phi(\beta, \alpha)]$ where $\alpha < \alpha + \phi(\beta, \alpha)$.

Definition 2.2. *(see Refs. [21, 29]) A function ψ defined on an invex set C is preinvex with respect to ϕ if for all $a, b \in C$ and $\rho \in [0, 1]$,*

$$\psi\left(a + \rho\phi(b, a)\right) \leq (1 - \rho)\psi(a) + \rho\psi(b).$$

Every convex function is preinvex, but not vice versa. For instance, the function $\psi(\xi) = -|\xi|$ is preinvex with respect to ϕ where

$$\phi(b, a) = \begin{cases} b - a, & \text{if } a, b \leq 0 \text{ or } a, b \geq 0, \\ a - b, & \text{otherwise.} \end{cases}$$

Further generalization to the concept of preinvexity was made by Antczak [5] (see also [30]). He introduced and studied the class of *r-preinvex functions* defined over invex sets. We define this as follows.

Definition 2.3. *(see [5]) A positive function ψ on an invex set C is r-preinvex with respect to ϕ if for all $a, b \in C$ and $\rho \in [0,1]$,*

$$\psi\left(a+\rho\phi\left(b,a\right)\right) \leq \begin{cases} \left[(1-\rho)\psi^r(a)+\rho\psi^r(b)\right]^{\frac{1}{r}}, & r>0, \\ \left(\psi(a)\right)^{1-\rho}\left(\psi(b)\right)^{\rho}, & r=0. \end{cases}$$

Remark 2.4. *Note that*

- *For $r = 0$, Definition 2.3 reduces to log-preinvexity; for $r = 1$, it recovers classical preinvexity.*
- *For $\phi(b,a) = b - a$ in Definition 2.3, r-preinvexity becomes r-convexity.*
- *Moreover, $\phi(b,a) = b - a$ reduces log-preinvexity to log-convexity, and $r = 1$ reduces r-convexity to classical convexity. See e.g., Noor et al. [23].*
- *The function ψ^r is preinvex for every $r > 0$ (see Antczak [5]). Applications of r-preinvexity are discussed in [28].*

Now we introduce the concept of $r-$prequasi-invex functions.

Definition 2.5. *A positive function ψ on an invex set C is r-prequasi-invex with respect to ϕ if for all $a, b \in C$ and $\rho \in [0,1]$,*

$$\psi\left(a+\rho\phi\left(b,a\right)\right) \leq \begin{cases} \left[\max\left\{\psi^r(a),\psi^r(b)\right\}\right]^{\frac{1}{r}}, & r>0, \\ \max\left\{\left(\psi(a)\right)^{1-\rho},\left(\psi(b)\right)^{\rho}\right\}, & r=0. \end{cases}$$

Remark 2.6. *As usual*

- *For $r = 0$, Definition 2.5 reduces to log-prequasi-invexity; while for $r = 1$, one recovers classical prequasi-invexity.*
- *For $\phi(b,a) = b - a$ in Definition 2.5, the r-prequasi-invexity becomes r-quasi-convexity.*
- *Further, $\phi(b,a) = b - a$ reduces log-prequasi-invexity to log-quasi-convexity, and $r = 1$ reduces r-quasi-convexity to quasi-convexity. See e.g., Noor et al. [23].*

Lemma 2.7. *(see Ref. [25]) Let $C \subseteq \mathbb{R}$ be an open invex set with respect to $\phi : C \times C \to \mathbb{R}_+$ and let $\psi : C \to \mathbb{R}$ be absolutely continuous on C. If ψ' is integrable on the ϕ-path $P_{\alpha\gamma}$ $\left(\gamma = \alpha + \phi(\beta,\alpha)\right)$, then*

$$\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(b,a)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right| = \phi(\beta,\alpha)\times$$

$$\int_0^1 \chi(\rho)\psi'(\alpha+\rho\phi(\beta,\alpha))d\rho,$$

where

$$\chi(\rho) = \begin{cases} \rho-\frac{1}{6}, & \rho\in\left[0,\frac{1}{2}\right), \\ \\ \rho-\frac{5}{6}, & \rho\in\left[\frac{1}{2},1\right]. \end{cases}$$

The proof of the above lemma is straightforward and interested readers may refer to [25].

3. Main Results

Here, we obtain several new Simpson's type inequalities for functions whose first-order derivatives in absolute value are r-preinvex and r-prequasi-invex.

3.1. Simpson inequalities for r-preinvex functions

Theorem 3.1. *Let $\phi : C \times C \to \mathbb{R}_+$ and $\psi : C \to \mathbb{R}$ be an absolutely continuous mapping on open invex set $C \subseteq \mathbb{R}$ with $\alpha, \beta \in C$. Suppose that $|\psi'(\alpha)| \neq |\psi'(\beta)|$ and $\phi(\beta, \alpha) \neq 0$. If $|\psi'|$ is r-preinvex then the following inequality holds:*

$$\left|\frac{1}{6}\left[\psi(\alpha) + 4\psi\left(\frac{2\alpha + \phi(\beta,\alpha)}{2}\right) + \psi\left(\alpha + \phi(\beta,\alpha)\right)\right] - \frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)} \psi(\xi)d\xi\right| \quad \leq \phi(\beta,\alpha)\left(\frac{rM(\alpha,\beta,r)}{1+r}\right),$$

where

$$M(\alpha,\beta,r) = M_1(\alpha,\beta,r) + M_2(\alpha,\beta,r) + M_3(\alpha,\beta,r) - M_4(\alpha,\beta,r) - M_5(\alpha,\beta,r).$$

and

$$M_1(\alpha,\beta,r) = \frac{\left|\psi'(\beta)\right|^{r(1+\frac{1}{r})} - \left|\psi'(\alpha)\right|^{r(1+\frac{1}{r})}}{6\left(\left|\psi'(\beta)\right|^r - \left|\psi'(\alpha)\right|^r\right)},$$

