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Error bounds of Boole's formula type inequalities for generalized coordinated convex functions with computational analysis and applications

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Abstract. This study aims to enhance the error bounds for Boole's formula-type inequalities applied to generalized coordinate convex functions. A solid mathematical foundation is established to provide improved error estimates for two-variable integrals that surpass those derived from traditional coordinate convex functions. The motivation is twofold: to extend results from convex functions to the broader class of s-convex functions and to offer optimal approximations that enhance numerical method accuracy. Moreover, Boole's formula offers better absolute error bounds for higher-degree polynomials, especially in cases where other methods, such as Simpson's rule, fail to provide the required precision. This is due to Boole's formula's ability to approximate polynomials up to degree five exactly, whereas Simpson's rule is limited to polynomials of degree three. Through computational analysis, key results are presented, demonstrating that the improved error bounds provide greater reliability for numerical integration applications. This research broadens the theoretical framework of mathematical analysis and optimization while providing practical methodologies to tackle real-world challenges with accuracy and effectiveness. Future extensions of this work may include applications to q-calculus, symmetrized q-calculus, fractional calculus, and multidimensional spaces, providing deeper insights into the behavior of Boole's formula and its error bounds.

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1. Introduction and Preliminaries

Convexity is essential in optimization theory and is applicable in various fields of pure and applied sciences. A convex function can be defined as

Definition 1.1. *[1] Let I be a convex subset of a real vector space and $F : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$F(\tau\theta + (1-\tau)\vartheta) \leq \tau F(\theta) + (1-\tau)F(\vartheta), \tag{1}$$

for all $\tau \in [0,1]$, and $\theta, \vartheta \in I$.

The Hermite-Hadamard inequality is a key result in mathematical analysis, reflecting the profound developments in the field. For more details about the contributions related to this inequality, you can refer to [2, 3]. The inequality stated as follow:

Theorem 1.2. *A function $F : [\theta, \vartheta] \to \mathbb{R}$ is convex, then*

$$F(\frac{\theta+\vartheta}{2}) \leq \frac{1}{\vartheta-\theta}\int_\theta^\vartheta F(x)dx \leq \frac{F(\theta)+F(\vartheta)}{2}. \tag{2}$$

Numerous generalizations of convexity have emerged in recent decades, as detailed in [4–8]. Convex functions of two variables are vital in optimization across engineering, economics, machine learning, and operations research, often serving as objective functions. In [9], Dragomir expanded the concept of convex functions to include coordinate convex functions.

Definition 1.3. *[9] Let the bidimensional interval $[\theta, \vartheta] \times [\omega, z]$ in $[0, \infty) \times [0, \infty)$. A function $F : [\theta, \vartheta] \times [\omega, z] \subset \mathbb{R}^2 \to \mathbb{R}$ is coordinated convex if the partial mappings $F_y : [\theta, \vartheta] \to \mathbb{R}$ defined as $F_y(\mu) = F(\mu, y)$ and $F_x : [\omega, z] \to \mathbb{R}$ defined as $F_x(v) = F(x, \tau)$ are convex for all $x \in [\theta, \vartheta]$ and $y \in [\omega, z]$.*

Dragomir demonstrated that every convex function is also coordinate convex, though the reverse is not universally applicable [9]. Coordinate convexity is a specific form of convexity in a multivariable context, where a set or function is considered coordinate convex if it is convex with respect to each coordinate individually while keeping the others fixed. This concept is frequently applied in high-dimensional spaces, allowing problems to be decomposed into lower-dimensional components, which is defined as follows:

Definition 1.4. *[10] A mapping $F : \Lambda \subset \mathbb{R}^2 \to \mathbb{R}$ is convex on coordinates if the following inequality holds*

$$F(\tau\theta + (1-\tau)\vartheta, \tau\omega + (1-\tau)z) \leq \tau F(\theta, \vartheta) + (1-\tau)F(\omega, z), \tag{3}$$

for all (θ, ϑ), $(\omega, z) \in \Lambda$ and $\tau \in [0,1]$.

Definition 1.5. *[11] A mapping $F : \triangle \subset \mathbb{R}^2 \to \mathbb{R}$ is convex on coordinates if the following inequality holds*

$$\begin{aligned} &F(\tau\theta + (1-\tau)\vartheta, \mu\omega + (1-\mu)z) \\ &\leq \tau\mu F(\theta, \omega) + \tau(1-\mu)F(\theta, z) + (1-\tau)\mu F(\vartheta, \omega) + (1-\tau)(1-\mu)F(\vartheta, z) \end{aligned} \tag{4}$$

for all (θ, ϑ), $(\omega, z) \in \triangle$ and $\tau, \mu \in [0,1]$.

Furthermore, Dragomir extends the class of convex functions on coordinates and proved Hermite-Hadamard type inequality for coordinated convex functions on the rectangle is as follows:

Theorem 1.6. *[10] Let a function $F : \triangle \subset \mathbb{R}^2 \to \mathbb{R}$ is convex on coordinates. Then one has the inequalities:*

$$\begin{aligned}
&\frac{F(\theta,\omega)+F(\theta,z)+F(\vartheta,\omega)+F(\vartheta,z)}{4}\\
\geq\ & \frac{1}{4}\left[\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}F(x,\omega)\,dx+\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}F(x,z)\,dx\right.\\
&\left.+\frac{1}{z-\omega}\int_{\omega}^{z}F(\theta,y)\,dy+\frac{1}{z-\omega}\int_{\omega}^{z}F(\vartheta,y)\,dy\right]\\
\geq\ & \frac{1}{(\vartheta-\theta)(z-\omega)}\int_{\theta}^{\vartheta}\int_{\omega}^{z}F(x,y)\,dydx\\
\geq\ & \frac{1}{2}\left[\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}F\left(x,\frac{\omega+z}{2}\right)dx+\frac{1}{z-\omega}\int_{\omega}^{z}F\left(\frac{\theta+\vartheta}{2},y\right)dy\right]\\
\geq\ & F\left(\frac{\theta+\vartheta}{2},\frac{\omega+z}{2}\right). \qquad (5)
\end{aligned}$$

For more results in the field of coordinated convex, one can consult for more details [10–16] and reference therein.

In 1979 [17], Breckner introduced generalized convex functions, leading to significant advancements in mathematical analysis. Hudzik et al. [18], explored s-convexity in the first sense, while Pycia [19] proved Breckner's results in 2001. This class is defined as follows:

Definition 1.7. *[18] A function $F : [0,\infty) \subset \mathbb{R}^+ \to \mathbb{R}$ is said to be s-convex in the second sense if*

$$F(\tau\theta+(1-\tau)\vartheta) \leq \tau^s F(\theta)+(1-\tau)^s F(\vartheta),\ \forall\,\theta,\vartheta\in[0,\infty),\ \tau\in[0,1], \qquad (6)$$

for some fixed $s \in (0,1]$. The s-convexity reduce to ordinary convex when $s = 1$ in above inequality (6) which is defined on $[0,\infty)$.

The following definitions pertain to the concept of coordinate s-convex functions [13].

Definition 1.8. *[13] Let the bidimensional interval $[\theta,\vartheta]\times[\omega,z]$ in $[0,\infty)\times[0,\infty)$. A function $F : [\theta,\vartheta]\times[\omega,z] \subset \mathbb{R}^2 \to \mathbb{R}$ is coordinated s-convex if the partial mappings $F_y : [\theta,\vartheta] \to \mathbb{R}$ defined as $F_y(\mu) = F(\mu,y)$ and $F_x : [\omega,z] \to \mathbb{R}$ defined as $F_x(v) = F(x,\tau)$ are s-convex for all $x \in [\theta,\vartheta]$ and $y \in [\omega,z]$ with some fixed $s \in (0,1]$.*

A formal definition for the coordinated s-convex functions can be stated as:

Definition 1.9. *[20] A mapping $F : [\theta,\vartheta]\times[\omega,z] \subset \mathbb{R}^2 \to \mathbb{R}$ is coordinates s-convex with some fixed $0 < s \leq 1$, if*

$$\begin{aligned}
&F(\tau\theta+(1-\tau)\vartheta,\mu\omega+(1-\mu)z)\\
&\leq \tau^s\mu^s F(\theta,\omega)+\tau^s(1-\mu)^s F(\theta,z)+(1-\tau)^s\mu^s F(\vartheta,\omega)+(1-\tau)^s(1-\mu)^s F(\vartheta,z), \qquad (7)
\end{aligned}$$

holds for all $\tau,\mu \in [0,1]$.

Remark 1.10. *Darus and Alomari demonstrated an intriguing result, every s-convex function is coordinated s-convex. However, the converse is not generally true [20].*

Remark 1.11. *By setting $s = 1$, then inequality (7) reduces to inequality (4).*

In [21], the following Hadamard's inequality in term of s-convex functions in the second sense holds:

Theorem 1.12. *If $F : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense where $0 < s < 1$ and let $\theta,\vartheta \in [0,\infty)$, $\theta < \vartheta$. If $F \in L_1[\theta,\vartheta]$, then the following inequalities hold:*

$$2^{s-1}F\left(\frac{\theta+\vartheta}{2}\right) \leq \frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}F(\omega)\,d\omega \leq \frac{F(\theta)+F(\vartheta)}{s+1}. \qquad (8)$$

For recent findings and insights on Hadamard's inequality, refer to [22–37]. Sarikaya et al. established new trapezoidal-type inequalities for differentiable coordinate convex functions on rectangles in [38]. Latif and Dragomir derived error bounds for the midpoint formula applied to coordinate convex functions [39]. In [40], Ozdemir et al. presented Simpson's type inequalities for coordinate convex functions, while Sabah Iftikhar et al. contributed Simpson's second type inequalities and their applications to cubature formulas in [41].

Numerical analysis has origins spanning over 2,000 years, with techniques like linear interpolation predating modern computing. Key algorithms from influential mathematicians include Euler's method [42], Gaussian elimination, Lagrange interpolation, and Newton's method. The term "numerical integration" was introduced in 1915 by David Gibb in his book A Course in Interpolation and Numerical Integration for the Mathematical Laboratory [43]. Today, numerical integration, also called numerical quadrature, is crucial in scientific computing, engineering, and data analysis. It approximates definite integrals using points and weights, with methods ranging from simple (Euler integration) to advanced (Gaussian quadrature) [44]. Quadrature rules are based on interpolating polynomials, like the midpoint rule (constant), trapezoidal rule (linear), and Simpson's rules (second-degree), which improve accuracy but not error order. Boole's rule offers lower error bounds and works well for sixth-degree polynomials, providing effective approximations for smooth and oscillatory functions.

$$\int_{\theta}^{\vartheta} F(x)\,dx = \frac{2h}{45}\left[7F(\theta) + 32F\left(\frac{\vartheta+3\theta}{4}\right) + 12F\left(\frac{\vartheta+\theta}{2}\right) + 32F\left(\frac{3\vartheta+\theta}{4}\right) + 7F(\vartheta)\right] + E(F), \tag{9}$$

where $E(F)$ is the error term and is defined as:

$$|E(F)| \leq \frac{-8h^7}{945}\left|F^{(6)}(x)\right|, \text{ for all } x \in [\theta, \vartheta].$$

To establish error bounds for numerical methods, functions must be bounded and at least twice differentiable. Dragomir and Agarwal derived an error estimate for the trapezoidal rule using only the first derivative [45], while Kirmaci proposed a similar term for the midpoint rule [46]. Sarikaya et al. expanded these results by formulating Simpson-type inequalities for differentiable convex functions, with identified error bounds [47]. This article provides error estimates for Boole's rule applied to functions of two variables, and for higher-dimensional integrals, employs advanced cubature techniques, a method for calculating multiple integrals. The midpoint formula for functions of two variables is

$$\begin{aligned}\int_{\theta}^{\vartheta}\int_{\omega}^{z} F(x,y)\,dxdy &= \frac{(\vartheta-\theta)(z-\omega)}{4}\left[F\left(\frac{3\theta+\vartheta}{4}, \frac{3\omega+z}{4}\right)\right.\\ &\left.+F\left(\frac{3\theta+\vartheta}{4}, \frac{\omega+3z}{4}\right) + F\left(\frac{\theta+3\vartheta}{4}, \frac{3\omega+z}{4}\right) + F\left(\frac{\theta+3\vartheta}{4}, \frac{\omega+3z}{4}\right)\right] + E(F),\end{aligned} \tag{10}$$

with error term:

$$E(F) \leq \frac{M(\vartheta-\theta)^2(d-c)}{8} + \frac{M(\vartheta-\theta)(d-c)^2}{8} + \frac{N(\vartheta-\theta)^2(d-c)}{16},$$

The Trapezoidal rule for functions of two variables is

$$\begin{aligned}&\int_{\theta}^{\vartheta}\int_{\omega}^{z} F(x,y)\,dxdy\\ &= \frac{(\vartheta-\theta)(z-\omega)}{16}\left[F(\theta,\omega) + F(\theta,z) + F(\vartheta,\omega) + F(\vartheta,z)\right.\\ &+ 2\left(F\left(\frac{\theta+\vartheta}{2},\omega\right) + F\left(\frac{\theta+\vartheta}{2},z\right) + F\left(\theta,\frac{\omega+z}{2}\right) + F\left(\vartheta,\frac{\omega+z}{2}\right)\right)\\ &\left.+4F\left(\frac{\theta+\vartheta}{2}, \frac{\omega+z}{2}\right)\right] + E(F),\end{aligned} \tag{11}$$

with error term:

$$E(F) \leq \frac{M(\vartheta-\theta)^2(d-c)}{4} + \frac{M(\vartheta-\theta)(d-c)^2}{4} + \frac{N(\vartheta-\theta)^2(d-c)}{16}.$$

where $|\nabla F| \leq M$ and $\left|F_{xy}\right| \leq N$. To describe the importance of $E(F)$. In practice, an approximation is useful only if we know its accuracy. When we compute a particular approximation to an integral, the error is the difference between the approximation and the exact value of the integral.

