On ℱ-semi-transitive families of operators

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Abstract. In this paper, we present the concept of disjoint Furstenberg-semi-transitivity. Then we provide sufficient conditions for cosine operator functions generated by adjoints of weighted composition operators to be Furstenberg-semi-transitive on the space of Radon measures on a locally compact, non-compact Hausdorff space. Also, we characterize disjoint Furstenberg-semi-transitive compositions of a right multiplier and a generalized bilateral weighted shift on the standard Hilbert module over the C*-algebra of compact operators on a separable Hilbert space. As a special case, we consider the case when the corresponding Furstenberg family is the family of all infinite subsets of the set of natural numbers. Finally, we illustrate our results with concrete examples.

1. Introduction

Supercyclicity is a fundamental concept in the dynamics of operators and numerous papers have been written on the topics related to supercyclicity of operators. For example, Hilden and Wallen in [22] proved that any unilateral backward weighted shift is supercyclic. Afterwards, Salas in [41] characterized supercyclic bilateral weighted shift operators on $l^p(\mathbb{Z})$ in terms of a supercyclicity criterion. Supercyclicity of several kinds of operators has also been studied in for instance [17, 30, 34, 36, 40, 44]. Moreover, there is a close connection between supercyclicity and semi-Fredholm theory, see [1, 10, 18]. On the other hand, Furstenberg topological transitivity as a generalization of topological transitivity and hypercyclicity has been studied in several papers, see [2, 3, 8, 20, 21, 33]. Motivated by these facts, in this paper we consider the concept of ℱ−semi-transitivity, as a generalization of the concept of topological transitivity for supercyclicity given in [34] and characterize ℱ−semi-transitivity classes of operators on various Banach spaces.

The research on the dynamics of weighted composition operators has started from pioneering papers by Salas on the dynamics on weighted shifts, see [38, 41]. Since then, numerous papers on the dynamics of weighted composition operators on various function spaces have been published. In particular, the dynamics of cosine operator functions generated by weighted composition operators has been studied in for instance [11–15, 25, 43]. Moreover, the dynamics of cosine operator functions generated by some other

operators has been investigated in for instance [29, 31, 32]. Finally, quite recently, topological transitivity of the adjoints of weighted composition operators has been investigated and studied in [29]. These adjoints considered in [29] act on the space of Radon measures on a locally compact, non-compact Hausdorff space. Although there are several published papers on the dynamics of cosine operator functions, supercyclicity of cosine operator functions has not been studied in any of these published papers. All these facts naturally lead us to study in this paper supercyclicity and $\mathcal{F}$−semi-transitivity (as a generalization of supercyclicity) of cosine operator functions generated by the adjoints of weighted composition operators. In Section 3 of this paper, after presenting the concept of $\mathcal{F}$−semi-transitivity, we obtain sufficient conditions for cosine operator functions generated by the adjoints of weighted composition operators to be $\mathcal{F}$−semi-transitive on the space of Radon measures (on a locally compact, non-compact Hausdorff space) equipped with the total variation norm (Theorem 3.2). A special case of this theory is the case when the corresponding Furstenberg family is the family of all infinite subsets of $\mathbb{N}$, as given in Corollary 3.3. We provide also concrete examples where the sufficient conditions of Theorem 3.2 and Corollary 3.3 in this paper are satisfied, but where the sufficient conditions of [29, Proposition 3.3] are not satisfied, illustrating thus the difference between (ordinary) topological transitivity and $\mathcal{F}$−semi-transitivity of cosine operator functions.

After pioneering papers by Salas on the dynamics on weighted shifts, several papers on the dynamics of various kinds of generalized weighted shifts have been published, see [26, 28, 45]. In particular, the dynamics of generalized bilateral weighted shifts on the standard Hilbert module over the C*-algebra of compact operators on a separable Hilbert space has been studied in [28, 39], however, the concepts such as supercyclicity and disjoint supercyclicity have not been considered in [28, 39]. This motivated us to study disjoint supercyclicity and related concepts for the generalized weighted shift operator considered in [28]. In Section 4 of this paper, we present first the concept of disjoint $\mathcal{F}$-semi-transitivity as a generalization of disjoint topological transitivity for supercyclicity introduced in [34], and then, as a continuation and extension of the results in [28, 39], in Theorem 4.2 we characterize disjoint $\mathcal{F}$-semi-transitive generalized weighted bilateral shift operators on the standard Hilbert module over the C*-algebra of compact operators on a separable Hilbert space in the case when the corresponding $\mathcal{F}$-family is finitely invariant. Once again it turns out that a special case of this theory is the case when the $\mathcal{F}$-family under consideration is the family of all infinite subsets of $\mathbb{N}$. This special case is treated in Corollary 4.4 and Proposition 4.6 in this paper. Also, we give concrete examples of $\mathcal{F}$-semi-transitive generalized weighted bilateral shift operators (Example 4.8) where we also illustrate the difference between $\mathcal{F}$-semi-transitive (and thus also supercyclic) generalized weighted bilateral shift operators on Hilbert modules and supercyclic classical weighted bilateral shifts on the sequence space ℓ^2. In addition, we introduce a new concept of $\mathcal{F}$-C*-semi-transitivity for generalized bilateral shift operators on the standard Hilbert C*-module and we characterize disjoint $\mathcal{F}$-C*-semi-transitive generalized weighted bilateral shifts. This new concept of $\mathcal{F}$-C*-semi-transitivity is a generalization of $\mathcal{F}$-semi-transitivity, so $\mathcal{F}$-semi-transitivity implies $\mathcal{F}$-C*-semi-transitivity, however, the converse is not true in general, as illustrated in Example 4.9 in this paper.
Next, hypercyclicity and supercyclicity of a composition of a left and a right multiplier on the space of compact operators have been studied in for instance [19, 37]. Motivated by those results, in Proposition 4.10 and Corollary 4.11 we provide sufficient conditions for disjoint $\mathcal{F}$-semi-transitivity of compositions of right multipliers and generalized weighted bilateral shift operators (where the corresponding weights are multiplied from the left) on the standard Hilbert module over the C*-algebra of compact operators on a separable Hilbert space and we illustrate these results with concrete example where these sufficient conditions are satisfied (Example 4.12). Finally, in Remark 4.13 we explain that our results from Section 4 remain valid also if we instead of a separable Hilbert space consider any standard Hilbert module over a unital C*-algebra.

2. Preliminaries

In this section we will first recall some fundamental concepts from linear dynamics of operators.

Definition 2.1. *Let X be a Banach space and $T \in B(X)$, that is T is a bounded linear operator on X. We say that T is topologically transitive on X if for each pair of open non-empty subsets O_1 and O_2 of X there exists some $n \in \mathbb{N}$ such*

that $T^n(O_1) \cap O_2 \neq \emptyset$. Similarly, we say that T is topologically semi-transitive on X if for each pair of open non-empty subsets O_1 and O_2 of X there exists some $n \in \mathbb{N}$ and some $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda T^n(O_1) \cap O_2 \neq \emptyset$.

Remark 2.2. *In [34, Definition 1.2], topological semi-transitivity is actually called topological transitivity for supercyclicity. Topological semi-transitivity of weighted composition operators on various classes of function spaces has been studied also recently in [30].*

In this paper, given a set $\mathcal{S}$, $\mathcal{F}$ represents a Furstenberg family of subsets of $\mathcal{S}$, which is a non-empty collection of subsets of $\mathcal{S}$ satisfying the following conditions:

1. The empty set $\emptyset$ is not an element of $\mathcal{F}$.
2. If A belongs to $\mathcal{F}$ and $A \subseteq B \subseteq \mathcal{S}$, then B also belongs to $\mathcal{F}$.

Now we recall also definition of $\mathcal{F}$-transitivity, which is a generalization of topological transitivity.

Definition 2.3. *[2, Definition 1.1] Let X be a Banach space and B(X) be the space of all bounded linear operators on X. Assume that $\mathcal{S}$ is a set and $\mathcal{F}$ is a Furstenberg family of subsets of $\mathcal{S}$. A family $\{T_t\}_{t \in \mathcal{S}} \subseteq B(X)$ is called $\mathcal{F}$-transitive if for every pair of non-empty subsets $U, V \subseteq X$ we have*

$$\{t \in \mathcal{S} : U \cap T_t^{-1}(V) \neq \emptyset\} \in \mathcal{F}.$$

Throughout this paper, we let Ω be a locally compact non-compact Hausdorff space and α be a homeomorphism from Ω to Ω. As usual, $C_0(\Omega)$ denotes the space of all continuous functions on Ω vanishing at infinity, $C_b(\Omega)$ denotes the space of bounded continuous functions on Ω, whereas $C_c(\Omega)$ stands for the set of all continuous compactly supported functions on Ω. Both $C_0(\Omega)$ and $C_b(\Omega)$ are equipped with the supremum norm. Moreover, we let w be a positive continuous bounded function on Ω such that also $w^{-1} := \frac{1}{w} \in C_b(\Omega)$. Such function will be called *weight* throughout the paper. We put then $T_{\alpha,w}$ to be the weighted composition operator on $C_0(\Omega)$ with respect to α and w, that is $T_{\alpha,w}(f) = w \cdot (f \circ \alpha)$ for all $f \in C_0(\Omega)$. Easily, one can see that by the above assumptions $T_{\alpha,w}$ is well-defined and $\|T_{\alpha,w}\| \leq \|w\|_{\sup}$. Since $\frac{1}{w}$ is also bounded, then $T_{\alpha,w}$ is invertible and we have

$$T_{\alpha,w}^{-1} f = \frac{f \circ \alpha^{-1}}{w \circ \alpha^{-1}}, \quad (f \in C_0(\Omega)).$$

Simply we denote $S_{\alpha,w} := T_{\alpha,w}^{-1}$.

By some calculation one can see that for each $n \in \mathbb{N}$ and $f \in C_0(\Omega)$,

$$T_{\alpha,w}^n f = \left(\prod_{j=0}^{n-1} (w \circ \alpha^j) \right) \cdot (f \circ \alpha^n) \quad (1)$$

and

$$S_{\alpha,w}^n f = \left(\prod_{j=1}^{n} (w \circ \alpha^{-j}) \right)^{-1} \cdot (f \circ \alpha^{-n}). \quad (2)$$

The adjoint $T_{\alpha,w}^*$ is a bounded operator on $M(\Omega)$ where $M(\Omega)$ stands for the Banach space of all Radon measures on Ω equipped with the total variation norm. It is straightforward to check that

$$T_{\alpha,w}^*(\mu)(E) = \int_E w \circ \alpha^{-1} \, d(\mu \circ \alpha^{-1})$$

for every $\mu \in M(\Omega)$ and every measurable subset E of Ω. Here $\mu \circ \alpha^{-1}(E) = \mu(\alpha^{-1}(E))$ for every $\mu \in M(\Omega)$ and every measurable subset E of Ω. By (1) and (2) it follows that for every $n \in \mathbb{N}$, $\mu \in M(\Omega)$ and Borel measurable subset $E \subseteq \Omega$ we have

$$T_{\alpha,w}^{*n}(\mu)(E) = \int_E \prod_{j=0}^{n-1} w \circ \alpha^{j-n} \, d(\mu \circ \alpha^{-n})$$

and

$$S_{\alpha,w}^{*n}(\mu)(E) = \int_E \prod_{j=1}^{n} (w \circ \alpha^{n-j})^{-1} \, d(\mu \circ \alpha^n).$$

For every Radon measure μ on Ω, we let as usual $|\mu|$ denote the total variation of μ. Also, we let $M_c(\Omega)$ denote the space of all compactly supported Radon measures on Ω.

