Filomat 40:7 (2026), 2483–2508
https://doi.org/10.2298/FIL2607483M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Analysis of a nonlinear Caputo fractional coupled system with integral boundary conditions in Banach spaces

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Abstract. In this work, we investigate a nonlinear coupled system of Caputo fractional differential equations subject to integral boundary conditions in the framework of Banach spaces. The analysis is conducted through a fixed point approach. First, the existence of mild solutions is established by employing the Kuratowski measure of noncompactness together with the Generalized Darbo's theorem involving a nondecreasing control function. Uniqueness of the solution is then obtained using the Banach contraction principle. In addition, we study the continuous dependence of the solutions on the input functions, which ensures the stability of the model under perturbations. The Ulam-Hyers stability of the system is also investigated, ensuring that small perturbations in the initial functions lead to proportionally small changes in the solution. These results contribute to the well-posedness and robustness of the proposed fractional model. Finally, a concrete example is provided to illustrate the applicability and effectiveness of the theoretical results obtained.

1. Introduction

In recent decades, growing evidence from various scientific disciplines has demonstrated that classical differential equations often fall short in capturing the complexities of systems governed by memory effects, hereditary mechanisms, or nonlocal temporal behavior. These limitations have led to the rise of fractional differential equations (FDEs) a class of mathematical models that generalize ordinary derivatives to non-integer (real or complex) orders. Unlike classical derivatives, fractional operators possess an inherent nonlocal character, making them ideally suited for modeling systems where the entire history influences present dynamics [3, 10–12, 16, 20, 22, 25].

FDEs have found profound applications across numerous fields. In geophysics, they describe wave propagation and diffusion in media with memory. In climatology, they help quantify persistent dependencies in atmospheric and oceanic data [23, 25]. Control theory has benefited from fractional controllers capable of capturing hysteresis and damping in complex systems [27]. In energy science, such equations

2020 *Mathematics Subject Classification*. Primary 26A33, 34A08, 34A12; Secondary 45G05, 47H10.
Keywords. Fixed point theorems, Caputo fractional differential systems, mild solutions, existence, uniqueness, continuous dependence, Ulam stability.
Received: 08 December 2025; Accepted: 10 December 2025
Communicated by Maria Alessandra Ragusa
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model anomalous transport in batteries and capacitors [21]. They are also prevalent in epidemiological models that reflect memory in transmission rates, and in neuroscience, where they help describe memory-based plasticity and non-instantaneous neural interactions [5, 11, 30].

Among the various definitions of fractional derivatives, the Caputo derivative is widely adopted in applied problems, particularly those involving initial value problems, as it accommodates classical initial conditions naturally [1, 28, 31]. Its formulation has proven effective in incorporating temporal memory and delays into time-fractional systems.

Furthermore, coupled systems of fractional differential equations where multiple dependent variables interact dynamically provide a more realistic representation of many physical and biological systems. When paired with integral (nonlocal) boundary conditions, these models can account for spatial averaging effects, control constraints, and other global influences frequently encountered in engineering and physics [6, 21, 23].

To address such systems rigorously, it is advantageous to formulate them within a Banach space framework, which allows for greater generality and access to functional analysis tools. In this setting, existence and uniqueness results are often derived using fixed point theory. While classical tools like Banach's contraction principle are effective for strict contraction mappings, the Kuratowski measure of noncompactness offers a powerful alternative in dealing with more complex, nonlinear, or weakly compact settings [4, 5, 8, 29].

Beyond existence, a well-posed model should ensure continuous dependence on input data an essential property for both numerical simulations and physical consistency. To further strengthen model robustness, Ulam-Hyers stability offers a way to quantify how solutions respond to small perturbations, thus reinforcing the reliability of the model under data uncertainty or modeling errors [19, 26, 30].

In [2], Ahmed Alsaedi, Muhammed Aldhuain and Bashir Ahmad considered a coupled system of nonlinear fractional differential equations of Caputo type, described by the representative equation

$$\begin{cases} {}^{c}D^{\alpha}\varphi(t) = f_1(t,\varphi(t),\psi(t)),\ \alpha \in (2,3),\ t \in [0,Q], \\ {}^{c}D^{\beta}\psi(t) = f_2(t,\varphi(t),\psi(t)),\ \beta \in (2,3),\ t \in [0,Q], \end{cases}$$

supplemented with the following integral boundary conditions

$$\begin{cases} \varphi(0) = 0,\ \varphi'(Q) = 0,\ \varphi(Q) = \int_0^{\xi} [p_1\psi(s) + Qp_2\psi'(s)]\,ds,\ p_1,p_2 \in \mathbb{R}, \\ \psi(0) = 0,\ \psi'(Q) = 0,\ \psi(Q) = \int_0^{\xi} [r_1\varphi(s) + Qr_2\varphi'(s)]\,ds,\ r_1,r_2 \in \mathbb{R}, \end{cases}$$

where $\xi \in I$, ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ are the Caputo fractional derivatives of order α and β respectively, and $f_1, f_2 : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. They proved the existence of solutions using the Leray-Schauder nonlinear alternative and established uniqueness via the Banach fixed point theorem.

Motivated by these developments, this paper investigates a nonlinear coupled system of Caputo-type fractional differential equations subject to integral boundary conditions in a Banach space setting. We provide sufficient conditions for the existence, continuous dependence, uniqueness, and Ulam-Hyers stability of mild solutions, employing a blend of fixed point theorems and techniques from the theory of noncompactness highlighting modern approaches in the analysis of memory-governed systems

$$\begin{cases} {}^{c}D^{\alpha}(\varphi(t) - g_1(t,\varphi(t),\psi(t))) = f_1(t,\varphi(t),\psi(t)),\ t \in I = [0,Q], \\ {}^{c}D^{\beta}(\psi(t) - g_2(t,\varphi(t),\psi(t))) = f_2(t,\varphi(t),\psi(t)),\ t \in I = [0,Q], \end{cases} \tag{1}$$

supplemented with the following integral boundary conditions

$$\begin{cases} \varphi'(0) = 0,\ \varphi(0) + \varphi'(Q) = 0,\ \varphi(Q) = p_1\int_0^{\xi}\psi(s)\,ds + Qp_2\psi(\xi),\ p_1,p_2 \in \mathbb{R}, \\ \psi'(0) = 0,\ \psi(0) + \psi'(Q) = 0,\ \psi(Q) = r_1\int_0^{\xi}\varphi(s)\,ds + Qr_2\varphi(\xi),\ r_1,r_2 \in \mathbb{R}, \end{cases} \tag{2}$$

where $\alpha, \beta \in (2,3)$, ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ are the Caputo fractional derivatives of order α and β respectively, $(E, \|.\|)$ is a real Banach space, and $f_1, f_2, g_1, g_2 : I \times E \times E \to E$ are continuous functions. The analysis of this coupled system with boundary conditions is carried out by transforming the system into an operator equation defined over a Banach space. The existence of solutions is established through analytical methods founded on the Kuratowski measure of noncompactness, in conjunction with the Generalized Darbo's theorem involving a nondecreasing control function. These methods are particularly well-suited for dealing with nonlinearities in infinite-dimensional settings. Uniqueness is established using Banach's contraction mapping theorem, which ensures a single mild solution under suitable conditions. Additionally, attention is given to the sensitivity of solutions with respect to changes in initial conditions, and their stability is discussed in the framework of Ulam-Hyers theory. Altogether, this study highlights the role of functional analytic techniques in understanding the structure and stability properties of fractional differential systems.

2. Preliminaries

In the subsequent sections of this work, we denote by

- $C(I,E)$: the space of continuous functions defined from I into the Banach space E, with the norm

$$\|\varphi\|_{\infty} = \sup_{t\in I} \|\varphi(t)\|, \ \varphi \in C(I,E).$$

- $C(I\times E\times E,E)$: the space of continuous functions defined from the product space $I\times E\times E$ into E, along with the norm defined below

$$\|f\|_{c} = \sup_{(t,u,v)\in I\times E\times E} \|f(t,u,v)\|, \ f\in C(I\times E\times E,E).$$

- $C(I,E)\times C(I,E)$: denotes the Banach space of all continuous function pairs on $I\times E$, furnished with the norm

$$\|(\varphi,\psi)\|_{c\times c} = \|\varphi\|_{\infty} + \|\psi\|_{\infty}, \ (\varphi,\psi)\in C(I\times E)\times C(I\times E).$$

Definition 2.1 ([14, 15, 20]). *Consider a function φ from $[a,b]$ into E which is Bochner integrable, and let $\alpha > 0$. The order α left-sided Riemann-Liouville fractional integral of φ is defined as follows*

$$\left(I_{a^+}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\varphi(s)\,ds, \ t>a,$$

where $\Gamma(\alpha)$ is the Euler Gamma function.

Definition 2.2 ([14, 15, 20]). *Consider a function φ from $[a,b]$ into E witch is n-times differentiable in the Bochner sense, and let $\alpha\in(n-1,n)$ with $n=[\alpha]+1\in\mathbb{N}$. The left-sided Caputo fractional derivative of order α is defined as*

$$\left({}^cD_{a^+}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\varphi^{(n)}(s)\,ds, \ t>a.$$

Remark 2.3 ([14, 15, 20]). *One of the most important identities involving the Caputo derivative is*

$$\left(I_{a^+}^{\alpha}\,{}^cD_{a^+}^{\alpha}\varphi\right)(t) = \varphi(t) - \sum_{k=0}^{n-1}\frac{\varphi^{(k)}(a)}{k!}(t-a)^k.$$

Definition 2.4. *Considering the following coupled system*

$$\begin{cases} {}^cD^{\alpha}\varphi(t) = F(t), \ t\in[0,Q], \ 2<\alpha<3, \\ {}^cD^{\beta}\psi(t) = G(t), \ t\in[0,Q], \ 2<\beta<3, \end{cases} \tag{3}$$

where $F,G\in C(I,E)$.
A function $(\varphi,\psi)\in C(I,E)\times C(I,E)$ is called a mild solution of (3) and (2) if it satisfies the corresponding integral equation of (3) and (2).

Lemma 2.5. *The coupled system (3) with conditions (2) has a unique mild solution (φ, ψ) in $C(I,E) \times C(I,E)$, given by the following pair of integral equations*

$$\begin{aligned}\varphi(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds + \frac{1}{\Delta}\int_0^Q \left(\gamma_4(t)\frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} - \gamma_1(t)\frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right) G(s)ds \\ &+ \frac{1}{\Delta}\int_0^Q \left(\gamma_3(t)\frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \gamma_2(t)\frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right) F(s)ds \\ &+ \frac{1}{\Delta}\int_0^\xi \left(\gamma_2(t) Q p_2 \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + p_1\gamma_2(t)\frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right) G(s)ds \\ &+ \frac{1}{\Delta}\int_0^\xi \left(\gamma_1(t) Q r_2 \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + r_1\gamma_1(t)\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right) F(s)ds, \end{aligned} \tag{4}$$

and

$$\begin{aligned}\psi(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} G(s)ds + \frac{1}{\Delta}\int_0^Q \left(\kappa_3(t)\frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \kappa_2(t)\frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right) F(s)ds \\ &+ \frac{1}{\Delta}\int_0^Q \left(\kappa_4(t)\frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} - \kappa_1(t)\frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right) G(s)ds \\ &+ \frac{1}{\Delta}\int_0^\xi \left(\kappa_1(t) Q r_2 \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + \kappa_1(t) r_1 \frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right) F(s)ds \\ &+ \frac{1}{\Delta}\int_0^\xi \left(\kappa_2(t) Q p_2 \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + \kappa_2(t) p_1 \frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right) G(s)ds, \end{aligned} \tag{5}$$

where

$$\begin{aligned} &\gamma_1(t) = (\mathrm{A}_2 - 2Q\mathrm{C}_2)t^2 + \left(Q^2 - 2Q\right),\ \gamma_2(t) = \left(Q^2 - 2Q\right)t^2 + (\mathrm{A}_1 - 2Q\mathrm{C}_1), \\ &\gamma_3(t) = -\left(2Q - Q^2 + \mathrm{A}_2\mathrm{C}_1 - 2Q\mathrm{C}_1\mathrm{C}_2\right)t^2 - \left(Q^2\mathrm{C}_1 - \mathrm{A}_1\right), \\ &\gamma_4(t) = -\left(Q^2\mathrm{C}_2 - \mathrm{A}_2\right)t^2 - \left(\mathrm{A}_1\mathrm{C}_2 - Q^2 - 2Q\mathrm{C}_1\mathrm{C}_2 + 2Q\right), \\ &\kappa_1(t) = \left(4Q^2\mathrm{C}_2 - 2Q\mathrm{A}_2\right)t^2 + \left(4Q^2 - 2Q^3\right),\ \kappa_2(t) = \left(4Q^2 - 2Q^3\right)t^2 + \left(4Q^2\mathrm{C}_1 - 2Q\mathrm{A}_1\right), \\ &\kappa_3(t) = -\left(Q^4 - 2Q^3 - \mathrm{A}_1\mathrm{A}_2 + 2Q\mathrm{A}_1\mathrm{C}_2\right)t^2 - \left(2Q\mathrm{A}_1 - 2Q^3\mathrm{C}_1\right), \\ &\kappa_4(t) = -\left(2Q\mathrm{A}_2 - 2Q^3\mathrm{C}_2\right)t^2 - \left(Q^4 - 2Q^3 - \mathrm{A}_1\mathrm{A}_2 + 2Q\mathrm{A}_2\mathrm{C}_1\right), \end{aligned}$$

and

$$\Delta = A_1(2QC_2 - A_2) + 2QC_1(A_2 - 2QC_2) + Q^4 - 4Q^3 + 4Q^2,$$

with

$$\begin{aligned} &\mathrm{A}_1 = \frac{r_1\xi^3}{3} + Qr_2\xi^2,\ \mathrm{C}_1 = r_1\xi + Qr_2,\ E_1 = -\int_0^Q \frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)}F(s)ds, \\ &\mathrm{A}_2 = \frac{p_1\xi^3}{3} + Qp_2\xi^2,\ \mathrm{C}_2 = p_1\xi + Qp_2,\ E_2 = -\int_0^Q \frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)}G(s)ds, \\ &D_1 = -\int_0^Q \frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}G(s)ds + \frac{Qr_2}{\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1}F(s)ds + \frac{r_1}{\Gamma(\alpha+1)}\int_0^\xi (\xi-s)^{\alpha}F(s)ds, \\ &D_2 = -\int_0^Q \frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds + \frac{Qp_2}{\Gamma(\beta)}\int_0^\xi (\xi-s)^{\beta-1}G(s)ds + \frac{p_1}{\Gamma(\beta+1)}\int_0^\xi (\xi-s)^{\beta}G(s)ds. \end{aligned}$$