While previous research has explored Boole's type inequalities in the context of classical [48] and multiplicative calculus [49], there is limited work on these inequalities. For the other Boole type inequalities, please refer to [50–55] This study is motivated by the necessity to improve error estimation in numerical integration, particularly for functions of two variables. Establishing Boole's rule-type inequalities for generalized coordinate convex functions provides a broader range of error bounds compared to traditional methods. The concept of s-convexity enhances the flexibility and robustness of these inequalities, facilitating more accurate approximations in double integrals. This work supports applications in various fields, including physics and engineering, where analytical integration is often impractical.

The structure of the paper is outlined as follows: Section 2 details the core findings regarding Boole's formula for differentiable coordinate s-convex functions in the second sense. Section 3 delves into the applications of these results within the context of numerical integration. Section 4 offers a thorough computational analysis of the inequalities associated with generalized convex functions, accompanied by mathematical examples that illustrate and support our results. Finally, Section 5 presents concluding remarks on the implications of our study and proposes directions for future exploration.

2. Main Results

The main results, presented in this section, rely on the following lemma.

Lemma 2.1. *Let $F : [\theta,\vartheta] \times [\omega,z] \subset \mathbb{R}^2 \to \mathbb{R}$ is a partial differentiable mapping on $\triangle^\circ$. If $\frac{\partial^2 F}{\partial\tau\partial\mu} \in L(\triangle^\circ)$ then following equality holds:*

$$\begin{aligned}
&\frac{1}{90\times 90}\{7[7(F(\theta,\omega)+F(\theta,z)+F(\vartheta,\omega)+F(\vartheta,z))\\
&+32\left(F\left(\frac{3\theta+\vartheta}{4},\omega\right)+F\left(\theta,\frac{3\omega+z}{4}\right)+F\left(\frac{3\theta+\vartheta}{4},z\right)+F\left(\vartheta,\frac{3\omega+z}{4}\right)\right)\\
&+12\left(F\left(\frac{\theta+\vartheta}{2},\omega\right)+F\left(\frac{\theta+\vartheta}{2},z\right)+F\left(\theta,\frac{\omega+z}{2}\right)+F\left(\vartheta,\frac{\omega+z}{2}\right)\right)\\
&+32\left(F\left(\frac{\theta+3\vartheta}{4},\omega\right)+F\left(\theta,\frac{\omega+3z}{4}\right)+F\left(\frac{\theta+3\vartheta}{4},z\right)+F\left(\vartheta,\frac{\omega+3z}{4}\right)\right)\Bigg]\\
&+32\left[32\left(F\left(\frac{3\theta+\vartheta}{4},\frac{3\omega+z}{4}\right)+F\left(\frac{3\theta+\vartheta}{4},\frac{\omega+3z}{4}\right)\right.\right.\\
&\left.+F\left(\frac{\theta+3\vartheta}{4},\frac{3\omega+z}{4}\right)+F\left(\frac{\theta+3\vartheta}{4},\frac{\omega+3z}{4}\right)\right)+12\left(F\left(\frac{3\theta+\vartheta}{4},\frac{\omega+z}{2}\right)\right.\\
&\left.\left.+F\left(\frac{\theta+3\vartheta}{4},\frac{\omega+z}{2}\right)+F\left(\frac{\theta+\vartheta}{2},\frac{3\omega+z}{4}\right)+F\left(\frac{\theta+\vartheta}{2},\frac{\omega+3z}{4}\right)\right)\right]\\
&\left.+12\left[12F\left(\frac{\theta+\vartheta}{2},\frac{\omega+z}{2}\right)\right]\right\}\\
&-\frac{1}{90(\vartheta-\theta)}\int_\theta^\vartheta\left[7F(x,\omega)+32F\left(x,\frac{3\omega+z}{4}\right)\right.\\
&\left.+12F\left(x,\frac{\omega+z}{2}\right)+32F\left(x,\frac{\omega+3z}{4}\right)+7F(x,z)\right]dx\\
&-\frac{1}{90(z-\omega)}\int_\omega^z\left[7F(\theta,y)+32F\left(\frac{3\theta+\vartheta}{4},y\right)\right.
\end{aligned}$$

$$+12F\left(\frac{\theta+\vartheta}{2},y\right)+32F\left(\frac{\theta+3\vartheta}{4},y\right)+7F(\vartheta,y)\Big]dy$$
$$+\frac{1}{(\vartheta-\theta)(z-\omega)}\int_{\theta}^{\vartheta}\int_{\omega}^{z}F(x,y)\,dydx$$
$$=(\vartheta-\theta)(z-\omega)\int_0^1\int_0^1\Upsilon(\mu,y)\,\lambda(x,\tau)\,\frac{\partial^2}{\partial\tau\partial\mu}F(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega)\,d\tau d\mu, \tag{12}$$

where

$$\Upsilon(\mu,y)=\begin{cases}\left(\mu-\frac{7}{90}\right); & \mu\in\left[0,\frac{1}{4}\right)\\ \left(\mu-\frac{39}{90}\right); & \mu\in\left[\frac{1}{4},\frac{1}{2}\right)\\ \left(\mu-\frac{51}{90}\right); & \mu\in\left[\frac{1}{2},\frac{3}{4}\right)\\ \left(\mu-\frac{83}{90}\right); & \mu\in\left[\frac{3}{4},1\right]\end{cases},$$

and

$$\lambda(x,\tau)=\begin{cases}\left(\tau-\frac{7}{90}\right); & \tau\in\left[0,\frac{1}{4}\right)\\ \left(\tau-\frac{39}{90}\right); & \tau\in\left[\frac{1}{4},\frac{1}{2}\right)\\ \left(\tau-\frac{51}{90}\right); & \tau\in\left[\frac{1}{2},\frac{3}{4}\right)\\ \left(\tau-\frac{83}{90}\right); & \tau\in\left[\frac{3}{4},1\right]\end{cases}$$

Proof. From the left-hand side, we have

$$T=(\vartheta-\theta)(z-\omega)\int_0^1\int_0^1\Upsilon(\mu,y)\,\lambda(x,\tau)\,\frac{\partial^2}{\partial\tau\partial\mu}F(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega)\,d\tau d\mu$$
$$=\int_0^1\Upsilon(\mu,y)\Big[\int_0^{\frac{1}{4}}\left(\tau-\frac{7}{90}\right)\frac{\partial^2}{\partial\tau\partial\mu}F(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega)\,d\tau$$
$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tau-\frac{39}{90}\right)\frac{\partial^2}{\partial\tau\partial\mu}F(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega)\,d\tau$$
$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\tau-\frac{51}{90}\right)\frac{\partial^2}{\partial\tau\partial\mu}F(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega)\,d\tau$$
$$+\int_{\frac{3}{4}}^{1}\left(\tau-\frac{83}{90}\right)\frac{\partial^2}{\partial\tau\partial\mu}F(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega)\,d\tau\Big]d\mu$$
$$=\int_0^{\frac{1}{4}}\left(\mu-\frac{7}{90}\right)\Big[\int_0^{\frac{1}{4}}\left(\tau-\frac{7}{90}\right)\frac{\partial^2}{\partial\tau\partial\mu}F(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega)\,d\tau$$
$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tau-\frac{39}{90}\right)\frac{\partial^2}{\partial\tau\partial\mu}F(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega)\,d\tau$$
$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\tau-\frac{51}{90}\right)\frac{\partial^2}{\partial\tau\partial\mu}F(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega)\,d\tau$$

$$
\begin{aligned}
&+\int_{\frac{3}{4}}^{1}\left(\tau-\frac{83}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\Bigg]d\mu\\
&+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\mu-\frac{39}{90}\right)\Bigg[\int_{0}^{\frac{1}{4}}\left(\tau-\frac{7}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\\
&+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tau-\frac{39}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\tau-\frac{51}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\\
&+\int_{\frac{3}{4}}^{1}\left(\tau-\frac{83}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\Bigg]d\mu\\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\mu-\frac{51}{90}\right)\Bigg[\int_{0}^{\frac{1}{4}}\left(\tau-\frac{7}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\\
&+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tau-\frac{39}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\tau-\frac{51}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\\
&+\int_{\frac{3}{4}}^{1}\left(\tau-\frac{83}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\Bigg]d\mu\\
&+\int_{\frac{3}{4}}^{1}\left(\mu-\frac{83}{90}\right)\Bigg[\int_{0}^{\frac{1}{4}}\left(\tau-\frac{7}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\\
&+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tau-\frac{39}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\tau-\frac{51}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\\
&+\int_{\frac{3}{4}}^{1}\left(\tau-\frac{83}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\Bigg]d\mu\\
&=T_{1}+T_{2}+T_{3}+T_{4}.
\end{aligned}
\tag{13}
$$

Solving T_1, we have

$$
\begin{aligned}
T_{1}=&\int_{0}^{\frac{1}{4}}\left(\mu-\frac{7}{90}\right)\Bigg[\int_{0}^{\frac{1}{4}}\left(\tau-\frac{7}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\\
&+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tau-\frac{39}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\tau-\frac{51}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\\
&+\int_{\frac{3}{4}}^{1}\left(\tau-\frac{83}{90}\right)\frac{\partial^{2}}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)d\tau\Bigg]d\mu
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \left(\mu - \frac{7}{90}\right)\left(\tau - \frac{7}{90}\right) \frac{\partial^2}{\partial\tau\partial\mu} F\left(\tau\vartheta + (1-\tau)\theta, \mu z + (1-\mu)\omega\right) d\tau d\mu \\
&+ \int_0^{\frac{1}{4}} \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\mu - \frac{7}{90}\right)\left(\tau - \frac{39}{90}\right) \frac{\partial^2}{\partial\tau\partial\mu} F\left(\tau\vartheta + (1-\tau)\theta, \mu z + (1-\mu)\omega\right) d\tau d\mu \\
&+ \int_0^{\frac{1}{4}} \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\mu - \frac{7}{90}\right)\left(\tau - \frac{51}{90}\right) \frac{\partial^2}{\partial\tau\partial\mu} F\left(\tau\vartheta + (1-\tau)\theta, \mu z + (1-\mu)\omega\right) d\tau d\mu \\
&+ \int_0^{\frac{1}{4}} \int_{\frac{3}{4}}^{1} \left(\mu - \frac{7}{90}\right)\left(\tau - \frac{83}{90}\right) \frac{\partial^2}{\partial\tau\partial\mu} F\left(\tau\vartheta + (1-\tau)\theta, \mu z + (1-\mu)\omega\right) d\tau d\mu \\
&= T_{11} + T_{12} + I_{13} + T_{14}.
\end{aligned} \tag{14}
$$

