



Statistical concentration and failure measures under project-type refinements on atomless probability spaces

Ivana D. Petrović^{a,*}, Jelena M. Višnjić^a

^aDepartment of Mathematics and Informatics, Medical faculty, University of Niš, Serbia

Abstract. In this paper we give a simple measure-theoretic reformulation of the refinement procedure of Ilić and Veličković (2019) on a general atomless probability space. We work with finite measurable partitions and show that suitable project-type refinements, which split project cells of maximal probability into smaller pieces of the same total probability, strictly decrease the project failure measure. Moreover, for any given threshold the failure measure can be made no larger than this threshold after finitely many such refinements, and along suitable infinite refinement sequences it converges to zero. We then introduce a natural concentration functional which measures how strongly the total failure probability is concentrated in a few project cells, and we prove that it is also strictly decreased by the same project-type refinements and can be made arbitrarily small. Several small project examples are given to illustrate the effect of refinements on concrete failure profiles.

1. Introduction

Large projects in engineering, information technology, healthcare, and public infrastructure are typically realized through a finite collection of interdependent tasks. In practice, each task carries a certain amount of uncertainty: it can be delayed, it can fail completely, or it can cause hidden side effects that propagate through the project. Empirical studies repeatedly show that a substantial proportion of projects are delivered late, over budget, or with reduced scope, despite the existence of detailed methodologies, standards and best-practice guidelines (see, for example, recent discussions in the project management and quality management literature and the quantitative risk-prioritisation framework of Acebes et al. (2024)).¹⁾

Standard project management frameworks (such as process standards and bodies of knowledge in engineering and IT) provide systematic procedures for planning, risk registers, and qualitative or semi-quantitative risk assessment. However, the underlying uncertainty is rarely treated within a rigorous probabilistic framework. From the mathematical point of view, it is natural to describe a completed project

2020 *Mathematics Subject Classification.* Primary 60A10; Secondary 28A20, 94A17.

Keywords. atomless probability space; statistical concentration, finite measurable partition; project-type refinement; failure measure; Rényi-type entropy; Herfindahl–Hirschman index.

Received: 27 November 2025; Revised: 29 December 2025; Accepted: 03 January 2026

Communicated by Aleksandar Nastić

Authors are supported by Grant No. 451-03-34/2026-03/200113 of the Ministry of Science, Technological Development and Innovation, Republic of Serbia.

* Corresponding author: Ivana D. Petrović

Email addresses: ivanailicc3@gmail.com (Ivana D. Petrović), jvisnjic@gmail.com (Jelena M. Višnjić)

ORCID iDs: <https://orcid.org/0000-0002-6887-3268> (Ivana D. Petrović), <https://orcid.org/0000-0002-1580-6969> (Jelena M. Višnjić)

¹⁾We do not attempt a full survey here; our focus is on the mathematical structure behind such phenomena.

as a structured object whose failure or success can be encoded by a probability measure on a suitable state space, and to ask how project revisions transform this measure. In this way, the language of probability offers a universal and precise tool for describing risk and for analysing the effect of improvements on the overall failure profile.

A first step in this direction was taken in Ilić and Veličković (2019), where a completed project is represented by a finite set of tasks Z equipped with quality indices. These indices induce the *difference indicators* $P(x)$, $x \in Z$, which form a probability distribution on Z and quantify the relative contribution of each task to the project failure. The project failure measure is defined there as

$$\mu(P) = \max_{x \in Z} P(x),$$

so that a large value of $\mu(P)$ indicates that the risk is heavily concentrated in a single task. Project revisions are modelled by replacing a problematic task by a finite family of new subtasks with improved quality indices, while preserving or increasing the total index of that task. On the probabilistic level this corresponds to splitting the atom $\{x\}$ of the finite probability space $(Z, 2^Z, P)$ into smaller atoms. Using a construction based on optional random sequences and decompositions of atomless probability spaces (Billingsley (1995)), it is shown in Ilić and Veličković (2019) that, by a suitable sequence of such revisions, the failure measure $\mu(P)$ can be made arbitrarily small and that all difference indicators fall below a prescribed threshold $\varepsilon > 0$.

A complementary line of work appears in Ilić and Višnjić (2025), where projects are studied at a more general and abstract level. There, the aim is to build axiomatic foundations for Project Theory by representing projects as set-theoretic structures with well-founded relations and by analysing their properties via tools from mathematical logic and set theory. Central roles are played by the Axiom of Compatibility, ensuring consistency across subprojects, and the Axiom of Regularity, which prevents infinite regress in project hierarchies.