$$M_2(\alpha,\beta,r) = \frac{2r\left(\frac{5}{6}\left|\psi'(\alpha)\right|^r + \frac{1}{6}\left|\psi'(\beta)\right|^r\right)^{2+\frac{1}{r}}}{(1+2r)\left(\left|\psi'(\alpha)\right|^r - \left|\psi'(\beta)\right|^r\right)^2},$$

$$M_3(\alpha,\beta,r) = \frac{2r\left(\frac{1}{6}\left|\psi'(\alpha)\right|^r + \frac{5}{6}\left|\psi'(\beta)\right|^r\right)^{2+\frac{1}{r}}}{(1+2r)\left(\left|\psi'(\alpha)\right|^r - \left|\psi'(\beta)\right|^r\right)^2},$$

$$M_4(\alpha,\beta,r) = \frac{2r\left(\frac{\left|\psi'(\alpha)\right|^r + \left|\psi'(\beta)\right|^r}{2}\right)^{2+\frac{1}{r}}}{(1+2r)\left(\left|\psi'(\alpha)\right|^r - \left|\psi'(\beta)\right|^r\right)^2},$$

$$M_5(\alpha,\beta,r) = \frac{r\left(\left|\psi'(\alpha)\right|^{r\left(2+\frac{1}{r}\right)} + \left|\psi'(\beta)\right|^{r\left(2+\frac{1}{r}\right)}\right)}{(1+2r)\left(\left|\psi'(\alpha)\right|^r - \left|\psi'(\beta)\right|^r\right)^2}.$$

Proof. Using the r-preinvexity of $\left|\psi'\right|$ together with Lemma 2.7, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\alpha) + 4\psi\left(\frac{2\alpha + \phi(\beta,\alpha)}{2}\right) + \psi\left(\alpha + \phi(\beta,\alpha)\right)\right] - \frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)} \psi(\xi)\mathrm{d}\xi\right| \\
\leq\;& \phi(\beta,\alpha)\left\{\int_0^{\frac{1}{2}}\left|\rho - \frac{1}{6}\right|\left|\psi'(\alpha + \rho\phi(\beta,\alpha))\right|\mathrm{d}\rho + \int_{\frac{1}{2}}^1\left|\rho - \frac{5}{6}\right|\left|\psi'(\alpha + \rho\phi(\beta,\alpha))\right|\mathrm{d}\rho\right\} \\
\leq\;& \phi(\beta,\alpha)\left\{\int_0^{\frac{1}{2}}\left|\rho - \frac{1}{6}\right|\left[(1-\rho)\left|\psi'(\alpha)\right|^r + \rho\left|\psi'(\beta)\right|^r\right]^{\frac{1}{r}}\mathrm{d}\rho + \int_{\frac{1}{2}}^1\left|\rho - \frac{5}{6}\right|\left[(1-\rho)\left|\psi'(\alpha)\right|^r + \rho\left|\psi'(\beta)\right|^r\right]^{\frac{1}{r}}\mathrm{d}\rho\right\} \\
\leq\;& \phi(\beta,\alpha)\left\{I_1 + I_2\right\},
\end{aligned}$$

where

$$I_1 = \int_0^{\frac{1}{6}}\left(\frac{1}{6} - \rho\right)\left[(1-\rho)\left|\psi'(\alpha)\right|^r + \rho\left|\psi'(\beta)\right|^r\right]^{\frac{1}{r}}\mathrm{d}\rho + \int_{\frac{1}{6}}^{\frac{1}{2}}\left(\rho - \frac{1}{6}\right)\left[(1-\rho)\left|\psi'(\alpha)\right|^r + \rho\left|\psi'(\beta)\right|^r\right]^{\frac{1}{r}}\mathrm{d}\rho,$$

$$I_2 = \int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6} - \rho\right)\left[(1-\rho)\left|\psi'(\alpha)\right|^r + \rho\left|\psi'(\beta)\right|^r\right]^{\frac{1}{r}}\mathrm{d}\rho + \int_{\frac{5}{6}}^1\left(\rho - \frac{5}{6}\right)\left[(1-\rho)\left|\psi'(\alpha)\right|^r + \rho\left|\psi'(\beta)\right|^r\right]^{\frac{1}{r}}\mathrm{d}\rho.$$

Computing each integral separately using substitution and algebraic manipulation (details are provided in Appendix A), we obtain the stated result. □

Corollary 3.2. *For $r = 1$, the inequality in Theorem 3.1 reduces to the following*

$$\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right|\leq\frac{5\phi(\beta,\alpha)}{72}\left(\left|\psi'(\alpha)\right|+\left|\psi'(\beta)\right|\right).$$

This inequality holds for all first order differentiable preinvex functions as proved by Özdemir and Ardic [25].

Theorem 3.3. *Let $\phi : C\times C \to \mathbb{R}_+$ and $\psi : C \to \mathbb{R}$ be an absolutely continuous mapping on open invex set $C \subseteq \mathbb{R}$ with $\alpha, \beta \in C$. Suppose that $|\psi'(\alpha)| \neq |\psi'(\beta)|$ and $\phi(\beta, \alpha) \neq 0$. If $|\psi'|^\nu$ is r-preinvex for some fixed $\nu > 1$ and μ such that $1/\mu + 1/\nu = 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right|\leq\phi(\beta,\alpha)N(r,\alpha,\beta,\mu,\nu),$$

where

$$N(r,\alpha,\beta,\mu,\nu)=N_1(r,\alpha,\beta,\mu,\nu)\left[N_2(r,\alpha,\beta,\nu)+N_3(r,\alpha,\beta,\nu)\right],$$

and

$$N_1(r,\alpha,\beta,\mu,\nu)=\left(\frac{1+2^{\mu+1}}{6^{\mu+1}(1+\mu)}\right)^{\frac{1}{\mu}}\left(\frac{r}{1+r}\left[\frac{1}{\left|\psi'(\alpha)\right|^{r\nu}-\left|\psi'(\beta)\right|^{r\nu}}\right]\right)^{\frac{1}{\nu}},$$

$$N_2(r,\alpha,\beta,\nu)=\left(2^{-\left(1+\frac{1}{r}\right)}\left(\left|\psi'(\alpha)\right|^{r\nu}+\left|\psi'(\beta)\right|^{r\nu}\right)^{1+\frac{1}{r}}-\left|\psi'(\alpha)\right|^{r\nu\left(1+\frac{1}{r}\right)}\right)^{\frac{1}{\nu}},$$

$$N_3(r,\alpha,\beta,\nu)=\left(\left|\psi'(\beta)\right|^{r\nu\left(1+\frac{1}{r}\right)}-2^{-\left(1+\frac{1}{r}\right)}\left(\left|\psi'(\alpha)\right|^{r\nu}+\left|\psi'(\beta)\right|^{r\nu}\right)^{1+\frac{1}{r}}\right)^{\frac{1}{\nu}}.$$