Solving T_{11}, we have

$$
\begin{aligned}
T_{11} &= \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \left(\mu - \frac{7}{90}\right)\left(\tau - \frac{7}{90}\right) \frac{\partial^2}{\partial\tau\partial\mu} F\left(\tau\vartheta + (1-\tau)\theta, \mu z + (1-\mu)\omega\right) d\tau d\mu \\
&= \int_0^{\frac{1}{4}} \left(\mu - \frac{7}{90}\right) \Bigg[\frac{1}{\vartheta - \theta} \left(\tau - \frac{7}{90}\right) \frac{\partial}{\partial\mu} F\left(\tau\vartheta + (1-\tau)\theta, \mu z + (1-\mu)\omega\right)\Bigg|_0^{\frac{1}{4}} \\
&- \frac{1}{\vartheta - \theta} \int_0^{\frac{1}{4}} \frac{\partial}{\partial\mu} F\left(\tau\vartheta + (1-\tau)\theta, \mu z + (1-\mu)\omega\right) d\tau \Bigg] d\mu \\
&= \frac{31}{180(\vartheta - \theta)} \int_0^{\frac{1}{4}} \left(\mu - \frac{7}{90}\right) \frac{\partial}{\partial\mu} F\left(\frac{\vartheta + 3\theta}{4}, \mu z + (1-\mu)\omega\right) d\mu \\
&+ \frac{7}{90(\vartheta - \theta)} \int_0^{\frac{1}{4}} \left(\mu - \frac{7}{90}\right) \frac{\partial}{\partial\mu} F\left(\theta, \mu z + (1-\mu)\omega\right) d\mu \\
&- \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \left(\mu - \frac{7}{90}\right) \frac{\partial}{\partial\mu} F\left(\tau\vartheta + (1-\tau)\theta, \mu z + (1-\mu)\omega\right) d\tau d\mu \\
&= \frac{1}{(\vartheta - \theta)(z - \omega)} \Bigg[\frac{31 \times 31}{180 \times 180} F\left(\frac{\vartheta + 3\theta}{4}, \frac{z + 3\omega}{4}\right) + \frac{31 \times 7}{180 \times 90} F\left(\frac{\vartheta + 3\theta}{4}, \omega\right) \\
&+ \frac{31 \times 7}{180 \times 90} F\left(\theta, \frac{z + 3\omega}{4}\right) + \frac{7 \times 7}{90 \times 90} F(\theta, \omega) \Bigg] \\
&- \frac{1}{(z - \omega)} \left[\frac{31}{180} \int_0^{\frac{1}{4}} F\left(\frac{\vartheta + 3\theta}{4}, \mu z + (1-\mu)\omega\right) d\mu + \frac{7}{90} \int_0^{\frac{1}{4}} F\left(\theta, \mu z + (1-\mu)\omega\right) d\mu \right] \\
&- \frac{1}{(\vartheta - \theta)} \left[\frac{31}{180} \int_0^{\frac{1}{4}} F\left(\tau\vartheta + (1-\tau)\theta, \frac{z + 3\omega}{4}\right) d\tau + \frac{7}{90} \int_0^{\frac{1}{4}} F\left(\tau\vartheta + (1-\tau)\theta, \omega\right) d\tau \right] \\
&+ \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} F\left(\tau\vartheta + (1-\tau)\theta, \mu z + (1-\mu)\omega\right) d\tau d\mu.
\end{aligned} \tag{15}
$$

Similarly,

$$
\begin{aligned}
T_{12} &= \frac{1}{(\vartheta - \theta)(z - \omega)} \Bigg[\frac{31 \times 6}{180 \times 90} F\left(\frac{\vartheta + 3\theta}{4}, \frac{z + \omega}{2}\right) + \frac{31 \times 33}{180 \times 180} F\left(\frac{\vartheta + 3\theta}{4}, \frac{z + 3\omega}{4}\right) \\
&+ \frac{6 \times 7}{90 \times 90} F\left(\theta, \frac{z + \omega}{2}\right) + \frac{7 \times 33}{90 \times 180} F\left(\theta, \frac{z + 3\omega}{4}\right) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{(z-\omega)}\left[\frac{31}{180}\int_{\frac{1}{4}}^{\frac{1}{2}} F\left(\frac{\vartheta+3\theta}{4}, \mu z+(1-\mu)\omega\right) d\mu+\frac{7}{90}\int_{\frac{1}{4}}^{\frac{1}{2}} F(\theta, \mu z+(1-\mu)\omega)\, d\mu\right]\\
&-\frac{1}{(\vartheta-\theta)}\left[\frac{6}{90}\int_0^{\frac{1}{4}} F\left(\tau\vartheta+(1-\tau)\theta, \frac{z+\omega}{2}\right) d\tau\right.\\
&\left.+\frac{33}{180}\int_0^{\frac{1}{4}} F\left(\tau\vartheta+(1-\tau)\theta, \frac{z+3\omega}{4}\right) d\tau\right]\\
&+\int_0^{\frac{1}{4}}\int_{\frac{1}{4}}^{\frac{1}{2}} F(\tau\vartheta+(1-\tau)\theta, \mu z+(1-\mu)\omega)\, d\tau d\mu, \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
T_{13}=&\frac{1}{(\vartheta-\theta)(z-\omega)}\left[\frac{31\times 33}{180\times 180}F\left(\frac{\vartheta+3\theta}{4}, \frac{3z+\omega}{2}\right)+\frac{31\times 6}{180\times 90}F\left(\frac{\vartheta+3\theta}{4}, \frac{z+\omega}{2}\right)\right.\\
&\left.+\frac{33\times 7}{180\times 90}F\left(\theta, \frac{3z+\omega}{4}\right)+\frac{7\times 6}{90\times 90}F\left(\theta, \frac{z+\omega}{2}\right)\right]\\
&-\frac{1}{(z-\omega)}\left[\frac{31}{180}\int_{\frac{1}{2}}^{\frac{3}{4}} F\left(\frac{\vartheta+3\theta}{4}, \mu z+(1-\mu)\omega\right) d\mu+\frac{7}{90}\int_{\frac{1}{2}}^{\frac{3}{4}} F(\theta, \mu z+(1-\mu)\omega)\, d\mu\right]\\
&-\frac{1}{(\vartheta-\theta)}\left[\frac{33}{180}\int_0^{\frac{1}{4}} F\left(\tau\vartheta+(1-\tau)\theta, \frac{3z+\omega}{4}\right) d\tau\right.\\
&\left.+\frac{6}{90}\int_0^{\frac{1}{4}} F\left(\tau\vartheta+(1-\tau)\theta, \frac{z+\omega}{2}\right) d\tau\right]\\
&+\int_0^{\frac{1}{4}}\int_{\frac{1}{2}}^{\frac{3}{4}} F(\tau\vartheta+(1-\tau)\theta, \mu z+(1-\mu)\omega)\, d\tau d\mu, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
T_{14}=&\frac{1}{(\vartheta-\theta)(z-\omega)}\left[\frac{31\times 7}{180\times 90}F\left(\frac{\vartheta+3\theta}{4}, z\right)+\frac{31\times 31}{180\times 180}F\left(\frac{\vartheta+3\theta}{4}, \frac{3z+\omega}{2}\right)\right.\\
&\left.+\frac{7\times 7}{90\times 90}F(\theta, z)+\frac{7\times 31}{90\times 180}F\left(\theta, \frac{3z+\omega}{4}\right)\right]\\
&-\frac{1}{(z-\omega)}\left[\frac{31}{180}\int_{\frac{3}{4}}^{1} F\left(\frac{\vartheta+3\theta}{4}, \mu z+(1-\mu)\omega\right) d\mu+\frac{7}{90}\int_{\frac{3}{4}}^{1} F(\theta, \mu z+(1-\mu)\omega)\, d\mu\right]\\
&-\frac{1}{(\vartheta-\theta)}\left[\frac{7}{90}\int_0^{\frac{1}{4}} F(\tau\vartheta+(1-\tau)\theta, z)\, d\tau+\frac{31}{180}\int_0^{\frac{1}{4}} F\left(\tau\vartheta+(1-\tau)\theta, \frac{3z+\omega}{4}\right) d\tau\right]\\
&+\int_0^{\frac{1}{4}}\int_{\frac{3}{4}}^{1} F(\tau\vartheta+(1-\tau)\theta, \mu z+(1-\mu)\omega)\, d\tau d\mu. \qquad (18)
\end{aligned}
$$

By adding (15) to (18), we get (14). Similarly solve T_2, T_3 and T_4 also, by change of variables $x=\tau\vartheta+(1-\tau)\theta$, $y=\mu z+(1-\mu)\omega$ and multiply $(\vartheta-\theta)(z-\omega)$ then simplify by simple mathematical operations we get the desired result. Hence the proof of Lemma 2.1 has been completed. □

Theorem 2.2. *Let $F:[\theta,\vartheta]\times[\omega,z]\subset\mathbb{R}^2\to\mathbb{R}$ is a partial differentiable mapping on $\triangle^\circ$. If $\left|\frac{\partial^2 F}{\partial\tau\partial\mu}\right|$ is coordinated* *s-convex function on* $[\theta,\vartheta]\times[\omega,z]$, *for some fixed* $s\in(0,1]$ *then we have the following inequality:*

$$\begin{aligned}
&\left|\Lambda(\theta,\vartheta,x;\omega,z,y)\right|\\
&\leq \frac{4^{-2s-3}45^{-2(s+2)}(\vartheta-\theta)(z-\omega)}{(s+1)^2(s+2)^2}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s-19\,4^{s+2}45^{s+1}\right.\\
&\left.-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)^2\\
&\times\left[\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,\omega)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,z)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,z)\right|\right],
\end{aligned}\tag{19}$$

where

$$\begin{aligned}
&\Lambda(\theta,\vartheta,x;\omega,z,y)\\
&=\frac{1}{90\times 90}\{7[7(F(\theta,\omega)+F(\theta,z)+F(\vartheta,\omega)+F(\vartheta,z))\\
&+32\left(F\left(\frac{3\theta+\vartheta}{4},\omega\right)+F\left(\theta,\frac{3\omega+z}{4}\right)+F\left(\frac{3\theta+\vartheta}{4},z\right)+F\left(\vartheta,\frac{3\omega+z}{4}\right)\right)\\
&+12\left(F\left(\frac{\theta+\vartheta}{2},\omega\right)+F\left(\frac{\theta+\vartheta}{2},z\right)+F\left(\theta,\frac{\omega+z}{2}\right)+F\left(\vartheta,\frac{\omega+z}{2}\right)\right)\\
&+32\left(F\left(\frac{\theta+3\vartheta}{4},\omega\right)+F\left(\theta,\frac{\omega+3z}{4}\right)+F\left(\frac{\theta+3\vartheta}{4},z\right)+F\left(\vartheta,\frac{\omega+3z}{4}\right)\right)\Bigg]\\
&+32\left[32\left(F\left(\frac{3\theta+\vartheta}{4},\frac{3\omega+z}{4}\right)+F\left(\frac{3\theta+\vartheta}{4},\frac{\omega+3z}{4}\right)+F\left(\frac{\theta+3\vartheta}{4},\frac{3\omega+z}{4}\right)\right.\right.\\
&\left.+F\left(\frac{\theta+3\vartheta}{4},\frac{\omega+3z}{4}\right)\right)+12\left(F\left(\frac{3\theta+\vartheta}{4},\frac{\omega+z}{2}\right)+F\left(\frac{\theta+3\vartheta}{4},\frac{\omega+z}{2}\right)\right.\\
&\left.\left.+F\left(\frac{\theta+\vartheta}{2},\frac{3\omega+z}{4}\right)+F\left(\frac{\theta+\vartheta}{2},\frac{\omega+3z}{4}\right)\right)\right]+12\left[12F\left(\frac{\theta+\vartheta}{2},\frac{\omega+z}{2}\right)\right]\Bigg\}\\
&-\frac{1}{90(\vartheta-\theta)}\int_\theta^\vartheta\left[7F(x,\omega)+32F\left(x,\frac{3\omega+z}{4}\right)+12F\left(x,\frac{\omega+z}{2}\right)\right.\\
&\left.+32F\left(x,\frac{\omega+3z}{4}\right)+7F(x,z)\right]dx-\frac{1}{90(z-\omega)}\int_\omega^z\left[7F(\theta,y)+32F\left(\frac{3\theta+\vartheta}{4},y\right)\right.\\
&\left.+12F\left(\frac{\theta+\vartheta}{2},y\right)+32F\left(\frac{\theta+3\vartheta}{4},y\right)+7F(\vartheta,y)\right]dy\\
&+\frac{1}{(\vartheta-\theta)(z-\omega)}\int_\theta^\vartheta\int_\omega^z F(x,y)\,dydx.
\end{aligned}$$