This axiomatic framework provides robust structural foundations for project modelling, but it does not focus on probabilistic measures of failure or on quantitative refinement procedures.

The present paper lies at the intersection of these two approaches. Our starting point is a general atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, in the standard sense of measure theory (see, for example, Billingsley (1995, Chapters 1–3)), and a finite measurable partition $\mathcal{P} = \{A_1, \dots, A_k\}$ of Ω .

The sets A_i are interpreted as classes of tasks or failure modes, and the probabilities $\mathbb{P}(A_i)$ as their associated difference indicators. For any such partition \mathcal{P} we define the project failure measure:

$$\mu(\mathcal{P}) = \max_{1 \leq i \leq k} \mathbb{P}(A_i),$$

which is the direct analogue of $\mu(P)$ in the discrete model of Ilić and Veličković (2019). A *project-type refinement* of \mathcal{P} is then defined as an operation which replaces one set A_i by finitely many measurable subsets of A_i whose total measure equals $\mathbb{P}(A_i)$, while all other sets are kept unchanged. This operation preserves the total probability mass and refines the partition, but it redistributes the mass of A_i among smaller pieces.

Comment 1.1. Throughout the paper we write P for probability measures on the finite task set Z in the discrete project model of Ilić–Veličković (2019), \mathbb{P} for the underlying atomless probability measure on (Ω, \mathcal{F}) , and \mathcal{P} for finite measurable partitions of Ω . The notation μ is used consistently in both discrete and measure-theoretic contexts, with the domain understood from the argument.

Our first aim is to show that the process of project quality improvement from Ilić and Veličković (2019) admits a particularly clean measure-theoretic reformulation in this framework. We prove that every project-type refinement which splits all sets of maximal mass strictly decreases the failure measure $\mu(\mathcal{P})$, and that for every $\varepsilon > 0$ there exists a finite sequence of such refinements which produces a partition \mathcal{P} with $\mu(\mathcal{P}) \leq \varepsilon$. Equivalently, one can construct an infinite refining sequence $(\mathcal{P}_n)_{n \geq 0}$ of finite partitions with $\mu(\mathcal{P}_n) \rightarrow 0$ as $n \rightarrow \infty$. In contrast to the original proof in Ilić and Veličković (2019), which relies on optional sequences and general decompositions, our argument is based on explicit local splitting operations on the sets of a finite partition.

As a second step, we go beyond the maximal set mass and introduce, for $q > 1$, the concentration functional:

$$\Phi_q(\mathcal{P}) = \sum_{A \in \mathcal{P}} \mathbb{P}(A)^q.$$

This quantity is well known as a basic concentration index in probability and statistics and, for $q = 2$, it coincides with the Herfindahl index of concentration (for more details on Herfindahl–Hirschman Index or HHI, see for example Rhoades (1993)). In our context, $\Phi_q(\mathcal{P})$ measures how strongly the total failure mass is concentrated in a few sets of the partition. We show that the same project–type refinements which decrease $\mu(\mathcal{P})$ also strictly decrease $\Phi_q(\mathcal{P})$ and that, along suitable refining sequences (\mathcal{P}_n) , one has $\Phi_q(\mathcal{P}_n) \rightarrow 0$. In particular, both the maximal failure probability and the basic concentration index $\Phi_2(\mathcal{P})$ can be made arbitrarily small by finitely many elementary splits of sets.

The contribution of this paper is therefore twofold. First, it provides a simple measure–theoretic reformulation and clarification of the refinement procedure implicit in Ilić and Veličković (2019), by working directly with finite partitions on an atomless probability space and with explicit splitting operations. Second, it extends the analysis to a family of concentration functionals $(\Phi_q)_{q>1}$, giving a probabilistic interpretation of these indices in terms of project revisions and illustrating their behaviour on small project examples.

The paper is organized as follows. In Section 2 we recall basic facts about atomless probability spaces and finite measurable partitions, and we introduce notation for project–type refinements. We then present our main results for the failure measure $\mu(\mathcal{P})$ and the concentration functional $\Phi_q(\mathcal{P})$, showing that both are strictly decreased by suitable refinements.

Section 3 is devoted to illustrative project examples, where we compute $\mu(\mathcal{P})$ and $\Phi_2(\mathcal{P})$ before and after refinement. Finally, Section 4 contains concluding remarks and some directions for further work.