3. $\mathcal{F}$-semi-transitive cosine operator functions

Assume that $\mathcal{S}$ is a set and $\mathcal{F}$ is a proper Furstenberg family of subsets of $\mathcal{S}$.

Let $(\alpha_z)_{z \in \mathcal{S}}$ be a family of homeomorphisms from Ω to Ω and $(w_z)_{z \in \mathcal{S}}$ be a family of weights on Ω. For each $z \in \mathcal{S}$, we let $T_z^* := T_{\alpha_z,w_z}^*$ and let $C_z^* := \frac{1}{2}(T_z^* + S_z^*)$. We will say that $\{C_z^*\}_{z \in \mathcal{S}}$ is $\mathcal{F}$-semi-transitive if for every non-empty open subsets U_1 and U_2 of $M(\Omega)$, the set

$$\{z \in \mathcal{S} : \exists \lambda_z \in \mathbb{R}^+ \text{ such that } \lambda_z C_z^*(U_1) \cap U_2 \neq \emptyset\}$$

is an element of $\mathcal{F}$.

Remark 3.1. *We observe that $\mathcal{F}$-semi-transitivity is in fact an extension of $\mathcal{F}$-transitivity. While $\mathcal{F}$-transitivity is a well known concept in the literature, see [2, 8, 20, 21, 33], the concept of $\mathcal{F}$-semi-transitivity is a new concept. Moreover, while $\mathcal{F}$-transitivity generalizes topological transitivity, $\mathcal{F}$-semi-transitivity generalizes topological semi-transitivity.*

The next theorem gives some sufficient conditions for the family $\{C_z^*\}_{z \in \mathcal{S}}$ to be $\mathcal{F}$-semi-transitive.

Theorem 3.2. *We have (ii) $\Rightarrow$ (i).*

(i) $\{C_z^*\}_{z \in \mathcal{S}}$ *is $\mathcal{F}$-semi-transitive;*
(ii) *for each compact subset K of Ω, each $\varepsilon > 0$, and any two non-zero measures μ $\nu \in M(\Omega)$ with $|\mu|(K^c) = |\nu|(K^c) = 0$, there exists some $F \in \mathcal{F}$ and collections $\{A_z\}_{z \in F}, \{D_z\}_{z \in F}, \{E_z\}_{z \in F}$ of Borel subsets of K such that*

$$|\mu|(A_z) < \varepsilon, \ |\nu|(A_z) < \varepsilon, \ D_z \cap E_z = \emptyset, \ A_z^c \cap K = D_z \cup E_z,$$

for all $z \in F$, and

$$\left(\sup_{t \in A_z^c \cap K} w_z(t) \right) \cdot \left(\sup_{t \in D_z} w_z(t) \right) < \varepsilon,$$

$$\left(\sup_{t \in A_z^c \cap K} w_z(t) \right) \cdot \left(\sup_{t \in E_z} \left(w_z \circ \alpha^{-1} \right)^{-1} (t) \right) < \varepsilon,$$

$$\left(\sup_{t \in A_z^c \cap K} \left(w_z \circ \alpha^{-1} \right)^{-1} (t) \right) \cdot \left(\sup_{t \in D_z} w_z(t) \right) < \varepsilon,$$

$$\left(\sup_{t \in A_z^c \cap K} \left(w_z \circ \alpha^{-1} \right)^{-1} (t) \right) \cdot \left(\sup_{t \in E_z} \left(w_z \circ \alpha^{-1} \right)^{-1} (t) \right) < \varepsilon,$$

$$\sup_{t \in D_z} \left(\prod_{j=0}^{1} \left(w_z \circ \alpha_z^j \right) (t) \right) < \varepsilon,$$

$$\sup_{t \in E_z} \left(\prod_{j=1}^{2} \left(w_z \circ \alpha_z^{-j} \right)^{-1} (t) \right) < \varepsilon \quad \text{for all } z \in F.$$

Proof. Let U_1 and U_2 be two open non-empty subsets of $M(\Omega)$. Since $M_c(\Omega)$ is dense in $M(\Omega)$, we can find two measures $\mu \in U_1 \setminus \{0\}$, $\nu \in U_2 \setminus \{0\}$, and a compact subset K of Ω such that $|\mu|(K^c) = |\nu|(K^c) = 0$. Given $\delta > 0$, let $B(\mu, \delta)$ and $B(\nu, \delta)$ denote the open balls in $M(\Omega)$ with radius δ and centres in μ and ν, respectively. We can find some $\delta > 0$ such that $B(\mu, \delta) \subset U_1 \setminus \{0\}$ and $B(\nu, \delta) \subset U_2 \setminus \{0\}$. Choose $F \in \mathcal{F}$ and collections of Borel subsets $\{A_z\}_{z \in F}$, $\{D_z\}_{z \in F}$, $\{E_z\}_{z \in F}$ satisfying the assumptions of (*ii*) with respect to μ, ν, and

$$\varepsilon := \min\left\{\frac{\delta}{4}, \frac{\delta^2}{64\|\mu\|\|\nu\|}, \frac{\delta}{8\|\nu\|}\right\}.$$

For each $z \in F$, let $\tilde{\mu}_z, \tilde{\nu}_z, \tilde{\tilde{\nu}}_z \in M(\Omega)$ be the elements of $M(\Omega)$ given by $\tilde{\mu}_z(B) = \mu(B \cap A_z^c \cap K)$, $\tilde{\nu}_z(B) = \nu(B \cap D_z)$, and $\tilde{\tilde{\nu}}_z(B) = \nu(B \cap E_z)$ for every $z \in F$ and any Borel subset B of Ω. Then

$$|\mu - \tilde{\mu}_z|(\Omega) = |\mu|(A_z) < \varepsilon < \delta \quad \text{and} \quad |\nu - \tilde{\nu}_z - \tilde{\tilde{\nu}}_z|(\Omega) = |\nu|(A_z) < \varepsilon < \delta,$$

so $\tilde{\mu}_z \in U_1 \setminus \{0\}$ and $\tilde{\nu}_z + \tilde{\tilde{\nu}}_z \in U_2 \setminus \{0\}$ for all $z \in F$. Since T_z^* and S_z^* are invertible for all $z \in F$, we must have that $T_z^*(\tilde{\mu}_z) \neq 0$ and $S_z^*(\tilde{\mu}_z) \neq 0$ for all $z \in F$. Moreover, since $\tilde{\nu}_z + \tilde{\tilde{\nu}}_z \neq 0$ for all $z \in F$, for each $z \in F$, either $\tilde{\nu}_z$ or $\tilde{\tilde{\nu}}_z$ (or both) are nonzero, and hence $T_z^*(\tilde{\nu}_z)$ and $S_z^*(\tilde{\tilde{\nu}}_z)$ (or both) are nonzero for each $z \in F$. Therefore, for all $z \in F$, we have

$$\|T_z^*(\tilde{\mu}_z)\| + \|S_z^*(\tilde{\mu}_z)\| > 0 \quad \text{and} \quad \|T_z^*(\tilde{\nu}_z)\| + \|S_z^*(\tilde{\tilde{\nu}}_z)\| > 0.$$

For each $z \in F$, put

$$\phi_z := \tilde{\mu}_z + \frac{2\sqrt{\|T_z^*(\tilde{\mu}_z)\| + \|S_z^*(\tilde{\mu}_z)\|}}{\sqrt{\|T_z^*(\tilde{\nu}_z)\| + \|S_z^*(\tilde{\tilde{\nu}}_z)\|}} \left(T_z^*(\tilde{\nu}_z) + S_z^*(\tilde{\tilde{\nu}}_z)\right).$$

Then

$$\|\phi_z - \mu\| \leq \|\mu - \tilde{\mu}_z\| + 2\sqrt{\|T_z^*(\tilde{\mu}_z)\| + \|S_z^*(\tilde{\mu}_z)\|}\sqrt{\|T_z^*(\tilde{\nu}_z)\| + \|S_z^*(\tilde{\tilde{\nu}}_z)\|}$$

and

$$\left\|\frac{\sqrt{\|T_z^*(\tilde{\nu}_z)\| + \|S_z^*(\tilde{\tilde{\nu}}_z)\|}}{\sqrt{\|T_z^*(\tilde{\mu}_z)\| + \|S_z^*(\tilde{\mu}_z)\|}} C_z^*(\phi_z) - \nu\right\| =$$

$$= \left\|\frac{\sqrt{\|T_z^*(\tilde{\nu}_z)\| + \|S_z^*(\tilde{\tilde{\nu}}_z)\|}}{\sqrt{\|T_z^*(\tilde{\mu}_z)\| + \|S_z^*(\tilde{\mu}_z)\|}} C_z^*(\tilde{\mu}_z) + (T_z^*)^2(\tilde{\nu}_z) + (S_z^*)^2(\tilde{\tilde{\nu}}_z) + \tilde{\nu}_z + \tilde{\tilde{\nu}}_z - \nu\right\| \leq$$

$$\leq \frac{\sqrt{\|T_z^*(\tilde{\nu}_z)\| + \|S_z^*(\tilde{\tilde{\nu}}_z)\|}}{\sqrt{\|T_z^*(\tilde{\mu}_z)\| + \|S_z^*(\tilde{\mu}_z)\|}} \|C_z^*(\tilde{\mu}_z)\| + \|(T_z^*)^2(\tilde{\nu}_z)\| + \|(S_z^*)^2(\tilde{\tilde{\nu}}_z)\| + \|\tilde{\nu}_z + \tilde{\tilde{\nu}}_z - \nu\| \leq$$

$$\leq \frac{\sqrt{\|T_z^*(\tilde{\nu}_z)\| + \|S_z^*(\tilde{\tilde{\nu}}_z)\|}}{2\sqrt{\|T_z^*(\tilde{\mu}_z)\| + \|S_z^*(\tilde{\mu}_z)\|}} \left(\|T_z^*(\tilde{\mu}_z)\| + \|S_z^*(\tilde{\mu}_z)\|\right) +$$

$$+ \|(T_z^*)^2(\tilde{\nu}_z)\| + \|(S_z^*)^2(\tilde{\tilde{\nu}}_z)\| + \|\tilde{\nu}_z + \tilde{\tilde{\nu}}_z - \nu\| =$$

$$= \frac{1}{2}\sqrt{\|T_z^*(\tilde{\nu}_z)\| + \|S_z^*(\tilde{\tilde{\nu}}_z)\|}\sqrt{\|T_z^*(\tilde{\mu}_z)\| + \|S_z^*(\tilde{\mu}_z)\|} +$$