Proof. By Lemma 2.1 and using the property of modulus, we can write as

$$\begin{aligned}
&\left|\Lambda(\theta,\vartheta,x;\omega,z,y)\right|\\
&\leq(\vartheta-\theta)(z-\omega)\int_0^1\int_0^1\left|\Upsilon(\mu,y)\lambda(x,\tau)\right|\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega)\right|d\tau d\mu.
\end{aligned}\tag{20}$$

Since F is coordinated s-convex functions, we have

$$\begin{aligned}
\left|\Lambda(\theta,\vartheta,x;\omega,z,y)\right|\leq(\vartheta-\theta)(z-\omega)&\int_0^1\left|\Upsilon(\mu,y)\right|\left[\int_0^1|\lambda(x,\tau)|\right.\\
&\times\left\{\tau^s\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,\mu z+(1-\mu)\omega)\right|\right.\\
&\left.\left.+(1-\tau)^s\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,\mu z+(1-\mu)\omega)\right|\right\}d\tau\right]d\mu.
\end{aligned}\tag{21}$$

Now, solving integral on the right-hand side of the above inequality.

$$\int_0^1 |\lambda(x,\tau)| \left\{ \tau^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\vartheta, \mu z + (1-\mu)\omega) \right| + (1-\tau)^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\theta, \mu z + (1-\mu)\omega) \right| \right\} d\tau$$
$$= \int_0^{\frac{1}{4}} \left| \tau - \frac{7}{90} \right| \left\{ \tau^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\vartheta, \mu z + (1-\mu)\omega) \right| \right.$$
$$\left. + (1-\tau)^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\theta, \mu z + (1-\mu)\omega) \right| \right\} d\tau$$
$$+ \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \tau - \frac{39}{90} \right| \left\{ \tau^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\vartheta, \mu z + (1-\mu)\omega) \right| \right.$$
$$\left. + (1-\tau)^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\theta, \mu z + (1-\mu)\omega) \right| \right\} d\tau$$
$$+ \int_{\frac{1}{2}}^{\frac{3}{4}} \left| \tau - \frac{51}{90} \right| \left\{ \tau^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\vartheta, \mu z + (1-\mu)\omega) \right| \right.$$
$$\left. + (1-\tau)^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\theta, \mu z + (1-\mu)\omega) \right| \right\} d\tau$$
$$+ \int_{\frac{3}{4}}^{1} \left| \tau - \frac{83}{90} \right| \left\{ \tau^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\vartheta, \mu z + (1-\mu)\omega) \right| \right.$$
$$\left. + (1-\tau)^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\theta, \mu z + (1-\mu)\omega) \right| \right\} d\tau$$
$$= \frac{2^{-2s-3}45^{-s-2}}{(s+1)(s+2)} \left(45^{s+1} \left(3^{s+1} + 7\ 4^{s+1} - 1 \right) s - 19\ 4^{s+2}45^{s+1} \right.$$
$$\left. -47\ 45^{s+1} - 133\ 135^{s+1} + 14^{s+2} + 78^{s+2} - 90^{s+2} + 102^{s+2} + 166^{s+2} \right)$$
$$\times \left(\left| \frac{\partial^2}{\partial\tau\partial\mu} F(\vartheta, \mu z + (1-\mu)\omega) \right| + \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\theta, \mu z + (1-\mu)\omega) \right| \right), \tag{22}$$

put eq. (22) into (21), we obtain

$$\left| \Lambda(\theta, \vartheta, x; \omega, z, y) \right|$$
$$\le \frac{2^{-2s-3}45^{-s-2}(\vartheta-\theta)(z-\omega)}{(s+1)(s+2)} \left(45^{s+1} \left(3^{s+1} + 7\ 4^{s+1} - 1 \right) s - 19\ 4^{s+2}45^{s+1} \right.$$
$$\left. -47\ 45^{s+1} - 133\ 135^{s+1} + 14^{s+2} + 78^{s+2} - 90^{s+2} + 102^{s+2} + 166^{s+2} \right)$$
$$\times \int_0^1 \left| \Upsilon(\mu, y) \right| \left\{ \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\vartheta, \mu z + (1-\mu)\omega) \right| \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\theta, \mu z + (1-\mu)\omega) \right| \right\} d\mu. \tag{23}$$

Similarly,

$$\int_0^1 \left| \Upsilon(\mu, y) \right| \left\{ \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\vartheta, \mu z + (1-\mu)\omega) \right| \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\theta, \mu z + (1-\mu)\omega) \right| \right\} d\mu$$
$$= \int_0^{\frac{1}{4}} \left| \mu - \frac{7}{90} \right| \left\{ \mu^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\vartheta, z) \right| + (1-\mu)^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\vartheta, \omega) \right| \right\} d\mu$$
$$\int_0^{\frac{1}{4}} \left| \mu - \frac{7}{90} \right| \left\{ \mu^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\theta, z) \right| + (1-\mu)^s \left| \frac{\partial^2}{\partial\tau\partial\mu} F(\theta, \omega) \right| \right\} d\mu$$

$$
\begin{aligned}
&+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\mu-\frac{39}{90}\right|\left\{\mu^{s}\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,z)\right|+(1-\mu)^{s}\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|\right\}d\mu\\
&+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\mu-\frac{39}{90}\right|\left\{\mu^{s}\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,z)\right|+(1-\mu)^{s}\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,\omega)\right|\right\}d\mu\\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\mu-\frac{51}{90}\right|\left\{\mu^{s}\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,z)\right|+(1-\mu)^{s}\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|\right\}d\mu\\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\mu-\frac{51}{90}\right|\left\{\mu^{s}\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,z)\right|+(1-\mu)^{s}\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,\omega)\right|\right\}d\mu\\
&+\int_{\frac{3}{4}}^{1}\left|\mu-\frac{83}{90}\right|\left\{\mu^{s}\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,z)\right|+(1-\mu)^{s}\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|\right\}d\mu\\
&+\int_{\frac{3}{4}}^{1}\left|\mu-\frac{83}{90}\right|\left\{\mu^{s}\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,z)\right|+(1-\mu)^{s}\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,\omega)\right|\right\}d\mu\\
&=\frac{2^{-2s-3}45^{-s-2}}{(s+1)(s+2)}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s-19\,4^{s+2}45^{s+1}\right.\\
&\left.-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)\\
&\times\left[\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,z)\right|+\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|+\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,z)\right|+\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,\omega)\right|\right]. \qquad (24)
\end{aligned}
$$

By putting (24) into (23). Thus the proof of Theorem 2.2 has been finalized. □

Corollary 2.3. *By setting $s = 1$ in Theorem 2.2, then have*

$$
\begin{aligned}
&\left|\Lambda(\theta,\vartheta,x;\omega,z,y)\right|\leq\frac{57121\,(\vartheta-\theta)(z-\omega)}{6480\times6480}\\
&\times\left[\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,z)\right|+\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|+\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,z)\right|+\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,\omega)\right|\right],
\end{aligned}
$$

where $\Lambda(\theta,\vartheta,x;\omega,z,y)$ is defined in Theorem 2.2.

Theorem 2.4. *Let $F:[\theta,\vartheta]\times[\omega,z]\subset\mathbb{R}^2\to\mathbb{R}$ is a partial differentiable mapping on $\triangle^\circ$. If $\left|\frac{\partial^2 F}{\partial\tau\partial\mu}\right|^q$ is coordinated s-convex function on $[\theta,\vartheta]\times[\omega,z]$, for some fixed $s\in(0,1]$, and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have the following inequality:*

$$
\begin{aligned}
&\left|\Lambda(\theta,\vartheta,x;\omega,z,y)\right|\\
&\leq\frac{(\vartheta-\theta)(z-\omega)}{(s+1)^{2/q}}\left(\frac{4^{-2p-1}2025^{-p-1}\left(12^{p+1}+14^{p+1}+31^{p+1}+33^{p+1}\right)^{2}}{(p+1)^{2}}\right)^{\frac{1}{p}}\\
&\times\left[\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,\omega)\right|^{q}+\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,z)\right|^{q}+\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|^{q}+\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,z)\right|^{q}\right]^{\frac{1}{q}}, \qquad (25)
\end{aligned}
$$

where $\Lambda(\theta,\vartheta,x;\omega,z,y)$ is defined in Theorem 2.2.

Proof. From Lemma 2.1 and using Hölder inequality for a double integral after taking the property of modulus, we can write as

$$\begin{aligned}
&\left|\Lambda\left(\theta,\vartheta,x;\omega,z,y\right)\right| \\
&\leq\left(\vartheta-\theta\right)\left(z-\omega\right)\int_0^1\int_0^1\left|\Upsilon\left(\mu,y\right)\lambda\left(x,\tau\right)\right|\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\tau\vartheta+\left(1-\tau\right)\theta,\mu z+\left(1-\mu\right)\omega\right)\right|d\tau d\mu. \\
&\leq\left(\vartheta-\theta\right)\left(z-\omega\right)\left(\int_0^1\int_0^1\left(\left|\Upsilon\left(\mu,y\right)\lambda\left(x,\tau\right)\right|\right)^p d\tau d\mu\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\tau\vartheta+\left(1-\tau\right)\theta,\mu z+\left(1-\mu\right)\omega\right)\right|^q d\tau d\mu\right)^{\frac{1}{q}}.
\end{aligned}\tag{26}$$

Since, $\left|\frac{\partial^2 F}{\partial\tau\partial\mu}\right|^q$ is coordinated s-convex function, we have

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\tau\vartheta+\left(1-\tau\right)\theta,\mu z+\left(1-\mu\right)\omega\right)\right|^q d\tau d\mu \\
&\leq\int_0^1\int_0^1\left[\tau^s\mu^s\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\vartheta,z\right)\right|^q+\tau^s\left(1-\mu\right)^s\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\vartheta,\omega\right)\right|^q\right. \\
&\left.+\left(1-\tau\right)^s\mu^s\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\theta,z\right)\right|^q+\left(1-\tau\right)^s\left(1-\mu\right)^s\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\theta,\omega\right)\right|^q\right]d\tau d\mu \\
&=\frac{1}{\left(s+1\right)^2}\left[\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\vartheta,z\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\vartheta,\omega\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\theta,z\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\theta,\omega\right)\right|^q\right].
\end{aligned}\tag{27}$$

Also, we have to solve:

$$\begin{aligned}
&\int_0^1\int_0^1\left(\left|\Upsilon\left(\mu,y\right)\lambda\left(x,\tau\right)\right|\right)^p d\tau d\mu \\
&=\int_0^1\left|\Upsilon\left(\mu,y\right)\right|^p\left[\int_0^{\frac{1}{4}}\left|\tau-\frac{7}{90}\right|^p d\tau+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\tau-\frac{39}{90}\right|^p d\tau+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\tau-\frac{51}{90}\right|^p d\tau\right. \\
&\left.+\int_{\frac{3}{4}}^1\left|\tau-\frac{83}{90}\right|^p d\tau\right]d\mu \\
&=\int_0^{\frac{1}{4}}\left|\mu-\frac{7}{90}\right|^p\left[\int_0^{\frac{1}{4}}\left|\tau-\frac{7}{90}\right|^p d\tau+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\tau-\frac{39}{90}\right|^p d\tau+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\tau-\frac{51}{90}\right|^p d\tau\right. \\
&\left.+\int_{\frac{3}{4}}^1\left|\tau-\frac{83}{90}\right|^p d\tau\right]d\mu+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\mu-\frac{39}{90}\right|^p\left[\int_0^{\frac{1}{4}}\left|\tau-\frac{7}{90}\right|^p d\tau+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\tau-\frac{39}{90}\right|^p d\tau\right. \\
&\left.+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\tau-\frac{51}{90}\right|^p d\tau+\int_{\frac{3}{4}}^1\left|\tau-\frac{83}{90}\right|^p d\tau\right]d\mu+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\mu-\frac{51}{90}\right|^p\left[\int_0^{\frac{1}{4}}\left|\tau-\frac{7}{90}\right|^p d\tau\right. \\
&\left.+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\tau-\frac{39}{90}\right|^p d\tau+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\tau-\frac{51}{90}\right|^p d\tau+\int_{\frac{3}{4}}^1\left|\tau-\frac{83}{90}\right|^p d\tau\right]d\mu \\
&+\int_{\frac{3}{4}}^1\left|\mu-\frac{83}{90}\right|^p\left[\int_0^{\frac{1}{4}}\left|\tau-\frac{7}{90}\right|^p d\tau+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\tau-\frac{39}{90}\right|^p d\tau+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\tau-\frac{51}{90}\right|^p d\tau\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\frac{3}{4}}^{1}\left|\tau-\frac{83}{90}\right|^{p}d\tau\Bigg]d\mu \\
&=\int_{0}^{\frac{1}{4}}\int_{0}^{\frac{1}{4}}\left|\mu-\frac{7}{90}\right|^{p}\left|\tau-\frac{7}{90}\right|^{p}d\tau d\mu+\int_{0}^{\frac{1}{4}}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\mu-\frac{7}{90}\right|^{p}\left|\tau-\frac{39}{90}\right|^{p}d\tau d\mu \\
&+\int_{0}^{\frac{1}{4}}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\mu-\frac{7}{90}\right|^{p}\left|\tau-\frac{51}{90}\right|^{p}d\tau d\mu+\int_{0}^{\frac{1}{4}}\int_{\frac{3}{4}}^{1}\left|\mu-\frac{7}{90}\right|^{p}\left|\tau-\frac{83}{90}\right|^{p}d\tau d\mu \\
&+\int_{\frac{1}{4}}^{\frac{1}{2}}\int_{0}^{\frac{1}{4}}\left|\mu-\frac{39}{90}\right|^{p}\left|\tau-\frac{7}{90}\right|^{p}d\tau d\mu+\int_{\frac{1}{4}}^{\frac{1}{2}}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\mu-\frac{39}{90}\right|^{p}\left|\tau-\frac{39}{90}\right|^{p}d\tau d\mu \\
&+\int_{\frac{1}{4}}^{\frac{1}{2}}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\mu-\frac{39}{90}\right|^{p}\left|\tau-\frac{51}{90}\right|^{p}d\tau d\mu+\int_{\frac{1}{4}}^{\frac{1}{2}}\int_{\frac{3}{4}}^{1}\left|\mu-\frac{39}{90}\right|^{p}\left|\tau-\frac{83}{90}\right|^{p}d\tau d\mu \\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\int_{0}^{\frac{1}{4}}\left|\mu-\frac{51}{90}\right|^{p}\left|\tau-\frac{7}{90}\right|^{p}d\tau d\mu+\int_{\frac{1}{2}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\mu-\frac{51}{90}\right|^{p}\left|\tau-\frac{51}{90}\right|^{p}d\tau d\mu \\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\mu-\frac{51}{90}\right|^{p}\left|\tau-\frac{51}{90}\right|^{p}d\tau d\mu+\int_{\frac{1}{2}}^{\frac{3}{4}}\int_{\frac{3}{4}}^{1}\left|\mu-\frac{51}{90}\right|^{p}\left|\tau-\frac{83}{90}\right|^{p}d\tau d\mu \\
&+\int_{\frac{3}{4}}^{1}\int_{0}^{\frac{1}{4}}\left|\mu-\frac{83}{90}\right|^{p}\left|\tau-\frac{7}{90}\right|^{p}d\tau d\mu+\int_{\frac{3}{4}}^{1}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\mu-\frac{83}{90}\right|^{p}\left|\tau-\frac{39}{90}\right|^{p}d\tau d\mu \\
&+\int_{\frac{3}{4}}^{1}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\mu-\frac{83}{90}\right|^{p}\left|\tau-\frac{51}{90}\right|^{p}d\tau d\mu+\int_{\frac{3}{4}}^{1}\int_{\frac{3}{4}}^{1}\left|\mu-\frac{83}{90}\right|^{p}\left|\tau-\frac{83}{90}\right|^{p}d\tau d\mu \\
&=\frac{4^{-2p-1}2025^{-p-1}\left(12^{p+1}+14^{p+1}+31^{p+1}+33^{p+1}\right)^{2}}{(p+1)^{2}}. \qquad (28)
\end{aligned}
$$

Putting (27) and (28) into (26), we get the desired result. Thus, the Theorem 2.4 has been finalized. □

Corollary 2.5. *By setting $s = 1$ in Theorem 2.4, then have*

$$
\begin{aligned}
&\left|\Lambda(\theta,\vartheta,x;\omega,z,y)\right| \\
&\leq\frac{(\vartheta-\theta)(z-\omega)}{(4)^{\frac{1}{q}}}\left(\frac{4^{-2p-1}2025^{-p-1}\left(12^{p+1}+14^{p+1}+31^{p+1}+33^{p+1}\right)^{2}}{(p+1)^{2}}\right)^{\frac{1}{p}} \\
&\times\left[\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,z)\right|^{q}+\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|^{q}+\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,z)\right|^{q}+\left|\frac{\partial^{2}}{\partial\tau\partial\mu}F(\theta,\omega)\right|^{q}\right]^{\frac{1}{q}},
\end{aligned}
$$

where $\Lambda(\theta,\vartheta,x;\omega,z,y)$ is defined in Theorem 2.2.

Theorem 2.6. *Let $F:[\theta,\vartheta]\times[\omega,z]\subset\mathbb{R}^{2}\to\mathbb{R}$ is a partial differentiable mapping on $\triangle^{\circ}$. If $\left|\frac{\partial^{2}F}{\partial\tau\partial\mu}\right|^{q}$ is co-ordinated* *s-convex function on $[\theta,\vartheta]\times[\omega,z]$, for some fixed $s\in(0,1]$, and $q\geq 1$, then we have the following inequality:*

$$\begin{aligned}&\left|\Lambda\left(\theta,\vartheta,x;\omega,z,y\right)\right|\leq\left(\vartheta-\theta\right)\left(z-\omega\right)\left(\frac{57121}{10497600}\right)^{1-\frac{1}{q}}\\&\times\left[\frac{4^{-2s-3}45^{-2(s+2)}\left(\vartheta-\theta\right)\left(z-\omega\right)}{(s+1)^2(s+2)^2}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s-19\,4^{s+2}45^{s+1}\right.\right.\\&\left.-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)^2\\&\left.\times\left(\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\theta,\omega\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\theta,z\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\vartheta,\omega\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\vartheta,z\right)\right|^q\right)\right]^{\frac{1}{q}},\end{aligned}\tag{29}$$

where $\Lambda\left(\theta,\vartheta,x;\omega,z,y\right)$ is defined in Theorem 2.2.

Proof. From Lemma 2.1 and using power mean inequality for a double integral after taking the property of modulus, we have

$$\begin{aligned}&\left|\Lambda\left(\theta,\vartheta,x;\omega,z,y\right)\right|\\&\leq\left(\vartheta-\theta\right)\left(z-\omega\right)\int_0^1\int_0^1\left|\Upsilon\left(\mu,y\right)\lambda\left(x,\tau\right)\right|\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)\right|d\tau d\mu.\end{aligned}\tag{30}$$

$$\begin{aligned}&\leq\left(\vartheta-\theta\right)\left(z-\omega\right)\left(\int_0^1\int_0^1\left|\Upsilon\left(\mu,y\right)\lambda\left(x,\tau\right)\right|d\tau d\mu\right)^{1-\frac{1}{q}}\\&\times\left(\int_0^1\int_0^1\left|\Upsilon\left(\mu,y\right)\lambda\left(x,\tau\right)\right|\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)\right|^q d\tau d\mu\right)^{\frac{1}{q}}.\end{aligned}\tag{31}$$

Since, $\left|\frac{\partial^2F}{\partial\tau\partial\mu}\right|^q$ is coordinated s-convex function, we have

$$\begin{aligned}&\int_0^1\int_0^1\left|\Upsilon\left(\mu,y\right)\lambda\left(x,\tau\right)\right|\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\tau\vartheta+(1-\tau)\theta,\mu z+(1-\mu)\omega\right)\right|^q d\tau d\mu\\&\leq\int_0^1\int_0^1\left|\Upsilon\left(\mu,y\right)\lambda\left(x,\tau\right)\right|\left[\tau^s\mu^s\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\vartheta,z\right)\right|^q+\tau^s\left(1-\mu\right)^s\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\vartheta,\omega\right)\right|^q\right.\\&\left.+\left(1-\tau\right)^s\mu^s\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\theta,z\right)\right|^q+\left(1-\tau\right)^s\left(1-\mu\right)^s\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\theta,\omega\right)\right|^q\right]d\tau d\mu\\&=\frac{4^{-2s-3}45^{-2(s+2)}}{(s+1)^2(s+2)^2}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s-19\,4^{s+2}45^{s+1}\right.\\&\left.-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)^2\\&\times\left[\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\vartheta,z\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\vartheta,\omega\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\theta,z\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\theta,\omega\right)\right|^q\right].\end{aligned}\tag{32}$$

And solving,

$$\begin{aligned}
&\int_0^1\int_0^1 \left|\Upsilon(\mu,y)\,\lambda(x,\tau)\right| d\tau d\mu = \int_0^1 \left|\Upsilon(\mu,y)\right| \left[\int_0^{\frac14}\left|\tau-\frac{7}{90}\right|d\tau + \int_{\frac14}^{\frac12}\left|\tau-\frac{39}{90}\right|d\tau\right.\\
&\left.+\int_{\frac12}^{\frac34}\left|\tau-\frac{51}{90}\right|d\tau + \int_{\frac34}^{1}\left|\tau-\frac{83}{90}\right|d\tau\right]d\mu\\
&=\int_0^{\frac14}\left|\mu-\frac{7}{90}\right|\left[\int_0^{\frac14}\left|\tau-\frac{7}{90}\right|d\tau + \int_{\frac14}^{\frac12}\left|\tau-\frac{39}{90}\right|d\tau + \int_{\frac12}^{\frac34}\left|\tau-\frac{51}{90}\right|d\tau\right.\\
&\left.+\int_{\frac34}^{1}\left|\tau-\frac{83}{90}\right|d\tau\right]d\mu + \int_{\frac14}^{\frac12}\left|\mu-\frac{39}{90}\right|\left[\int_0^{\frac14}\left|\tau-\frac{7}{90}\right|d\tau + \int_{\frac14}^{\frac12}\left|\tau-\frac{39}{90}\right|d\tau\right.\\
&\left.+\int_{\frac12}^{\frac34}\left|\tau-\frac{51}{90}\right|d\tau + \int_{\frac34}^{1}\left|\tau-\frac{83}{90}\right|d\tau\right]d\mu + \int_{\frac12}^{\frac34}\left|\mu-\frac{51}{90}\right|\left[\int_0^{\frac14}\left|\tau-\frac{7}{90}\right|d\tau\right.\\
&\left.+\int_{\frac14}^{\frac12}\left|\tau-\frac{39}{90}\right|d\tau + \int_{\frac12}^{\frac34}\left|\tau-\frac{51}{90}\right|d\tau + \int_{\frac34}^{1}\left|\tau-\frac{83}{90}\right|d\tau\right]d\mu\\
&+\int_{\frac34}^{1}\left|\mu-\frac{83}{90}\right|\left[\int_0^{\frac14}\left|\tau-\frac{7}{90}\right|d\tau + \int_{\frac14}^{\frac12}\left|\tau-\frac{39}{90}\right|d\tau + \int_{\frac12}^{\frac34}\left|\tau-\frac{51}{90}\right|d\tau\right.\\
&\left.+\int_{\frac34}^{1}\left|\tau-\frac{83}{90}\right|d\tau\right]d\mu\\
&=\int_0^{\frac14}\int_0^{\frac14}\left|\mu-\frac{7}{90}\right|\left|\tau-\frac{7}{90}\right|d\tau d\mu + \int_0^{\frac14}\int_{\frac14}^{\frac12}\left|\mu-\frac{7}{90}\right|\left|\tau-\frac{39}{90}\right|d\tau d\mu\\
&+\int_0^{\frac14}\int_{\frac12}^{\frac34}\left|\mu-\frac{7}{90}\right|\left|\tau-\frac{51}{90}\right|d\tau d\mu + \int_0^{\frac14}\int_{\frac34}^{1}\left|\mu-\frac{7}{90}\right|\left|\tau-\frac{83}{90}\right|d\tau d\mu\\
&+\int_{\frac14}^{\frac12}\int_0^{\frac14}\left|\mu-\frac{39}{90}\right|\left|\tau-\frac{7}{90}\right|d\tau d\mu + \int_{\frac14}^{\frac12}\int_{\frac14}^{\frac12}\left|\mu-\frac{39}{90}\right|\left|\tau-\frac{39}{90}\right|d\tau d\mu\\
&+\int_{\frac14}^{\frac12}\int_{\frac12}^{\frac34}\left|\mu-\frac{39}{90}\right|\left|\tau-\frac{51}{90}\right|d\tau d\mu + \int_{\frac14}^{\frac12}\int_{\frac34}^{1}\left|\mu-\frac{39}{90}\right|\left|\tau-\frac{83}{90}\right|d\tau d\mu\\
&+\int_{\frac12}^{\frac34}\int_0^{\frac14}\left|\mu-\frac{51}{90}\right|\left|\tau-\frac{7}{90}\right|d\tau d\mu + \int_{\frac12}^{\frac34}\int_{\frac14}^{\frac12}\left|\mu-\frac{51}{90}\right|\left|\tau-\frac{51}{90}\right|d\tau d\mu\\
&+\int_{\frac12}^{\frac34}\int_{\frac12}^{\frac34}\left|\mu-\frac{51}{90}\right|\left|\tau-\frac{51}{90}\right|d\tau d\mu + \int_{\frac12}^{\frac34}\int_{\frac34}^{1}\left|\mu-\frac{51}{90}\right|\left|\tau-\frac{83}{90}\right|d\tau d\mu\\
&+\int_{\frac34}^{1}\int_0^{\frac14}\left|\mu-\frac{83}{90}\right|\left|\tau-\frac{7}{90}\right|d\tau d\mu + \int_{\frac34}^{1}\int_{\frac14}^{\frac12}\left|\mu-\frac{83}{90}\right|\left|\tau-\frac{39}{90}\right|d\tau d\mu\\
&+\int_{\frac34}^{1}\int_{\frac12}^{\frac34}\left|\mu-\frac{83}{90}\right|\left|\tau-\frac{51}{90}\right|d\tau d\mu + \int_{\frac34}^{1}\int_{\frac34}^{1}\left|\mu-\frac{83}{90}\right|\left|\tau-\frac{83}{90}\right|d\tau d\mu\\
&=\frac{57121}{10497600}.
\end{aligned} \tag{33}$$