Proof. Using Lemma 2.7 and Hölder inequality (with conjugate exponents μ and ν) in conjunction with the r-preinvexity of $\left|\psi'\right|^\nu$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)\mathrm{d}\xi\right|\\
\leq\ &\phi(\beta,\alpha)\left[\left(\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|^{\mu}\mathrm{d}\rho\right)^{\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\left|\psi'(\alpha+\rho\phi(\beta,\alpha))\right|^{\nu}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|^{\mu}\mathrm{d}\rho\right)^{\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^{1}\left|\psi'(\alpha+\rho\phi(\beta,\alpha))\right|^{\nu}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right]\\
\leq\ &\phi(\beta,\alpha)\left[\left(\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|^{\mu}\mathrm{d}\rho\right)^{\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\left[(1-\rho)\left|\psi'(\alpha)\right|^{r\nu}+\rho\left|\psi'(\beta)\right|^{r\nu}\right]^{\frac{1}{r}}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|^{\mu}\mathrm{d}\rho\right)^{\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^{1}\left[(1-\rho)\left|\psi'(\alpha)\right|^{r\nu}+\rho\left|\psi'(\beta)\right|^{r\nu}\right]^{\frac{1}{r}}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right]\\
\leq\ &\phi(\beta,\alpha)\left\{I_1J_1+I_2J_2\right\},
\end{aligned}$$

where

$$I_1 = \left(\int_0^{\frac{1}{6}} \left(\frac{1}{6} - \rho \right)^{\mu} \mathrm{d}\rho + \int_{\frac{1}{6}}^{\frac{1}{2}} \left(\rho - \frac{1}{6} \right)^{\mu} \mathrm{d}\rho \right)^{\frac{1}{\mu}},$$

$$J_1 = \left(\int_0^{\frac{1}{2}} \left[(1-\rho) \left| \psi'(\alpha) \right|^{r\nu} + \rho \left| \psi'(\beta) \right|^{r\nu} \right]^{\frac{1}{r}} \mathrm{d}\rho \right)^{\frac{1}{\nu}},$$

$$I_2 = \left(\int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6} - \rho \right)^{\mu} d\rho + \int_{\frac{5}{6}}^{1} \left(\rho - \frac{5}{6} \right)^{\mu} \mathrm{d}\rho \right)^{\frac{1}{\mu}},$$

$$J_2 = \left(\int_{\frac{1}{2}}^{1} \left[(1-\rho) \left| \psi'(\alpha) \right|^{r\nu} + \rho \left| \psi'(\beta) \right|^{r\nu} \right]^{\frac{1}{r}} \mathrm{d}\rho \right)^{\frac{1}{\nu}}.$$

Evaluating the integrals (via power rule and substitution) and simplifying the resulting expressions yields the desired inequality. □

Corollary 3.4. *On putting $r = 1$, the inequality in Theorem 3.3 yields the following:*

$$\begin{aligned} &\left| \frac{1}{6} \left[\psi(\alpha) + 4\psi \left(\frac{2\alpha + \phi(\beta,\alpha)}{2} \right) + \psi\left(\alpha + \phi(\beta,\alpha)\right) \right] - \frac{1}{\phi(\beta,\alpha)} \int_{\alpha}^{\alpha+\phi(\beta,\alpha)} \psi(\xi) d\xi \right| \\ \leq\ & \phi(\beta,\alpha) \left(\frac{1+2^{\mu+1}}{6^{\mu+1}(1+\mu)} \right)^{\frac{1}{\mu}} \left[\left(\frac{3}{8} \left| \psi'(\alpha) \right|^{\nu} + \frac{1}{8} \left| \psi'(\beta) \right|^{\nu} \right)^{\frac{1}{\nu}} + \left(\frac{1}{8} \left| \psi'(\alpha) \right|^{\nu} + \frac{3}{8} \left| \psi'(\beta) \right|^{\nu} \right)^{\frac{1}{\nu}} \right]. \end{aligned}$$

This inequality holds for all differentiable preinvex functions as shown by Özdemir and Ardic [25].

Theorem 3.5. *Under the assumptions of Theorem 3.3 (including $|\psi'(\alpha)| \neq |\psi'(\beta)|$, $\phi(\beta,\alpha) \neq 0$, and Hölder conjugates μ and ν), the following inequality holds:*

$$\left| \frac{1}{6} \left[\psi(\alpha) + 4\psi \left(\frac{2\alpha + \phi(\beta,\alpha)}{2} \right) + \psi\left(\alpha + \phi(\beta,\alpha)\right) \right] - \frac{1}{\phi(\beta,\alpha)} \int_{\alpha}^{\alpha+\phi(\beta,\alpha)} \psi(\xi) d\xi \right| \leq \phi(\beta,\alpha) Q(r,\alpha,\beta,\mu,\nu),$$

where

$$Q(r,\alpha,\beta,\mu,\nu) = \left(\frac{2(1+2^{\mu+1})}{6^{\mu+1}(1+\mu)} \right)^{\frac{1}{\mu}} \left(\frac{r}{1+r} \left[\frac{1}{\left| \psi'(\alpha) \right|^{r\nu} - \left| \psi'(\beta) \right|^{r\nu}} \right] \left[\left| \psi'(\alpha) \right|^{r\nu\left(1+\frac{1}{r}\right)} - \left| \psi'(\beta) \right|^{r\nu\left(1+\frac{1}{r}\right)} \right] \right)^{\frac{1}{\nu}}.$$