Proof. Let $(\varphi, \psi) \in C(I,E) \times C(I,E)$ be a solution of (3) and (2), we have

$$\begin{cases} {}^cD^{\alpha}\varphi(t) = F(t), \ t \in [0,Q], \ 2 < \alpha < 3, \\ {}^cD^{\beta}\psi(t) = G(t), \ t \in [0,Q], \ 2 < \beta < 3. \end{cases}$$

Applying the operators I^{α} and I^{β} on the equations ${}^cD^{\alpha}\varphi(t) = F(t)$ and ${}^cD^{\beta}\psi(t) = G(t)$, respectively, produces

$$\begin{cases} \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s)\,ds + a_1 t^2 + c_1, \\ \psi(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} G(s)\,ds + a_2 t^2 + c_2, \end{cases} \tag{6}$$

where $a_1 = \varphi''(0)$, $a_2 = \psi''(0)$, $c_1 = \varphi(0)$ and $c_2 = \psi(0)$. Thus, we have

$$\begin{cases} \varphi'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} F(s)\,ds + 2a_1 t, \\ \psi'(t) = \frac{1}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-2} G(s)\,ds + 2a_2 t. \end{cases} \tag{7}$$

Given the conditions $\varphi(0) + \varphi'(Q) = 0$ and $\psi(0) + \psi'(Q) = 0$, it follows that

$$\begin{cases} 2a_1 Q + c_1 = E_1, \\ 2a_2 Q + c_2 = E_2. \end{cases} \tag{8}$$

We have

$$\begin{aligned} \int_0^{\xi} \varphi(s)\,ds &= \frac{1}{\Gamma(\alpha)} \int_0^{\xi} \int_0^s (s-v)^{\alpha-1} F(v)dvds + \int_0^{\xi} \left(a_1 s^2 + c_1\right) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{\xi} \int_0^s (s-v)^{\alpha-1} F(v)dvds + a_1 \frac{\xi^3}{3} + c_1 \xi \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{\xi} F(v) \int_v^{\xi} (s-v)^{\alpha-1} dsdv + a_1 \frac{\xi^3}{3} + c_1 \xi \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^{\xi} (\xi-v)^{\alpha} F(v)dv + a_1 \frac{\xi^3}{3} + c_1 \xi, \end{aligned}$$

then, we obtain

$$\begin{cases} \int_0^{\xi} \varphi(s)\,ds = \frac{1}{\Gamma(\alpha+1)} \int_0^{\xi} (\xi-s)^{\alpha} F(s)ds + a_1 \frac{\xi^3}{3} + c_1 \xi, \\ \int_0^{\xi} \psi(s)\,ds = \frac{1}{\Gamma(\beta+1)} \int_0^{\xi} (\xi-s)^{\beta} G(s)ds + a_2 \frac{\xi^3}{3} + c_2 \xi. \end{cases} \tag{9}$$

From the conditions $\varphi(Q) = p_1 \int_0^{\xi} \psi(s)\,ds + Qp_2\psi(\xi)$ and $\psi(Q) = r_1 \int_0^{\xi} \varphi(s)\,ds + Qr_2\varphi(\xi)$, it can be deduced that

$$\begin{cases} a_1 A_1 - c_1 C_1 + a_2 Q^2 + c_2 = D_1, \\ a_1 Q^2 + c_1 - a_2 A_2 - c_2 C_2 = D_2. \end{cases} \tag{10}$$

By (8) and (10), we have

$$\begin{cases} 2a_1 Q + c_1 = E_1, \\ 2a_2 Q + c_2 = E_2, \\ -a_1 A_1 - c_1 C_1 + a_2 Q^2 + c_2 = D_1, \\ a_1 Q^2 + c_1 - a_2 A_2 - c_2 C_2 = D_2. \end{cases}$$

We rewrite this representation in matrix form $MV = W$, where

$$M = \begin{pmatrix} 2Q & 1 & 0 & 0 \\ 0 & 0 & 2Q & 1 \\ -A_1 & -C_1 & Q^2 & 1 \\ Q^2 & 1 & -A_2 & -C_2 \end{pmatrix}, \ V = \begin{pmatrix} a_1 \\ c_1 \\ a_2 \\ c_2 \end{pmatrix} \text{ and } W = \begin{pmatrix} E_1 \\ E_2 \\ D_1 \\ D_2 \end{pmatrix}.$$

By applying Cramer's Rule, we obtain the values of the constants a_1, a_2, c_1 and c_2, from which we deduce the integral expressions for $\varphi(t)$ and $\psi(t)$. Hence, the lemma is proven. □

Definition 2.6 (The Kuratowski measure of noncompactness [18]). *Let $(E, \|.\|)$ be a Banach space and let B_E denote the collection of bounded subsets of E excluding the empty set.*
The Kuratowski measure of noncompactness is the mapping $\Omega : B_E \to [0, +\infty)$ defined for $A \in B_E$ by

$$\Omega(A) = \inf\left\{d > 0 : \exists n \in \mathbb{N},\ \exists A_i \subset E,\ i = \overline{1,n},\ \text{such that } A \subset \cup_{i=1}^{n} A_i,\ \text{and } diam(Ai) \leq d,\ \forall i\right\},$$

where

$$diam(A_i) = \sup\left\{\left\|\varphi - \psi\right\| : \varphi, \psi \in A_i\right\}.$$

Lemma 2.7 ([18]). *For bounded sets $A, B \subset E$, the following properties hold.*
Monotonicity: if $A \subset B$, then $\Omega(A) \leq \Omega(B)$.
Closure invariance: $\Omega(\bar{A}) = \Omega(A)$.
Convex hull invariance: $\Omega(conv(A)) = \Omega(A)$.
Homogeneity: $\Omega(\lambda A) = |\lambda|\,\Omega(A)$ for all scalars $\lambda \in \mathbb{R}$.
Subadditivity: $\Omega(A + B) \leq \Omega(A) + \Omega(B)$, where $A + B = \{a + b : a \in A,\ b \in B\}$.
Zero characterization: $\Omega(A) = 0 \Leftrightarrow A$ is relatively compact in E.

Lemma 2.8 ([18]). *Let $A \subset C(I, E)$ be a bounded and equicontinuous set.*
For each $t \in I$, define $A(t) = \{\varphi(t) : \varphi \in A\} \subset E$. Then, the function $t \to \Omega(A(t))$ is continuous on I where

$$\Omega_c(A) = \sup_{t \in I} \Omega(A(t)),$$

and

$$\Omega\left(\int_I \varphi(s)\,ds, \varphi \in A\right) \leq \int_I \Omega(A(s))\,ds.$$

Theorem 2.9 ([7]). *Let Ω_1, Ω_2, ..., Ω_n be measures of noncompactness on the Banach spaces E_1, E_2, ..., E_n, respectively. Consider a convex mapping $F : (\mathbb{R}^+)^n \to \mathbb{R}^+$ satisfying $F(t_1, t_2 ..., t_n) = 0$ if and only if $t_i = 0$ for all $i = 1, 2, ...n$. Then, for any bounded set $A \subset E_1 \times E_2 \times ... \times E_n$, the quantity*

$$\tilde{\Omega}(A) = F(\Omega_1(A_1), \Omega_2(A_2), ..., \Omega_n(A_n)),$$

defines a measure of noncompactness on the product space $E_1 \times E_2 \times ... \times E_n$ where A_i denotes the canonical projection of A into E_i for each $i = 1, 2, ..., n$.

Example 2.10 ([7]). *Let Ω be a measure of non-compactness in E. Consider the mapping $F : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ defined by $F(t, s) = t + s$, it is clear that F satisfies all the hypotheses stated in Theorem 2.9. Consequently, the functional*

$$\tilde{\Omega}(A) = \Omega(A_1) + \Omega(A_2),$$

defines a measure of non-compactness in the product space $E \times E$, where A_1, A_2 denote the canonical projections of A.

Lemma 2.11 ([9]). *Let $v : \mathbb{R}^+ \to \mathbb{R}^+$ be an upper semicontinuous and nondecreasing function. The following are equivalent*

- *$v(t) < t$, for every $t > 0$.*
- *for each $t \geq 0$, the iterates $v^n(t)$ converge to zero as $n \to +\infty$, that is $\lim_{n \to +\infty} v^n(t) = 0$.*

Theorem 2.12 (Generalized Darbo's fixed point theorem [9]). *Let $M \neq \phi$ be a closed, bounded, and convex subset of a Banach space E. Consider a continuous mapping $N : M \to M$ such that*

$$\Omega(NA) \leq v(\Omega(A)), \tag{11}$$

for all nonempty subset $A \subset M$, where Ω is any measure of non-compactness and $v : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function satisfying $\lim_{n\to\infty} v^n(t) = 0$, for all $t \in \mathbb{R}^+$, with v^n denoting the n-th iterate of v. Under these assumptions, N possesses at least one fixed point in M.

Theorem 2.13 (Contraction [24]). *Let T be an operator defined from a Banach space E into itself. We say that T is a contraction if there exists a constant $k \in (0, 1)$ such that*

$$\left\|T\psi - T\varphi\right\| \leq K\left\|\psi - \varphi\right\|, \ \forall \varphi, \psi \in E.$$

Theorem 2.14 (Banach fixed point theorem [24]). *Let T be a contraction operator defined from a Banach space E into itself. Then T has a unique fixed point φ in E, that is*

$$\exists! \varphi \in E \text{ such that } T\varphi = \varphi.$$

3. Main results

To establish the upcoming results, we define the operator $\Psi : C(I \times E) \times C(I \times E) \to C(I \times E) \times C(I \times E)$ as follows

$$\Psi(\varphi, \psi) = \begin{pmatrix} \Psi_1(\varphi, \psi) \\ \Psi_2(\varphi, \psi) \end{pmatrix}, \tag{12}$$

where Ψ_1 and Ψ_2 are defined from $C(I \times E) \times C(I \times E)$ into $C(I \times E)$ by the following

$$\begin{aligned} \Psi_1(\varphi, \psi)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \varphi(s), \psi(s))\, ds + g_1(t, \varphi(t), \psi(t)) \\ &+ \frac{1}{\Delta} \int_0^Q \left(\gamma_4(t) \frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} - \gamma_1(t) \frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right) f_2(s, \varphi(s), \psi(s))\, ds \\ &+ \frac{1}{\Delta} \int_0^Q \left(\gamma_3(t) \frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \gamma_2(t) \frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right) f_1(s, \varphi(s), \psi(s))\, ds \\ &+ \frac{1}{\Delta} \int_0^\xi \left(\gamma_2(t) Q p_2 \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + p_1 \gamma_2(t) \frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right) f_2(s, \varphi(s), \psi(s))\, ds \\ &+ \frac{1}{\Delta} \int_0^\xi \left(\gamma_1(t) Q r_2 \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + r_1 \gamma_1(t) \frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right) f_1(s, \varphi(s), \psi(s))\, ds, \end{aligned} \tag{13}$$

and

$$\begin{aligned} \Psi_2(\varphi, \psi)(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f_2(s, \varphi(s), \psi(s))\, ds + g_2(t, \varphi(t), \psi(t)) \\ &+ \frac{1}{\Delta} \int_0^Q \left(\kappa_3(t) \frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \kappa_2(t) \frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right) f_1(s, \varphi(s), \psi(s))\, ds \\ &+ \frac{1}{\Delta} \int_0^Q \left(\kappa_4(t) \frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} - \kappa_1(t) \frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right) f_2(s, \varphi(s), \psi(s))\, ds \\ &+ \frac{1}{\Delta} \int_0^\xi \left(\kappa_1(t) Q r_2 \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + \kappa_1(t) r_1 \frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right) f_1(s, \varphi(s), \psi(s))\, ds \\ &+ \frac{1}{\Delta} \int_0^\xi \left(\kappa_2(t) Q p_2 \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + \kappa_2(t) p_1 \frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right) f_2(s, \varphi(s), \psi(s))\, ds, \end{aligned} \tag{14}$$

for any $t \in I$.

It is evident that the fixed points of the operator (12) correspond to the mild solutions of the coupled system (1) with conditions (2).

3.1. Existence and uniqueness

In this section, we first employ the Kuratowski measure of noncompactness together with the Generalized Darbo's theorem involving a nondecreasing control function to prove the existence of solutions, and then apply the Banach contraction principle to establish existence and uniqueness results.

3.1.1. Existence of solutions

Let Υ be the set of functions defined as follows

$$\Upsilon = \left\{ v : \mathbb{R}^+ \to \mathbb{R}^+ : v \text{ is a nondecreasing mapping satisfying } \lim_{n\to\infty} v^n(t) = 0, \text{ for all } t \in \mathbb{R}^+ \right\}.$$

Remark 3.1. *The following properties hold for the set Υ :*

- *If $\lambda \in [0,1]$, $v \in \Upsilon$, then $\lambda v \in \Upsilon$.*
- *If $v \in \Upsilon$, then it follows immediately that $v(t) < t$ for all $t > 0$.*

Define

$$B_\omega = \left\{ (\varphi, \psi) \in C(I,E) \times C(I,E) : \left\| (\varphi,\psi) \right\|_{c\times c} \le \omega \right\},$$

to be the closed ball of radius ω centered at the origin in the product space $C(I,E) \times C(I,E)$.