So,

$$\left(\int_0^1\int_0^1\left(\left|\Upsilon(\mu,y)\,\lambda(x,\tau)\right|\right)d\tau d\mu\right)^{1-\frac{1}{q}} = \left(\frac{57121}{10497600}\right)^{1-\frac{1}{q}}. \tag{34}$$

Putting (32) and (34) in (31). Thus the proof of Theorem 2.6 has been finalized. □

Corollary 2.7. *By setting $s = 1$ in Theorem 2.6, then have*

$$\begin{aligned}&\left|\Lambda(\theta,\vartheta,x;\omega,z,y)\right| \leq (\vartheta-\theta)(z-\omega)\left(\frac{57121}{6480\times 6480}\right)^{\frac{1}{q}}\left(\frac{57121}{10497600}\right)^{1-\frac{1}{q}}\\&\times\left[\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,z)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,z)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,\omega)\right|^q\right]^{\frac{1}{q}},\end{aligned}$$

where $\Lambda(\theta,\vartheta,x;\omega,z,y)$ is defined in Theorem 2.2.

3. Application to Quadrature Formula

Mathematicians have employed cubature formulas to validate Boole's rule for numerical integration involving functions of two variables. In this section, we apply the integral inequalities derived in the main results to obtain estimates for Boole's cubature formula. Assume that, β_N and β_M is the partition given by

$$\beta_N : \theta = \zeta_0 < \zeta_1 < ... < \zeta_{N-1} < \zeta_N = \vartheta,$$

also

$$\beta_M : \omega = \xi_0 < \xi_1 < ... < \xi_{M-1} < \xi_M = z,$$

$$\begin{aligned}h_i &= \frac{(\zeta_{i+1}-\zeta_i)}{\mathbf{N}}, \text{ for all } i = 0,1,2,...,N-1,\\ k_j &= \frac{\left(\xi_{j+1}-\xi_j\right)}{\mathbf{M}}, \text{ for all } j = 0,1,2,...,M-1,\end{aligned}$$

where **N** and **M** must be divisible by 4. Then we have

$$\int_\theta^\vartheta\int_\omega^z F(\zeta,\xi)\,d\xi d\zeta = S_{N,M}(\beta_N,\beta_M,F) + R_{N,M}(\beta_N,\beta_M,F),$$

where

$$\begin{aligned}&S_{N,M}(\beta_N,\beta_M,F)\\&=\frac{1}{90\times 90}\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}(\zeta_{i+1}-\zeta_i)\left(\xi_{j+1}-\xi_j\right)\Big\{7\Big[7\Big(F\left(\zeta_i,\xi_j\right)+F\left(\zeta_i,\xi_{j+1}\right)+F\left(\zeta_{i+1},\xi_j\right)\\&+F\left(\zeta_{i+1},\xi_{j+1}\right)\Big)+32\Big(F\left(\zeta_i+h,\xi_j\right)+F\left(\zeta_i,\xi_j+k\right)+F\left(\zeta_i+h,\xi_{j+1}\right)+F\left(\zeta_{i+1},\xi_j+k\right)\Big)\\&+12\Big(F\left(\zeta_i+2h,\xi_j\right)+F\left(\zeta_i+2h,\xi_{j+1}\right)+F\left(\zeta_i,\xi_j+2k\right)+F\left(\zeta_{i+1},\xi_j+2k\right)\Big)\\&+32\Big(F\left(\zeta_i+3h,\xi_j\right)+F\left(\zeta_i,\xi_j+3k\right)+F\left(\zeta_i+3h,\xi_{j+1}\right)+F\left(\zeta_{i+1},\xi_j+3k\right)\Big)\Big]\\&+32\Big[32\Big(F\left(\zeta_i+h,\xi_j+k\right)+F\left(\zeta_i+h,\xi_j+3k\right)+F\left(\zeta_i+3h,\xi_j+k\right)\\&+F\left(\zeta_i+3h,\xi_j+3k\right)\Big)+12\Big(F\left(\zeta_i+h,\xi_j+2k\right)+F\left(\zeta_i+3h,\xi_j+2k\right)\\&+F\left(\zeta_i+2h,\xi_j+k\right)+F\left(\zeta_i+2h,\xi_j+3k\right)\Big)\Big]+12\Big[12F\left(\zeta_i+2h,\xi_j+2k\right)\Big]\Big\}.\end{aligned}$$

The remainder term $R_{N,M}(\beta_N, \beta_M, F)$ satisfy the estimation:

$$\begin{aligned}
&\left|R_{N,M}(\beta_N, \beta_M, F)\right| \\
&\leq \frac{4^{-2s-3}45^{-2(s+2)}}{(s+1)^2(s+2)^2}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s-19\,4^{s+2}45^{s+1}\right. \\
&\left.-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)^2 \\
&\times\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\left|F''_{i+1,j+1}\right|\frac{(\zeta_{i+1}-\zeta_i)^2\left(\xi_{j+1}-\xi_j\right)^2}{\mathbf{M}\times\mathbf{N}}. \qquad (35)
\end{aligned}$$

If we vary the parameter s, the remainder term gives a different estimation. We obtained a good estimation when $s \to 0^+$. For $s = 1$ the remainder term satisfies the estimation:

$$\left|R_{N,M}(\beta_N, \beta_M, F)\right| \leq \frac{57121}{(6480)^2}\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\left|F''_{i+1,j+1}\right|\frac{(\zeta_{i+1}-\zeta_i)^2\left(\xi_{j+1}-\xi_j\right)^2}{\mathbf{M}\times\mathbf{N}}. \qquad (36)$$

We prove the following proposition for error estimation bounds for Boole's rule.

Proposition 3.1. *Assume that $F : [\theta, \vartheta] \times [\omega, z] \to \mathbb{R}$ is a partial differentiable mapping on $\triangle^\circ$. Then, the remainder term $R_{N,M}(\beta_N, \beta_M, F)$ satisfy the estimation:*

$$\begin{aligned}
&\left|R_{N,M}(\beta_N, \beta_M, F)\right| \\
&\leq \frac{4^{-2s-3}45^{-2(s+2)}}{(s+1)^2(s+2)^2}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s-19\,4^{s+2}45^{s+1}\right. \\
&\left.-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)^2 \\
&\times\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\frac{(\zeta_{i+1}-\zeta_i)^2\left(\xi_{j+1}-\xi_j\right)^2}{16}\left\{\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_i, \xi_j\right)\right|\right. \\
&\left.+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_i, \xi_{j+1}\right)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_{i+1}, \xi_j\right)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_{i+1}, \xi_{j+1}\right)\right|\right\},
\end{aligned}$$

for all $i = 0, 1, 2, ..., N-1$, and $j = 0, 1, 2, ..., M-1$.

Proof. Applying Theorem 2.2 and arrange things for our result

$$\begin{aligned}
\theta &= \zeta_i, \vartheta = \zeta_{i+1}, 4h_i = \zeta_{i+1}-\zeta_i, \\
\omega &= \xi_j, z = \xi_{j+1}, 4K_j = \xi_{j+1}-\xi_j.
\end{aligned}$$

Then we have the estimation

$$
\begin{aligned}
&\left|\frac{(\zeta_{i+1}-\zeta_i)\left(\xi_{j+1}-\xi_j\right)}{90\times 90}\left\{7\left[7\left(F\left(\zeta_i,\xi_j\right)+F\left(\zeta_i,\xi_{j+1}\right)+F\left(\zeta_{i+1},\xi_j\right)+F\left(\zeta_{i+1},\xi_{j+1}\right)\right)\right.\right.\right.\\
&+32\left(F\left(\zeta_i+h,\xi_j\right)+F\left(\zeta_i,\xi_j+k\right)+F\left(\zeta_i+h,\xi_{j+1}\right)+F\left(\zeta_{i+1},\xi_j+k\right)\right)\\
&+12\left(F\left(\zeta_i+2h,\xi_j\right)+F\left(\zeta_i+2h,\xi_{j+1}\right)+F\left(\zeta_i,\xi_j+2k\right)+F\left(\zeta_{i+1},\xi_j+2k\right)\right)\\
&+32\left.\left(F\left(\zeta_i+3h,\xi_j\right)+F\left(\zeta_i,\xi_j+3k\right)+F\left(\zeta_i+3h,\xi_{j+1}\right)+F\left(\zeta_{i+1},\xi_j+3k\right)\right)\right]\\
&+32\left[32\left(F\left(\zeta_i+h,\xi_j+k\right)+F\left(\zeta_i+h,\xi_j+3k\right)+F\left(\zeta_i+3h,\xi_j+k\right)\right.\right.\\
&\left.+F\left(\zeta_i+3h,\xi_j+3k\right)\right)+12\left(F\left(\zeta_i+h,\xi_j+2k\right)+F\left(\zeta_i+3h,\xi_j+2k\right)\right.\\
&\left.\left.\left.+F\left(\zeta_i+2h,\xi_j+k\right)+F\left(\zeta_i+2h,\xi_j+3k\right)\right)\right]+12\left[12F\left(\zeta_i+2h,\xi_j+2k\right)\right]\right\}\\
&\left.-\int_{\zeta_i}^{\zeta_{i+1}}\int_{\xi_j}^{\xi_{j+1}}F(\zeta,\xi)\,d\xi d\zeta\right|\leq\frac{4^{-2s-3}45^{-2(s+2)}}{(s+1)^2(s+2)^2}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s\right.\\
&-19\,4^{s+2}45^{s+1}-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2} \qquad (37)\\
&\left.+166^{s+2}\right)^2\left(\frac{(\zeta_{i+1}-\zeta_i)^2\left(\xi_{j+1}-\xi_j\right)^2}{16}\right)\left[\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_i,\xi_j\right)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_i,\xi_{j+1}\right)\right|\right.\\
&\left.+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_{i+1},\xi_j\right)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_{i+1},\xi_{j+1}\right)\right|\right]. \qquad (38)
\end{aligned}
$$