Proof. Using Lemma 2.7 and Hölder inequality (with conjugate exponents μ and ν) in conjunction with the r-preinvexity of $\left| \psi' \right|^{\nu}$, we have

$$\begin{aligned} &\left| \frac{1}{6} \left[\psi(\alpha) + 4\psi \left(\frac{2\alpha + \phi(\beta,\alpha)}{2} \right) + \psi\left(\alpha + \phi(\beta,\alpha)\right) \right] - \frac{1}{\phi(\beta,\alpha)} \int_{\alpha}^{\alpha+\phi(\beta,\alpha)} \psi(\xi) \mathrm{d}\xi \right| \\ \leq\ & \phi(\beta,\alpha) \left\{ \left(\int_0^1 \left| \chi(\rho) \right|^{\mu} \mathrm{d}\rho \right)^{\frac{1}{\mu}} \left(\int_0^1 \left| \psi'\left(\alpha + \rho\phi(\beta,\alpha)\right) \right|^{\nu} \mathrm{d}\rho \right)^{\frac{1}{\nu}} \right\} \\ \leq\ & \phi(\beta,\alpha) \left\{ \left(\int_0^{\frac{1}{2}} \left| \rho - \frac{1}{6} \right|^{\mu} \mathrm{d}\rho + \int_{\frac{1}{2}}^{1} \left| \rho - \frac{5}{6} \right|^{\mu} \mathrm{d}\rho \right)^{\frac{1}{\mu}} \left(\int_0^1 \left[(1-\rho) \left| \psi'(\alpha) \right|^{r\nu} + \rho \left| \psi'(\beta) \right|^{r\nu} \right]^{\frac{1}{r}} \mathrm{d}\rho \right)^{\frac{1}{\nu}} \right\}. \end{aligned}$$

Using the symmetry of the integrals:

$$\int_0^{\frac{1}{2}} \left| \rho - \frac{1}{6} \right|^{\mu} \mathrm{d}\rho = \int_{\frac{1}{2}}^{1} \left| \rho - \frac{5}{6} \right|^{\mu} \mathrm{d}\rho = \frac{1+2^{\mu+1}}{6^{\mu+1}(\mu+1)}.$$

and evaluating the remaining integral over $[0,1]$ using substitution, we obtain the stated result. □

Corollary 3.6. *On substituting $r = 1$, the inequality in Theorem 3.5 reduces to:*

$$\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right|$$
$$\leq \phi(\beta,\alpha)\left(\frac{2(1+2^{\mu+1})}{6^{\mu+1}(\mu+1)}\right)^{\frac{1}{\mu}}\left(\frac{\left|\psi'(\alpha)\right|^{\nu}+\left|\psi'(\beta)\right|^{\nu}}{2}\right)^{\frac{1}{\nu}}.$$

This holds for all differentiable preinvex functions and aligns with [25].

Theorem 3.7. *Let $\phi : C \times C \to \mathbb{R}_+$ and $\psi : C \to \mathbb{R}$ be an absolutely continuous mapping on an open invex set $C \subseteq \mathbb{R}$ with $\alpha, \beta \in C$. Suppose that $|\psi'(\alpha)| \neq |\psi'(\beta)|$ and $\phi(\beta,\alpha) \neq 0$. If $|\psi'|^{\nu}$ is $r-$preinvex for some fixed $\nu \geq 1$ then the following inequality holds:*

$$\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right|$$
$$\leq \phi(\beta,\alpha)\left(\frac{5}{72}\right)^{1-\frac{1}{\nu}} S(r,\alpha,\beta,\nu),$$

where

$$S(r,\alpha,\beta,\nu) = (S_1(r,\alpha,\beta,\nu)S_2(r,\alpha,\beta,\nu)+S_3(r,\alpha,\beta,\nu)S_4(r,\alpha,\beta,\nu))^{1/\nu}$$
$$+(S_1(r,\alpha,\beta,\nu)S_5(r,\alpha,\beta,\nu)+S_3(r,\alpha,\beta,\nu)S_6(r,\alpha,\beta,\nu))^{1/\nu},$$

and

$$S_1(r,\alpha,\beta,\nu) = \frac{r}{3(1+r)(\left|\psi'(\beta)\right|^{r\nu}-\left|\psi'(\alpha)\right|^{r\nu})},$$
$$S_2(r,\alpha,\beta,\nu) = \left(\frac{\left|\psi'(\beta)\right|^{r\nu}+\left|\psi'(\alpha)\right|^{r\nu}}{2}\right)^{1+\frac{1}{r}}-\frac{\left|\psi'(\alpha)\right|^{r\nu(1+\frac{1}{r})}}{2},$$
$$S_3(r,\alpha,\beta,\nu) = \frac{r^2}{(1+r)(1+2r)(\left|\psi'(\alpha)\right|^{r\nu}-\left|\psi'(\beta)\right|^{r\nu})^2},$$
$$S_4(r,\alpha,\beta,\nu) = 2\left(\frac{5}{6}\left|\psi'(\alpha)\right|^{r\nu}+\frac{1}{6}\left|\psi'(\alpha)\right|^{r\nu}\right)^{2+\frac{1}{r}}-\left|\psi'(\alpha)\right|^{r\nu(1+\frac{1}{r})}-\left(\frac{\left|\psi'(\beta)\right|^{r\nu}+\left|\psi'(\alpha)\right|^{r\nu}}{2}\right)^{2+\frac{1}{r}},$$
$$S_5(r,\alpha,\beta,\nu) = \frac{\left|\psi'(\alpha)\right|^{r\nu(1+\frac{1}{r})}}{2}-\left(\frac{\left|\psi'(\beta)\right|^{r\nu}+\left|\psi'(\alpha)\right|^{r\nu}}{2}\right)^{1+\frac{1}{r}},$$
$$S_6(r,\alpha,\beta,\nu) = 2\left(\frac{1}{6}\left|\psi'(\alpha)\right|^{r\nu}+\frac{5}{6}\left|\psi'(\alpha)\right|^{r\nu}\right)^{2+\frac{1}{r}}-\left(\frac{\left|\psi'(\alpha)\right|^{r\nu}+\left|\psi'(\alpha)\right|^{r\nu}}{2}\right)^{2+\frac{1}{r}}-\left|\psi'(\beta)\right|^{r\nu(2+\frac{1}{r})}.$$