To simplify the forthcoming computations, we introduce the following constants

$$\widehat{f_i} = \sup_{t\in I} \left\| f_i(t,0,0) \right\|, \; \widehat{g_i} = \sup_{t\in I} \left\| g_i(t,0,0) \right\|, \; i \in \{1,2\},$$
$$\widehat{\gamma_i} = \sup_{t\in I} \left| \gamma_i(t) \right|, \; i \in \{1,2,3,4\},$$
$$\widehat{\kappa_i} = \sup_{t\in I} \left| \kappa_i(t) \right|, \; i \in \{1,2,3,4\}.$$

Consider the following assumptions:

- There exist four continuous functions $\tau_1, \tau_2, \varrho_1, \varrho_2 \in \Upsilon$ such that

$$\left\| f_i(t,\varphi_2,\psi_2) - f_i(t,\varphi_1,\psi_1) \right\| \le \tau_i\left(\left\| \varphi_2 - \varphi_1 \right\| + \left\| \psi_2 - \psi_1 \right\| \right), \; i = 1,2, \tag{15}$$

  and

$$\left\| g_i(t,\varphi_2,\psi_2) - g_i(t,\varphi_1,\psi_1) \right\| \le \varrho_i\left(\left\| \varphi_2 - \varphi_1 \right\| + \left\| \psi_2 - \psi_1 \right\| \right), \; i = 1,2. \tag{16}$$

- There exist two constants $k_1, k_2 \in \mathbb{R}_+^*$ with $\frac{1}{k_1} + \frac{1}{k_2} \le 1$, such that

$$\begin{aligned}
\rho_1 = {} & \frac{1}{|\Delta|}\left(\tau_1(\omega) + \widehat{f_1}\right)\left(\frac{|\Delta|\, Q^\alpha}{\Gamma(\alpha+1)} + \frac{\widehat{\gamma_3} Q^{\alpha-1}}{\Gamma(\alpha)} + \frac{\widehat{\gamma_2} Q^\alpha}{\Gamma(\alpha+1)} + \widehat{\gamma_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)} + \frac{|r_1|}{\Gamma(\alpha+2)} \right) Q^{\alpha+1} \right) \\
& + \frac{1}{|\Delta|}\left(\tau_2(\omega) + \widehat{f_2}\right)\left(\frac{\widehat{\gamma_4} Q^{\beta-1}}{\Gamma(\beta)} + \frac{\widehat{\gamma_1} Q^\beta}{\Gamma(\beta+1)} + \widehat{\gamma_2}\left(\frac{|p_2|}{\Gamma(\beta+1)} + \frac{|p_1|}{\Gamma(\beta+2)} \right) Q^{\beta+1} \right) + \varrho_1(\omega) + \widehat{g_1} \\
& \le \frac{1}{k_1}\omega,
\end{aligned} \tag{17}$$

and

$$\rho_2 = \frac{1}{|\Delta|}\left(\tau_1(\omega)+\widehat{f_1}\right)\left(\frac{\widehat{\kappa_3}Q^{\alpha-1}}{\Gamma(\alpha)}+\frac{\widehat{\kappa_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\widehat{\kappa_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}\right)+\varrho_2(\omega)+\widehat{g_2}$$
$$+\frac{1}{|\Delta|}\left(\tau_2(\omega)+\widehat{f_2}\right)\left(\frac{|\Delta|Q^{\beta}}{\Gamma(\beta+1)}+\frac{\widehat{\kappa_4}Q^{\beta-1}}{\Gamma(\beta)}+\frac{\widehat{\kappa_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\kappa_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}\right)$$
$$\leq \frac{1}{K_2}\omega. \tag{18}$$

Remark 3.2. *In view of assumptions (15) and (16), we deduce that for all $A_1, A_2 \subset C(I,E)$ and for all $t \in I$, we have*

$$\Omega\left(f_i(t,A_1,A_2)\right) \leq \tau_i\left(\Omega(A_1)+\Omega(A_2)\right), \tag{19}$$

and

$$\Omega\left(g_i(t,A_1,A_2)\right) \leq \varrho_i\left(\Omega(A_1)+\Omega(A_2)\right). \tag{20}$$

Theorem 3.3 (Existence). *Suppose that conditions (15), (16), (17) and (18) hold. Then, the coupled system (1) with conditions (2) has at least one mild solution.*

Proof. For the sake of clarity and logical flow, we structure the proof into a number of successive steps.
Step 1. Ψ maps B_ω into itself.
Let $(\varphi,\psi) \in B_\omega$ and $t \in I$, from (15) and (16) we have

$$\begin{aligned}\left\|f_i(s,\varphi(s),\psi(s))\right\| &\leq \left\|f_i(s,\varphi(s),\psi(s)) - f_i(s,0,0)\right\| + \left\|f_i(s,0,0)\right\| \\ &\leq \tau_i\left(\left\|\varphi(s)\right\|+\left\|\psi(s)\right\|\right)+\widehat{f_i} \\ &\leq \tau_i(\omega)+\widehat{f_i},\end{aligned} \tag{21}$$

and

$$\begin{aligned}\left\|g_i(s,\varphi(s),\psi(s))\right\| &\leq \left\|g_i(s,\varphi(s),\psi(s)) - g_i(s,0,0)\right\| + \left\|g_i(s,0,0)\right\| \\ &\leq \varrho_i\left(\left\|\varphi(s)\right\|+\left\|\psi(s)\right\|\right)+\widehat{g_i} \\ &\leq \varrho_i(\omega)+\widehat{g_i}.\end{aligned} \tag{22}$$

Then, we get

$$\begin{aligned}\left\|\Psi_1(\varphi,\psi)(t)\right\| \leq{}& \left(\tau_1(\omega)+\widehat{f_1}\right)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\varrho_1(\omega)+\widehat{g_1} \\ &+\frac{1}{|\Delta|}\left(\tau_2(\omega)+\widehat{f_2}\right)\int_0^Q\left(\left|\gamma_4(t)\right|\frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)}+\left|\gamma_1(t)\right|\frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right)ds \\ &+\frac{1}{|\Delta|}\left(\tau_1(\omega)+\widehat{f_1}\right)\int_0^Q\left(\left|\gamma_3(t)\right|\frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\left|\gamma_2(t)\right|\frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right)ds \\ &+\frac{1}{|\Delta|}\left(\tau_2(\omega)+\widehat{f_2}\right)\int_0^\xi\left(\left|p_2\gamma_2(t)\right|Q\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}+\left|p_1\gamma_2(t)\right|\frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right)ds \\ &+\frac{1}{|\Delta|}\left(\tau_1(\omega)+\widehat{f_1}\right)\int_0^\xi\left(\left|r_2\gamma_1(t)\right|Q\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}+\left|r_1\gamma_1(t)\right|\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right)ds.\end{aligned}$$

Thus

$$\left\|\Psi_1(\varphi,\psi)(t)\right\| \le \frac{1}{|\Delta|}\left(\tau_1(\omega)+\widehat{f_1}\right)\left(\frac{|\Delta| Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\gamma_3}Q^{\alpha-1}}{\Gamma(\alpha)}+\frac{\widehat{\gamma_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\widehat{\gamma_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}\right)$$
$$+\frac{1}{|\Delta|}\left(\tau_2(\omega)+\widehat{f_2}\right)\left(\frac{\widehat{\gamma_4}Q^{\beta-1}}{\Gamma(\beta)}+\frac{\widehat{\gamma_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\gamma_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}\right)+\varrho_1(\omega)+\widehat{g_1}.$$

Taking the supremum over $t \in I$ and then using (17), we obtain

$$\left\|\Psi_1(\varphi,\psi)\right\|_{\infty} \le \frac{1}{|\Delta|}\left(\tau_1(\omega)+\widehat{f_1}\right)\left(\frac{|\Delta| Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\gamma_3}Q^{\alpha-1}}{\Gamma(\alpha)}+\frac{\widehat{\gamma_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\widehat{\gamma_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}\right)$$
$$+\frac{1}{|\Delta|}\left(\tau_2(\omega)+\widehat{f_2}\right)\left(\frac{\widehat{\gamma_4}Q^{\beta-1}}{\Gamma(\beta)}+\frac{\widehat{\gamma_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\gamma_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}\right)+\varrho_1(\omega)+\widehat{g_1}$$
$$\le \frac{1}{k_1}\omega.$$

Similarly, using (18) we obtain

$$\left\|\Psi_2(\varphi,\psi)\right\|_{\infty} \le \frac{1}{|\Delta|}\left(\tau_1(\omega)+\widehat{f_1}\right)\left(\frac{\widehat{\kappa_3}Q^{\alpha-1}}{\Gamma(\alpha)}+\frac{\widehat{\kappa_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\widehat{\kappa_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}\right)+\varrho_2(\omega)+\widehat{g_2}$$
$$+\frac{1}{|\Delta|}\left(\tau_2(\omega)+\widehat{f_2}\right)\left(\frac{|\Delta| Q^{\beta}}{\Gamma(\beta+1)}+\frac{\widehat{\kappa_4}Q^{\beta-1}}{\Gamma(\beta)}+\frac{\widehat{\kappa_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\kappa_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}\right)$$
$$\le \frac{1}{K_2}\omega.$$

Consequently

$$\left\|\Psi(\varphi,\psi)\right\|_{c\times c} = \left\|\Psi_1(\varphi,\psi)\right\|_{\infty}+\left\|\Psi_2(\varphi,\psi)\right\|_{\infty} \le \omega.$$

Then, Ψ maps B_ω into itself.

Step 2. Ψ maps every bounded subset of B_ω into an equicontinuous set.

Let $B_{\varpi} = \left\{(\varphi,\psi) \in C(I,E)\times C(I,E) : \left\|(\varphi,\psi)\right\|_{c\times c} \le \varpi\right\} \subset B_\omega$ be a bounded set. By using (15), for any $(\varphi,\psi) \in B_{\varpi}$ and $t_1, t_2 \in I$ with $t_1 < t_2$, we have

$$\left\|g_i(t_2,\varphi(t_2),\psi(t_2)) - g_i(t_1,\varphi(t_1),\psi(t_1))\right\|$$
$$\le \tau_i\left(\left\|\varphi(t_2)-\varphi(t_1)\right\|+\left\|\psi(t_2)-\psi(t_1)\right\|\right)+\left\|g_i(t_2,\varphi(t_1),\psi(t_1))-g_i(t_1,\varphi(t_1),\psi(t_1))\right\|.$$

Since $\tau_i(t) > 0$, for all $t > 0$, we have $\tau_i(0) = 0$, and $\lim_{t\to 0}\tau_i(t) = 0$. Therefore, τ_i is continuous at $t = 0$, in addition, from the fact that φ and ψ are continuous, we get

$$\tau_i\left(\left\|\varphi(t_2)-\varphi(t_1)\right\|+\left\|\psi(t_2)-\psi(t_1)\right\|\right) \to 0 \text{ as } t_1 \to t_2. \tag{23}$$

On another hand, for all $\delta > 0$, we have

$$\left\|g_i(t_2,\varphi(t_1),\psi(t_1)) - g_i(t_1,\varphi(t_1),\psi(t_1))\right\|$$
$$\le \sup\left\{\left\|g_i(a,p,q)-g_i(b,p,q)\right\|,\ |a-b| \le \delta,\ (p,q) \in E\times E \text{ with } \left\|p\right\| \le \varpi, \left\|q\right\| \le \varpi\right\}.$$

Since $t_2 \to t_1$, we get

$$\left\|g_i(t_2,\varphi(t_1),\psi(t_1)) - g_i(t_1,\varphi(t_1),\psi(t_1))\right\|$$
$$\le \sup_{\delta\to 0}\left\{\left\|g_i(a,p,q)-g_i(b,p,q)\right\|,\ |a-b| \le \delta,\ (p,q) \in E\times E \text{ with } \left\|p\right\| \le \varpi, \left\|q\right\| \le \varpi\right\}.$$

Keeping in mind that g_i is uniformly continuous on any bounded subset of $I \times E \times E$, we have

$$\sup_{\delta \to 0} \left\{ \left\| g_i(a,p,q) - g_i(b,p,q) \right\|,\ |a-b| \le \delta, (p,q) \in E \times E \text{ with } \left\|p\right\| \le \varpi,\ \left\|q\right\| \le \varpi \right\} \to 0.$$

That is

$$\left\| g_i(t_2, \varphi(t_1), \psi(t_1)) - g_i(t_1, \varphi(t_1), \psi(t_1)) \right\| \to 0 \text{ as } t_1 \to t_2. \tag{24}$$

From (23) and (24), we deduce that

$$\left\| g_i(t_2, \varphi(t_2), \psi(t_2)) - g_i(t_1, \varphi(t_1), \psi(t_1)) \right\| \to 0 \text{ as } t_1 \to t_2. \tag{25}$$

Now, we have

$$\begin{aligned}
&\left\| \Psi_1(\varphi,\psi)(t_2) - \Psi_1(\varphi,\psi)(t_1) \right\| \\
&\le \int_0^{t_1} \left(\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \left\| f_1(s,\varphi(s),\psi(s)) \right\| ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \left\| f_1(s,\varphi(s),\psi(s)) \right\| ds + \left\| g_1(t_2,\varphi(t_2),\psi(t_2)) - g_1(t_1,\varphi(t_1),\psi(t_1)) \right\| \\
&+ \frac{1}{\Delta} \int_0^{Q} \left(\left|\gamma_4(t_2) - \gamma_4(t_1)\right| \frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} + \left|\gamma_1(t_2) - \gamma_1(t_1)\right| \frac{(Q-s)^{\beta-1}}{\Gamma(\beta)} \right) \left\| f_2(s,\varphi(s),\psi(s)) \right\| ds \\
&+ \frac{1}{\Delta} \int_0^{Q} \left(\left|\gamma_3(t_2) - \gamma_3(t_1)\right| \frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \left|\gamma_2(t_2) - \gamma_2(t_1)\right| \frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \left\| f_1(s,\varphi(s),\psi(s)) \right\| ds \\
&+ \frac{1}{\Delta} \int_0^{\xi} \left(\left|p_2(\gamma_2(t_2) - \gamma_2(t_1))\right| \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} Q + \left|p_1(\gamma_2(t_2) - \gamma_2(t_1))\right| \frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)} \right) \left\| f_2(s,\varphi(s),\psi(s)) \right\| ds \\
&+ \frac{1}{\Delta} \int_0^{\xi} \left(\left|r_2(\gamma_1(t_2) - \gamma_1(t_1))\right| \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} Q + \left|r_1(\gamma_1(t_2) - \gamma_1(t_2))\right| \frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)} \right) \left\| f_1(s,\varphi(s),\psi(s)) \right\| ds.
\end{aligned}$$