After applying summation on both sides of inequality (38) and triangular inequality, we have

$$
\begin{aligned}
&\left|\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\frac{(\zeta_{i+1}-\zeta_i)\left(\xi_{j+1}-\xi_j\right)}{90\times 90}\left\{7\left[7\left(F\left(\zeta_i,\xi_j\right)+F\left(\zeta_i,\xi_{j+1}\right)+F\left(\zeta_{i+1},\xi_j\right)\right.\right.\right.\right.\\
&\left.+F\left(\zeta_{i+1},\xi_{j+1}\right)\right)+32\left(F\left(\zeta_i+h,\xi_j\right)+F\left(\zeta_i,\xi_j+k\right)+F\left(\zeta_i+h,\xi_{j+1}\right)+F\left(\zeta_{i+1},\xi_j+k\right)\right)\\
&+12\left(F\left(\zeta_i+2h,\xi_j\right)+F\left(\zeta_i+2h,\xi_{j+1}\right)+F\left(\zeta_i,\xi_j+2k\right)+F\left(\zeta_{i+1},\xi_j+2k\right)\right)\\
&+32\left.\left(F\left(\zeta_i+3h,\xi_j\right)+F\left(\zeta_i,\xi_j+3k\right)+F\left(\zeta_i+3h,\xi_{j+1}\right)+F\left(\zeta_{i+1},\xi_j+3k\right)\right)\right]\\
&+32\left[32\left(F\left(\zeta_i+h,\xi_j+k\right)+F\left(\zeta_i+h,\xi_j+3k\right)+F\left(\zeta_i+3h,\xi_j+k\right)\right.\right.\\
&\left.+F\left(\zeta_i+3h,\xi_j+3k\right)\right)+12\left(F\left(\zeta_i+h,\xi_j+2k\right)+F\left(\zeta_i+3h,\xi_j+2k\right)\right.\\
&\left.\left.\left.+F\left(\zeta_i+2h,\xi_j+k\right)+F\left(\zeta_i+2h,\xi_j+3k\right)\right)\right]+12\left[12F\left(\zeta_i+2h,\xi_j+2k\right)\right]\right\}\\
&\left.-\int_{\zeta_i}^{\zeta_{i+1}}\int_{\xi_j}^{\xi_{j+1}}F(\zeta,\xi)\,d\xi d\zeta\right|\leq\frac{4^{-2s-3}45^{-2(s+2)}}{(s+1)^2(s+2)^2}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s\right.\\
&-19\,4^{s+2}45^{s+1}-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}\\
&\left.+166^{s+2}\right)^2\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\left(\frac{(\zeta_{i+1}-\zeta_i)^2\left(\xi_{j+1}-\xi_j\right)^2}{16}\right)\left[\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_i,\xi_j\right)\right|\right.\\
&\left.+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_i,\xi_{j+1}\right)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_{i+1},\xi_j\right)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_{i+1},\xi_{j+1}\right)\right|\right]. \qquad (39)
\end{aligned}
$$

For $s = 1$ in (39), we have

$$\left|\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\frac{(\zeta_{i+1}-\zeta_i)\left(\xi_{j+1}-\xi_j\right)}{90\times 90}\left\{7\left[7\left(F\left(\zeta_i,\xi_j\right)+F\left(\zeta_i,\xi_{j+1}\right)+F\left(\zeta_{i+1},\xi_j\right)\right.\right.\right.\right.$$
$$\left.+F\left(\zeta_{i+1},\xi_{j+1}\right)\right)+32\left(F\left(\zeta_i+h,\xi_j\right)+F\left(\zeta_i,\xi_j+k\right)+F\left(\zeta_i+h,\xi_{j+1}\right)+F\left(\zeta_{i+1},\xi_j+k\right)\right)$$
$$+12\left(F\left(\zeta_i+2h,\xi_j\right)+F\left(\zeta_i+2h,\xi_{j+1}\right)+F\left(\zeta_i,\xi_j+2k\right)+F\left(\zeta_{i+1},\xi_j+2k\right)\right)$$
$$\left.+32\left(F\left(\zeta_i+3h,\xi_j\right)+F\left(\zeta_i,\xi_j+3k\right)+F\left(\zeta_i+3h,\xi_{j+1}\right)+F\left(\zeta_{i+1},\xi_j+3k\right)\right)\right]$$
$$+32\left[32\left(F\left(\zeta_i+h,\xi_j+k\right)+F\left(\zeta_i+h,\xi_j+3k\right)+F\left(\zeta_i+3h,\xi_j+k\right)\right.\right.$$
$$\left.+F\left(\zeta_i+3h,\xi_j+3k\right)\right)+12\left(F\left(\zeta_i+h,\xi_j+2k\right)+F\left(\zeta_i+3h,\xi_j+2k\right)\right.$$
$$\left.\left.+F\left(\zeta_i+2h,\xi_j+k\right)+F\left(\zeta_i+2h,\xi_j+3k\right)\right)\right]+12\left[12F\left(\zeta_i+2h,\xi_j+2k\right)\right]\Big\}$$
$$\left.-\int_\theta^\vartheta\int_\omega^z F(\zeta,\xi)\,d\xi d\zeta\right|\le\frac{57121}{(6480)^2}\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\left(\frac{(\zeta_{i+1}-\zeta_i)^2\left(\xi_{j+1}-\xi_j\right)^2}{16}\right)\left[\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_i,\xi_j\right)\right|\right.$$
$$\left.+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_i,\xi_{j+1}\right)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_{i+1},\xi_j\right)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_{i+1},\xi_{j+1}\right)\right|\right],$$

which completes the proof. □

Proposition 3.2. *Assume that $F:[\theta,\vartheta]\times[\omega,z]\to\mathbb{R}$ is a partial differentiable mapping on $\triangle^\circ$. Then, the remainder term $R_{n,m}(I_n,I_m,F)$ satisfy the expression below:*

$$\left|R_{N,M}(\beta_N,\beta_M,F)\right|$$
$$\le\frac{1}{(s+1)^{\frac{2}{q}}}\left(\frac{4^{-2p-1}2025^{-p-1}\left(12^{p+1}+14^{p+1}+31^{p+1}+33^{p+1}\right)^2}{(p+1)^2}\right)^{\frac{1}{p}}$$
$$\times\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\frac{(\zeta_{i+1}-\zeta_i)^2\left(\xi_{j+1}-\xi_j\right)^2}{16}\left\{\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_i,\xi_j\right)\right|^q\right.$$
$$\left.+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_i,\xi_{j+1}\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_{i+1},\xi_j\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F\left(\zeta_{i+1},\xi_{j+1}\right)\right|^q\right\}^{\frac{1}{q}}. \tag{40}$$

Proof. Applying Theorem 2.4, the proof is similar to the Proposition 3.1. Thus, the required result was obtained. □

Remark 3.3. *With the variation of parameter s in (39) and (40), one can achieve different and good estimations. When $s = 1$, then results become for classical convex functions.*

4. Numerical Examples and Computational Analysis

In this Section, we performed a detailed numerical analysis to validate the effectiveness of our newly derived results. Through various computational experiments, we demonstrated the practical utility of the proposed inequalities, particularly in approximating integrals of differentiable s-convex functions. Applying our newly proved results offers practical tools for solving optimization problems and analyzing data.

Example 4.1. *Consider $F(\tau,\mu)=\tau^{6s}\mu^{6s}$ is partial differentiable function and $\left|\frac{\partial^2 F}{\partial\tau\partial\mu}\right|$ is co-ordinated s-convex on $[\theta,\vartheta]\times[\omega,z]=[1,2]\times[1,2]$. Then solve by Mathematica, the left-hand side of inequality (19), for $s=\frac{1}{2}$ become:*

$$\begin{aligned}
&\left|\Lambda(\theta,\vartheta,x;\omega,z,y)\right| \\
&=\left|\frac{1}{90\times 90}\left\{7\left[7\left(F(\theta,\omega)+F(\theta,z)+F(\vartheta,\omega)+F(\vartheta,z)\right)\right.\right.\right. \\
&+32\left(F\left(\frac{3\theta+\vartheta}{4},\omega\right)+F\left(\theta,\frac{3\omega+z}{4}\right)+F\left(\frac{3\theta+\vartheta}{4},z\right)+F\left(\vartheta,\frac{3\omega+z}{4}\right)\right) \\
&+12\left(F\left(\frac{\theta+\vartheta}{2},\omega\right)+F\left(\frac{\theta+\vartheta}{2},z\right)+F\left(\theta,\frac{\omega+z}{2}\right)+F\left(\vartheta,\frac{\omega+z}{2}\right)\right) \\
&\left.+32\left(F\left(\frac{\theta+3\vartheta}{4},\omega\right)+F\left(\theta,\frac{\omega+3z}{4}\right)+F\left(\frac{\theta+3\vartheta}{4},z\right)+F\left(\vartheta,\frac{\omega+3z}{4}\right)\right)\right] \\
&+32\left[32\left(F\left(\frac{3\theta+\vartheta}{4},\frac{3\omega+z}{4}\right)+F\left(\frac{3\theta+\vartheta}{4},\frac{\omega+3z}{4}\right)+F\left(\frac{\theta+3\vartheta}{4},\frac{3\omega+z}{4}\right)\right.\right. \\
&\left.+F\left(\frac{\theta+3\vartheta}{4},\frac{\omega+3z}{4}\right)\right)+12\left(F\left(\frac{3\theta+\vartheta}{4},\frac{\omega+z}{2}\right)+F\left(\frac{\theta+3\vartheta}{4},\frac{\omega+z}{2}\right)\right. \\
&\left.\left.\left.+F\left(\frac{\theta+\vartheta}{2},\frac{3\omega+z}{4}\right)+F\left(\frac{\theta+\vartheta}{2},\frac{\omega+3z}{4}\right)\right)\right]+12\left[12F\left(\frac{\theta+\vartheta}{2},\frac{\omega+z}{2}\right)\right]\right\} \\
&-\frac{1}{90(\vartheta-\theta)}\int_\theta^\vartheta\left[7F(x,\omega)+32F\left(x,\frac{3\omega+z}{4}\right)+12F\left(x,\frac{\omega+z}{2}\right)\right. \\
&\left.+32F\left(x,\frac{\omega+3z}{4}\right)+7F(x,z)\right]dx-\frac{1}{90(z-\omega)}\int_\omega^z\left[7F(\theta,y)+32F\left(\frac{3\theta+\vartheta}{4},y\right)\right. \\
&\left.+12F\left(\frac{\theta+\vartheta}{2},y\right)+32F\left(\frac{\theta+3\vartheta}{4},y\right)+7F(\vartheta,y)\right]dy \\
&\left.+\frac{1}{(\vartheta-\theta)(z-\omega)}\int_\theta^\vartheta\int_\omega^z F(x,y)\,dydx\right|=6.375\times 10^{-8}.
\end{aligned}\tag{41}$$

The right-hand side of inequality (19) for $s=\frac{1}{2}$ become:

$$\begin{aligned}
&\frac{4^{-2s-3}45^{-2(s+2)}(\vartheta-\theta)(z-\omega)}{(s+1)^2(s+2)^2}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s-19\,4^{s+2}45^{s+1}\right. \\
&\left.-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)^2 \\
&\times\left[\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,\omega)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,z)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,z)\right|\right]=24.1702.
\end{aligned}\tag{42}$$

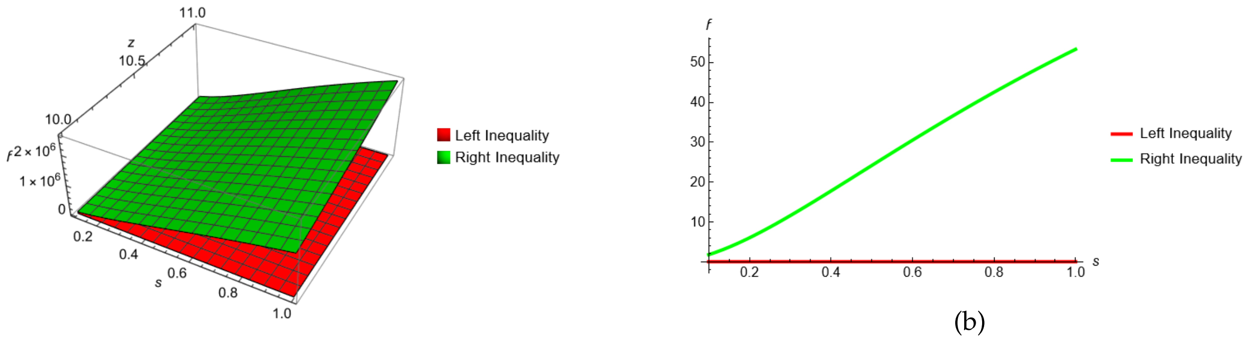
From Eq. (41) and Eq. (42), it is observed

$$6.375\times 10^{-8}<24.1702.$$

The above inequality show that the inequality (19) of Theorem 2.2, is valid for co-ordinated s-convex functions.