2. Preliminaries and Main results

We work throughout on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that a measurable set $A \in \mathcal{F}$ is called an atom if $\mathbb{P}(A) > 0$ and for every measurable $B \subset A$ one has either $\mathbb{P}(B) = 0$ or $\mathbb{P}(B) = \mathbb{P}(A)$. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *atomless* if it has no atoms, that is, if for every $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ there exists a measurable $B \subset A$ with $0 < \mathbb{P}(B) < \mathbb{P}(A)$ (see, for example, Billingsley (1995)).

Definition 2.1 (Project cell and project partition). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A project cell is a measurable subset $C \in \mathcal{F}$ with $\mathbb{P}(C) > 0$ that represents an elementary measurable component of a project within $(\Omega, \mathcal{F}, \mathbb{P})$. Unlike measure-theoretic atoms, a project cell need not be minimal with respect to \mathbb{P} ; in an atomless space, each project cell can be further subdivided into smaller measurable subsets of positive probability.*

A project partition is a finite collection of project cells

$$\mathcal{P} = \{C_1, C_2, \dots, C_k\}, \quad C_i \in \mathcal{F},$$

such that

$$C_i \cap C_j = \emptyset \text{ for } i \neq j, \quad \text{and} \quad \bigcup_{i=1}^k C_i = \Omega.$$

Each C_i represents one measurable unit (cell) of the overall project structure.

In this setting a completed project is represented by a finite measurable partition $\mathcal{P} = \{A_1, \dots, A_k\}$ of Ω , and we define its failure measure by:

$$\mu(\mathcal{P}) = \max_{A \in \mathcal{P}} \mathbb{P}(A).$$

A *project–type refinement* of \mathcal{P} is an operation which replaces finitely many project cells A_i by finite measurable partitions of A_i with the same total probability, while leaving all other project cells unchanged.

Thus, project cells provide the measurable building blocks of a project, generalizing the notion of atomic elements from discrete to atomless probability spaces.

Lemma 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space and let $\mathcal{P} = \{A_1, \dots, A_k\}$ be a finite measurable partition with $\mathbb{P}(A_i) > 0$ for all i . Set $\mu(\mathcal{P}) = \max_{1 \leq i \leq k} \mathbb{P}(A_i)$ and let $I = \{i : \mathbb{P}(A_i) = \mu(\mathcal{P})\}$ be the set of indices of maximal project cells. For each $i \in I$ choose an integer $m_i \geq 2$ and split A_i into a finite partition $A_i = A_{i,1} \cup \dots \cup A_{i,m_i}$ with $\mathbb{P}(A_{i,j}) = \mathbb{P}(A_i)/m_i$ for $j = 1, \dots, m_i$. Define a new finite partition:

$$\mathcal{P}' = \{A_j : j \notin I\} \cup \{A_{i,j} : i \in I, j = 1, \dots, m_i\}.$$

Then $\mu(\mathcal{P}') < \mu(\mathcal{P})$.

Proof. Since the space is atomless, for each $i \in I$ and each $m_i \geq 2$ there exists a measurable partition $A_i = A_{i,1} \cup \dots \cup A_{i,m_i}$ with $\mathbb{P}(A_{i,j}) = \mathbb{P}(A_i)/m_i$ for all j . By construction, for $i \in I$ we have

$$\mathbb{P}(A_{i,j}) = \frac{\mathbb{P}(A_i)}{m_i} = \frac{\mu(\mathcal{P})}{m_i} < \mu(\mathcal{P}), \quad j = 1, \dots, m_i.$$

If $j \notin I$, then $\mathbb{P}(A_j) < \mu(\mathcal{P})$ by the definition of I , and these sets are kept unchanged in \mathcal{P}' . Hence no project cell of \mathcal{P}' has measure equal to $\mu(\mathcal{P})$, so

$$\mu(\mathcal{P}') = \max_{B \in \mathcal{P}'} \mathbb{P}(B) < \mu(\mathcal{P}),$$

as claimed. \square

Theorem 2.1 (Main project-refinement theorem). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space and let

$$\mathcal{P}_0 = \{A_1^{(0)}, \dots, A_{k_0}^{(0)}\}$$

be a finite measurable partition with $\mathbb{P}(A_i^{(0)}) > 0$ for all i . Define

$$\mu(\mathcal{P}) = \max_{B \in \mathcal{P}} \mathbb{P}(B)$$

for any finite partition \mathcal{P} .