Using (21), we get

$$\begin{aligned}
&\left\| \Psi_1(\varphi,\psi)(t_2) - \Psi_1(\varphi,\psi)(t_1) \right\| \\
&\le \frac{\left(\tau_1(\omega) + \widehat{f_1}\right)}{\Gamma(\alpha+1)} \left(t_2^{\alpha} - t_1^{\alpha}\right) + \left\| g_1(t_2,\varphi(t_2),\psi(t_2)) - g_1(t_1,\varphi(t_1),\psi(t_1)) \right\| \\
&+ \frac{1}{\Delta} \left(\tau_2(\omega) + \widehat{f_2}\right) \frac{\left|\gamma_4(t_2) - \gamma_4(t_1)\right| Q^{\beta-1}}{\Gamma(\beta)} + \frac{\left|\gamma_1(t_2) - \gamma_1(t_1)\right| Q^{\beta}}{\Gamma(\beta+1)} \\
&+ \frac{1}{\Delta} \left(\tau_1(\omega) + \widehat{f_1}\right) \frac{\left|\gamma_3(t_2) - \gamma_3(t_1)\right| Q^{\alpha-1}}{\Gamma(\alpha)} + \frac{\left|\gamma_2(t_2) - \gamma_2(t_1)\right| Q^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \frac{1}{\Delta} \left(\tau_2(\omega) + \widehat{f_2}\right) \frac{\left|p_2(\gamma_2(t_2) - \gamma_2(t_1))\right| Q^{\beta+1}}{\Gamma(\beta+1)} + \frac{\left|p_1(\gamma_2(t_2) - \gamma_2(t_1))\right| Q^{\beta+1}}{\Gamma(\beta+2)} \\
&+ \frac{1}{\Delta} \left(\tau_1(\omega) + \widehat{f_1}\right) \frac{\left|r_2(\gamma_1(t_2) - \gamma_1(t_1))\right| Q^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{\left|r_1(\gamma_1(t_2) - \gamma_1(t_2))\right| Q^{\alpha+1}}{\Gamma(\alpha+2)}.
\end{aligned}$$

From (25), we deduce that

$$\left\| \Psi_1(\varphi,\psi)(t_2) - \Psi_1(\varphi,\psi)(t_1) \right\| \to 0 \text{ as } t_1 \to t_2,$$

this implies that $\Psi_1(B_\varpi)$ is equicontinuous.

Using the same reasoning as before, we obtain that

$$\left\|\Psi_2(\varphi,\psi)(t_2) - \Psi_2(\varphi,\psi)(t_1)\right\| \to 0 \text{ as } t_1 \to t_2,$$

which means that $\Psi_2(B_\omega)$ is also equicontinuous.

Consequently, the operator Ψ maps bounded sets into equicontinuous sets.

Step 3. The operator Ψ is continuous on B_ω.

Let $(\varphi_n,\psi_n) \subset B_\omega$ be a sequence such that $(\varphi_n,\psi_n) \to (\varphi,\psi)$ in B_ω. Then, we have

$$\begin{aligned}
&\left\|\Psi_1(\varphi_n,\psi_n)(t) - \Psi_1(\varphi,\psi)(t)\right\| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\|f_1(s,\varphi_n(s),\psi_n(s)) - f_1(s,\varphi(s),\psi(s))\right\| ds + \left\|g_1(t,\varphi_n(t),\psi_n(t)) - g_1(t,\varphi(t),\psi(t))\right\| \\
&+ \frac{1}{|\Delta|}\int_0^Q \left(\left|\gamma_4(t)\right| \frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} + \left|\gamma_1(t)\right| \frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right) \left\|f_2(s,\varphi_n(s),\psi_n(s)) - f_2(s,\varphi(s),\psi(s))\right\| ds \\
&+ \frac{1}{|\Delta|}\int_0^Q \left(\left|\gamma_3(t)\right| \frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \left|\gamma_2(t)\right| \frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right) \left\|f_1(s,\varphi_n(s),\psi_n(s)) - f_1(s,\varphi(s),\psi(s))\right\| ds \\
&+ \frac{1}{|\Delta|}\int_0^\xi \left(\left|p_2\gamma_2(t)\right| Q\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + \left|p_1\gamma_2(t)\right| \frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right) \left\|f_2(s,\varphi_n(s),\psi_n(s)) - f_2(s,\varphi(s),\psi(s))\right\| ds \\
&+ \frac{1}{|\Delta|}\int_0^\xi \left(\left|r_2\gamma_1(t)\right| Q\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + \left|r_1\gamma_1(t)\right| \frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right) \left\|f_1(s,\varphi_n(s),\psi_n(s)) - f_1(s,\varphi(s),\psi(s))\right\| ds.
\end{aligned}$$

Using the Dominated Convergence Theorem (DCT), it follows that

$$\left\|\Psi_1(\varphi_n,\psi_n)(t) - \Psi_1(\varphi,\psi)(t)\right\| \to 0 \text{ as } n \to \infty,$$

similarly

$$\left\|\Psi_2(\varphi_n,\psi_n)(t) - \Psi_2(\varphi,\psi)(t)\right\| \to 0 \text{ as } n \to \infty.$$

Having established the pointwise convergence of the sequences $\Psi_1(\varphi_n,\psi_n) \to \Psi_1(\varphi,\psi)$ and $\Psi_2(\varphi_n,\psi_n) \to \Psi_2(\varphi,\psi)$, the final step is to show that they are equicontinuous, allowing us to invoke the Arzelà-Ascoli theorem.

From Step 2, Ψ maps bounded sets into equicontinuous sets. Hence, $\{\Psi(\varphi_n,\psi_n)\}_{n\in\mathbb{N}}$ is equicontinuous, and together with the pointwise convergence, we conclude that the convergence is uniform. Therefore, we conclude that Ψ is continuous.

Step 4. The operator Ψ satisfies inequality (11).

Let $G = \overline{conv}(\Psi B_\omega)$. It is clear that G is convex, closed, and bounded subset of B_ω. Since $G \subset B_\omega$, then $\Psi(G) \subset \Psi(B_\omega) \subset G$. From Steps 1 and 3, we conclude that the operator $\Psi : G \to G$ is both bounded and continuous, and from Step 2, we conclude that G is equicontinuous. It follows from Lemma 2.8 that $t \to \widetilde{\Omega}(G(t))$ is continuous on I. Using Example 2.10 along with the properties of the measure of noncompactness $\widetilde{\Omega}$, we deduce that, for each $t \in I$

$$\widetilde{\Omega}(\Psi(G)(t)) = \Omega(\Psi_1(G)(t)) + \Omega(\Psi_2(G)(t)).$$

On the one hand, we have

$$\begin{aligned}\Omega(\Psi_1(G)(t)) \leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Omega(f_1(s,G_1(s),G_2(s)))\,ds + \Omega(g_1(t,G_1(t),G_2(t)))\\ &+ \frac{1}{|\Delta|}\int_0^Q \left(\left|\gamma_4(t)\right|\frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} + \left|\gamma_1(t)\right|\frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right)\Omega(f_2(s,G_1(s),G_2(s)))\,ds\\ &+ \frac{1}{|\Delta|}\int_0^Q \left(\left|\gamma_3(t)\right|\frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \left|\gamma_2(t)\right|\frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right)\Omega(f_1(s,G_1(s),G_1(s)))\,ds\\ &+ \frac{1}{|\Delta|}\int_0^\xi \left(\left|p_2\gamma_2(t)\right|Q\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + \left|p_1\gamma_2(t)\right|\frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right)\Omega(f_2(s,G_1(s),G_2(s)))\,ds\\ &+ \frac{1}{|\Delta|}\int_0^\xi \left(\left|r_2\gamma_1(t)\right|Q\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + \left|r_1\gamma_1(t)\right|\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right)\Omega(f_1(s,G_1(s),G_2(s)))\,ds,\end{aligned}$$

where G_1 and G_2 be the canonical projection of $G \subset B_\omega$ over $C(I,E)$.
Using (25) and (20), we get

$$\begin{aligned}\Omega(\Psi_1(G)(t)) \leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\tau_1(\Omega(G_1(s)+\Omega(G_2(s))))\,ds + \varrho_1(\Omega(G_1(t)+\Omega(G_2(t))))\\ &+ \frac{1}{|\Delta|}\int_0^Q \left(\left|\gamma_4(t)\right|\frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} + \left|\gamma_1(t)\right|\frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right)\tau_2(\Omega(G_1(s)+\Omega(G_2(s))))\,ds\\ &+ \frac{1}{|\Delta|}\int_0^Q \left(\left|\gamma_3(t)\right|\frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \left|\gamma_2(t)\right|\frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right)\tau_1(\Omega(G_1(s)+\Omega(G_2(s))))\,ds\\ &+ \frac{1}{|\Delta|}\int_0^\xi \left(\left|p_2\gamma_2(t)\right|Q\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + \left|p_1\gamma_2(t)\right|\frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right)\tau_2(\Omega(G_1(s)+\Omega(G_2(s))))\,ds\\ &+ \frac{1}{|\Delta|}\int_0^\xi \left(\left|r_2\gamma_1(t)\right|Q\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + \left|r_1\gamma_1(t)\right|\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right)\tau_1(\Omega(G_1(s)+\Omega(G_2(s))))\,ds.\end{aligned}$$

Thus, by Lemma 2.8, we get

$$\begin{aligned}&\Omega_c(\Psi_1(G))\\ &\leq \varrho_1\left(\widetilde{\Omega}_c(G)\right) + \frac{1}{|\Delta|}\tau_1\left(\widetilde{\Omega}_c(G)\right)\left(\frac{|\Delta|\,Q^\alpha}{\Gamma(\alpha+1)} + \widehat{\gamma_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)} + \frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1} + \frac{\widehat{\gamma_2}Q^\alpha}{\Gamma(\alpha+1)} + \frac{\widehat{\gamma_3}Q^{\alpha-1}}{\Gamma(\alpha)}\right)\\ &+ \frac{1}{|\Delta|}\tau_2\left(\widetilde{\Omega}_c(G)\right)\left(\frac{\widehat{\gamma_1}Q^\beta}{\Gamma(\beta+1)} + \widehat{\gamma_2}\left(\frac{\left|p_2\right|}{\Gamma(\beta+1)} + \frac{\left|p_1\right|}{\Gamma(\beta+2)}\right)Q^{\beta+1} + \frac{\widehat{\gamma_4}Q^{\beta-1}}{\Gamma(\beta)}\right).\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}&\Omega_c(\Psi_2(G))\\ &\leq \varrho_2\left(\widetilde{\Omega}_c(G)\right) + \frac{1}{|\Delta|}\tau_2\left(\widetilde{\Omega}_c(G)\right)\left(\frac{|\Delta|\,Q^\beta}{\Gamma(\beta+1)} + \frac{\widehat{\kappa_1}Q^\beta}{\Gamma(\beta+1)} + \widehat{\kappa_2}\left(\frac{\left|p_2\right|}{\Gamma(\beta+1)} + \frac{\left|p_1\right|}{\Gamma(\beta+2)}\right)Q^{\beta+1} + \frac{\widehat{\kappa_4}Q^{\beta-1}}{\Gamma(\beta)}\right)\\ &+ \frac{1}{|\Delta|}\tau_1\left(\widetilde{\Omega}_c(G)\right)\left(\widehat{\kappa_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)} + \frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1} + \frac{\widehat{\kappa_2}Q^\alpha}{\Gamma(\alpha+1)} + \frac{\widehat{\kappa_3}Q^{\alpha-1}}{\Gamma(\alpha)}\right).\end{aligned}$$

Consequently, we obtain

$$
\begin{aligned}
&\widetilde{\Omega}_c (\Psi (G)) \\
&\leq \varrho_1 \left(\widetilde{\Omega}_c (G)\right) + \varrho_2 \left(\widetilde{\Omega}_c (G)\right) \\
&+ \frac{1}{|\Delta|}\tau_1 \left(\widetilde{\Omega}_c (G)\right)\left(\frac{|\Delta| Q^{\alpha}}{\Gamma(\alpha+1)} + \left(\widehat{\gamma_1} + \widehat{\kappa_1}\right)\left(\frac{|r_2|}{\Gamma(\alpha+1)} + \frac{|r_1|}{\Gamma(\alpha+2)}\right) Q^{\alpha+1} + \frac{\left(\widehat{\gamma_2} + \widehat{\kappa_2}\right) Q^{\alpha}}{\Gamma(\alpha+1)} + \frac{\left(\widehat{\gamma_3} + \widehat{\kappa_3}\right) Q^{\alpha-1}}{\Gamma(\alpha)}\right) \\
&+ \frac{1}{|\Delta|}\tau_2 \left(\widetilde{\Omega}_c (G)\right)\left(\frac{|\Delta| Q^{\beta}}{\Gamma(\beta+1)} + \frac{\left(\widehat{\gamma_1} + \widehat{\kappa_1}\right) Q^{\beta}}{\Gamma(\beta+1)} + \left(\widehat{\gamma_2} + \widehat{\kappa_2}\right)\left(\frac{\left|p_2\right|}{\Gamma(\beta+1)} + \frac{\left|p_1\right|}{\Gamma(\beta+2)}\right) Q^{\beta+1} + \frac{\left(\widehat{\gamma_4} + \widehat{\kappa_4}\right) Q^{\beta-1}}{\Gamma(\beta)}\right).
\end{aligned}
$$