$$+ \|(T_z^*)^2(\tilde{\nu}_z)\| + \|(S_z^*)^2(\tilde{\tilde{\nu}}_z)\| + \|\tilde{\nu}_z + \tilde{\tilde{\nu}}_z - \nu\|$$

for all $z \in F$. Now, by exactly the same arguments as in the proof of [29, Proposition 3.1] part (*ii*) $\Rightarrow$ (*i*), one can show that for each $z \in F$ we have

$$\left\|T_z^*(\tilde{\mu}_z)\right\| \leq \sup_{t \in A_z^c \cap K}\{w_z(t)\|\mu\|\},$$

$$\left\| S_z^*(\tilde{\mu}_z) \right\| \leq \sup_{t \in A_z^c \cap K} \left\{ (w_z \circ \alpha_z^{-1})^{-1}(t) \|\mu\| \right\},$$

$$\left\| T_z^*(\tilde{v}_z) \right\| \leq \sup_{t \in D_z} \left\{ w_z(t) \|v\| \right\},$$

$$\left\| S_z^*(\tilde{v}_z) \right\| \leq \sup_{t \in E_z} \left\{ (w_z \circ \alpha_z^{-1})^{-1}(t) \|v\| \right\},$$

$$\left\| (T_z^*)^2(\tilde{v}_z) \right\| \leq \sup_{t \in D_z} \left\{ \prod_{j=0}^{1} (w_z \circ \alpha_z^j)(t) \|v\| \right\},$$

and

$$\left\| (S_z^*)^2(\tilde{v}_z) \right\| \leq \sup_{t \in E_z} \left\{ \prod_{j=1}^{2} (w_z \circ \alpha_z^{-j})^{-1}(t) \|v\| \right\}.$$

Indeed, as noticed in the proof of [29, Proposition 3.1] part $(ii) \Rightarrow (i)$, for all $z \in F$ and $n \in \mathbb{N}$ we have

$$T_z^{*n}(\tilde{\mu}_z)(B) = \int_B \prod_{j=0}^{n-1} (w_z \circ \alpha_z^{j-n}) \, d(\tilde{\mu}_z \circ \alpha_z^{-n})$$

for every measurable subset B of Ω. However,

$$\int_B \prod_{j=0}^{n-1} (w_z \circ \alpha_z^{j-n}) \, d(\tilde{\mu}_z \circ \alpha_z^{-n}) = \int_{B \cap \alpha_z^n(A_z^c \cap K)} \prod_{j=0}^{n-1} (w_z \circ \alpha_z^{j-n}) \, d(\tilde{\mu}_z \circ \alpha_z^{-n}) =$$

$$= \int_{\alpha_z^{-n}(B) \cap A_z^c \cap K} \prod_{j=0}^{n-1} (w_z \circ \alpha_z^j) \, d\tilde{\mu}_z,$$

so we get

$$|T_z^{*n}(\tilde{\mu}_z)(B)| \leq \sup_{t \in A_z^c \cap K} \prod_{j=0}^{n-1} (w_z \circ \alpha_z^j)(t) |\tilde{\mu}_z|(\alpha_z^{-n}(B) \cap A_z^c \cap K) =$$

$$= \sup_{t \in A_z^c \cap K} \prod_{j=0}^{n-1} (w_z \circ \alpha_z^j)(t) |\mu|(\alpha_z^{-n}(B) \cap A_z^c \cap K)$$

for each $n \in \mathbb{N}$, $z \in F$ and every measurable subset B of Ω. It follows that

$$\| T_z^{*n}(\tilde{\mu}_z) \| \leq \sup_{t \in A_z^c \cap K} \left(\prod_{j=0}^{n-1} (w_z \circ \alpha_z^j)(t) \right) \| \mu \|$$

for each $n \in \mathbb{N}$, $z \in F$. In particular, for all $z \in F$ we have

$$\left\| T_z^*(\tilde{\mu}_z) \right\| \leq \sup_{t \in A_z^c \cap K} \left\{ w_z(t) \|\mu\| \right\}.$$

Other inequalities above can be proved in a similar way. Utilizing all these inequalities above together with assumptions (ii) with respect to

$$\varepsilon = \min \left\{ \frac{\delta}{4}, \frac{\delta^2}{64\|\mu\|\|v\|}, \frac{\delta}{8\|v\|} \right\},$$

and the fact that $B(\mu, \delta) \subset U_1$ and $B(\nu, \delta) \subset U_2$, we deduce that $\phi_z \in U_1$ and

$$\frac{\sqrt{\|T_z^*(\tilde{\nu}_z)\| + \|S_z^*(\tilde{\nu}_z)\|}}{\sqrt{\|T_z^*(\tilde{\mu}_z)\| + \|S_z^*(\tilde{\mu}_z)\|}} C_z^*(\phi_z) \in U_2$$

for all $z \in F$. Hence

$$F \subseteq \{z \in \mathcal{S} : \exists \lambda_z \in \mathbb{R}^+ \text{ such that } \lambda_z C_z^*(U_1) \cap U_2 \neq \emptyset\}.$$

Since $\mathcal{F}$ is Furstenberg , we obtain that

$$\{z \in \mathcal{S} : \exists \lambda_z \in \mathbb{R}^+ \text{ such that } \lambda_z C_z^*(U_1) \cap U_2 \neq \emptyset\} \in \mathcal{F}.$$

Hence, $\{C_z^*\}_{z \in \mathcal{S}}$ is $\mathcal{F}$-semi-transitive since U_1 and U_2 were arbitrary. $\quad\square$

Corollary 3.3. *Let w be a weight on Ω and α be a homeomorphism from Ω to Ω. Suppose that for each compact subset K of Ω it holds that*

$$\lim_{n \longrightarrow \infty} \left(\sup_{t \in K} \prod_{j=0}^{n-1} \left(w \circ \alpha^j \right)(t) \right) \left(\sup_{t \in K} \prod_{j=1}^{n} \left(w \circ \alpha^{-j} \right)^{-1}(t) \right) = 0$$

and either

$$\lim_{n \longrightarrow \infty} \left(\sup_{t \in K} \prod_{j=0}^{2n-1} \left(w \circ \alpha^j \right)(t) \right) = 0$$

or

$$\lim_{n \longrightarrow \infty} \left(\sup_{t \in K} \prod_{j=1}^{2n} \left(w \circ \alpha^{-j} \right)^{-1}(t) \right) = 0.$$

Then, for each non-empty open subsets U_1 and U_2 of $M(\Omega)$, there exists some $N \in \mathbb{N}$ and some sequence $\{\lambda_n\}_{n=N}^{\infty}$ of positive numbers such that $\lambda_n C_n^*(U_1) \cap U_2 \neq \emptyset$ for all $n \geq N$, where $C_n^* = \frac{1}{2}\left(T_{\alpha,w}^{*n} + S_{\alpha,w}^{*n} \right)$.

Proof. Let $S = \mathbb{N}$ and $\mathcal{F}$ be the collection of all $A \subseteq \mathbb{N}$ such that for some $N \in \mathbb{N}$ we have $\{N, N+1, \dots\} \subseteq A$. For each $n \in \mathbb{N}$, we let $\alpha_n := \alpha^n$ and $w_n := \prod_{j=0}^{n-1} \left(w \circ \alpha^j \right)$. Then we consider the case when $A_n = E_n = \emptyset$ and $D_n = K$ (such that $\tilde{\nu}_n = 0$) for all $n \in \mathbb{N}$, and the case when $A_n = D_n = \emptyset$ and $E_n = K$ for all $n \in \mathbb{N}$, (such that $\tilde{\nu}_n = 0$ for all $n \in \mathbb{N}$). By adapting the proof of Theorem 3.2 for these special cases, we can deduce the statement of corollary. $\quad\square$

Now we will provide a concrete example where the conditions of Corollary 3.3 are satisfied.

Example 3.4. *Let $\Omega = \mathbb{R}$ and M, δ be positive constants such that $M \geq 2 + 2\delta$ and $\delta \geq 1$. Put α to be the function on $\mathbb{R}$ given by $\alpha(t) = t + 1$ for all $t \in \mathbb{R}$. Set*

$$w(t) = \begin{cases} M \text{ for } t \leq -1, \\ M + \frac{t+1}{2}(1 + \delta - M) \text{ for } t \in [-1, 1], \\ 1 + \delta \text{ for } t \geq 1. \end{cases} \quad .$$

By some calculations one can check that for each compact subset K of Ω it holds

$$\lim_{n \longrightarrow \infty} \left(\sup_{t \in K} \prod_{j=0}^{n-1} \left(w \circ \alpha^j \right)(t) \right) \left(\sup_{t \in K} \prod_{j=1}^{n} \left(w \circ \alpha^{-j} \right)^{-1}(t) \right) = 0$$

and

$$\lim_{n \longrightarrow \infty} \left(\sup_{t \in K} \prod_{j=1}^{2n} \left(w \circ \alpha^{-j} \right)^{-1} (t) \right) = 0,$$

so the conditions of Corollary 3.3 are satisfied. However, it is not hard to check by some calculations that for every strictly increasing sequence $\{n_k\} \subseteq \mathbb{N}$ and every $t \in \mathbb{R}$ it holds that

$$\lim_{k \to \infty} \prod_{j=0}^{n_k-1} (w \circ \alpha^j)(t) = \infty,$$

hence the conditions of [29, Proposition 3.3] are not satisfied. This illustrates the difference between the sufficient conditions of Corollary 3.3 and the sufficient conditions of [29, Proposition 3.3].

4. Disjoint $\mathcal{F}$-semi-transitive generalized weighted shifts

We start with the following definition.