Proof. From the power mean inequality applied to the r-preinvex function $\left|\psi'\right|^{\nu}$ in conjunction with Lemma

2.7, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)\mathrm{d}\xi\right|\\
\leq\;& \phi(\beta,\alpha)\left\{\left(\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|\mathrm{d}\rho\right)^{1-\frac{1}{\nu}}\left(\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|\left|\psi'(\alpha+\rho\phi(\beta,\alpha))\right|^{\nu}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|\mathrm{d}\rho\right)^{1-\frac{1}{\nu}}\left(\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|\left|\psi'(\alpha+\rho\phi(\beta,\alpha))\right|^{\nu}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right\}\\
\leq\;& \phi(\beta,\alpha)\left\{\left(\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|\mathrm{d}\rho\right)^{1-\frac{1}{\nu}}(I_1+I_2)^{\frac{1}{\nu}}+\left(\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|\mathrm{d}\rho\right)^{1-\frac{1}{\nu}}(I_3+I_4)^{\frac{1}{\nu}}\right\},
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{6}}\left(\frac{1}{6}-\rho\right)\left[(1-\rho)\left|\psi'(\alpha)\right|^{r\nu}+\rho\left|\psi'(\beta)\right|^{r\nu}\right]^{\frac{1}{r}}\mathrm{d}\rho,\\
I_2 &= \int_{\frac{1}{6}}^{\frac{1}{2}}\left(\rho-\frac{1}{6}\right)\left[(1-\rho)\left|\psi'(\alpha)\right|^{r\nu}+\rho\left|\psi'(\beta)\right|^{r\nu}\right]^{\frac{1}{r}}\mathrm{d}\rho,\\
I_3 &= \int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-\rho\right)\left[(1-\rho)\left|\psi'(\alpha)\right|^{r\nu}+\rho\left|\psi'(\beta)\right|^{r\nu}\right]^{\frac{1}{r}}\mathrm{d}\rho,\\
I_4 &= \int_{\frac{5}{6}}^{1}\left(\rho-\frac{5}{6}\right)\left[(1-\rho)\left|\psi'(\alpha)\right|^{r\nu}+\rho\left|\psi'(\beta)\right|^{r\nu}\right]^{\frac{1}{r}}\mathrm{d}\rho.
\end{aligned}$$

Evaluating the integrals (via substitution and algebraic expansion) and simplifying together with

$$\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|\mathrm{d}\rho=\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|\mathrm{d}\rho=\frac{5}{72}$$

yields the desired result. This completes the proof. □

Corollary 3.8. *For $r = 1$, the inequality in Theorem 3.7 gives the following:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right|\\
\leq\;& \phi(\beta,\alpha)\left(\frac{5}{72}\right)^{1-\frac{1}{\nu}}\left[\left(\frac{61\left|\psi'(\alpha)\right|^{\nu}+29\left|\psi'(\beta)\right|^{\nu}}{1296}\right)^{\frac{1}{\nu}}+\left(\frac{29\left|\psi'(\alpha)\right|^{\nu}+61\left|\psi'(\beta)\right|^{\nu}}{1296}\right)^{\frac{1}{\nu}}\right].
\end{aligned}$$

This result is true for all differentiable preinvex functions, as shown in [25].

Remark 3.9. *By letting $\phi(\beta,\alpha) = \beta - \alpha$ in Theorems 3.1, 3.3, 3.5, and 3.7, the results reduce to inequalities for differentiable r-convex functions. Similarly, setting $\phi(\beta,\alpha) = \beta - \alpha$ in Corollaries 3.2, 3.4, 3.6, and 3.8 yields corresponding results for classical convex functions. Furthermore, by setting*

$$\psi(\alpha)=\psi(\beta)=\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right),$$

in the stated theorems and corollaries immediately gives mid-point inequalities for differentiable r-preinvex and preinvex functions respectively.

3.2. Simpson inequalities for r-prequasi-invex functions

Theorem 3.10. *Let $\phi : C\times C \to \mathbb{R}_+$ and $\psi : C \to \mathbb{R}$ be an absolutely continuous mapping on open invex set $C \subseteq \mathbb{R}$ with $\alpha,\beta \in C$. Suppose that $|\psi'(\alpha)| \neq |\psi'(\beta)|$ and $\phi(\beta,\alpha) \neq 0$. If $|\psi'|$ is r-prequasi-invex then the following holds:*

$$\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right|\leq\phi(\beta,\alpha)\mathcal{M}(\alpha,\beta,r),$$

where

$$\mathcal{M}(\alpha,\beta,r)=\frac{5}{36}\left[\max\left\{\left|\psi'(\alpha)\right|^{r},\left|\psi'(\beta)\right|^{r}\right\}\right]^{1/r}.$$

Proof. Using the r-prequasi-invexity of $\left|\psi'\right|$ together with Lemma 2.7, we can have

$$\begin{aligned}&\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)\mathrm{d}\xi\right|\\ \leq\;&\phi(\beta,\alpha)\left\{\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|\left|\psi'(\alpha+\rho\phi(\beta,\alpha))\right|\mathrm{d}\rho+\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|\left|\psi'(\alpha+\rho\phi(\beta,\alpha))\right|\mathrm{d}\rho\right\}\\ \leq\;&\phi(\beta,\alpha)\left[\max\left\{\left|\psi'(\alpha)\right|^{r},\left|\psi'(\beta)\right|^{r}\right\}\right]^{1/r}\left\{\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|\mathrm{d}\rho+\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|\mathrm{d}\rho\right\}.\end{aligned}$$

Now computing each integral separately, we obtain the desired result. □

Corollary 3.11. *For $r=1$ in Theorem 3.10 gives the following inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right|\\ &\leq\frac{5}{36}\phi(\beta,\alpha)\max\left\{\left|\psi'(\alpha)\right|,\left|\psi'(\beta)\right|\right\}.\end{aligned}$$

This inequality holds for all first order differentiable prequasi-invex functions.