By setting

$$
\begin{aligned}
\varepsilon = \varrho_1 + \varrho_2 + \frac{1}{|\Delta|}&\left(\frac{|\Delta| Q^{\alpha}}{\Gamma(\alpha+1)} + \left(\widehat{\gamma_1} + \widehat{\kappa_1}\right)\left(\frac{|r_2|}{\Gamma(\alpha+1)} + \frac{|r_1|}{\Gamma(\alpha+2)}\right) Q^{\alpha+1} + \frac{\left(\widehat{\gamma_2} + \widehat{\kappa_2}\right) Q^{\alpha}}{\Gamma(\alpha+1)} + \frac{\left(\widehat{\gamma_3} + \widehat{\kappa_3}\right) Q^{\alpha-1}}{\Gamma(\alpha)}\right)\tau_1 \\
+ \frac{1}{|\Delta|}&\left(\frac{|\Delta| Q^{\beta}}{\Gamma(\beta+1)} + \frac{\left(\widehat{\gamma_1} + \widehat{\kappa_1}\right) Q^{\beta}}{\Gamma(\beta+1)} + \left(\widehat{\gamma_2} + \widehat{\kappa_2}\right)\left(\frac{\left|p_2\right|}{\Gamma(\beta+1)} + \frac{\left|p_1\right|}{\Gamma(\beta+2)}\right) Q^{\beta+1} + \frac{\left(\widehat{\gamma_4} + \widehat{\kappa_4}\right) Q^{\beta-1}}{\Gamma(\beta)}\right)\tau_2,
\end{aligned}
$$

we find that

$$\widetilde{\Omega}_c (\Psi (G)) \leq \varepsilon \left(\widetilde{\Omega}_c (G)\right).$$

It follows from Remark 3.1 that $\varepsilon \in \Upsilon$.
Hence, by Theorem 2.12 and Lemma 2.4, it follows that the coupled system (1) with conditions (2) has at least one mild solution (φ, ψ) in G. □

Example 3.4. *Let*

$$E = l^1 (\mathbb{R}) = \left\{ \varphi = \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \\ \varphi_n \\ \vdots \end{pmatrix},\ \varphi_i \in \mathbb{R},\ i \in \mathbb{N}^*,\ \sum_{i=0}^{\infty} \left|\varphi_i\right| < \infty \right\},$$

be the Banach space of absolutely convergent real sequences, endowed with the norm

$$\left\|\varphi\right\| = \sum_{i=0}^{\infty} \left|\varphi_i\right|,\ \varphi \in E.$$

Consider the following coupled system of nonlinear Caputo fractional differential equations

$$\begin{cases} {}^cD^{\alpha} (\varphi (t) - g_1 (t, \varphi (t), \psi (t))) = f_1 (t, \varphi (t), \psi (t)),\ t \in [0, Q], \\ {}^cD^{\beta} (\psi (t) - g_2 (t, \varphi (t), \psi (t))) = f_2 (t, \varphi (t), \psi (t)),\ t \in [0, Q], \end{cases} \tag{26}$$

with the following integral boundary conditions

$$\begin{cases} \varphi' (0) = 0,\ \varphi (0) + \varphi' (Q) = 0,\ \varphi (2) = 1.3 \int_0^{0.9} \psi (s)\, ds + 1.2Q\psi (0.7), \\ \psi' (0) = 0,\ \psi (0) + \psi' (Q) = 0,\ \psi (2) = 1.1 \int_0^{0.9} \varphi (s)\, ds + 1.3Q\varphi (0.7), \end{cases} \tag{27}$$

where $Q = 1$, $\alpha = 2.7$, $\beta = 2.5$,

$$
\begin{aligned}
&f_1 (t, \varphi, \psi) = \left\{\left(\frac{1}{10} \sin (t)\right)\varphi_n + \frac{1}{30} e^{-t} \psi_n\right\}_{n\in\mathbb{N}^*},\ f_2 (t, \varphi, \psi) = \left\{\frac{2}{25}\varphi_n + \frac{1}{10} \cos^2 (t)\, \psi_n\right\}_{n\in\mathbb{N}^*}, \\
&g_1 (t, \varphi, \psi) = \left\{\frac{2}{t+20}\varphi_n + \frac{1}{50}\psi_n\right\}_{n\in\mathbb{N}^*},\ g_2 (t, \varphi, \psi) = \left\{\frac{1}{t^2+10}\varphi_n + \frac{3}{40} \sin^2 (t)\, \psi_n\right\}_{n\in\mathbb{N}^*}.
\end{aligned}
$$

For all $(\varphi,\psi),\left(\tilde{\varphi},\tilde{\psi}\right)\in E\times E$, *we have*

$$\begin{aligned}\left\|f_1(t,\varphi,\psi)-f_1\left(t,\tilde{\varphi},\tilde{\psi}\right)\right\| &\le \left(\frac{1}{10}\sin(t)\right)\|\varphi-\tilde{\varphi}\|+\frac{1}{30}e^{-t}\left\|\psi-\tilde{\psi}\right\| \\ &\le \frac{1}{10}\left(\|\varphi-\tilde{\varphi}\|+\left\|\psi-\tilde{\psi}\right\|\right),\end{aligned}$$

$$\begin{aligned}\left\|f_2(t,\varphi,\psi)-f_2\left(t,\tilde{\varphi},\tilde{\psi}\right)\right\| &\le \frac{2}{25}\|\varphi-\tilde{\varphi}\|+\frac{1}{10}\cos^2(t)\left\|\psi-\tilde{\psi}\right\| \\ &\le \frac{1}{10}\left(\|\varphi-\tilde{\varphi}\|+\left\|\psi-\tilde{\psi}\right\|\right),\end{aligned}$$

$$\begin{aligned}\left\|g_1(t,\varphi,\psi)-g_1\left(t,\tilde{\varphi},\tilde{\psi}\right)\right\| &\le \frac{2}{t+20}\|\varphi-\tilde{\varphi}\|+\frac{1}{50}\left\|\psi-\tilde{\psi}\right\| \\ &\le \frac{1}{10}\left(\|\varphi-\tilde{\varphi}\|+\left\|\psi-\tilde{\psi}\right\|\right),\end{aligned}$$

and

$$\begin{aligned}\left\|g_2(t,\varphi,\psi)-g_2\left(t,\tilde{\varphi},\tilde{\psi}\right)\right\| &\le \frac{1}{t^2+10}\|\varphi-\tilde{\varphi}\|+\frac{3}{40}\sin^2(t)\left\|\psi-\tilde{\psi}\right\| \\ &\le \frac{1}{10}\left(\|\varphi-\tilde{\varphi}\|+\left\|\psi-\tilde{\psi}\right\|\right),\end{aligned}$$

then for $i\in\{1,2\}$, *we have* $\tau_i(t)=\varrho_i(t)=\frac{1}{10}t$, *are nondecreasing and upper semicontinuous function satisfying* $\tau_i(t)<t$, *and* $\varrho_i(t)<t$, *for all* $t>0$, *then by Lemma 2.11 we conclude that* $\tau_i,\varrho_i\in\Upsilon$.
We set ω *and* $k_1=k_2=\frac{1}{2}$. *Then we obtain the associated inequalities*

$$\begin{aligned}\rho_1 &= \frac{1}{21.1137}\left(\frac{1}{10}\times 2\right)\left(\frac{21.1137}{\Gamma(3.7)}+\frac{53.911}{\Gamma(2.7)}+\frac{10.2725}{\Gamma(3.7)}+15.102\left(\frac{2}{\Gamma(3.7)}+\frac{2.5}{\Gamma(4.7)}\right)\right) \\ &\quad+\frac{1}{21.1137}\left(\frac{1}{10}\times 2\right)\left(\frac{13.7404}{\Gamma(2.5)}+\frac{15.102}{\Gamma(3.5)}+10.2725\left(\frac{1}{\Gamma(3.5)}+\frac{1.5}{\Gamma(4.5)}\right)\right)+\frac{1}{10}\times 2 \\ &\simeq 0.8764\le 1=\frac{1}{2}\omega,\end{aligned}$$

and

$$\begin{aligned}\rho_2 &= \frac{1}{21.1137}\left(\frac{1}{10}\times 2\right)\left(\frac{23.3672}{\Gamma(2.7)}+\frac{20.545}{\Gamma(3.7)}+30.204\left(\frac{2}{\Gamma(3.7)}+\frac{2.5}{\Gamma(4.7)}\right)\right)+\frac{1}{10}\times 2 \\ &\quad+\frac{1}{21.1137}\left(\frac{1}{10}\times 2\right)\left(\frac{21.1137}{\Gamma(3.5)}+\frac{6.3671}{\Gamma(2.5)}+\frac{30.204}{\Gamma(3.5)}+20.545\left(\frac{1}{\Gamma(3.5)}+\frac{1.5}{\Gamma(4.5)}\right)\right) \\ &\simeq 0.8488\le 1=\frac{1}{2}\omega.\end{aligned}$$

Hence, it follows that all the assumptions required by Theorem 3.3 are satisfied. Consequently, the coupled system (26) with conditions (27) has at least one mild solution (φ,ψ) *in* B_2.

3.1.2. *Uniqueness of solutions using the Banach theorem*

For the purpose of proving our main results, we adopt the following set of hypotheses:
(H1) There exist four constants $L_{f_i},L_{gi}\in\mathbb{R}_+^*$, with $i\in\{1,2\}$, such that

$$\begin{aligned}&\left\|f_i(t,\varphi_2,\psi_2)-f_i(t,\varphi_1,\psi_1)\right\|\le L_{fi}\left(\|\varphi_2-\varphi_1\|+\|\psi_2-\psi_1\|\right), \\ &\left\|g_i(t,\varphi_2,\psi_2)-g_i(t,\varphi_1,\psi_1)\right\|\le L_{gi}\left(\|\varphi_2-\varphi_1\|+\|\psi_2-\psi_1\|\right).\end{aligned}$$

(H2) Suppose the following inequality holds

$$\rho = \frac{1}{|\Delta|} L_{f_1} \left((\widehat{\gamma_1} + \widehat{\kappa_1}) \left(\frac{|r_2|}{\Gamma(\alpha+1)} + \frac{|r_1|}{\Gamma(\alpha+2)} \right) Q^{\alpha+1} + \frac{(\widehat{\gamma_2} + \widehat{\kappa_2}) Q^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\widehat{\gamma_3} + \widehat{\kappa_3}) Q^{\alpha-1}}{\Gamma(\alpha)} \right)$$
$$+ \frac{1}{|\Delta|} L_{f2} \left(\frac{(\widehat{\gamma_1} + \widehat{\kappa_1}) Q^{\beta}}{\Gamma(\beta+1)} + (\widehat{\gamma_2} + \widehat{\kappa_2}) \left(\frac{|p_2|}{\Gamma(\beta+1)} + \frac{|p_1|}{\Gamma(\beta+2)} \right) Q^{\beta+1} + \frac{(\widehat{\gamma_4} + \widehat{\kappa_4}) Q^{\beta-1}}{\Gamma(\beta)} \right)$$
$$+ \frac{L_{f_1} Q^{\alpha}}{\Gamma(\alpha+1)} + \frac{L_{f2} Q^{\beta}}{\Gamma(\beta+1)} + L_{g1} + L_{g2} < 1,$$

where $\widehat{\gamma_i} = \sup_{t \in I} |\gamma_i(t)|$, $\widehat{\kappa_i} = \sup_{t \in I} |\kappa_i(t)|$, $i \in \{1, 2, 3, 4\}$.

Theorem 3.5. *Suppose that hypotheses (H1) and (H2) hold. Then, the coupled system (1) with conditions (2) has a unique mild solution (φ, ψ) in $C(I \times E) \times C(I \times E)$.*

Proof. Let (φ_1, ψ_1) and (φ_2, ψ_2) be two elements of the product space $C(I \times E) \times C(I \times E)$, then we have

$$\left\| \Psi(\varphi_2, \psi_2) - \Psi(\varphi_1, \psi_1) \right\|_{c \times c} = \left\| \Psi_1(\varphi_2, \psi_2) - \Psi_1(\varphi_1, \psi_1) \right\|_{\infty} + \left\| \Psi_2(\varphi_2, \psi_2) - \Psi_2(\varphi_1, \psi_1) \right\|_{\infty}.$$

On the one hand, we have

$$\left\| \Psi_1(\varphi_2, \psi_2)(t) - \Psi_1(\varphi_1, \psi_1)(t) \right\|$$
$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\| f_1(s, \varphi_2(s), \psi_2(s)) - f_1(s, \varphi_1(s), \psi_1(s)) \right\| ds$$
$$+ \left\| g_1(t, \varphi_2(t), \psi_2(t)) - g_1(t, \varphi_1(t), \psi_1(t)) \right\|$$
$$+ \frac{1}{|\Delta|} \int_0^Q \left(\left| \widehat{\gamma_3}(t) \right| \frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} + |\gamma_2(t)| \frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \left\| f_1(s, \varphi_2(s), \psi_2(s)) - f_1(s, \varphi_1(s), \psi_1(s)) \right\| ds$$
$$+ \frac{1}{|\Delta|} \int_0^{\xi} \left(|r_2 \gamma_1(t)| Q \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + |r_1 \gamma_1(t)| \frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)} \right) \left\| f_1(s, \varphi_2(s), \psi_2(s)) - f_1(s, \varphi_1(s), \psi_1(s)) \right\| ds$$
$$+ \frac{1}{|\Delta|} \int_0^{\xi} \left(|p_2 \gamma_2(t)| Q \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + |p_1 \gamma_2(t)| \frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)} \right) \left\| f_2(s, \varphi_2(s), \psi_2(s)) - f_2(s, \varphi_1(s), \psi_1(s)) \right\| ds$$
$$+ \frac{1}{|\Delta|} \int_0^Q \left(|\gamma_4(t)| \frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} + |\gamma_1(t)| \frac{(Q-s)^{\beta-1}}{\Gamma(\beta)} \right) \left\| f_2(s, \varphi_2(s), \psi_2(s)) - f_2(s, \varphi_1(s), \psi_1(s)) \right\| ds.$$