Definition 4.1. *Let $\mathcal{S}$ be a set, $\mathcal{F}$ be a Furstenberg family of subsets of $\mathcal{S}$ and $\{T_{t,1}\}_{t \in \mathcal{S}}, \ldots, \{T_{t,N}\}_{t \in \mathcal{S}}$ be families of bounded linear operators on a Banach space X. We say that $\{T_{t,1}\}_{t \in \mathcal{S}}, \ldots, \{T_{t,N}\}_{t \in \mathcal{S}}$ are disjoint $\mathcal{F}$-semi-transitive, or shortly d$\mathcal{F}$-semi-transitive, if for every collection of non-empty open subsets $O, V_1, \ldots, V_N$ of X, there exists some $F \in \mathcal{F}$ such that for all $t \in F$ there exists some $\lambda_t \in \mathbb{R}^+$ satisfying*

$$O \cap \lambda_t T_{t,1}^{-1}(V_1) \cap \cdots \cap \lambda_t T_{t,N}^{-1}(V_N) \neq \emptyset.$$

We recall also that the families $\{T_{t,1}\}_{t \in \mathcal{S}}, \ldots, \{T_{t,N}\}_{t \in \mathcal{S}}$ are said to be *disjoint $\mathcal{F}$-transitive, or shortly d$\mathcal{F}$-transitive*, if for every collection of non-empty open subsets $O, V_1, \ldots, V_N$ of X, there exists some $F \in \mathcal{F}$ such that for all $t \in F$ we have

$$O \cap T_{t,1}^{-1}(V_1) \cap \cdots \cap T_{t,N}^{-1}(V_N) \neq \emptyset.$$

In this section, for a separable Hilbert space H we let $B(H)$ be the space of all bounded linear operators on H, and C be the C^*-algebra of compact operators on H. For an orthonormal basis $\{e_j\}_{j \in \mathbb{Z}}$ for H, we let for each $m \in \mathbb{N}$, P_m be the orthogonal projection onto $\text{Span}\{e_{-m}, \ldots, e_m\}$. We will denote by $\ell_2(C)$ the standard (right) Hilbert module over C, see [35, Example 1.3.5]. Notice that $\ell_2(C)$ is a Banach algebra. Indeed, we can define multiplication on $\ell_2(C)$ as pointwise multiplication, i.e., if $\{x_j\}_{j \in \mathbb{Z}}, \{y_j\}_{j \in \mathbb{Z}} \in \ell_2(C)$, then

$$\{x_j\}_{j \in \mathbb{Z}} \cdot \{y_j\}_{j \in \mathbb{Z}} = \{x_j y_j\}_{j \in \mathbb{Z}}.$$

To see that $\{x_j y_j\}_{j \in \mathbb{Z}}$ belongs to $\ell_2(C)$, it suffices to observe that

$$\sum_j y_j^* x_j^* x_j y_j \leq \sum_j y_j^* \|x_j\|^2 y_j \leq \| \left(\sum_j x_j^* x_j \right) \| \sum_j y_j^* y_j.$$

For each $m, J \in \mathbb{N}$, we let $\tilde{p}_{J,m} \in \ell_2(C)$ be given by

$$(\tilde{p}_{J,m})_i = \begin{cases} P_m, & \text{if } -J \leq i \leq J, \\ 0, & \text{else.} \end{cases}$$

As shown in the proof of [28, Proposition 3.1], the set $\{\tilde{p}_{J,m}\}_{(J,m) \in \mathbb{N}^2}$ forms a left approximate unit for $\ell_2(C)$.

For $N \in \mathbb{N}$, let $r_1, \ldots, r_N \in \mathbb{N}$ with $r_1 < r_2 < \cdots < r_N$. Further, for each $n \in \mathbb{N}$ and $s \in \{1, \ldots, N\}$, let $\{W_{n,j}^{(s)}\}_{j \in \mathbb{Z}}$ be a family of operators in $B(H)$ which is uniformly bounded in norm and such that each $W_{n,j}^{(s)}$ has

a bounded inverse. Also, assume that $\mathcal{U}_1, \ldots, \mathcal{U}_N$ are unitary operators on H. Then, for each $n \in \mathbb{N}$ and $s \in \{1, \ldots, N\}$, we let $T_{n,s}$ be the operator on $\ell_2(C)$ defined as

$$\left(T_{n,s}(\{x_i\}_{i \in \mathbb{Z}})\right) = \{W_{n,j}^{(s)} x_{j-r_s n} \mathcal{U}_s^{r_s n}\}_{j \in \mathbb{Z}}.$$

By [28], $T_{n,s}$ is a bounded linear operator on $\ell_2(C)$ for each $n \in \mathbb{N}$ and $s \in \{1, \ldots, N\}$.

In what follows, we will assume that $\mathcal{F}$ is a family of subsets of $\mathbb{N}$ which is finitely invariant, that is, for each $n \in \mathbb{N}$ and $F \in \mathcal{F}$ we have $F \setminus \{1, \ldots, n\} \in \mathcal{F}$ too.

Now we are ready to give the first result of this section.

Theorem 4.2. *Under the above notation and assumptions, the following statements are equivalent.*

(1) The families of operators $\{T_{n,1}\}_{n \in \mathbb{N}}, \ldots, \{T_{n,N}\}_{n \in \mathbb{N}}$ are $d\mathcal{F}$-semi-transitive.

(2) For every $J, m \in \mathbb{N}$ and $\varepsilon > 0$, there exists some $F \in \mathcal{F}$ such that for each $j \in \mathbb{Z}$ with $-J \leq j \leq J$ we can find families $\{D_{n,j}\}_{n \in F}$, $\{G_{n,j}^{(1)}\}_{n \in F}, \ldots, \{G_{n,j}^{(N)}\}_{n \in F}$ of compact operators on H satisfying

$$\|D_{n,j} - P_m\| < \varepsilon, \quad \|G_{n,j}^{(s)} - P_m\| < \varepsilon,$$

and

$$\|W_{n,j+r_\ell n}^{(\ell)} D_{n,j}\| \, \|W_{n,j}^{(s)-1} G_{n,j}^{(s)}\| < \varepsilon^2 \quad \text{for all } n \in F \text{ and } \ell, s \in \{1, \ldots, N\},$$

and satisfying in addition that for each distinct $s, \ell \in \{1, \ldots, N\}$ and all $n \in F$

$$\|W_{n,j-r_s n+r_\ell n}^{(\ell)} W_{n,j}^{(s)-1} G_{n,j}^{(s)}\| < \varepsilon.$$

Proof. (1) $\Rightarrow$ (2) Let $J, m \in \mathbb{N}$ be given. Since the families $\{T_{n,1}\}_{n \in \mathbb{N}}, \ldots, \{T_{n,N}\}_{n \in \mathbb{N}}$ are disjoint $\mathcal{F}$-semi-transitive and $\mathcal{F}$ is finitely invariant. given $\varepsilon > 0$, we can find some $F \subseteq \mathcal{F}$ with $\{1, \ldots, 2J\} \cap F = \emptyset$ such that for each $n \in F$, there exists some $x_n \in \ell_2(C)$ and some $\lambda_n \in \mathbb{R}^+$ with

$$\|(x_n) - \tilde{p}_{J,m}\| < \varepsilon,$$

$$\|\lambda_n(T_{n,\ell} x_n) - \tilde{p}_{J,m}\| < \varepsilon \text{ for all } \ell \in \{1, \ldots, N\}.$$

This gives further that

$$\|(x_n)_j - P_m\| < \varepsilon,$$

and

$$\|\lambda_n W_{n,j+r_\ell n}^{(\ell)}(x_n)_j \mathcal{U}_\ell^{r_\ell n}\| < \varepsilon.$$

for all $j \in \mathbb{Z}$ with $-J \leq j \leq J$ and $\ell \in \{1, \ldots, N\}$, because

$$(T_{n,\ell}(x_n))_{j+r_\ell n} = W_{n,j+r_\ell n}^{(\ell)} (x_n)_j \mathcal{U}_\ell^{r_\ell n},$$

and $j + r_\ell n > J$ for all $j \in \mathbb{Z}$ with $-J \leq j \leq J$ and $n \in F$.

Hence

$$|\lambda_n| \, \|W_{n,j+r_\ell n}^{(\ell)}(x_n)_j\| = \|\lambda_n W_{n,j+r_\ell n}^{(\ell)}(x_n)_j\| =$$

$$= \|\lambda_n W_{n,j+r_\ell n}^{(\ell)}(x_n)_j \mathcal{U}_\ell^{r_\ell n}\| < \varepsilon$$

for all $n \in F$, $j \in \mathbb{Z}$ with $-J \leq j \leq J$ and $\ell \in \{1, \ldots, N\}$. Further, since

$$\|\lambda_n(T_{n,s} x_n)_j - P_m\| < \varepsilon \quad \text{for all } j \in \mathbb{Z} \text{ with } -J \leq j \leq J \text{ and } s \in \{1, \ldots, N\},$$

it follows that

$$\|\lambda_n W_{n,j}^{(s)}(x_n)_{j-r_s n} \mathcal{U}_s^{r_s n} - P_m\| < \varepsilon,$$

however,

$$\frac{1}{|\lambda_n|}\|W_{n,j}^{(s)-1}(\lambda_n W_{n,j}^{(s)}(x_n)_{j-r_sn}\,\mathcal{U}_s^{r_sn})\| = \|W_{n,j}^{(s)-1}\frac{1}{\lambda_n}(\lambda_n W_{n,j}^{(s)}(x_n)_{j-r_sn}\,\mathcal{U}_s^{r_sn})\|$$

$$= \|(x_n)_{j-r_sn}\,\mathcal{U}_s^{r_sn}\| = \|(x_n)_{j-r_sn}\| \le \|x_n - \tilde{p}_{J,m}\| < \varepsilon,$$

once again because $j - r_sn < J - 2J \le -J$. For each $n \in F$, $\ell, s \in \{1,\ldots,N\}$ and $j \in \mathbb{Z}$ with $-J \le j \le J$, define

$$D_{n,j} := (x_n)_j \quad \text{and} \quad G_{n,j}^{(s)} = \lambda_n W_{n,j}^{(s)}(x_n)_{j-r_sn}\,\mathcal{U}_s^{r_sn}.$$

Then

$$\|D_{n,j} - P_m\| < \varepsilon, \quad \|G_{n,j}^{(s)} - P_m\| < \varepsilon,$$

and

$$|\lambda_n|\,\|W_{n,j+r_\ell n}^{(\ell)}D_{n,j}\| = \|\lambda_n W_{n,j+r_\ell n}^{(\ell)}D_{n,j}\| < \varepsilon,$$

$$\tfrac{1}{|\lambda_n|}\|W_{n,j}^{(s)-1}G_{n,j}^{(s)}\| = \|W_{n,j}^{(s)-1}\tfrac{1}{\lambda_n}G_{n,j}^{(s)}\| < \varepsilon,$$

so

$$\|W_{n,j+r_\ell n}^{(\ell)}D_{n,j}\|\|W_{n,j}^{(s)-1}G_{n,j}^{(s)}\| < \varepsilon^2.$$

Finally, if $\ell, s \in \{1,\ldots,N\}$ with $\ell \ne s$, then $|r_\ell - r_s|n > 2J$ for all $n \in F$. Hence, either $j - r_sn + r_\ell n < -J$ or $j - r_sn + r_\ell n > J$ for any $j \in \mathbb{Z}$ with $-J \le j \le J$. Therefore, $(\tilde{p}_{J,m})_{j-r_sn+r_\ell n} = 0$, hence

$$\|W_{n,j-r_sn+r_\ell n}^{(\ell)}\,W_{n,j}^{(s)-1}\,G_{n,j}^{(s)}\| =$$

$$= \|\lambda_n W_{n,j-r_sn+r_\ell n}^{(\ell)}\,W_{n,j}^{(s)-1}\tfrac{1}{\lambda_n}G_{n,j}^{(s)}\| = \left\|\lambda_n W_{n,j-r_sn+r_\ell n}^{(\ell)}(x_n)_{j-r_sn}\,\mathcal{U}_s^{r_sn}\right\|$$

$$= \left\|\lambda_n W_{n,j-r_sn+r_\ell n}^{(\ell)}(x_n)_{j-r_sn}\right\| = \left\|\lambda_n W_{n,j-r_sn+r_\ell n}^{(\ell)}(x_n)_{j-r_sn}\,\mathcal{U}_\ell^{r_\ell n}\right\|$$

$$= \left\|\lambda_n (T_{n,\ell}(x_n))_{j-r_sn+r_\ell n}\right\| = \left\|(\lambda_n T_{n,\ell}(x_n) - \tilde{p}_{J,m})_{j-r_sn+r_\ell n}\right\| < \varepsilon.$$

for all $n \in F$, $j \in \mathbb{Z}$ with $-J \le j \le J$, and $s, \ell \in \{1,\ldots,N\}$ with $s \ne \ell$.