(i) For every $\varepsilon > 0$ there exists a finite sequence of finite partitions

$$\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_N$$

such that for each $n = 0, \dots, N-1$ the partition \mathcal{P}_{n+1} is either obtained from \mathcal{P}_n by splitting finitely many project cells into finitely many measurable subsets (as in Lemma 2.1), or else $\mathcal{P}_{n+1} = \mathcal{P}_n$. Whenever $\mu(\mathcal{P}_n) > \varepsilon$, we have

$$\mu(\mathcal{P}_{n+1}) < \mu(\mathcal{P}_n).$$

Moreover,

$$\mu(\mathcal{P}_N) \leq \varepsilon.$$

(ii) In particular, there exists an infinite refining sequence of finite partitions $(\mathcal{P}_n)_{n \geq 0}$ with \mathcal{P}_{n+1} obtained from \mathcal{P}_n by splitting finitely many project cells, such that

$$\mu(\mathcal{P}_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. (i) Fix $\varepsilon > 0$ and consider the given finite partition

$$\mathcal{P}_0 = \{A_1^{(0)}, \dots, A_{k_0}^{(0)}\}$$

with $\mathbb{P}(A_i^{(0)}) > 0$ for all i . For each i define

$$N_i = \max\left\{0, \left\lceil \log_2 \frac{\mathbb{P}(A_i^{(0)})}{\varepsilon} \right\rceil\right\}.$$

Here $\lceil x \rceil$ denotes the smallest integer greater than or equal to x .

Put

$$N = \max_{1 \leq i \leq k_0} N_i + 1.$$

We construct inductively a finite sequence of partitions

$$\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_N$$

as follows. Given \mathcal{P}_n , if $\mu(\mathcal{P}_n) \leq \varepsilon$, we set $\mathcal{P}_{n+1} = \mathcal{P}_n$ for all remaining indices and stop. Otherwise, let \mathcal{B}_n be the collection of project cells $B \in \mathcal{P}_n$ with $\mathbb{P}(B) > \varepsilon$. For each $B \in \mathcal{B}_n$ we use the atomlessness of $(\Omega, \mathcal{F}, \mathbb{P})$ to split B into two measurable subsets B' and B'' with $\mathbb{P}(B') = \mathbb{P}(B'') = \mathbb{P}(B)/2$. All project cells $B \in \mathcal{P}_n$ with $\mathbb{P}(B) \leq \varepsilon$ are left unchanged. We denote by \mathcal{P}_{n+1} the resulting finite partition. By construction, \mathcal{P}_{n+1} is obtained from \mathcal{P}_n by a project-type refinement.

If $\mu(\mathcal{P}_n) > \varepsilon$, there exists at least one project cell $B^* \in \mathcal{P}_n$ with $\mathbb{P}(B^*) = \mu(\mathcal{P}_n)$. Such a project cell necessarily belongs to \mathcal{B}_n and is therefore split into two project cells of measure $\mu(\mathcal{P}_n)/2$. All other project cells of \mathcal{P}_n have measure at most $\mu(\mathcal{P}_n)$ and are either left unchanged or replaced by project cells of strictly smaller measure. Hence,

$$\mu(\mathcal{P}_{n+1}) < \mu(\mathcal{P}_n)$$

whenever $\mu(\mathcal{P}_n) > \varepsilon$.

Next we show that after at most N steps we must have $\mu(\mathcal{P}_N) \leq \varepsilon$. Fix $i \in \{1, \dots, k_0\}$ and consider the descendants of the initial project cell $A_i^{(0)}$ in the refining sequence (\mathcal{P}_n) . Whenever a descendant of $A_i^{(0)}$ is split in the above construction, its measure is exactly halved. Hence any descendant of $A_i^{(0)}$ that has been split k times has measure $\mathbb{P}(A_i^{(0)})/2^k$. By the definition of N_i we have

$$\frac{\mathbb{P}(A_i^{(0)})}{2^{N_i}} \leq \varepsilon,$$

so after at most N_i splits along any branch the measure of every descendant of $A_i^{(0)}$ is at most ε and no further splitting of these descendants takes place. Since $N > N_i$ for all i , it follows that after at most N refinement steps every project cell of \mathcal{P}_N has measure at most ε , that is:

$$\mu(\mathcal{P}_N) \leq \varepsilon.$$

This completes the proof of (i).