Using (H1), we find

$$\left\| \Psi_1(\varphi_2, \psi_2)(t) - \Psi_1(\varphi_1, \psi_1)(t) \right\|$$
$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(\|\varphi_2(s) - \varphi_1(s)\| + \|\psi_2(s) - \psi_1(s)\| \right) ds + \left(\|\varphi_2(s) - \varphi_1(s)\| + \|\psi_2(s) - \psi_1(s)\| \right)$$
$$+ \frac{1}{|\Delta|} \int_0^Q \left(|\gamma_3(t)| \frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} + |\gamma_2(t)| \frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \left(\|\varphi_2(s) - \varphi_1(s)\| + \|\psi_2(s) - \psi_1(s)\| \right) ds$$
$$+ \frac{1}{|\Delta|} \int_0^{\xi} \left(|r_2 \gamma_1(t)| Q \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + |r_1 \gamma_1(t)| \frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)} \right) \left(\|\varphi_2(s) - \varphi_1(s)\| + \|\psi_2(s) - \psi_1(s)\| \right) ds$$
$$+ \frac{1}{|\Delta|} \int_0^{\xi} \left(|p_2 \gamma_2(t)| Q \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + |p_1 \gamma_2(t)| \frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)} \right) \left(\|\varphi_2(s) - \varphi_1(s)\| + \|\psi_2(s) - \psi_1(s)\| \right) ds$$
$$+ \frac{1}{|\Delta|} \int_0^Q \left(|\gamma_4(t)| \frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} + |\gamma_1(t)| \frac{(Q-s)^{\beta-1}}{\Gamma(\beta)} \right) \left(\|\varphi_2(s) - \varphi_1(s)\| + \|\psi_2(s) - \psi_1(s)\| \right) ds.$$

Thus, we get

$$\begin{aligned}
&\left\|\Psi_1(\varphi_2,\psi_2)(t)-\Psi_1(\varphi_1,\psi_1)(t)\right\| \\
&\leq \left[\frac{1}{|\Delta|}L_{f_1}\left(\frac{\Delta Q^{\alpha}}{\Gamma(\alpha+1)}+\widehat{\gamma_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\gamma_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\gamma_3}Q^{\alpha-1}}{\Gamma(\alpha)}\right)\right. \\
&\left.+\frac{1}{|\Delta|}L_{f2}\left(\frac{\widehat{\gamma_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\gamma_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\gamma_4}Q^{\beta-1}}{\Gamma(\beta)}\right)+L_{g1}\right]\left\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\right\|_{c\times c}. \qquad (28)
\end{aligned}$$

Taking the supremum over $t\in I$, we obtain

$$\begin{aligned}
&\left\|\Psi_1(\varphi_2,\psi_2)-\Psi_1(\varphi_1,\psi_1)\right\|_{\infty} \\
&\leq \left[\frac{1}{|\Delta|}L_{f_1}\left(\frac{|\Delta| Q^{\alpha}}{\Gamma(\alpha+1)}+\widehat{\gamma_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\gamma_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\gamma_3}Q^{\alpha-1}}{\Gamma(\alpha)}\right)\right. \\
&\left.+\frac{1}{|\Delta|}L_{f2}\left(\frac{\widehat{\gamma_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\gamma_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\gamma_4}Q^{\beta-1}}{\Gamma(\beta)}\right)+L_{g1}\right]\left\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\right\|_{c\times c}.
\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}
&\left\|\Psi_2(\varphi_2,\psi_2)(t)-\Psi_2(\varphi_1,\psi_1)(t)\right\| \\
&\leq \left[\frac{1}{|\Delta|}L_{f_1}\left(\widehat{\kappa_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\kappa_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\kappa_3}Q^{\alpha-1}}{\Gamma(\alpha)}\right)+L_{g2}\right. \\
&\left.+\frac{1}{|\Delta|}L_{f2}\left(\frac{\Delta Q^{\beta}}{\Gamma(\beta+1)}+\frac{\widehat{\kappa_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\kappa_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\kappa_4}Q^{\beta-1}}{\Gamma(\beta)}\right)\right] \\
&\times\left(\left\|\varphi_2-\varphi_1\right\|_{\infty}+\left\|\psi_2-\psi_1\right\|_{\infty}\right). \qquad (29)
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\left\|\Psi_2(\varphi_2,\psi_2)-\Psi_2(\varphi_1,\psi_1)\right\|_{\infty} \\
&\leq \left[\frac{1}{|\Delta|}L_{f_1}\left(\widehat{\kappa_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\kappa_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\kappa_3}Q^{\alpha-1}}{\Gamma(\alpha)}\right)+L_{g2}\right. \\
&\left.+\frac{1}{|\Delta|}L_{f2}\left(\frac{\Delta Q^{\beta}}{\Gamma(\beta+1)}+\frac{\widehat{\kappa_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\kappa_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\kappa_4}Q^{\beta-1}}{\Gamma(\beta)}\right)\right] \\
&\times\left(\left\|\varphi_2-\varphi_1\right\|_{\infty}+\left\|\psi_2-\psi_1\right\|_{\infty}\right).
\end{aligned}$$

We conclude that

$$\begin{aligned}
&\left\|\Psi(\varphi_2,\psi_2)-\Psi(\varphi_1,\psi_1)\right\|_{c\times c} \\
&\leq \left[\frac{1}{|\Delta|}L_{f_1}\left((\widehat{\gamma_1}+\widehat{\kappa_1})\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{(\widehat{\gamma_2}+\widehat{\kappa_2})Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\widehat{\gamma_3}+\widehat{\kappa_3})Q^{\alpha-1}}{\Gamma(\alpha)}\right)\right. \\
&+\frac{1}{|\Delta|}L_{f2}\left(\frac{(\widehat{\gamma_1}+\widehat{\kappa_1})Q^{\beta}}{\Gamma(\beta+1)}+(\widehat{\gamma_2}+\widehat{\kappa_2})\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{(\widehat{\gamma_4}+\widehat{\kappa_4})Q^{\beta-1}}{\Gamma(\beta)}\right) \\
&\left.+\frac{L_{f_1}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{L_{f2}Q^{\beta}}{\Gamma(\beta+1)}+L_{g1}+L_{g2}\right]\left\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\right\|_{c\times c} \\
&\leq \rho\left\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\right\|_{c\times c}.
\end{aligned}$$

Using (H2), we obtain that Ψ is a contraction.

Finally, according to Theorem (2.14), the operator Ψ possesses a unique fixed point (φ, ψ) in $C(I \times E) \times C(I \times E)$, which corresponds precisely to the unique mild solution of the coupled system (1) with conditions (2). □

3.2. Continuous dependence

Definition 3.6 (Continuous dependence [13]). *Consider two mild solutions (φ, ψ) and (φ^*, ψ^*) of the coupled system (1) with conditions (2), corresponding to the sets of functions (f_1, f_2, g_1, g_2) and $\left(f_1^*, f_2^*, g_1^*, g_2^*\right)$ respectively. We say that the mild solution depend continuously on the functions f_1, f_2, g_1 and g_2,if there exist four constants $k_1, k_2, k_3, k_4 \in \mathbb{R}_+^*$, such that*

$$\left\|(\varphi, \psi) - (\varphi^*, \psi^*)\right\| \leq k_1 \left\|f_1 - f_1^*\right\|_c + k_2 \left\|f_2 - f_2^*\right\|_c + k_3 \left\|g_1 - g_1^*\right\|_c + k_4 \left\|g_2 - g_2^*\right\|_c.$$

Theorem 3.7. *Assume that hypotheses (H1) and (H2) hold. Then, the unique mild solution $(\varphi, \psi) \in C(I \times E) \times C(I \times E)$ of the system (1) with conditions (2) depends continuously on the functions f_1, f_2, g_1 and g_2.*

Proof. Let $f_1, f_2, g_1, g_2, f_1^*, f_2^*, g_1^*, g_2^* : I \times E \times E \to E$, continuous in their respective arguments.

According to Theorem 3.5, the two following problems

$$\begin{cases} {}^cD^\alpha (\varphi(t) - g_1(t, \varphi(t), \psi(t))) = f_1(t, \varphi(t), \psi(t)), \ t \in [0, Q], \\ {}^cD^\beta (\psi(t) - g_2(t, \varphi(t), \psi(t))) = f_2(t, \varphi(t), \psi(t)), \ t \in [0, Q], \end{cases}$$

and

$$\begin{cases} {}^cD^\alpha \left(\varphi(t) - g_1^*(t, \varphi(t), \psi(t))\right) = f_1^*(t, \varphi(t), \psi(t)), \ t \in [0, Q], \\ {}^cD^\beta \left(\psi(t) - g_2^*(t, \varphi(t), \psi(t))\right) = f_2^*(t, \varphi(t), \psi(t)), \ t \in [0, Q], \end{cases}$$

together with conditions (2), admit two mild solutions (φ, ψ) and (φ^*, ψ^*) in $C(I \times E) \times C(I \times E)$ respectively, such that

$$\begin{aligned} \varphi(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \varphi(s), \psi(s)) \, ds + g_1(t, \varphi(t), \psi(t)) \\ & + \frac{1}{\Delta} \int_0^Q \left(\gamma_4(t) \frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} - \gamma_1(t) \frac{(Q-s)^{\beta-1}}{\Gamma(\beta)} \right) f_2(s, \varphi(s), \psi(s)) \, ds \\ & + \frac{1}{\Delta} \int_0^Q \left(\gamma_3(t) \frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \gamma_2(t) \frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)} \right) f_1(s, \varphi(s), \psi(s)) \, ds \\ & + \frac{1}{\Delta} \int_0^\xi \left(Qp_2\gamma_2(t) \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + p_1\gamma_2(t) \frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)} \right) f_2(s, \varphi(s), \psi(s)) \, ds \\ & + \frac{1}{\Delta} \int_0^\xi \left(Qr_2\gamma_1(t) \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + r_1\gamma_1(t) \frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)} \right) f_1(s, \varphi(s), \psi(s)) \, ds, \end{aligned}$$

$$\begin{aligned} \varphi^*(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1^*(s, \varphi^*(s), \psi^*(s)) \, ds + g_1^*(t, \varphi^*(t), \psi^*(t)) \\ & + \frac{1}{\Delta} \int_0^Q \left(\gamma_4(t) \frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} - \gamma_1(t) \frac{(Q-s)^{\beta-1}}{\Gamma(\beta)} \right) f_2^*(s, \varphi^*(s), \psi^*(s)) \, ds \\ & + \frac{1}{\Delta} \int_0^Q \left(\gamma_3(t) \frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \gamma_2(t) \frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)} \right) f_1^*(s, \varphi^*(s), \psi^*(s)) \, ds \\ & + \frac{1}{\Delta} \int_0^\xi \left(Qp_2\gamma_2(t) \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + p_1\gamma_2(t) \frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)} \right) f_2^*(s, \varphi^*(s), \psi^*(s)) \, ds \\ & + \frac{1}{\Delta} \int_0^\xi \left(Qr_2\gamma_1(t) \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + r_1\gamma_1(t) \frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)} \right) f_1^*(s, \varphi^*(s), \psi^*(s)) \, ds, \end{aligned}$$

$$\begin{aligned}\psi(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f_2(s,\varphi(s),\psi(s))\,ds + g_2(t,\varphi(t),\psi(t))\\&+\frac{1}{\Delta}\int_0^Q\left(\kappa_3(t)\frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\kappa_2(t)\frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right)f_1(s,\varphi(s),\psi(s))\,ds\\&+\frac{1}{\Delta}\int_0^Q\left(\kappa_4(t)\frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)}-\kappa_1(t)\frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right)f_2(s,\varphi(s),\psi(s))\,ds\\&+\frac{1}{\Delta}\int_0^\xi\left(Qr_2\kappa_1(t)\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}+r_1\kappa_1(t)\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right)f_1(s,\varphi(s),\psi(s))\,ds\\&+\frac{1}{\Delta}\int_0^\xi\left(Qp_2\kappa_2(t)\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}+p_1\kappa_2(t)\frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right)f_2(s,\varphi(s),\psi(s))\,ds,\end{aligned}$$

and

$$\begin{aligned}\psi^*(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f_2^*(s,\varphi^*(s),\psi^*(s))\,ds + g_2^*(t,\varphi^*(t),\psi^*(t))\\&+\frac{1}{\Delta}\int_0^Q\left(\kappa_3(t)\frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\kappa_2(t)\frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right)f_1^*(s,\varphi^*(s),\psi^*(s))\,ds\\&+\frac{1}{\Delta}\int_0^Q\left(\kappa_4(t)\frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)}-\kappa_1(t)\frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right)f_2^*(s,\varphi^*(s),\psi^*(s))\,ds\\&+\frac{1}{\Delta}\int_0^\xi\left(Qr_2\kappa_1(t)\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}+r_1\kappa_1(t)\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right)f_1^*(s,\varphi^*(s),\psi^*(s))\,ds\\&+\frac{1}{\Delta}\int_0^\xi\left(Qp_2\kappa_2(t)\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}+p_1\kappa_2(t)\frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right)f_2^*(s,\varphi^*(s),\psi^*(s))\,ds.\end{aligned}$$

For $i\in\{1,2,3,4\}$, we have the following inequalities

$$\begin{aligned}\left\|f_i(s,\varphi(s),\psi(s))-f_i^*(s,\varphi(s),\psi(s))\right\| &\le \left\|f_i(s,\varphi(s),\psi(s))-f_i(s,\varphi^*(s),\psi^*(s))\right\|\\&+\left\|f_i(s,\varphi^*(s),\psi^*(s))-f_i^*(s,\varphi^*(s),\psi^*(s))\right\|,\end{aligned}$$

and

$$\begin{aligned}\left\|g_i(s,\varphi(s),\psi(s))-g_i^*(s,\varphi(s),\psi(s))\right\| &\le \left\|g_i(s,\varphi(s),\psi(s))-g_i(s,\varphi^*(s),\psi^*(s))\right\|\\&+\left\|g_i(s,\varphi^*(s),\psi^*(s))-g_i^*(s,\varphi^*(s),\psi^*(s))\right\|.\end{aligned}$$

Using (H1), we get

$$\left\|f_i(s,\varphi(s),\psi(s))-f_i^*(s,\varphi(s),\psi(s))\right\| \le \left\|f_i-f_i^*\right\|_c + L_{fi}\left\|(\varphi,\psi)-(\varphi^*,\psi^*)\right\|_{c\times c}, \tag{30}$$

and

$$\left\|g_i(s,\varphi(s),\psi(s))-g_i^*(s,\varphi(s),\psi(s))\right\| \le \left\|g_i-g_i^*\right\|_c + L_{gi}\left\|(\varphi,\psi)-(\varphi^*,\psi^*)\right\|_{c\times c}. \tag{31}$$