Theorem 3.12. *Let $\phi : C\times C \to \mathbb{R}_+$ and $\psi : C \to \mathbb{R}$ be an absolutely continuous mapping on open invex set $C \subseteq \mathbb{R}$ with $\alpha,\beta \in C$. Suppose that $|\psi'(\alpha)| \neq |\psi'(\beta)|$ and $\phi(\beta,\alpha) \neq 0$. If $|\psi'|^{\nu}$ is r-prequasi-invex for some fixed $\nu > 1$ and μ such that $1/\mu + 1/\nu = 1$, then the following holds:*

$$\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right|\leq\phi(\beta,\alpha)\mathcal{N}(r,\alpha,\beta,\mu,\nu),$$

where

$$\mathcal{N}(r,\alpha,\beta,\mu,\nu)=\left(\frac{2(2^{\mu+1}+1)}{6^{\mu+1}(\mu+1)}\right)^{1/\mu}\left(\left[\max\left\{\left|\psi'(\alpha)\right|^{r\nu},\left|\psi'(\beta)\right|^{r\nu}\right\}\right]^{1/r}\right)^{1/\nu},$$

and $\mu=\nu/(\nu-1)$.

Proof. From the r-prequasi-invexity of $\left|\psi'\right|^{\nu}$ together with Lemma 2.7 and the Hölder inequality (with conjugate exponents μ and ν), we have

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)\mathrm{d}\xi\right|\\
\leq\;& \phi(\beta,\alpha)\left[\left(\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|^{\mu}\mathrm{d}\rho\right)^{\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\left|\psi'(\alpha+\rho\phi(\beta,\alpha))\right|^{\nu}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|^{\mu}\mathrm{d}\rho\right)^{\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^{1}\left|\psi'(\alpha+\rho\phi(\beta,\alpha))\right|^{\nu}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right]\\
\leq\;& \phi(\beta,\alpha)\left[\left(\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|^{\mu}\mathrm{d}\rho\right)^{\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\left[\max\left\{\left|\psi'(\alpha)\right|^{r\nu},\left|\psi'(\beta)\right|^{r\nu}\right\}\right]^{1/r}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|^{\mu}\mathrm{d}\rho\right)^{\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^{1}\left[\max\left\{\left|\psi'(\alpha)\right|^{r\nu},\left|\psi'(\beta)\right|^{r\nu}\right\}\right]^{1/r}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right]\\
\leq\;& \phi(\beta,\alpha)\frac{\left(\left[\max\left\{\left|\psi'(\alpha)\right|^{r\nu},\left|\psi'(\beta)\right|^{r\nu}\right\}\right]^{1/r}\right)^{1/\nu}}{2^{1/\nu}}\left[\left(\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|^{\mu}\mathrm{d}\rho\right)^{\frac{1}{\mu}}+\left(\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|^{\mu}\mathrm{d}\rho\right)^{\frac{1}{\mu}}\right].
\end{aligned}
$$

Evaluating the integrals (via power rule and substitution) together with some simplification yields the desired inequality. □

Corollary 3.13. *On putting $r = 1$, the inequality in Theorem 3.12 yields the following:*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right|\\
\leq\;& \phi(\beta,\alpha)\left(\frac{2(2^{\mu+1}+1)}{6^{\mu+1}(\mu+1)}\right)^{1/\mu}\left(\max\left\{\left|\psi'(\alpha)\right|^{\nu},\left|\psi'(\beta)\right|^{\nu}\right\}\right)^{1/\nu}.
\end{aligned}
$$

This inequality holds for all differentiable prequasi-invex functions as shown by Özdemir and Ardic [25].

Theorem 3.14. *Under the assumptions of Theorem 3.12 (including $|\psi'(\alpha)| \neq |\psi'(\beta)|$, $\phi(\beta,\alpha) \neq 0$, and Hölder conjugates μ and ν), then the following holds:*

$$
\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right|\leq\phi(\beta,\alpha)\boldsymbol{Q}(r,\alpha,\beta,\mu,\nu)
$$

where

$$
\boldsymbol{Q}(r,\alpha,\beta,\mu,\nu)=2\left(\frac{1+2^{\mu+1}}{6^{\mu+1}(\mu+1)}\right)^{\frac{1}{\mu}}\left(\frac{\left[\max\left\{\left|\psi'(\alpha)\right|^{r\nu},\left|\psi'(\beta)\right|^{r\nu}\right\}\right]^{\frac{1}{r}}}{2}\right)^{\frac{1}{\nu}}.
$$