(ii) To obtain an infinite refining sequence with $\mu(\mathcal{P}_n) \rightarrow 0$, apply part (i) successively with $\varepsilon = 1/m$ for $m = 1, 2, \dots$ and concatenate the resulting finite refinement sequences. More precisely, starting from \mathcal{P}_0 we first apply (i) with $\varepsilon = 1$ to obtain a finite sequence ending at a partition \mathcal{P}_{N_1} with $\mu(\mathcal{P}_{N_1}) \leq 1$. Then we apply (i) with $\varepsilon = 1/2$ to \mathcal{P}_{N_1} as the starting partition, obtaining a finite sequence ending at \mathcal{P}_{N_2} with $\mu(\mathcal{P}_{N_2}) \leq 1/2$, and so on. Concatenating these sequences yields a refining sequence $(\mathcal{P}_n)_{n \geq 0}$ with

$$\mu(\mathcal{P}_n) \leq \frac{1}{m} \quad \text{for all } n \geq N_m,$$

and hence $\mu(\mathcal{P}_n) \rightarrow 0$ as $n \rightarrow \infty$.

□

Remark 2.1. In the discrete project model of Ilić and Veličković (2019), a completed project is represented by a finite task set Z with difference indicators $p(x)$ on Z , and project revisions correspond to splitting atoms of the discrete probability space $(Z, 2^Z, \mathbb{P})$ into finer components. Theorem 2.1 shows that, in the general atomless case, one can always refine a project-type partition so that the failure measure

$$\mu(\mathcal{P}) = \max_{B \in \mathcal{P}} \mathbb{P}(B)$$

becomes arbitrarily small. Note that the discrete case of Ilić and Veličković (2019) involves atomic probability spaces, while our framework assumes atomlessness; the discrete case can nevertheless be represented within this setting by identifying atoms with project cells of the corresponding partition.

While $\mu(\mathcal{P})$ controls the worst-case failure class, it does not distinguish between partitions in which the remaining failure probability is evenly spread among many small classes and those in which it is still concentrated in a few medium-sized ones. To capture this complementary aspect, for $q > 1$ we introduce the concentration functional

$$\Phi_q(\mathcal{P}) = \sum_{A \in \mathcal{P}} \mathbb{P}(A)^q,$$

which quantifies the overall concentration of the failure probability within the project partition.

This functional assigns larger values to partitions where the total mass is concentrated in a small number of project cells and smaller values to partitions where the mass is more evenly spread among many project cells. As a function of the project cell probabilities, it is a standard power-sum concentration index, closely related to Rényi-type entropies; see Rényi (1961).

In particular, for $q = 2$ the quantity $\Phi_2(\mathcal{P}) = \sum_{A \in \mathcal{P}} \mathbb{P}(A)^2$ coincides with the classical Herfindahl–Hirschman index of concentration (see Rhoades (1993)).

Our next result shows that the same project-type refinements that decrease $\mu(\mathcal{P})$ also strictly decrease $\Phi_q(\mathcal{P})$, and that along suitable refining sequences both the maximal project cell mass and the concentration functional can be made arbitrarily small.

In this way we extend the discrete analysis of Ilić and Veličković (2019), where only the maximal difference indicator $\mu(p)$ is controlled, to a broader measure-theoretic framework in which project revisions improve not only the worst individual failure class but also the overall concentration profile of failure probability.

Lemma 2.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space and let $\mathcal{P} = \{A_1, \dots, A_k\}$ be a finite measurable partition with $\mathbb{P}(A_i) > 0$ for all i . Fix $q > 1$ and define

$$\Phi_q(\mathcal{P}) = \sum_{i=1}^k \mathbb{P}(A_i)^q.$$

Suppose that for some index i_0 we split A_{i_0} into a finite partition with $m \geq 2$:

$$A_{i_0} = B_1 \cup \dots \cup B_m,$$

where $\mathbb{P}(B_j) > 0$ for all j and $\sum_{j=1}^m \mathbb{P}(B_j) = \mathbb{P}(A_{i_0})$, and we leave all other project cells A_i with $i \neq i_0$ unchanged. Let \mathcal{P}' denote the resulting finite partition. Then:

$$\Phi_q(\mathcal{P}') < \Phi_q(\mathcal{P}).$$

Proof. By construction,

$$\Phi_q(\mathcal{P}') = \sum_{\substack{i=1 \\ i \neq i_0}}^k \mathbb{P}(A_i)^q + \sum_{j=1}^m \mathbb{P}(B_j)^q, \quad \Phi_q(\mathcal{P}) = \sum_{\substack{i=1 \\ i \neq i_0}}^k \mathbb{P}(A_i)^q + \mathbb{P}(A_{i_0})^q.$$

Thus it suffices to compare $\sum_{j=1}^m \mathbb{P}(B_j)^q$ with $\mathbb{P}(A_{i_0})^q$. Put $M = \mathbb{P}(A_{i_0}) > 0$ and $t_j = \mathbb{P}(B_j)$ for $j = 1, \dots, m$. Then $0 < t_j < M$ for all j and $\sum_{j=1}^m t_j = M$. Really, with $m \geq 2$ and all $t_j > 0$ with $\sum_j t_j = M$, none of the t_j can equal M ; hence $0 < t_j < M$ for every j .