We have

$$\left\|(\varphi,\psi)-(\varphi^*,\psi^*)\right\|_{c\times c} = \left\|(\varphi-\varphi^*,\psi-\psi^*)\right\|_{c\times c} = \left\|\varphi-\varphi^*\right\|_\infty + \left\|\psi-\psi^*\right\|_\infty.$$

On the one hand, we have

$$\begin{aligned}
&\left\|\varphi(t)-\varphi^{*}(t)\right\| \\
&=\int_{0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f_{1}(s,\varphi(s),\psi(s))-f_{1}^{*}(s,\varphi^{*}(s),\psi^{*}(s))\right\|ds \\
&+\left\|g_{1}(t,\varphi(t),\psi(t))-g_{1}^{*}(t,\varphi^{*}(t),\psi^{*}(t))\right\| \\
&+\frac{1}{|\Delta|}\int_{0}^{Q}\left(\left|\gamma_{1}(t)\right|\frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}+\left|\gamma_{4}(t)\right|\frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)}\right)\left\|f_{2}(s,\varphi(s),\psi(s))-f_{2}^{*}(s,\varphi^{*}(s),\psi^{*}(s))\right\|ds \\
&+\frac{1}{|\Delta|}\int_{0}^{Q}\left(\left|\gamma_{2}(t)\right|\frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}+\left|\gamma_{3}(t)\right|\frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)\left\|f_{1}(s,\varphi(s),\psi(s))-f_{1}^{*}(s,\varphi^{*}(s),\psi^{*}(s))\right\|ds \\
&+\frac{1}{|\Delta|}\int_{0}^{\xi}\left(\left|p_{2}\gamma_{2}(t)\right|Q\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}+\left|p_{1}\gamma_{2}(t)\right|\frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right)\left\|f_{2}(s,\varphi(s),\psi(s))-f_{2}^{*}(s,\varphi^{*}(s),\psi^{*}(s))\right\|ds \\
&+\frac{1}{|\Delta|}\int_{0}^{\xi}\left(\left|r_{2}\gamma_{1}(t)\right|Q\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}+\left|r_{1}\gamma_{1}(t)\right|\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right)\left\|f_{1}(s,\varphi(s),\psi(s))-f_{1}^{*}(s,\varphi^{*}(s),\psi^{*}(s))\right\|ds,
\end{aligned}$$

Using (30) and (31), and then taking the supremum over $t\in I$, we obtain

$$\begin{aligned}
\left\|\varphi-\varphi^{*}\right\|_{\infty}\leq&\left[\frac{1}{|\Delta|}L_{f1}\left(\frac{|\Delta|\,Q^{\alpha}}{\Gamma(\alpha+1)}+\widehat{\gamma_{1}}\left(\frac{r_{2}}{\Gamma(\alpha+1)}+\frac{r_{1}}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\gamma_{2}}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\gamma_{3}}Q^{\alpha-1}}{\Gamma(\alpha)}\right)\right. \\
&\left.+\frac{1}{|\Delta|}L_{f2}\left(\frac{\widehat{\gamma_{1}}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\gamma_{2}}\left(\frac{p_{2}}{\Gamma(\beta+1)}+\frac{p_{1}}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\gamma_{4}}Q^{\beta-1}}{\Gamma(\beta)}\right)+L_{g1}\right] \\
&\times\left\|(\varphi,\psi)-(\varphi^{*},\psi^{*})\right\|_{c\times c}+\left\|g_{1}-g_{1}^{*}\right\|_{c} \\
&+\frac{1}{|\Delta|}\left(\frac{|\Delta|\,Q^{\alpha}}{\Gamma(\alpha+1)}+\widehat{\gamma_{1}}\left(\frac{r_{2}}{\Gamma(\alpha+1)}+\frac{r_{1}}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\gamma_{2}}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\gamma_{3}}Q^{\alpha-1}}{\Gamma(\alpha)}\right)\left\|f_{1}-f_{1}^{*}\right\|_{c} \\
&+\frac{1}{|\Delta|}\left(\frac{\widehat{\gamma_{1}}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\gamma_{2}}\left(\frac{p_{2}}{\Gamma(\beta+1)}+\frac{p_{1}}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\gamma_{4}}Q^{\beta-1}}{\Gamma(\beta)}\right)\left\|f_{2}-f_{2}^{*}\right\|_{c}.
\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}
\left\|\psi-\psi^{*}\right\|_{\infty}\leq&\left[\frac{1}{|\Delta|}L_{f1}\left(\widehat{\kappa_{1}}\left(\frac{|r_{2}|}{\Gamma(\alpha+1)}+\frac{|r_{1}|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\kappa_{2}}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\kappa_{3}}Q^{\alpha-1}}{\Gamma(\alpha)}\right)+L_{g2}\right. \\
&\left.+\frac{1}{|\Delta|}L_{f2}\left(\frac{|\Delta|\,Q^{\beta}}{\Gamma(\beta+1)}+\frac{\widehat{\kappa_{1}}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\kappa_{2}}\left(\frac{\left|p_{2}\right|}{\Gamma(\beta+1)}+\frac{\left|p_{1}\right|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\kappa_{4}}Q^{\beta-1}}{\Gamma(\beta)}\right)\right] \\
&\times\left\|(\varphi,\psi)-(\varphi^{*},\psi^{*})\right\|_{c\times c}+\left\|g_{2}-g_{2}^{*}\right\|_{c} \\
&+\frac{1}{|\Delta|}\left(\widehat{\kappa_{1}}\left(\frac{|r_{2}|}{\Gamma(\alpha+1)}+\frac{|r_{1}|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\kappa_{2}}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\kappa_{3}}Q^{\alpha-1}}{\Gamma(\alpha)}\right)\left\|f_{1}-f_{1}^{*}\right\|_{c} \\
&+\frac{1}{|\Delta|}\left(\frac{|\Delta|\,Q^{\beta}}{\Gamma(\beta+1)}+\frac{\widehat{\kappa_{1}}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\kappa_{2}}\left(\frac{\left|p_{2}\right|}{\Gamma(\beta+1)}+\frac{\left|p_{1}\right|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\kappa_{4}}Q^{\beta-1}}{\Gamma(\beta)}\right)\left\|f_{2}-f_{2}^{*}\right\|_{c}.
\end{aligned}$$

Consequently

$$
\begin{aligned}
&\left\|(\varphi,\psi)-(\varphi^*,\psi^*)\right\|_{c\times c}\\
&\le\left[\frac{1}{|\Delta|}\left((\widehat{\gamma_1}+\widehat{\kappa_1})\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{(\widehat{\gamma_2}+\widehat{\kappa_2})Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\widehat{\gamma_3}+\widehat{\kappa_3})Q^{\alpha-1}}{\Gamma(\alpha)}\right)\right.\\
&\left.+\frac{Q^{\alpha}}{\Gamma(\alpha+1)}\right]\frac{1}{1-\rho}\left\|f_1-f_1^*\right\|_c\\
&+\left[\frac{1}{|\Delta|}\left(\frac{(\widehat{\gamma_1}+\widehat{\kappa_1})Q^{\beta}}{\Gamma(\beta+1)}+(\widehat{\gamma_2}+\widehat{\kappa_2})\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{(\widehat{\gamma_4}+\widehat{\kappa_4})Q^{\beta-1}}{\Gamma(\beta)}\right)\right.\\
&\left.+\frac{Q^{\beta}}{\Gamma(\beta+1)}\right]\frac{1}{1-\rho}\left\|f_2-f_2^*\right\|_c+\frac{1}{1-\rho}\left\|g_1-g_1^*\right\|_c+\frac{1}{1-\rho}\left\|g_2-g_2^*\right\|_c\\
&\le k_1\left\|f_1-f_1^*\right\|_c+k_2\left\|f_2-f_2^*\right\|_c+k_3\left\|g_1-g_1^*\right\|_c+k_4\left\|g_2-g_2^*\right\|_c.
\end{aligned}
$$

As a result, the claim is established, which completes the proof. □

3.3. Stability

Define the following nonlinear operators $\Phi_1,\Phi_2 : C(I\times E)\times C(I\times E)\to C(I\times E)$

$$
\begin{cases}
\Phi_1(\varphi,\psi)(t)={}^cD^{\alpha}(\varphi(t)-g_1(t,\varphi(t),\psi(t)))-f_1(t,\varphi(t),\psi(t)),\ t\in[0,Q],\ 2<\alpha\le 3,\\
\Phi_2(\varphi,\psi)(t)={}^cD^{\beta}(\psi(t)-g_2(t,\varphi(t),\psi(t)))-f_2(t,\varphi(t),\psi(t)),\ t\in[0,Q],\ 2<\beta\le 3.
\end{cases}
$$

Definition 3.8 (Ulam-Hyers stability [17, 32]). *The coupled system (1) with conditions (2) is said to be Ulam-Hyers stable, if there exists some positive constants M_1 and M_2, such that for some ϵ_1 and ϵ_2 that are greater than 0, and for each mild solution (φ^*,ψ^*) in $C(I\times E)\times C(I\times E)$ that satisfies the following inequality*

$$
\begin{cases}
\left\|\Phi_1(\varphi^*,\psi^*)\right\|_{\infty}\le\epsilon_1,\\
\left\|\Phi_2(\varphi^*,\psi^*)\right\|_{\infty}\le\epsilon_2,
\end{cases}\tag{32}
$$

there exists a unique mild solution (φ,ψ) in $C(I\times E)\times C(I\times E)$ of the coupled system (1) with conditions (2), such that

$$\left\|(\varphi,\psi)-(\varphi^*,\psi^*)\right\|_{c\times c}\le M_1\epsilon_1+M_2\epsilon_2.$$

Theorem 3.9. *Suppose that hypotheses (H1) and (H2) hold. Then, the coupled system (1) with conditions (2) is Hyers-Ulam-stable.*

Proof. Let $(\varphi,\psi)\in C(I\times E)\times C(I\times E)$ denote the unique mild solution of the coupled system (1) with conditions (2) satisfying (13) and (14), and let $(\varphi^*,\psi^*)\in C(I\times E)\times C(I\times E)$ be any solution satisfying:

$$
\begin{cases}
{}^cD^{\alpha}(\varphi^*(t)-g_1(t,\varphi^*(t),\psi^*(t)))=f_1(t,\varphi^*(t),\psi^*(t))+\Phi_1(\varphi^*,\psi^*)(t),\ t\in[0,Q],\\
{}^cD^{\beta}(\psi^*(t)-g_2(t,\varphi^*(t),\psi^*(t)))=f_2(t,\varphi^*(t),\psi^*(t))+\Phi_2(\varphi^*,\psi^*)(t),\ t\in[0,Q].
\end{cases}
$$

Then, we have

$$\begin{aligned}\varphi^{*}(t) &= \Psi_1(\varphi^{*},\psi^{*})(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Phi_1(\varphi^{*},\psi^{*})(s)\,ds\\
&+ \frac{1}{\Delta}\int_0^Q \left(\gamma_4(t)\frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} - \gamma_1(t)\frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right)\Phi_2(\varphi^{*},\psi^{*})(s)\,ds\\
&+ \frac{1}{\Delta}\int_0^Q \left(\gamma_3(t)\frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \gamma_2(t)\frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right)\Phi_1(\varphi^{*},\psi^{*})(s)\,ds\\
&+ \frac{1}{\Delta}\int_0^\xi \left(\gamma_2(t)Qp_2\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + p_1\gamma_2(t)\frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right)\Phi_2(\varphi^{*},\psi^{*})(s)\,ds\\
&+ \frac{1}{\Delta}\int_0^\xi \left(\gamma_1(t)Qr_2\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + r_1\gamma_1(t)\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right)\Phi_1(\varphi^{*},\psi^{*})(s)\,ds,\end{aligned}$$

and

$$\begin{aligned}\psi^{*}(t) &= \Psi_2(\varphi^{*},\psi^{*})(t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Phi_2(\varphi^{*},\psi^{*})(s)\,ds\\
&+ \frac{1}{\Delta}\int_0^Q \left(\kappa_3(t)\frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \kappa_2(t)\frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right)\Phi_1(\varphi^{*},\psi^{*})(s)\,ds\\
&+ \frac{1}{\Delta}\int_0^Q \left(\kappa_4(t)\frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} - \kappa_1(t)\frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right)\Phi_2(\varphi^{*},\psi^{*})(s)\,ds\\
&+ \frac{1}{\Delta}\int_0^\xi \left(\kappa_1(t)Qr_2\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + \kappa_1(t)r_1\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right)\Phi_1(\varphi^{*},\psi^{*})(s)\,ds\\
&+ \frac{1}{\Delta}\int_0^\xi \left(\kappa_2(t)Qp_2\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + \kappa_2(t)p_1\frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right)\Phi_2(\varphi^{*},\psi^{*})(s)\,ds.\end{aligned}$$

For $t \in I$, we have

$$\begin{aligned}&\left\|\Psi_1(\varphi^{*},\psi^{*})(t) - \varphi^{*}(t)\right\|\\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|\Phi_1(\varphi^{*},\psi^{*})(s)\right\|ds + \frac{1}{|\Delta|}\int_0^Q \left(\left|\gamma_4(t)\right|\frac{(Q-s)^{\beta-2}}{\Gamma(\beta-1)} + \left|\gamma_1(t)\right|\frac{(Q-s)^{\beta-1}}{\Gamma(\beta)}\right)\left\|\Phi_2(\varphi^{*},\psi^{*})(s)\right\|ds\\
&+ \frac{1}{|\Delta|}\int_0^Q \left(\left|\gamma_3(t)\right|\frac{(Q-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \left|\gamma_2(t)\right|\frac{(Q-s)^{\alpha-1}}{\Gamma(\alpha)}\right)\left\|\Phi_1(\varphi^{*},\psi^{*})(s)\right\|ds\\
&+ \frac{1}{|\Delta|}\int_0^\xi \left(\left|p_2\gamma_2(t)\right|Q\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} + \left|p_1\gamma(t)\right|\frac{(\xi-s)^{\beta}}{\Gamma(\beta+1)}\right)\left\|\Phi_2(\varphi^{*},\psi^{*})(s)\right\|ds\\
&+ \frac{1}{|\Delta|}\int_0^\xi \left(\left|r_2\gamma_1(t)\right|Q\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} + \left|r_1\gamma_1(t)\right|\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)}\right)\left\|\Phi_1(\varphi^{*},\psi^{*})(s)\right\|ds.\end{aligned}$$