Proof. From the r-preiquasi-invexity of $\left|\psi'\right|^{\nu}$ together with Lemma 2.7 and the Hölder inequality (with conjugate exponents μ and ν), we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)\mathrm{d}\xi\right| \\
\leq\ & \phi(\beta,\alpha)\left\{\left(\int_0^1\left|\chi(\rho)\right|^{\mu}\mathrm{d}\rho\right)^{\frac{1}{\mu}}\left(\int_0^1\left|\psi'\left(\alpha+\rho\phi(\beta,\alpha)\right)\right|^{\nu}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right\} \\
\leq\ & \phi(\beta,\alpha)\left\{\left(\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|^{\mu}\mathrm{d}\rho+\int_{\frac{1}{2}}^1\left|\rho-\frac{5}{6}\right|^{\mu}\mathrm{d}\rho\right)^{\frac{1}{\mu}}\left(\int_0^1\left[\max\left\{\left|\psi'(\alpha)\right|^{r\nu},\left|\psi'(\beta)\right|^{r\nu}\right\}\right]^{\frac{1}{r}}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right\} \\
\leq\ & 2\phi(\beta,\alpha)\left(\frac{1+2^{\mu+1}}{6^{\mu+1}(\mu+1)}\right)^{\frac{1}{\mu}}\left(\frac{\left[\max\left\{\left|\psi'(\alpha)\right|^{r\nu},\left|\psi'(\beta)\right|^{r\nu}\right\}\right]^{\frac{1}{r}}}{2}\right)^{\frac{1}{\nu}},
\end{aligned}$$

where we used the symmetry of the integrals:

$$\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|^{\mu}\mathrm{d}\rho=\int_{\frac{1}{2}}^1\left|\rho-\frac{5}{6}\right|^{\mu}\mathrm{d}\rho=\frac{1+2^{\mu+1}}{6^{\mu+1}(\mu+1)}.$$

□

Corollary 3.15. *On substituting $r=1$, the inequality in Theorem 3.14 reduces to the following:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right| \\
\leq\ & 2\phi(\beta,\alpha)\left(\frac{1+2^{\mu+1}}{6^{\mu+1}(\mu+1)}\right)^{\frac{1}{\mu}}\left(\frac{\max\left\{\left|\psi'(\alpha)\right|^{\nu},\left|\psi'(\beta)\right|^{\nu}\right\}}{2}\right)^{\frac{1}{\nu}}.
\end{aligned}$$

This holds for all differentiable prequasi-invex functions and aligns with [25].

Theorem 3.16. *Let $\phi: C\times C\to\mathbb{R}_+$ and $\psi: C\to\mathbb{R}$ be an absolutely continuous mapping on an open invex set $C\subseteq\mathbb{R}$ with $\alpha,\beta\in C$. Suppose that $|\psi'(\alpha)|\neq|\psi'(\beta)|$ and $\phi(\beta,\alpha)\neq 0$. If $|\psi'|^{\nu}$ is $r-$prequasi-invex for some fixed $\nu\geq 1$ then the following holds:*

$$\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right|\leq\phi(\beta,\alpha)\mathcal{P}(r,\alpha,\beta,\nu),$$

where

$$\mathcal{P}(r,\alpha,\beta,\nu)=\frac{5}{36}\left(\left[\max\left\{\left|\psi'(\alpha)\right|^{r\nu},\left|\psi'(\beta)\right|^{r\nu}\right\}\right]^{\frac{1}{r}}\right)^{\frac{1}{\nu}}.$$

Proof. From Lemma 2.7 and the power mean inequality applied to the r-prequasi-invex function $\left|\psi'\right|^{\nu}$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)\mathrm{d}\xi\right| \\
\leq\ & \phi(\beta,\alpha)\left\{\left(\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|\mathrm{d}\rho\right)^{1-\frac{1}{\nu}}\left(\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|\left|\psi'(\alpha+t\phi(\beta,\alpha))\right|^{\nu}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|d\rho\right)^{1-\frac{1}{\nu}}\left(\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|\left|\psi'(\alpha+\rho\phi(\beta,\alpha))\right|^{\nu}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right\} \\
\leq\ & \phi(\beta,\alpha)\left(\frac{5}{72}\right)^{1-\frac{1}{\nu}}\left\{\left(\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|\left[\max\left\{\left|\psi'(\alpha)\right|^{r\nu},\left|\psi'(\beta)\right|^{r\nu}\right\}\right]^{\frac{1}{r}}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|\left[\max\left\{\left|\psi'(\alpha)\right|^{r\nu},\left|\psi'(\beta)\right|^{r\nu}\right\}\right]^{\frac{1}{r}}\mathrm{d}\rho\right)^{\frac{1}{\nu}}\right\} \\
\leq\ & \frac{5}{36}\phi(\beta,\alpha)\left(\left[\max\left\{\left|\psi'(\alpha)\right|^{r\nu},\left|\psi'(\beta)\right|^{r\nu}\right\}\right]^{\frac{1}{r}}\right)^{\frac{1}{\nu}},
\end{aligned}$$

since

$$\int_0^{\frac{1}{2}}\left|\rho-\frac{1}{6}\right|\mathrm{d}\rho=\int_{\frac{1}{2}}^{1}\left|\rho-\frac{5}{6}\right|\mathrm{d}\rho=\frac{5}{72}.$$

This completes the proof. □

Corollary 3.17. *For $r = 1$, the inequality in Theorem 3.16 gives the following:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\alpha)+4\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right)+\psi\left(\alpha+\phi(\beta,\alpha)\right)\right]-\frac{1}{\phi(\beta,\alpha)}\int_{\alpha}^{\alpha+\phi(\beta,\alpha)}\psi(\xi)d\xi\right| \\
\leq\ & \left(\frac{5}{36}\right)\phi(\beta,\alpha)\left(\max\left\{\left|\psi'(\alpha)\right|^{\nu},\left|\psi'(\beta)\right|^{\nu}\right\}\right)^{\frac{1}{\nu}}.
\end{aligned}$$

This result is true for all differentiable prequasi-invex functions, as shown in [25].

Remark 3.18. *By letting $\phi(\beta,\alpha) = \beta-\alpha$ in Theorems 3.10, 3.12, 3.14, and 3.16, the results reduce to inequalities for differentiable r-quasi-convex functions. Also, setting $\phi(\beta,\alpha) = \beta-\alpha$ in Corollaries 3.11, 3.13, 3.15, and 3.17 yields corresponding results for classical quasi-convex functions. Additionally, by letting*

$$\psi(\alpha)=\psi(\beta)=\psi\left(\frac{2\alpha+\phi(\beta,\alpha)}{2}\right),$$

in the stated theorems and corollaries immediately gives mid-point inequalities for differentiable r-prequasi-invex and prequasi-invex functions respectively.