Example 4.2. *Consider $F(\tau,\mu)=e^{s\tau\mu}$ is partial differentiable function and $\left|\frac{\partial^2 F}{\partial\tau\partial\mu}\right|$ is co-ordinated s-convex on $[\theta,\vartheta]\times[\omega,z]=[1,2]\times[1,2]$. Then solve by Mathematica, the left-hand side of inequality (19), for $s=\frac{1}{2}$ become:*

$$\left|\Lambda(\theta,\vartheta,x;\omega,z,y)\right|=1.11474\times 10^{-6}.\tag{43}$$



(a) (b)

Figure 1: Numerical Analysis of Theorem 2.2: (a) 3D-plot, (b) 2D-plot in Example 4.1, computed and plotted with Mathematica.

And, the right-hand side of inequality (19), for $s=\frac{1}{2}$ *become:*

$$\begin{aligned}
&\frac{4^{-2s-3}45^{-2(s+2)}(\vartheta-\theta)(z-\omega)}{(s+1)^2(s+2)^2}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s-19\,4^{s+2}45^{s+1}\right.\\
&\left.-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)^2\\
&\times\left[\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,\omega)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,z)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,z)\right|\right]\\
&=0.0437897. \qquad (44)
\end{aligned}$$

From Eq. (43) and Eq. (44), it is observed

$$1.11474\times10^{-6}<0.0437897.$$

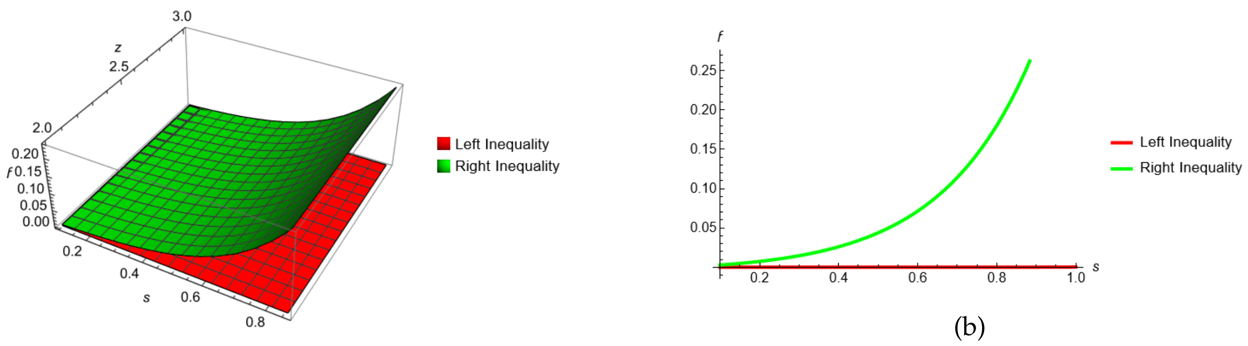
The above inequality show that the inequality (19) of Theorem 2.2, is valid for co-ordinated s-convex functions.

Example 4.3. *Consider* $F(\tau,\mu)=\tau^{4s}\mu^{4s}$ *is partial differentiable function and* $\left|\frac{\partial^2F}{\partial\tau\partial\mu}\right|^q$ *is co-ordinated s-convex on* $[\theta,\vartheta]\times[\omega,z]=[1,2]\times[1,2]$, *also* $p=q=2$. *Then solve by Mathematica, the left-hand side of inequality (25), for* $s=\frac{3}{4}$ *become:*

$$\left|\tau(\theta,\vartheta,x;\omega,z,y)\right|=6.375\times10^{-8}, \qquad (45)$$

the right-hand side of inequality (25), for $s=\frac{3}{4}$ *become:*

$$\begin{aligned}
&\frac{(\vartheta-\theta)(z-\omega)}{\left[(s+1)^2\right]^{\frac{1}{q}}}\left(\frac{4^{-2p-1}2025^{-p-1}\left(12^{p+1}+14^{p+1}+31^{p+1}+33^{p+1}\right)^2}{(p+1)^2}\right)^{\frac{1}{p}}\\
&\times\left[\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,\omega)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,z)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,z)\right|^q\right]^{\frac{1}{q}}\\
&=0.701587. \qquad (46)
\end{aligned}$$



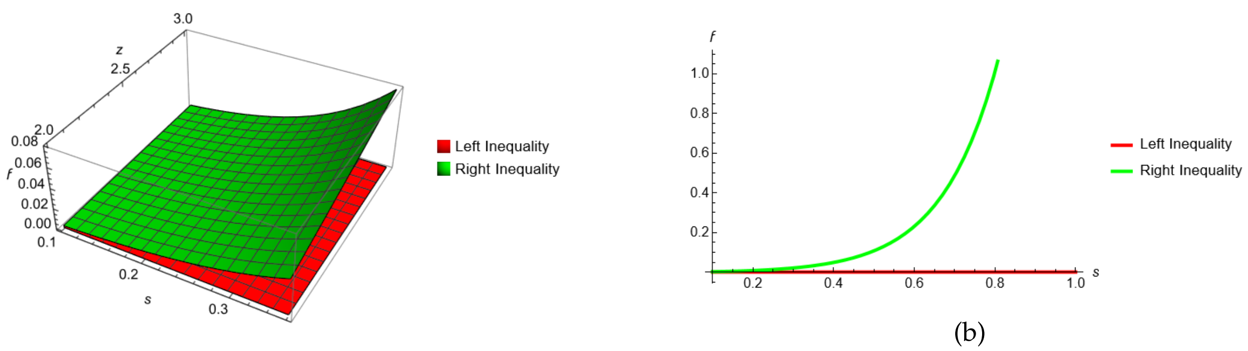
(a) (b)

Figure 2: Numerical Analysis of Theorem 2.2: (a) 3D-plot, (b) 2D-plot in Example 4.2, computed and plotted with Mathematica.

From Eq. (45) and Eq. (46), it is observed

$$6.375 \times 10^{-8} < 0.701587.$$

The above inequality show that the inequality (25) of Theorem 2.4, is valid for co-ordinated s-convex functions.



(a) (b)

Figure 3: Numerical Analysis of Theorem 2.4: (a) 3D-plot, (b) 2D-plot in Example 4.3, computed and plotted with Mathematica.

Example 4.4. *Consider* $F(\tau,\mu) = \tau^{6s}\mu^{6s}$ *is partial differentiable function and* $\left|\frac{\partial^2 F}{\partial\tau\partial\mu}\right|^q$ *is co-ordinated s-convex on* $[\theta,\vartheta]\times[\omega,z] = [1,2]\times[1,2]$, *and* $q = 2$. *Then solve by Mathematica, the left-hand side of inequality (29), for* $s = \frac{3}{4}$ *become:*

$$\left|\Lambda(\theta,\vartheta,x;\omega,z,y)\right| = 0.0000674731. \tag{47}$$

The right-hand side of inequality (29), for $s = \frac{3}{4}$ become:

$$\begin{aligned}
&(\vartheta-\theta)(z-\omega)\left(\frac{57121}{10497600}\right)^{1-\frac{1}{q}}\\
&\times\left[\frac{4^{-2s-3}45^{-2(s+2)}(\vartheta-\theta)(z-\omega)}{(s+1)^2(s+2)^2}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s-19\,4^{s+2}45^{s+1}\right.\right.\\
&\left.-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)^2\\
&\left.\times\left(\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,\omega)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,z)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,z)\right|^q\right)\right]^{\frac{1}{q}}\\
&=8.17114. \qquad (48)
\end{aligned}$$

From Eq. (47) and Eq. (48), it is observed

$$0.0000674731 < 8.17114.$$

The above inequality show that the inequality (29) of Theorem 2.6, is valid for co-ordinated s-convex functions.

Example 4.5. *Consider $F(\tau,\mu) = e^{s\tau\mu}$ is partial differentiable function and $\left|\frac{\partial^2 F}{\partial\tau\partial\mu}\right|$ is co-ordinated s-convex on $[\theta,\vartheta]\times[\omega,z]=[1,2]\times[1,2]$. Then solve by Mathematica, the left-hand side of inequality (29), for $s=\frac{3}{4}$ become:*

$$\left|\Lambda(\theta,\vartheta,x;\omega,z,y)\right| = 0.0000251054. \qquad (49)$$

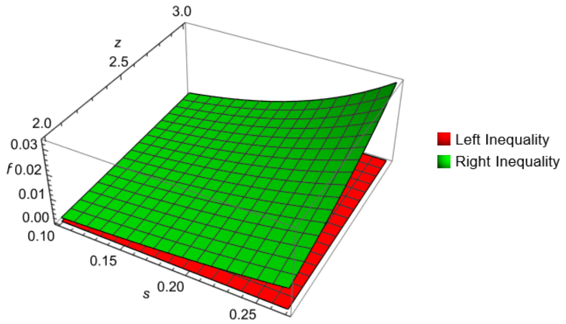
And, the right-hand side of inequality (29), for $s = \frac{3}{4}$ become:

$$\begin{aligned}
&(\vartheta-\theta)(z-\omega)\left(\frac{57121}{10497600}\right)^{1-\frac{1}{q}}\\
&\times\left[\frac{4^{-2s-3}45^{-2(s+2)}(\vartheta-\theta)(z-\omega)}{(s+1)^2(s+2)^2}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s-19\,4^{s+2}45^{s+1}\right.\right.\\
&\left.-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)^2\\
&\left.\times\left(\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,\omega)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\theta,z)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,\omega)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\mu}F(\vartheta,z)\right|^q\right)\right]^{\frac{1}{q}}\\
&=0.19231. \qquad (50)
\end{aligned}$$

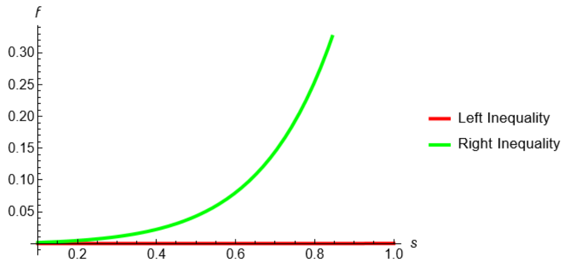
From Eq. (49) and Eq. (50), it is observed

$$0.0000251054 < 0.19231.$$

The above inequality show that the inequality (29) of Theorem 2.6, is valid for co-ordinated s-convex functions.



(a)



(b)

Figure 4: Numerical Analysis of Theorem 2.6: (a) 3D-plot, (b) 2D-plot in Example 4.5, computed and plotted with Mathematica.

s	Left Inequality	Right Inequality
0.1	2.44236×10^{-11}	0.00283137
0.2	2.03885×10^{-9}	0.00740893
0.3	3.034×10^{-8}	0.0146626
0.4	2.23032×10^{-7}	0.0260414
0.5	1.11474×10^{-6}	0.0437897
0.6	4.3674×10^{-6}	0.0713695
0.7	1.44702×10^{-5}	0.114108
0.8	4.2423×10^{-5}	0.180181
0.9	1.13314×10^{-4}	0.282115
1	2.81306×10^{-4}	0.439064

Table 1: Computational analysis between the left-hand and the right-hand sides of inequality (19) for discretization of "s" in Example 4.2.

s	Left Inequality	Right Inequality
0.1	2.39878×10^{-6}	0.00167529
0.2	1.11473×10^{-6}	0.00752334
0.3	1.00268×10^{-6}	0.0206178
0.4	1.64723×10^{-6}	0.0483851
0.5	2.1×10^{-8}	0.106996
0.6	2.38852×10^{-6}	0.230083
0.7	2.10876×10^{-6}	0.485757
0.8	3.57022×10^{-6}	1.00945
0.9	1.03931×10^{-5}	2.06677
1	2.046×10^{-7}	0.37279

Table 2: Computational analysis between the left-hand and the right-hand sides of inequality (25) for discretization of "s" in Example 4.3.

5. Conclusion

This article establishes new inequalities for Boole's formula related to generalized coordinate convex functions, enhancing the estimation of error bounds. The findings indicate that coordinate s-convex functions offer more accurate approximations than traditional coordinate convex functions. Practical applicability is demonstrated through integration into quadrature formulas. Numerical examples and computational analysis in Tables 1 and 2 show that the proposed inequalities are valid. When the parameter "s" is varied, the new bounds perform better than those of traditional coordinate convex functions, particularly when ($s = 1$). This research contributes valuable insights to mathematical analysis and optimization, with future investigations potentially incorporating q-calculus, fractional calculus, and multidimensional analysis. The results significantly advance integral inequalities and enhance the application of Boole's formula in practical scenarios.

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