(2) $\Rightarrow$ (1) Let $O, V_1,\ldots,V_N$ be non-empty open subsets of $\ell_2(C)$. Then there exist some $J, m \in \mathbb{N}$ and some $x \in O$, $y^{(1)} \in V_1,\ldots,y^{(N)} \in V_N$ such that $\tilde{p}_{J,m}x = x$ and $\tilde{p}_{J,m}y^{(\ell)} = y^{(\ell)}$ for each $\ell \in \{1,\ldots,N\}$ since the set $\{\tilde{p}_{J,m}\}_{(J,m)\in\mathbb{N}^2}$ forms a left approximate unit for $\ell_2(C)$.

For each $a \in \ell_2(C)$, let $B_\delta(a)$ denote the open ball with centre in a and radius δ. We can find some $\delta > 0$ such that $B_\delta(x) \subseteq O$ and $B_\delta(y^{(\ell)}) \subseteq V_\ell$ for each $\ell \in \{1,\ldots,N\}$. Put $C_y = \sum_{s=1}^{N}\|y^{(s)}\|$ and

$$\varepsilon := \frac{\delta}{2JN(\|x\| + 2C_y + \delta)}.$$

Choose $F \in \mathcal{F}$ and families

$$\{D_{n,j}\}_{n\in F}, \quad \{G_{n,j}^{(1)}\}_{n\in F},\ldots,\{G_{n,j}^{(N)}\}_{n\in F}$$

for each $j \in \mathbb{Z}$ with $-J \le j \le J$, satisfying the assumptions of (2) with respect to J, m, and ε.

For each $n \in F$, let $u_n, v_n^{(1)},\ldots,v_n^{(N)} \in \ell_2(C)$ be given by

$$(u_n)_j = \begin{cases} D_{n,j}x_j, & \text{if } -J \le j \le J, \\ 0, & \text{else.} \end{cases}$$

and

$$(v_n^{(\ell)})_{j-r_\ell n} = \begin{cases} W_{n,j}^{(\ell)-1}\,G_{n,j}^{(\ell)}(y^{(\ell)})_j\,\mathcal{U}_\ell^{*r_\ell n}, & \text{if } -J \le j \le J, \\ 0, & \text{else.} \end{cases}$$

Since $\|D_{n,j} - P_m\| < \varepsilon$ and $\|G^{(\ell)}_{n,j} - P_m\| < \varepsilon < 1$ for all $n \in F$, $\ell \in \{1, \ldots, N\}$ and $j \in \mathbb{Z}$ with $-J \le j \le J$, it follows that $D_{n,j} \ne 0$ and $G^{(\ell)}_{n,j} \ne 0$, hence $W^{(\ell)}_{n,j+r_\ell n} D_{n,j} \ne 0$, and $W^{(\ell)-1}_{n,j} G^{(\ell)}_{n,j} \ne 0$ for all $n \in F$, $\ell \in \{1, \ldots, N\}$ and $j \in \mathbb{Z}$ with $-J \le j \le J$. Now we notice that

$$\|T_{n,\ell}(u_n)\| \le \sum_{j=-J}^{J} \|W^{(\ell)}_{n,j+r_\ell n} D_{n,j}\| \, \|x\|$$

because

$$(T_{n,\ell}(u_n))_{j+r_\ell n} = \begin{cases} W^{(\ell)}_{n,j+r_\ell n} D_{n,j} x_j \, \mathcal{U}^{r_\ell n}_\ell, & \text{if } -J \le j \le J, \\ 0, & \text{else.} \end{cases}$$

Moreover, since for each distinct $s, \ell \in \{1, \ldots, N\}$ and all $n \in F$,

$$(T_{n,\ell}(v^{(s)}_n))_{j-r_s n + r_\ell n} =$$

$$= \begin{cases} W^{(\ell)}_{n,j-r_s n + r_\ell n} \, W^{(s)-1}_{n,j} \, G^{(s)}_{n,j}(y^{(s)})_j \, \mathcal{U}^{*\,r_s n}_s \, \mathcal{U}^{r_\ell n}_\ell, & \text{if } -J \le j \le J, \\ 0, & \text{else.} \end{cases}$$

it follows that

$$\|T_{n,\ell}(v^{(s)}_n)\| \le \sum_{j=-J}^{J} \|W^{(\ell)}_{n,j-r_s n + r_\ell n} \, W^{(s)-1}_{n,j} \, G^{(s)}_{n,j}\| \, \|y^{(s)}\|.$$

Finally, it is easy to see that by the construction of $v^{(\ell)}_n$, we have

$$T_{n,\ell}(v^{(\ell)}_n)_j = G^{(\ell)}_{n,j}(y^{(\ell)})_j \quad \text{if } -J \le j \le J, \quad \text{and } 0 \text{ else.}$$

For each $n \in F$, set

$$x_n := u_n + \frac{\sqrt{\sum_{\ell=1}^{N} \sum_{j=-J}^{J} \|W^{(\ell)}_{n,j+r_\ell n} D_{n,j}\|}}{\sqrt{\sum_{\ell=1}^{N} \sum_{j=-J}^{J} \|W^{(\ell)-1}_{n,j} G^{(\ell)}_{n,j}\|}} \sum_{\ell=1}^{N} v^{(\ell)}_n.$$

By the triangle inequality and the choice of ε, it is not hard to deduce by some calculations that

$$x_n \in O \quad \text{and} \quad \frac{\sqrt{\sum_{\ell=1}^{N} \sum_{j=-J}^{J} \|W^{(\ell)-1}_{n,j} G^{(\ell)}_{n,j}\|}}{\sqrt{\sum_{\ell=1}^{N} \sum_{j=-J}^{J} \|W^{(\ell)}_{n,j+r_\ell n} D_{n,j}\|}} T_{n,\ell}(x_n) \in V_\ell$$

for all $n \in F$ and $\ell \in \{1, \ldots, N\}$. $\square$

Next, let $\{S_n\}_{n \in \mathbb{N}}$ be a sequence of bounded invertible linear operators on H and $\{T_{n,1}\}_{n \in \mathbb{N}}, \ldots, \{T_{n,N}\}_{n \in \mathbb{N}}$ be families of generalized weighted bilateral shifts on $\ell_2(C)$ as defined in the beginning of this section. For each $n \in F$ and $s \in \{1, \ldots, N\}$ we let $S_n T_{n,s}$ be the operator on $\ell_2(C)$ given by

$$\left(S_n T_{n,s}(\{x_i\}_{i \in \mathbb{Z}})\right) := \{S_n W^{(s)}_{n,j} x_{j-r_s n} \mathcal{U}^{r_s n}_s\}_{j \in \mathbb{Z}}.$$

It is not hard to check that the operator $S_n T_{n,s}$ is a bounded linear operator on $\ell_2(C)$. In fact, $S_n T_{n,s}$ is again a generalized weighted bilateral shift operator on $\ell_2(C)$ with respective weights $\{S_n W^{(s)}_{n,j}\}_{j \in \mathbb{Z}}$ for every $n \in F$ and $s \in \{1, \ldots, N\}$.

We will say that $\{T_{n,1}\}_{n\in\mathbb{N}},\dots,\{T_{n,N}\}_{n\in\mathbb{N}}$ are *disjoint $\mathcal{F}$-C*-semi-transitive, or shortly $d\mathcal{F}$-C*-semi-transitive,* if for every collection of non-empty open subsets $O, V_1, \dots, V_N$ of $\ell_2(C)$, there exists some $F \in \mathcal{F}$ such that for all $n \in F$ there exists some invertible bounded linear operator S_n on H satisfying

$$O \cap (S_n T_{n,1})^{-1}(V_1) \cap \dots \cap (S_n T_{n,N})^{-1}(V_N) \neq \emptyset.$$

The next theorem can easily be deduced from Theorem 4.2.

Theorem 4.3. *Under the above notation and assumptions, the following statements are equivalent.*

(1) The families of operators $\{T_{n,1}\}_{n\in\mathbb{N}}, \dots, \{T_{n,N}\}_{n\in\mathbb{N}}$ are $d\mathcal{F}$-C*-semi-transitive.

(2) For every $J, m \in \mathbb{N}$ and $\varepsilon > 0$, there exists some $F \in \mathcal{F}$ and a sequence $\{S_n\}_{n\in F}$ of bounded invertible linear operators on H such that for each $j \in \mathbb{Z}$ with $-J \leq j \leq J$ we can find families $\{D_{n,j}\}_{n\in F}, \quad \{G_{n,j}^{(1)}\}_{n\in F}, \dots, \{G_{n,j}^{(N)}\}_{n\in F}$ of compact operators on H satisfying for all $n \in F$ and $\ell, s \in \{1, \dots, N\}$ that

$$\|D_{n,j} - P_m\| < \varepsilon, \quad \|G_{n,j}^{(s)} - P_m\| < \varepsilon,$$

$$\|S_n W_{n,j+r_\ell n}^{(\ell)} D_{n,j}\| < \varepsilon, \ \|W_{n,j}^{(s)-1} S_n^{-1} G_{n,j}^{(s)}\| < \varepsilon,$$

and satisfying in addition that for each distinct $s, \ell \in \{1, \dots, N\}$ and all $n \in F$

$$\|S_n W_{n,j-r_s n+r_\ell n}^{(\ell)} W_{n,j}^{(s)-1} S_n^{-1} G_{n,j}^{(s)}\| < \varepsilon.$$

Proof. For the proof of (1) $\Rightarrow$ (2), we can proceed in exactly the same way as in the proof of Theorem 4.2 part (1) $\Rightarrow$ (2) by considering S_n and S_n^{-1} instead of λ_n and $\frac{1}{\lambda_n}$, respectively.