Since $q > 1$ we have $t_j^{q-1} < M^{q-1}$ for each j , hence

$$t_j^q = t_j^{q-1} t_j < M^{q-1} t_j.$$

Summing over j yields

$$\sum_{j=1}^m t_j^q < M^{q-1} \sum_{j=1}^m t_j = M^q.$$

In terms of probabilities this means:

$$\sum_{j=1}^m \mathbb{P}(B_j)^q < \mathbb{P}(A_{i_0})^q.$$

Substituting into the expressions for $\Phi_q(\mathcal{P}')$ and $\Phi_q(\mathcal{P})$ gives $\Phi_q(\mathcal{P}') < \Phi_q(\mathcal{P})$, as claimed. \square

Theorem 2.2 (Project refinements reduce concentration). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space and let \mathcal{P}_0 be a finite measurable partition with $\mathbb{P}(A) > 0$ for every $A \in \mathcal{P}_0$. Let $(\mathcal{P}_n)_{n \geq 0}$ be a refining sequence of finite partitions constructed as in Theorem 2.1, so that each \mathcal{P}_{n+1} is obtained from \mathcal{P}_n by splitting finitely many project cells and*

$$\mu(\mathcal{P}_n) \xrightarrow{n \rightarrow \infty} 0.$$

Fix $q > 1$ and define $\Phi_q(\mathcal{P}_n) = \sum_{A \in \mathcal{P}_n} \mathbb{P}(A)^q$. Then:

- (a) for every n such that $\mathcal{P}_{n+1} \neq \mathcal{P}_n$ we have $\Phi_q(\mathcal{P}_{n+1}) < \Phi_q(\mathcal{P}_n)$;
- (b) $\Phi_q(\mathcal{P}_n) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. (a) By assumption, \mathcal{P}_{n+1} is obtained from \mathcal{P}_n by splitting finitely many project cells of \mathcal{P}_n into measurable subsets whose total measure is preserved. Applying Lemma 2.2 to each such split and observing that project cells which are not split remain unchanged, we see that each splitting step strictly decreases the value of Φ_q . Therefore $\Phi_q(\mathcal{P}_{n+1}) < \Phi_q(\mathcal{P}_n)$ whenever $\mathcal{P}_{n+1} \neq \mathcal{P}_n$.

(b) For any finite partition \mathcal{P} we have

$$\Phi_q(\mathcal{P}) = \sum_{A \in \mathcal{P}} \mathbb{P}(A)^q \leq \left(\max_{A \in \mathcal{P}} \mathbb{P}(A) \right)^{q-1} \sum_{A \in \mathcal{P}} \mathbb{P}(A) = \mu(\mathcal{P})^{q-1},$$

since $\sum_{A \in \mathcal{P}} \mathbb{P}(A) = \mathbb{P}(\Omega) = 1$. Applying this bound to $\mathcal{P} = \mathcal{P}_n$ yields

$$0 \leq \Phi_q(\mathcal{P}_n) \leq \mu(\mathcal{P}_n)^{q-1} \xrightarrow{n \rightarrow \infty} 0,$$

because $\mu(\mathcal{P}_n) \rightarrow 0$ by Theorem 2.1. This proves (b). \square

3. Illustrative examples

In this section we present a few small examples which illustrate the effect of project–type refinements on the failure measure $\mu(\mathcal{P})$ and on the concentration functional $\Phi_q(\mathcal{P})$. Throughout we work in the discrete setting of a finite task set Z with a probability mass function p on Z , so that a project partition is simply given by the singletons $\mathcal{P} = \{\{x\} : x \in Z\}$ and $\mathbb{P}(\{x\}) = p(x)$. Project revisions correspond to splitting some of the project cells $\{x\}$ into smaller project cells with the same total probability mass.