4. Conclusion

In this paper, we established new Simpson-type inequalities for differentiable functions whose first derivative in absolute value is r-preinvex ($0 < r \leq 1$). The results obtained in this study generalize classical convex functions, r-convex functions, and preinvex functions. Applications to numerical quadrature rules for r-preinvex, preinvex, r−convex, and classical convex functions can be explored analogously to Alomari et al. [3] and Dragomir et al. [8]. Several well-known results from the literature (see e.g., Remarks

2.4 and 2.6), including those of Özdemir and Ardic [25], are recovered as direct consequences of our main theorems. These generalizations highlight the flexibility of the r-preinvexity and r-prequasi-invexity framework in unifying and extending classical inequalities. Future work may explore applications to numerical quadrature rules or further generalizations to other classes of functions.

Declaration

Conflict of interest

The author declares no conflict of interest.

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Appendix A. Evaluation of the integrals I_1 and I_2

In this appendix, we provide a detailed evaluation of the integrals I_1 and I_2 appearing in proof of Theorem 3.1.

Let

$$A := |\psi'(\alpha)|^r, \qquad B := |\psi'(\beta)|^r,$$

and define

$$g(\rho) := \left[(1-\rho)A + \rho B\right]^{1/r} = \left[A + (B-A)\rho\right]^{1/r}, \qquad \rho \in [0,1].$$

To evaluate the integrals, we use the change of variables

$$u = A + (B-A)\rho, \qquad d\rho = \frac{du}{B-A},$$

which yields

$$g(\rho) = u^{1/r}, \qquad \rho = \frac{u-A}{B-A}.$$

Recall that

$$I_1 = \int_0^{\frac{1}{6}} \left(\frac{1}{6} - \rho\right) g(\rho)\mathrm{d}\rho + \int_{\frac{1}{6}}^{\frac{1}{2}} \left(\rho - \frac{1}{6}\right) g(\rho)\mathrm{d}\rho.$$

Applying the substitution above, the first integral becomes

$$\begin{aligned}\int_0^{\frac{1}{6}} \left(\frac{1}{6} - \rho\right) g(\rho)\mathrm{d}\rho &= \int_A^{A+\frac{B-A}{6}} \left(\frac{1}{6} - \frac{u-A}{B-A}\right) \frac{u^{1/r}}{B-A}\mathrm{d}u \\ &= \frac{1}{(B-A)^2} \int_A^{A+\frac{B-A}{6}} \left(\frac{B-A-6(u-A)}{6}\right) u^{1/r}\mathrm{d}u.\end{aligned}$$

Similarly, the second integral is transformed as

$$\begin{aligned}\int_{\frac{1}{6}}^{\frac{1}{2}} \left(\rho - \frac{1}{6}\right) g(\rho)\mathrm{d}\rho &= \int_{A+\frac{B-A}{6}}^{A+\frac{B-A}{2}} \left(\frac{u-A}{B-A} - \frac{1}{6}\right) \frac{u^{1/r}}{B-A}\mathrm{d}u \\ &= \frac{1}{(B-A)^2} \int_{A+\frac{B-A}{6}}^{A+\frac{B-A}{2}} \left((u-A) - \frac{B-A}{6}\right) u^{1/r}\mathrm{d}u.\end{aligned}$$

Therefore,

$$I_1 = \frac{1}{(B-A)^2}\left[\int_A^{A+\frac{B-A}{6}} \left(\frac{B-A-6(u-A)}{6}\right) u^{1/r}\mathrm{d}u + \int_{A+\frac{B-A}{6}}^{A+\frac{B-A}{2}} \left((u-A)-\frac{B-A}{6}\right) u^{1/r}\mathrm{d}u\right].$$

The integral I_2 is given by

$$I_2 = \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6}-\rho\right) g(\rho)\mathrm{d}\rho + \int_{\frac{5}{6}}^{1} \left(\rho-\frac{5}{6}\right) g(\rho)\mathrm{d}\rho.$$

Using the same substitution, the first term becomes

$$\begin{aligned}\int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6}-\rho\right) g(\rho)\mathrm{d}\rho &= \int_{A+\frac{B-A}{2}}^{A+\frac{5(B-A)}{6}} \left(\frac{5}{6}-\frac{u-A}{B-A}\right)\frac{u^{1/r}}{B-A}\mathrm{d}u \\ &= \frac{1}{(B-A)^2}\int_{A+\frac{B-A}{2}}^{A+\frac{5(B-A)}{6}} \left(\frac{5(B-A)-6(u-A)}{6}\right) u^{1/r}\mathrm{d}u.\end{aligned}$$

The second term is

$$\begin{aligned}\int_{\frac{5}{6}}^{1} \left(\rho-\frac{5}{6}\right) g(\rho)\mathrm{d}\rho &= \int_{A+\frac{5(B-A)}{6}}^{B} \left(\frac{u-A}{B-A}-\frac{5}{6}\right)\frac{u^{1/r}}{B-A}\mathrm{d}u \\ &= \frac{1}{(B-A)^2}\int_{A+\frac{5(B-A)}{6}}^{B} \left((u-A)-\frac{5(B-A)}{6}\right) u^{1/r}\mathrm{d}u.\end{aligned}$$

Consequently,

$$I_2 = \frac{1}{(B-A)^2}\left[\int_{A+\frac{B-A}{2}}^{A+\frac{5(B-A)}{6}} \left(\frac{5(B-A)-6(u-A)}{6}\right) u^{1/r}\mathrm{d}u + \int_{A+\frac{5(B-A)}{6}}^{B} \left((u-A)-\frac{5(B-A)}{6}\right) u^{1/r}\mathrm{d}u\right].$$

Each integral appearing above is a linear combination of the elementary integrals

$$\int u^{1/r}\mathrm{d}u = \frac{r}{r+1}u^{\frac{r+1}{r}}, \quad \int u^{1+\frac{1}{r}}\mathrm{d}u = \frac{r}{2r+1}u^{\frac{2r+1}{r}},$$

which upon back substituting A, B gives us the desired expressions.

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