For the proof of (2) $\Rightarrow$ (1), we can also proceed similarly as in the proof of Theorem 4.2 part (2) $\Rightarrow$ (1). However, the only difference is that in this case we let now for each $n \in F$, $v_n^{(1)}, \dots, v_n^{(N)} \in \ell_2(C)$ be given by

$$(v_n^{(\ell)})_{j-r_\ell n} = \begin{cases} W_{n,j}^{(\ell)-1} S_n^{-1} G_{n,j}^{(\ell)} (y^{(\ell)})_j \, \mathcal{U}_\ell^{*r_\ell n}, & \text{if } -J \leq j \leq J, \\ 0, & \text{else,} \end{cases}$$

which clearly gives that

$$\|(v_n^{(\ell)})\| \leq \sum_{j=-J}^{J} \|W_{n,j}^{(\ell)-1} S_n^{-1} G_{n,j}^{(\ell)}\| \|y^{(\ell)}\|.$$

Then it suffices to also notice in this case that for each $n \in F$ and $\ell \in \{1, \dots, N\}$ it holds that

$$\|S_n T_{n,\ell}(u_n)\| \leq \sum_{j=-J}^{J} \|S_n W_{n,j+r_\ell n}^{(\ell)} D_{n,j}\| \, \|x\|$$

(where u_n is the same as in the proof of Theorem 4.2 in the part (2) $\Rightarrow$ (1)) and and moreover, for each distinct $s, \ell \in \{1, \dots, N\}$ and $n \in F$ we have

$$\|S_n T_{n,\ell}(v_n^{(s)})\| \leq \sum_{j=-J}^{J} \|S_n W_{n,j-r_s n+r_\ell n}^{(\ell)} W_{n,j}^{(s)-1} S_n^{-1} G_{n,j}^{(s)}\| \, \|y^{(s)}\|.$$

Finally, by the construction of $v_n^{(\ell)}$, it follows that

$$S_n T_{n,\ell}(v_n^{(\ell)})_j = G_{n,j}^{(\ell)} (y^{(\ell)})_j \quad \text{if } -J \leq j \leq J, \quad \text{and } 0 \text{ else,}$$

for each $n \in F$ and $\ell \in \{1, \dots, N\}$.

For each $n \in F$, set

$$x_n := u_n + \sum_{\ell=1}^{N} v_n^{(\ell)}.$$

By the triangle inequality and the choice of ε, it is not hard to deduce by some calculations that

$$x_n \in O \quad \text{and} \quad S_n T_{n,\ell}(x_n) \in V_\ell$$

for all $n \in F$ and $\ell \in \{1, \ldots, N\}$. $\quad \square$

Now, let $\mathcal{F}$ be the family of all infinite subsets of $\mathbb{N}$ and

$$\{W_j^{(1)}\}_{j \in \mathbb{Z}}, \ldots, \{W_j^{(N)}\}_{j \in \mathbb{Z}} \subset B(H)$$

with each $W_j^{(\ell)}$ ($j \in \mathbb{Z}$, $1 \le \ell \le N$) invertible, and

$$\|W_j^{(\ell)}\| \le M \quad \text{for all } j \in \mathbb{Z},\ \ell \in \{1, \ldots, N\},$$

for some fixed $M > 0$. For each $n \in \mathbb{N}$, $i \in \mathbb{Z}$ and $\ell \in \{1, \ldots, N\}$, we let $W_{n,i}^{(l)} \in B(H)$ be given by

$$W_{n,i}^{(l)} = W_i^{(\ell)} W_{i-1}^{(\ell)} \cdots W_{i-r_\ell n+1}^{(\ell)}.$$

Finally, for each $\ell \in \{1, \ldots, N\}$ we let $\tilde{T}_\ell$ be the generalized bilateral weighted shift operator on $\ell_2(C)$ (introduced in [28]) with respect to the sequence $\{W_j^{(\ell)}\}_{j \in \mathbb{Z}}$ and unitary $\mathcal{U}_\ell$, i.e.

$$(\tilde{T}_\ell(a))_i = W_i^{(\ell)} a_{i-1} \mathcal{U}_\ell \quad \text{for all } i \in \mathbb{Z},\ a \in \ell_2(C).$$

Then it is not hard to check that for each $\ell \in \{1, \ldots, N\}$ it holds that

$$T_{n,\ell} = \tilde{T}_\ell^{r_\ell n} \quad \text{for all } n \in \mathbb{N}.$$

We will now present an application of Theorem 4.2 and Theorem 4.3 in this concrete case.

Corollary 4.4. *Under the above notation and assumptions, the following statements are equivalent.*
(1) The sequences of operators $\{\tilde{T}_1^{r_1 n}\}_{n \in \mathbb{N}}, \ldots, \{\tilde{T}_N^{r_N n}\}_{n \in \mathbb{N}}$ are $d\mathcal{F}$-semi-transitive.
(2) For every $J, m \in \mathbb{N}$ there exist a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ and sequences of operators in C

$$\{D_{-J}^{(k)}\}_{k \in \mathbb{N}}, \quad \{G_{-J,1}^{(k)}\}_{k \in \mathbb{N}}, \ldots, \{G_{-J,N}^{(k)}\}_{k \in \mathbb{N}},$$

$$\{D_{-J+1}^{(k)}\}_{k \in \mathbb{N}}, \quad \{G_{-J+1,1}^{(k)}\}_{k \in \mathbb{N}}, \ldots, \{G_{-J+1,N}^{(k)}\}_{k \in \mathbb{N}},$$

$$\vdots$$

$$\{D_J^{(k)}\}_{k \in \mathbb{N}}, \quad \{G_{J,1}^{(k)}\}_{k \in \mathbb{N}}, \ldots, \{G_{J,N}^{(k)}\}_{k \in \mathbb{N}},$$

such that

$$\lim_{k \to \infty} \|D_j^{(k)} - P_m\| = \lim_{k \to \infty} \|G_{j,\ell}^{(k)} - P_m\| = 0,$$

and

$$\lim_{k \to \infty} \|W_{j-r_s n_k+1}^{(s)\,-1} \cdots W_j^{(s)\,-1} G_{j,s}^{(k)}\|\|W_{j+r_\ell n_k}^{(\ell)} \cdots W_{j+1}^{(\ell)} D_j^{(k)}\| = 0$$

for all $J \in \mathbb{Z}$ with $-J \le j \le J$ and $\ell, s \in \{1, \ldots, N\}$. Moreover, for each distinct $\ell, s \in \{1, \ldots, N\}$ and all $j \in \mathbb{Z}$ with $-J \le j \le J$, it holds that

$$\lim_{k \to \infty} \|W_{j+r_\ell n_k-r_s n_k}^{(\ell)} \cdots W_{j-r_s n_k+1}^{(\ell)} W_{j-r_s n_k+1}^{(s)\,-1} \cdots W_j^{(s)\,-1} G_{j,s}^{(k)}\| = 0.$$

Corollary 4.5. *Under the above notation and assumptions, the following statements are equivalent.*

(1) The sequences of operators $\{\tilde{T}_1^{r_1 n}\}_{n \in \mathbb{N}}, \ldots, \{\tilde{T}_N^{r_N n}\}_{n \in \mathbb{N}}$ are $d\mathcal{F}$-C*-semi-transitive.

(2) For every $J, m \in \mathbb{N}$ there exist a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$, a sequence $\{S_{n_k}\}_{k \in \mathbb{N}}$ of bounded invertible linear operators on H and sequences of operators in C

$$\{D_{-J}^{(k)}\}_{k \in \mathbb{N}}, \quad \{G_{-J,1}^{(k)}\}_{k \in \mathbb{N}}, \ldots, \{G_{-J,N}^{(k)}\}_{k \in \mathbb{N}},$$

$$\{D_{-J+1}^{(k)}\}_{k \in \mathbb{N}}, \quad \{G_{-J+1,1}^{(k)}\}_{k \in \mathbb{N}}, \ldots, \{G_{-J+1,N}^{(k)}\}_{k \in \mathbb{N}},$$

$$\vdots$$

$$\{D_{J}^{(k)}\}_{k \in \mathbb{N}}, \quad \{G_{J,1}^{(k)}\}_{k \in \mathbb{N}}, \ldots, \{G_{J,N}^{(k)}\}_{k \in \mathbb{N}},$$

such that

$$\lim_{k \to \infty} \|D_j^{(k)} - P_m\| = \lim_{k \to \infty} \|G_{j,\ell}^{(k)} - P_m\| = 0,$$

and

$$\lim_{k \to \infty} \|W_{j-r_s n_k+1}^{(s)\,-1} \cdots W_j^{(s)\,-1} S_{n_k}^{-1} G_{j,s}^{(k)}\| = 0,$$

$$\lim_{k \to \infty} \|S_{n_k} W_{j+r_\ell n_k}^{(\ell)} \cdots W_{j+1}^{(\ell)} D_j^{(k)}\| = 0,$$

for all $J \in \mathbb{Z}$ with $-J \leq j \leq J$ and $\ell, s \in \{1, \ldots, N\}$ and satisfying in addition that for each distinct $\ell, s \in \{1, \ldots, N\}$ and all $j \in \mathbb{Z}$ with $-J \leq j \leq J$, it holds that

$$\lim_{k \to \infty} \|S_{n_k} W_{j+r_\ell n_k - r_s n_k}^{(\ell)} \cdots W_{j-r_s n_k+1}^{(\ell)} W_{j-r_s n_k+1}^{(s)\,-1} \cdots W_j^{(s)\,-1} S_{n_k}^{-1} G_{j,s}^{(k)}\| = 0.$$

In order to construct concrete examples satisfying the conditions in (2) of Corollary 4.4 and Corollary 4.5, we need first the following technical results.

Proposition 4.6. *Under the above notation and assumptions, if for each $j \in \mathbb{Z}$ there exist dense subsets $H_0^{(j)}, H_1^{(j)}, \ldots, H_N^{(j)}$ of H and a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that for each $s, \ell \in \{1, \ldots, N\}$ and every $x^{(j)} \in H_0^{(j)}, y_s^{(j)} \in H_s^{(j)}$ we have*

$$\lim_{k \to \infty} \left\| W_{j-r_s n_k+1}^{(s)\,-1} \cdots W_j^{(s)\,-1} y_s^{(j)} \right\| \left\| W_{j+r_\ell n_k}^{(\ell)} \cdots W_{j+1}^{(\ell)} x^{(j)} \right\| = 0,$$

and for each distinct $s, \ell \in \{1, \ldots, N\}$ and every $y_s^{(j)} \in H_s^{(j)}$, we have

$$\lim_{k \to \infty} \|W_{j+r_\ell n_k - r_s n_k}^{(\ell)} \cdots W_{j-r_s n_k+1}^{(\ell)} W_{j-r_s n_k+1}^{(s)\,-1} \cdots W_j^{(s)\,-1} y_s^{(j)}\| = 0,$$

then the sequences of operators $\{\tilde{T}_1^{r_1 n}\}_{n \in \mathbb{N}}, \ldots, \{\tilde{T}_N^{r_N n}\}_{n \in \mathbb{N}}$ are $d\mathcal{F}$-semi-transitive.