Example 3.1 (Reducing the worst–case failure class). Consider a project with three tasks, indexed by $Z = \{1, 2, 3\}$, and let

$$p(1) = \frac{3}{5}, \quad p(2) = \frac{3}{10}, \quad p(3) = \frac{1}{10}.$$

The corresponding project partition is

$$\mathcal{P}_0 = \{\{1\}, \{2\}, \{3\}\},$$

and the failure measure is

$$\mu(\mathcal{P}_0) = \max\{p(1), p(2), p(3)\} = \frac{3}{5}.$$

Thus task 1 is the dominant failure class.

Suppose that a project revision identifies two more specific failure modes within task 1, each with half of its original probability. On the level of the probability space this means that we replace the project cell $\{1\}$ by two project cells $\{1'\}$ and $\{1''\}$ with:

$$\mathbb{P}(\{1'\}) = \mathbb{P}(\{1''\}) = \frac{3}{10},$$

while leaving tasks 2 and 3 unchanged. The refined partition is:

$$\mathcal{P}_1 = \{\{1'\}, \{1''\}, \{2\}, \{3\}\},$$

and the new failure measure becomes:

$$\mu(\mathcal{P}_1) = \max\left\{\frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{10}\right\} = \frac{3}{10}.$$

Thus the worst–case failure probability has been reduced from $3/5$ to $3/10$ by a single project–type refinement. This is consistent with Theorem 2.1, which guarantees that finite sequences of such refinements can reduce $\mu(\mathcal{P})$ below any prescribed threshold.

Example 3.2 (Effect on the concentration functional). We continue with the setting of the previous example and take $q = 2$. For the initial partition \mathcal{P}_0 we obtain

$$\Phi_2(\mathcal{P}_0) = \sum_{A \in \mathcal{P}_0} \mathbb{P}(A)^2 = \left(\frac{3}{5}\right)^2 + \left(\frac{3}{10}\right)^2 + \left(\frac{1}{10}\right)^2 = \frac{36}{100} + \frac{9}{100} + \frac{1}{100} = \frac{46}{100}.$$

After the refinement described above (that is, after splitting task 1 into two sub–tasks of probability $3/10$ each) we have four project cells of probabilities $3/10, 3/10, 3/10, 1/10$, whence

$$\Phi_2(\mathcal{P}_1) = 3\left(\frac{3}{10}\right)^2 + \left(\frac{1}{10}\right)^2 = 3 \cdot \frac{9}{100} + \frac{1}{100} = \frac{27}{100} + \frac{1}{100} = \frac{28}{100}.$$

Both the failure measure and the concentration functional have decreased:

$$\mu(\mathcal{P}_1) = \frac{3}{10} < \frac{3}{5} = \mu(\mathcal{P}_0), \quad \Phi_2(\mathcal{P}_1) = \frac{28}{100} < \frac{46}{100} = \Phi_2(\mathcal{P}_0).$$

In this simple example the refinement has broken up the dominant task into several smaller failure classes of equal size, which both lowers the worst–case failure probability and makes the overall failure profile less concentrated. This illustrates the qualitative content of Theorem 2.1 and Theorem 2.2.

Example 3.3 (Refining a non-maximal task). *The next example shows that project-type refinements can also improve the concentration profile even when the maximal failure class is left unchanged. Consider a project with probabilities:*

$$p(1) = \frac{1}{2}, \quad p(2) = \frac{3}{10}, \quad p(3) = \frac{1}{5},$$

so that

$$\mu(\mathcal{P}_0) = \frac{1}{2}, \quad \Phi_2(\mathcal{P}_0) = \left(\frac{1}{2}\right)^2 + \left(\frac{3}{10}\right)^2 + \left(\frac{1}{5}\right)^2 = \frac{25}{100} + \frac{9}{100} + \frac{4}{100} = \frac{38}{100}.$$

Suppose now that a revision further decomposes task 2 into three sub-tasks of equal probability, each with probability $1/10$, while tasks 1 and 3 are left unchanged. The new partition has project cell probabilities

$$\frac{1}{2}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{5}$$

so that

$$\mu(\mathcal{P}_1) = \frac{1}{2}, \quad \Phi_2(\mathcal{P}_1) = \left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{10}\right)^2 + \left(\frac{1}{5}\right)^2 = \frac{25}{100} + 3 \cdot \frac{1}{100} + \frac{4}{100} = \frac{32}{100}.$$

In this case the project revision does not change the maximal failure class: $\mu(\mathcal{P}_1) = \mu(\mathcal{P}_0)$. However, the concentration of the failure probability is reduced, as seen from $\Phi_2(\mathcal{P}_1) < \Phi_2(\mathcal{P}_0)$. This behaviour is consistent with the general inequality of 2.2 and shows that project-type refinements may already improve the global risk profile even before the dominant failure class is addressed.