Proof. The main idea in this proof is motivated by the proofs of [28, Theorem 3.2] and [27, Proposition 2.7]. Given $m \in \mathbb{N}$ and $j \in \mathbb{Z}$ with $-J \leq j \leq J$, for every $\eta \in \{-m, \ldots, m\}$ we can find sequences

$$\{f_{i,j}^{(\eta)}\}_{i \in \mathbb{N}} \subseteq H_0^{(j)}, \{h_{i,l,j}^{(\eta)}\}_{i \in \mathbb{N}} \subseteq H_l^{(j)} \text{ for each } \ell \in \{1, \ldots, N\}$$

such that

$$f_{i,j}^{(\eta)} \to e_\eta, \text{ and } h_{i,\ell,j}^{(\eta)} \to e_\eta \quad \text{as } i \to \infty,$$

for all $\eta \in \{-m, \ldots, m\}$ and $\ell \in \{1, \ldots, N\}$. Since by the assumption we have for all $\ell, s \in \{1, \ldots, N\}$ and $\eta \in \{-m, \ldots, m\}$ that

$$\lim_{k \to \infty} \left\| W_{j-r_s n_k+1}^{(s)\,-1} \cdots W_j^{(s)\,-1} h_{i,s,j}^{(\eta)} \right\| \left\| W_{j+r_\ell n_k}^{(\ell)} \cdots W_{j+1}^{(\ell)} f_{i,j}^{(\eta)} \right\| = 0,$$

and for each distinct $s, \ell \in \{1, \ldots, N\}$ and $\eta \in \{-m, \ldots, m\}$ we have

$$\lim_{k \to \infty} \left\| W^{(\ell)}_{j+r_\ell n_k - r_s n_k} \cdots W^{(\ell)}_{j-r_\ell n_k+1} W^{(s)^{-1}}_{j-r_s n_k+1} \cdots W^{(s)^{-1}}_j h^{(\eta)}_{i,s,j} \right\| = 0,$$

we can inductively easily construct a subsequence $\{n_{k_i}\}_{i \in \mathbb{N}}$ such that

$$\left\| W^{(s)^{-1}}_{j-r_s n_{k_i}} \cdots W^{(s)^{-1}}_j h^{(\eta)}_{i,s,j} \right\| \left\| W^{(\ell)}_{j+r_\ell n_{k_i}} \cdots W^{(\ell)}_{j+1} f^{(\eta)}_{i,j} \right\| < \frac{1}{4m^2 4^i},$$

and

$$\left\| W^{(\ell)}_{j+r_\ell n_{k_i} - r_s n_{k_i}} \cdots W^{(\ell)}_{j-r_s n_{k_i}+1} W^{(s)^{-1}}_{j-r_s n_{k_i}+1} \cdots W^{(s)^{-1}}_j h^{(\eta)}_{i,s,j} \right\| < \frac{1}{2m 2^i}.$$

For each $i \in \mathbb{N}$ and $\ell \in \{1, \ldots, N\}$ we define operators $D^{(i)}_j$ and $G^{(i)}_{j,\ell}$ on H by

$$(D^{(i)}_j) e_\eta = \begin{cases} f^{(\eta)}_{i,j}, & \text{for } \eta \in \{-m, \ldots, m\}, \\ 0, & \text{else,} \end{cases} \qquad (G^{(i)}_{j,\ell}) e_\eta = \begin{cases} h^{(\eta)}_{i,\ell,j}, & \text{for } \eta \in \{-m, \ldots, m\}, \\ 0, & \text{else.} \end{cases}$$

By some calculations and arguments similar to those in the proof of [28, Theorem 3.2] and [27, Proposition 2.7], we can conclude that the sequences of operators

$$\{D^{(i)}_j\}_{i \in \mathbb{N}}, \quad \{G^{(i)}_{j,1}\}_{i \in \mathbb{N}}, \ldots, \{G^{(i)}_{j,N}\}_{i \in \mathbb{N}}$$

satisfy the conditions in (2) of Corollary 4.4. for every $j \in \mathbb{Z}$ with $-J \le j \le J$. $\quad\square$

Similarly, based on Theorem 4.3 and Corollary 4.5, we can prove the following proposition.

Proposition 4.7. *Under the above notation and assumptions, if for each $j \in \mathbb{Z}$ there exist dense subsets $H^{(j)}_0, H^{(j)}_1, \ldots, H^{(j)}_N$ of H, a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ and a sequence $\{S_{n_k}\}_{k \in \mathbb{N}}$ of bounded invertible linear operators on H such that for each $s, \ell \in \{1, \ldots, N\}$ and every $x^{(j)} \in H^{(j)}_0$, $y^{(j)}_s \in H^{(j)}_s$ we have*

$$\lim_{k \to \infty} \left\| W^{(s)^{-1}}_{j-r_s n_k+1} \cdots W^{(s)^{-1}}_j S^{-1}_{n_k} y^{(j)}_s \right\| = 0,$$

$$\lim_{k \to \infty} \left\| S_{n_k} W^{(\ell)}_{j+r_\ell n_k} \cdots W^{(\ell)}_{j+1} x^{(j)} \right\| = 0,$$

and for each distinct $s, \ell \in \{1, \ldots, N\}$ and every $y^{(j)}_s \in H^{(j)}_s$, we have

$$\lim_{k \to \infty} \| S_{n_k} W^{(\ell)}_{j+r_\ell n_k - r_s n_k} \cdots W^{(\ell)}_{j-r_s n_k+1} W^{(s)^{-1}}_{j-r_s n_k+1} \cdots W^{(s)^{-1}}_j S^{-1}_{n_k} y^{(j)}_s \| = 0,$$

then the sequences of operators $\{\tilde{T}^{r_1 n}_1\}_{n \in \mathbb{N}}, \ldots, \{\tilde{T}^{r_N n}_N\}_{n \in \mathbb{N}}$ are $d\mathcal{F}$-C*-semi-transitive.

Thanks to Proposition 4.6 we will now give an example where we will show that it can happen that $\|W_j\|, \|W^{-1}_j\| \ge 1$ for all $j \in \mathbb{Z}$ and that at a same time the sequence of operators $\{\tilde{T}^n\}_{n \in \mathbb{N}}$ is $\mathcal{F}$-semi-transitive, which illustrates the important difference between $\mathcal{F}$-semi-transitivity (and thus supercyclicity) of generalized bilateral weighted shifts on Hilbert C*-modules and supercyclicity of the classical bilateral weighted shifts on the sequence space ℓ^2.

Example 4.8. *Let $N = 1, r_1 = 1$ and $H = L^2(\mathbb{R})$. As in [28, Example 3.3], choose some $\{w_j\}_{j \in \mathbb{Z}} \subseteq L^\infty(\mathbb{R})$ such that $w_j > 0$, $w^{-1}_j \in L^\infty(\mathbb{R})$ and $\| w_j \| \le M$ for all $j \in \mathbb{Z}$ and some $M > 1$. Let $\{c_j\}_{j \in \mathbb{Z}} \subseteq \mathbb{R}^+$ be such that $c_j \ge C$ for all $j \in \mathbb{Z}$ and some $C > 0$. For each $j \in \mathbb{Z}$ let α_j be the translation on $\mathbb{R}$ given by $\alpha_j(t) = t - c_j$ for all $t \in \mathbb{R}$. If there exists an $\varepsilon > 0$ such that $w_j \chi_{[0,\infty)} \le 1 - \varepsilon, w_j \chi_{(-\infty,0)} = 1$ for all $j \ge 0$ and $w_j = 1$ for all $j < 0$; then by the same arguments as in [28, Example 3.3] one can show that the conditions of Proposition 4.6 are satisfied when $W_j(t) = w_j \cdot (f \circ \alpha_j)$ for all $f \in L^2(\mathbb{R}), j \in \mathbb{Z}$ and $H^{(j)}_0 = H^{(j)}_1 = C_c(\mathbb{R})$ for all $j \in \mathbb{Z}$. However, by the construction of the sequence $\{w_j\}$, it follows that $\|W_j\|, \|W^{-1}_j\| \ge 1$ for all $j \in \mathbb{Z}$. On the other hand, if there exists an $\varepsilon > 0$ such that $w_j \chi_{[0,\infty)} = 1, w_j \chi_{(-\infty,0)} \ge 1 + \varepsilon$ for all $j < 0$ and $w_j = 1$ for all $j \ge 0$, then again the conditions of Proposition 4.6 will be satisfied, however, in this case we would again have by the construction of the sequence $\{w_j\}$ that $\|W_j\|, \|W^{-1}_j\| \ge 1$ for all $j \in \mathbb{Z}$.*

By our definition, $d\mathcal{F}$-semi-transitivity implies $d\mathcal{F}$-C*-semi-transitivity, however, the converse is not true in general, as the next example illustrates.

Example 4.9. *Let again $N = 1, r_1 = 1, H = L^2(\mathbb{R})$ and α be the function on $\mathbb{R}$ given by $\alpha(t) = t - 1$ for all $t \in \mathbb{R}$. Let $v = 2\chi_{(-\infty,0)} + \frac{1}{2}\chi_{[0,\infty)}$ and $\tilde{S}$ be the weighted composition operator on H with respect to v and α, that is $\tilde{S}(f) = v \cdot (f \circ \alpha)$ for all $f \in H$. For each $n \in \mathbb{N}$, put $S_n := \tilde{S}^n$. If we let for each $j \in \mathbb{Z}$, W_j be the identity operator on H, then it is not hard to check that the conditions of Proposition 4.7 are satisfied with $H_0^{(j)} = H_1^{(j)} = C_c(\mathbb{R})$ for all $j \in \mathbb{Z}$. Thus, the generalized non-weighted bilateral shift operator on $\ell_2(C)$ is $\mathcal{F}$-C*-semi-transitive, however, by Theorem 4.2 it can not be $\mathcal{F}$-semi-transitive on $\ell_2(C)$.*

Hypercyclicity and supercyclicity of a composition of a left and a right multiplier on the space of compact operators have been studied in for instance [19, 37]. Motivated by those results, in the rest of this paper, we shall study the dynamics of a composition of a right multiplier and generalized weighted bilateral shift operator on $\ell_2(C)$ defined in the beginning of this section. To this end, let $\tilde{W}_1, \ldots, \tilde{W}_N$ be invertible, bounded linear operators on H. For each $s \in \{1, \ldots, N\}$ and $n \in \mathbb{N}$, we let $R_{s,n}$ be the operator on $\ell_2(C)$ given by

$$R_{s,n}(\{x_i\}_{i\in\mathbb{Z}}) = (\{x_i \cdot \mathcal{U}_s^{*\,r_s n} \tilde{W}_s^{\,r_s n}\}_{i\in\mathbb{Z}})$$

for all $\{x_i\}_{i\in\mathbb{Z}} \in \ell_2(C)$. By [35, Proposition 1.2.4 (i)], the operator $R_{s,n}$ is a bounded linear operator on $\ell_2(C)$ for each $s \in \{1, \ldots, N\}$ and $n \in \mathbb{N}$. Let $\{T_{n,1}\}_{n\in\mathbb{N}}, \ldots, \{T_{n,N}\}_{n\in\mathbb{N}}$ be the families of operators on $\ell_2(C)$ as defined in the beginning of this section. For each $s \in \{1, \ldots, N\}$ and $n \in \mathbb{N}$, we let $T_{n,s} \circ R_{s,n}$ be the composition of $T_{n,s}$ and $R_{s,n}$, that is

$$\left((T_{n,s} \circ R_{s,n})(\{x_i\}_{i\in\mathbb{Z}})\right) = \{W_{n,j}^{(s)} x_{j-r_s n} \tilde{W}_s^{\,r_s n}\}_{j\in\mathbb{Z}}.$$

We will now present an extension of Theorem 4.2 in this setting.

Proposition 4.10. *Under the above notation and assumptions, we have (2) $\Rightarrow$ (1).*
(1) The families of operators $\{T_{n,1} \circ R_{1,n}\}_{n\in\mathbb{N}}, \ldots, \{T_{n,N} \circ R_{N,n}\}_{n\in\mathbb{N}}$ are $d\mathcal{F}$-semi-transitive.
(2) For every $J, m \in \mathbb{N}$ and $\varepsilon > 0$, there exists some $F \in \mathcal{F}$ such that we can find families

$$\{\tilde{D}_n\}_{n\in F}, \{\tilde{G}_n^{\,(1)}\}_{n\in F}, \ldots, \{\tilde{G}_n^{\,(N)}\}_{n\in F},$$

$$\{D_{n,-J}\}_{n\in F}, \{G_{n,-J}^{(1)}\}_{n\in F}, \ldots, \{G_{n,-J}^{(N)}\}_{n\in F},$$

$$\{D_{n,-J+1}\}_{n\in F}, \{G_{n,-J+1}^{(1)}\}_{n\in F}, \ldots, \{G_{n,-J+1}^{(N)}\}_{n\in F},$$

$$\vdots$$

$$\{D_{n,J}\}_{n\in F}, \{G_{n,J}^{(1)}\}_{n\in F}, \ldots, \{G_{n,J}^{(N)}\}_{n\in F},$$

of compact operators on H satisfying

$$\|\tilde{D}_n - P_m\| < \varepsilon, \|D_{n,j} - P_m\| < \varepsilon, \|\tilde{G}_n^{\,(s)} - P_m\| < \varepsilon, \|G_{n,j}^{(s)} - P_m\| < \varepsilon,$$

and

$$\|W_{n,j+r_\ell n}^{(\ell)} D_{n,j}\| \|\tilde{D}_n \tilde{W}_\ell^{\,r_\ell n}\| \, \|W_{n,j}^{(s)-1} G_{n,j}^{(s)}\| \|\tilde{G}_n^{\,(s)} \tilde{W}_s^{\,-r_s n}\| < \varepsilon^2$$

for all $n \in F, j \in \mathbb{Z}$ with $-J \leq j \leq J$ and $\ell, s \in \{1, \ldots, N\}$; and satisfying in addition that for each distinct $s, \ell \in \{1, \ldots, N\}$ and all $n \in F, j \in \mathbb{Z}$ with $-J \leq j \leq J$ it holds that

$$\|W_{n,j-r_s n+r_\ell n}^{(\ell)} W_{n,j}^{(s)-1} G_{n,j}^{(s)}\| \|\tilde{G}_n^{\,(s)} \tilde{W}_s^{\,-r_s n} \tilde{W}_\ell^{\,r_\ell n}\| < \varepsilon.$$

Proof. Let $O, V_1, \ldots, V_N$ be non-empty open subsets of $\ell_2(C)$. Then there exist some $J, m_1 \in \mathbb{N}$ and some $x \in O$, $y^{(1)} \in V_1, \ldots, y^{(N)} \in V_N$ such that $\tilde{p}_{J,m_1} x = x$ and $\tilde{p}_{J,m_1} y^{(\ell)} = y^{(\ell)}$ for each $\ell \in \{1, \ldots, N\}$. Since $\{P_m\}_{m \in \mathbb{N}}$ forms also a right approximate unit for C by [35, Proposition 2.2.1], it follows that we can find some $m_2 \in \mathbb{N}$ such that $\tilde{p}_{J,m_1} x \tilde{p}_{J,m_2} \in O$ and $\tilde{p}_{J,m_1} y^{(\ell)} \tilde{p}_{J,m_2} \in V_l$ for each $\ell \in \{1, \ldots, N\}$. Set $m := \max\{m_1, m_2\}$. Then $\tilde{p}_{J,m} x \tilde{p}_{J,m} \in O$ and $\tilde{p}_{J,m} y^{(\ell)} \tilde{p}_{J,m} \in V_l$ for each $\ell \in \{1, \ldots, N\}$. Put $C_y = \sum_{s=1}^{N} \|y^{(s)}\|$ and

$$\varepsilon := \frac{\delta}{(2JN+3)(\|x\| + 2C_y + \delta)}.$$

Choose $F \in \mathcal{F}$ and families

$$\{\tilde{D}_n\}_{n \in F}, \{D_{n,j}\}_{n \in F}, \{\tilde{G}_n^{(1)}\}_{n \in F}, \{G_{n,j}^{(1)}\}_{n \in F}, \ldots, \{\tilde{G}_n^{(N)}\}_{n \in F}, \{G_{n,j}^{(N)}\}_{n \in F}$$

for each $j \in \mathbb{Z}$ with $-J \leq j \leq J$, satisfying the assumptions of (2) with respect to J, m, and ε. For each $n \in F$, let $u_n, v_n^{(1)}, \ldots, v_n^{(N)} \in \ell_2(C)$ be given by

$$(u_n)_j = \begin{cases} D_{n,j} x_j \tilde{D}_n, & \text{if } -J \leq j \leq J, \\ 0, & \text{else.} \end{cases}$$

and

$$(v_n^{(\ell)})_{j-r_\ell n} = \begin{cases} W_{n,j}^{(\ell)-1} G_{n,j}^{(\ell)} (y^{(\ell)})_j \tilde{G}_n^{(\ell)} \tilde{W}_\ell^{-r_\ell n}, & \text{if } -J \leq j \leq J, \\ 0, & \text{else.} \end{cases}$$

By similar arguments as in the proof of Theorem 4.2 part (2) $\Rightarrow$ (1), we can deduce that $\tilde{D}_n \neq 0$ and $\tilde{G}_n^{(s)} \neq 0$ for all $n \in F, s \in \{1, \ldots, N\}$, and moreover

$$\|T_{n,\ell}(u_n)\| \leq \sum_{j=-J}^{J} \|W_{n,j+r_\ell n}^{(\ell)} D_{n,j}\| \|\tilde{D}_n \tilde{W}_\ell^{r_\ell n}\| \, \|x\|$$

and

$$\|T_{n,\ell}(v_n^{(s)})\| \leq \sum_{j=-J}^{J} \|W_{n,j-r_s n+r_\ell n}^{(\ell)} W_{n,j}^{(s)-1} G_{n,j}^{(s)}\| \|\tilde{G}_n^{(s)} \tilde{W}_s^{-r_s n} \tilde{W}_\ell^{r_\ell n}\| \, \|y^{(s)}\|.$$

For each $n \in F$, set

$$x_n := u_n + \frac{\sqrt{\sum_{\ell=1}^{N} \sum_{j=-J}^{J} \|W_{n,j+r_\ell n}^{(\ell)} D_{n,j}\| \|\tilde{D}_n \tilde{W}_\ell^{r_\ell n}\|}}{\sqrt{\sum_{\ell=1}^{N} \sum_{j=-J}^{J} \|W_{n,j}^{(\ell)-1} G_{n,j}^{(\ell)}\| \|\tilde{G}_n^{(\ell)} \tilde{W}_\ell^{-r_\ell n}\|}} \sum_{\ell=1}^{N} v_n^{(\ell)}.$$

and proceed further as in the proof of Theorem 4.2 part (2) $\Rightarrow$ (1). $\quad\square$

Let again, for each $\ell \in \{1, \ldots, N\}$, $\tilde{T}_\ell$ be the generalized bilateral weighted shift operator on $\ell_2(C)$ with respect to the sequence $\{W_j^{(\ell)}\}_{j \in \mathbb{Z}}$ and unitary $\mathcal{U}_\ell$, as defined in the beginning of this section. Recalling that the involution on C is an isometry, by similar arguments as in the proof of Proposition 4.6, we easily obtain the following corollary of Proposition 4.10 in the special case when $\mathcal{F}$ is the family of all infinite subsets of $\mathbb{N}$.

Corollary 4.11. *Under the above notation and assumptions, if there exist dense subsets*

$$\tilde{H}_0, \tilde{H}_1, \ldots, \tilde{H}_N,$$

$$H_0^{(-J)}, H_1^{(-J)}, \ldots, H_N^{(-J)},$$

$$\vdots$$

$$H_0^{(J)}, H_1^{(J)}, \ldots, H_N^{(J)}$$

of H and a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that for each $s, \ell \in \{1, \ldots, N\}$, each $j \in \mathbb{Z}$ with $-J \le j \le J$ and every $\tilde{x} \in \tilde{H}_0, x^{(j)} \in H_0^{(j)}, \tilde{y}_s \in \tilde{H}_s, y_s^{(j)} \in H_s^{(j)}$ we have

$$\lim_{k\to\infty} \left\| W_{j-r_s n_k+1}^{(s)\,-1} \cdots W_j^{(s)\,-1} y_s^{(j)} \right\| \left\| ((\tilde{W}_s^{-r_s n_k})^* \tilde{y}_s \right\| \left\| W_{j+r_\ell n_k}^{(\ell)} \cdots W_{j+1}^{(\ell)} x^{(j)} \right\| \left\| (\tilde{W}_\ell^{r_\ell n_k})^* \tilde{x} \right\| = 0,$$

and for each distinct $s, \ell \in \{1, \ldots, N\}$ and every $y_s^{(j)} \in H_s^{(j)}$, we have

$$\lim_{k\to\infty} \| W_{j+r_\ell n_k - r_s n_k}^{(\ell)} \cdots W_{j-r_s n_k+1}^{(\ell)} W_{j-r_s n_k+1}^{(s)\,-1} \cdots W_j^{(s)\,-1} y_s^{(j)} \| \| (\tilde{W}_\ell^{r_\ell n_k})^* (\tilde{W}_s^{-r_s n_k})^* \tilde{y}_s \| = 0,$$

then the sequences of operators $\{\tilde{T}_1^{r_1 n} \circ R_{1,n}\}_{n\in\mathbb{N}}, \ldots, \{\tilde{T}_N^{r_N n} \circ R_{N,n}\}_{n\in\mathbb{N}}$ are $d\mathcal{F}$-semi-transitive on $\ell_2(C)$.

We end this paper with an example where the conditions of Corollary 4.11 are satisfied.

Example 4.12. *Let $N = 1$ and $H = L^2(\mathbb{R})$. For each $j \in \mathbb{Z}$, let W_j be the identity operator on H. Put $\tilde{\alpha}$ to be the function on $\mathbb{R}$ given by $\tilde{\alpha}(t) = t - 1$ for all $t \in \mathbb{R}$ and $\tilde{w} == 2\chi_{(-\infty,0)} + \frac{1}{2}\chi_{[0,\infty)}$. If we let $\tilde{W}$ be the weighted composition operator on H with respect to $\tilde{\alpha}$ and $\tilde{w}$, then by some calculations it can be checked that the the conditions of Corollary 4.11 are satisfied with $\tilde{H}_0 = \tilde{H}_1 = H_0^{(j)} = H_1^{(j)} = C_c(\mathbb{R})$ for all $j \in \mathbb{Z}$.*

Remark 4.13. *We notice that due to [35, Proposition 2.2.1] our results in this section remain valid if we instead of a separable Hilbert space H consider any standard Hilbert module $\ell_2(\mathcal{A})$ over a unital C*-algebra $\mathcal{A}$.*

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