These simple examples demonstrate how the abstract measure-theoretic results can be interpreted in concrete project settings. In particular, Theorem 2.1 formalizes the idea that, after finitely many suitable revisions, every individual failure mode of the project can be made sufficiently unlikely, while Theorem 2.2 shows that the same revisions simultaneously reduce the overall concentration of the failure probability.

4. Conclusion

In this paper we have revisited the process-based view of projects introduced in Ilić and Veličković (2019) and further developed in Ilić and Višnjić (2025), and we have shown that its essential features admit a simple and robust measure-theoretic formulation on atomless probability spaces. In our framework a completed project is represented by a finite measurable partition $\mathcal{P} = \{A_1, \dots, A_k\}$ of an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the failure profile of the project is encoded by the project cell probabilities $\mathbb{P}(A_i)$. The basic failure measure $\mu(\mathcal{P}) = \max_{A \in \mathcal{P}} \mathbb{P}(A)$ plays the role of a worst-case failure probability, while the functionals $\Phi_q(\mathcal{P}) = \sum_{A \in \mathcal{P}} \mathbb{P}(A)^q$ for $q > 1$ quantify the overall concentration of the failure probability across the project tasks, in close analogy with classical concentration indices and Rényi-type entropies.

Within this setting we have identified a natural class of project-type refinements, modeled as operations that split finitely many project cells of the partition into finitely many measurable sub-project cells of the same total probability. Our first main result, Theorem 2.1, shows that, on an atomless probability space, suitable finite sequences of such refinements can reduce the failure measure $\mu(\mathcal{P})$ below any prescribed threshold, and that one can construct refining sequences $(\mathcal{P}_n)_{n \geq 0}$ with $\mu(\mathcal{P}_n) \rightarrow 0$. This formalises, in a purely probabilistic language, the intuitive idea that a complex project can be revised and decomposed into sufficiently detailed sub-tasks so that no individual failure mode remains dominant.

Our second main result, Theorem 2.2 complements this worst-case control by showing that the same project-type refinements also decrease the concentration functionals $\Phi_q(\mathcal{P})$ for every fixed $q > 1$ and can drive them to zero along suitable refining sequences. In particular, for $q = 2$ this means that the Herfindahl–Hirschman index of the failure profile is reduced by iterated refinements. Thus, the refinements we consider simultaneously mitigate the maximal failure probability and redistribute the overall risk more evenly across

the project structure. The examples in Section 3 illustrate these effects on small discrete models and show how the abstract theory can be interpreted in concrete project settings.

The results obtained here are intentionally structural: we do not optimise over costs, resources or time, and we do not prescribe specific decision rules for choosing refinements. Several directions therefore suggest themselves for future work. One natural extension is to incorporate cost or resource weights into the definition of project-type refinements and to study trade-offs between reducing $\mu(\mathcal{P})$ or $\Phi_q(\mathcal{P})$ and the total refinement cost. Another direction is to combine the present measure-theoretic framework with statistical estimation of the project cell probabilities from data on past projects, thereby connecting the probabilistic model more directly with empirical project management practice. Finally, it would be of interest to investigate whether analogous refinement principles hold for more general classes of random partitions or for dynamic project models in which the partition itself evolves in time.

References

- [1] I. Ilić and V. M. Veličković, "A process of project quality improvement," *Filomat* **33**:6 (2019), 1833–1844.
- [2] I. D. Ilić and J. M. Višnjić, "Towards mathematical foundations of projects: Construction and formalization," *Filomat* **39**:18 (2025), 6103–6121.
- [3] P. Billingsley, *Probability and Measure*, 3rd ed., Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, New York, 1995.
- [4] A. Rényi, "On measures of entropy and information," in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. I, University of California Press, Berkeley, 1961, pp. 547–561.
- [5] S. A. Rhoades, "The Herfindahl–Hirschman index," *Federal Reserve Bulletin* **79** (1993), 188–189.
- [6] F. Acebes, J. M. González-Varona, A. López-Paredes and J. Pajares, "Beyond probability–impact matrices in project risk management: A quantitative methodology for risk prioritisation," *Humanities and Social Sciences Communications* **11** (2024), Article 670.