Thus

$$\begin{aligned}&\left\|\Psi_1(\varphi^{*},\psi^{*})(t) - \varphi^{*}(t)\right\|\\
&\leq \frac{Q^{\alpha}}{\Gamma(\alpha+1)}\left\|\Phi_1(\varphi^{*},\psi^{*})\right\|_\infty + \frac{1}{|\Delta|}\left(\frac{\widehat{\gamma_4}Q^{\beta-1}}{\Gamma(\beta)} + \frac{\widehat{\gamma_1}Q^{\beta}}{\Gamma(\beta+1)}\right)\left\|\Phi_2(\varphi^{*},\psi^{*})\right\|_\infty\\
&+ \frac{1}{|\Delta|}\left(\frac{\widehat{\gamma_3}Q^{\alpha-1}}{\Gamma(\alpha)} + \frac{\widehat{\gamma_2}Q^{\alpha}}{\Gamma(\alpha+1)}\right)\left\|\Phi_1(\varphi^{*},\psi^{*})\right\|_\infty + \frac{1}{|\Delta|}\widehat{\gamma_2}\left(\frac{\left|p_2\right|}{\Gamma(\beta+1)} + \frac{\left|p_1\right|}{\Gamma(\beta+2)}\right)Q^{\beta+1}\left\|\Phi_2(\varphi^{*},\psi^{*})\right\|_\infty\\
&+ \frac{1}{|\Delta|}\widehat{\gamma_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)} + \frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}\left\|\Phi_1(\varphi^{*},\psi^{*})\right\|_\infty.\end{aligned}$$

Using (32), we get

$$\begin{aligned}
&\left\|\Psi_1\left(\varphi^*,\psi^*\right)(t)-\varphi^*(t)\right\| \\
&\leq \frac{1}{|\Delta|}\left(\frac{|\Delta|\,Q^{\alpha}}{\Gamma(\alpha+1)}+\widehat{\gamma_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\gamma_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\gamma_3}Q^{\alpha-1}}{\Gamma(\alpha)}\right)\epsilon_1 \\
&+\frac{1}{|\Delta|}\left(\frac{\widehat{\gamma_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\gamma_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\gamma_4}Q^{\beta-1}}{\Gamma(\beta)}\right)\epsilon_2. \qquad (33)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\left\|\Psi_2\left(\varphi^*,\psi^*\right)(t)-\psi^*(t)\right\| \\
&\leq \frac{1}{|\Delta|}\left(\widehat{\kappa_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\kappa_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\kappa_3}Q^{\alpha-1}}{\Gamma(\alpha)}\right)\epsilon_1 \\
&+\frac{1}{|\Delta|}\left(\frac{|\Delta|\,Q^{\beta}}{\Gamma(\beta+1)}+\frac{\widehat{\kappa_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\kappa_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\kappa_4}Q^{\beta-1}}{\Gamma(\beta)}\right)\epsilon_2. \qquad (34)
\end{aligned}$$

Therefore, by (28) and (33) we find

$$\begin{aligned}
&\left\|\varphi(t)-\varphi^*(t)\right\| \\
&\leq \left\|\Psi_1(\varphi,\psi)(t)-\Psi_1\left(\varphi^*,\psi^*\right)(t)\right\|+\left\|\Psi_1\left(\varphi^*,\psi^*\right)(t)-\varphi^*(t)\right\| \\
&\leq \frac{1}{|\Delta|}\left(\frac{|\Delta|\,Q^{\alpha}}{\Gamma(\alpha+1)}+\widehat{\gamma_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\gamma_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\gamma_3}Q^{\alpha-1}}{\Gamma(\alpha)}\right)\epsilon_1 \\
&+\frac{1}{|\Delta|}\left(\frac{\widehat{\gamma_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\gamma_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\gamma_4}Q^{\beta-1}}{\Gamma(\beta)}\right)\epsilon_2 \\
&+\left[\frac{1}{|\Delta|}L_{f_1}\left(\frac{\Delta Q^{\alpha}}{\Gamma(\alpha+1)}+\widehat{\gamma_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\gamma_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\gamma_3}Q^{\alpha-1}}{\Gamma(\alpha)}\right)\right. \\
&\left.+\frac{1}{|\Delta|}L_{f2}\left(\frac{\widehat{\gamma_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\gamma_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\gamma_4}Q^{\beta-1}}{\Gamma(\beta)}\right)+L_{g1}\right] \\
&\times\left\|(\varphi,\psi)-\left(\varphi^*,\psi^*\right)\right\|_{c\times c}. \qquad (35)
\end{aligned}$$

and similarly, by (29) and (34) we get

$$\begin{aligned}
&\left\|\psi(t)-\psi^*(t)\right\| \\
&\leq \left\|\Psi_2(\varphi,\psi)(t)-\Psi_2\left(\varphi^*,\psi^*\right)(t)\right\|+\left\|\Psi_2\left(\varphi^*,\psi^*\right)(t)-\psi^*(t)\right\| \\
&\leq \frac{1}{|\Delta|}\left(\widehat{\kappa_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\kappa_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\kappa_3}Q^{\alpha-1}}{\Gamma(\alpha)}\right)\epsilon_1 \\
&+\frac{1}{|\Delta|}\left(\frac{|\Delta|\,Q^{\beta}}{\Gamma(\beta+1)}+\frac{\widehat{\kappa_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\kappa_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\kappa_4}Q^{\beta-1}}{\Gamma(\beta)}\right)\epsilon_2 \\
&\left[\frac{1}{|\Delta|}L_{f_1}\left(\widehat{\kappa_1}\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\widehat{\kappa_2}Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\widehat{\kappa_3}Q^{\alpha-1}}{\Gamma(\alpha)}\right)+L_{g2}\right. \\
&\left.+\frac{1}{|\Delta|}L_{f2}\left(\frac{\Delta Q^{\beta}}{\Gamma(\beta+1)}+\frac{\widehat{\kappa_1}Q^{\beta}}{\Gamma(\beta+1)}+\widehat{\kappa_2}\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\widehat{\kappa_4}Q^{\beta-1}}{\Gamma(\beta)}\right)\right] \\
&\times\left(\left\|\varphi-\varphi^*\right\|_{\infty}+\left\|\psi-\psi^*\right\|_{\infty}\right). \qquad (36)
\end{aligned}$$

From (35) and (36), we conclude that

$$\begin{aligned}
&\left\|(\varphi,\psi)-(\varphi^{*},\psi^{*})\right\|_{c\times c}\\
&\leq \frac{1}{1-\rho}\frac{1}{|\Delta|}\left(\left(\widehat{\gamma_1}+\widehat{\kappa_1}\right)\left(\frac{|r_2|}{\Gamma(\alpha+1)}+\frac{|r_1|}{\Gamma(\alpha+2)}\right)Q^{\alpha+1}+\frac{\left(\widehat{\gamma_2}+\widehat{\kappa_2}\right)Q^{\alpha}}{\Gamma(\alpha+1)}+\frac{\left(\widehat{\gamma_3}+\widehat{\kappa_3}\right)Q^{\alpha-1}}{\Gamma(\alpha)}+\frac{|\Delta|\,Q^{\alpha}}{\Gamma(\alpha+1)}\right)\epsilon_1\\
&+\frac{1}{1-\rho}\frac{1}{|\Delta|}\left(\left(\widehat{\gamma_2}+\widehat{\kappa_2}\right)\left(\frac{|p_2|}{\Gamma(\beta+1)}+\frac{|p_1|}{\Gamma(\beta+2)}\right)Q^{\beta+1}+\frac{\left(\widehat{\gamma_1}+\widehat{\kappa_1}\right)Q^{\beta}}{\Gamma(\beta+1)}+\frac{\left(\widehat{\gamma_4}+\widehat{\kappa_4}\right)Q^{\beta-1}}{\Gamma(\beta)}+\frac{|\Delta|\,Q^{\beta}}{\Gamma(\beta+1)}\right)\epsilon_2.
\end{aligned}$$

Consequently, the coupled system (1) with conditions (2) is Ulam-Hyers stable. □

Example 3.10. *Let*

$$E=l^{1}(\mathbb{R})=\left\{\varphi=\begin{pmatrix}\varphi_0\\ \varphi_1\\ \vdots\\ \varphi_n\\ \vdots\end{pmatrix},\ \varphi_i\in\mathbb{R},\ i\in\mathbb{N}^{*},\ \sum_{i=0}^{\infty}\left|\varphi_i\right|<\infty\right\},$$

be the Banach space of absolutely convergent real sequences, endowed with the norm

$$\left\|\varphi\right\|=\sum_{i=0}^{\infty}\left|\varphi_i\right|,\ \varphi\in E.$$

Consider the following coupled system of nonlinear Caputo fractional differential equations

$$\begin{cases} {}^{c}D^{\alpha}\left(\varphi(t)-g_1\left(t,\varphi(t),\psi(t)\right)\right)=f_1\left(t,\varphi(t),\psi(t)\right),\ t\in[0,Q],\\ {}^{c}D^{\beta}\left(\psi(t)-g_2\left(t,\varphi(t),\psi(t)\right)\right)=f_2\left(t,\varphi(t),\psi(t)\right),\ t\in[0,Q],\end{cases}\tag{37}$$

with the following integral boundary conditions

$$\begin{cases} \varphi'(0)=0,\ \varphi(0)+\varphi'(Q)=0,\ \varphi(2)=1.3\int_0^{0.8}\psi(s)\,ds+1.2Q\psi(0.8),\\ \psi'(0)=0,\ \psi(0)+\psi'(Q)=0,\ \psi(2)=1.2\int_0^{0.8}\varphi(s)\,ds+1.1Q\varphi(0.8),\end{cases}\tag{38}$$

where $Q=2$, $\alpha=2.8$, $\beta=2.9$,

$$\begin{aligned}
f_1(t,\varphi,\psi)&=\left\{\frac{6}{100\left(\cos\left(\frac{t}{2}\right)+1\right)}\left|\varphi_n\right|+\frac{1.5}{t+50}\left|\psi_n\right|+\frac{t+1}{n^{6}}\right\}_{n\in\mathbb{N}},\\
f_2(t,\varphi,\psi)&=\left\{\frac{0.225t^{2}}{30}\left|\varphi_n\right|+\frac{0.6t}{40}\left|\psi_n\right|+\frac{t^{2}\sin(t)}{n^{2}}\right\}_{n\in\mathbb{N}},\\
g_1(t,\varphi,\psi)&=\left\{\frac{\sin(2t)+1}{40}\left|\varphi_n\right|+0.05\left|\psi_n\right|+\frac{\cos\left(\frac{t}{2}\right)}{3n^{4}}\right\}_{n\in\mathbb{N}},\\
g_2(t,\varphi,\psi)&=\left\{0.05\left|\varphi_n\right|+\frac{5}{t+100}\left|\psi_n\right|+\frac{5\sqrt{t}}{2n^{7}}\right\}_{n\in\mathbb{N}}.
\end{aligned}$$

For all $(\varphi,\psi),\left(\tilde{\varphi},\tilde{\psi}\right)\in E\times E$, *we have*

$$\begin{aligned}
\left\|f_1(t,\varphi,\psi)-f_1\left(t,\tilde{\varphi},\tilde{\psi}\right)\right\|&\leq\frac{6}{100\left(\cos\left(\frac{t}{2}\right)+1\right)}\left\|\varphi-\tilde{\varphi}\right\|+\frac{1.5}{t+50}\left\|\psi-\tilde{\psi}\right\|\\
&\leq 0.03\left(\left\|\varphi-\tilde{\varphi}\right\|+\left\|\psi-\tilde{\psi}\right\|\right),
\end{aligned}$$

$$\begin{aligned}\left\|f_2(t,\varphi,\psi)-f_2\left(t,\tilde{\varphi},\tilde{\psi}\right)\right\| &\leq \frac{0.225t^2}{30}\left\|\varphi-\tilde{\varphi}\right\|+\frac{0.6t}{40}\left\|\psi-\tilde{\psi}\right\| \\ &\leq 0.03\left(\left\|\varphi-\tilde{\varphi}\right\|+\left\|\psi-\tilde{\psi}\right\|\right),\end{aligned}$$

$$\begin{aligned}\left\|g_1(t,\varphi,\psi)-g_1\left(t,\tilde{\varphi},\tilde{\psi}\right)\right\| &\leq \frac{\sin(2t)+1}{40}\left\|\varphi-\tilde{\varphi}\right\|+0.05\left\|\psi-\tilde{\psi}\right\| \\ &\leq 0.05\left(\left\|\varphi-\tilde{\varphi}\right\|+\left\|\psi-\tilde{\psi}\right\|\right),\end{aligned}$$

and

$$\begin{aligned}\left\|g_2(t,\varphi,\psi)-g_2\left(t,\tilde{\varphi},\tilde{\psi}\right)\right\| &\leq 0.05\left\|\varphi-\tilde{\varphi}\right\|+\frac{5}{t+100}\left\|\psi-\tilde{\psi}\right\| \\ &\leq 0.05\left(\left\|\varphi-\tilde{\varphi}\right\|+\left\|\psi-\tilde{\psi}\right\|\right),\end{aligned}$$

then $L_{f_1}=L_{f_2}=0.03$ *and* $L_{g_1}=L_{g_2}=0.05$.
By computing the associated inequality

$$\begin{aligned}\rho &= \frac{1}{132.3499}0.03\left((48.00853+192.0341)\left(\frac{1.1}{\Gamma(3.8)}+\frac{1.2}{\Gamma(4.8)}\right)2^{3.8}\right. \\ &\left.+\frac{(11.0272+44.1088)}{\Gamma(3.8)}2^{2.8}+\frac{(140.6798+44.1088)}{\Gamma(2.8)}2^{1.8}\right) \\ &+\frac{1}{132.3499}0.03\left(\frac{(48.00853+192.0341)}{\Gamma(3.9)}2^{2.9}+(11.0272+44.1088)\left(\frac{1.2}{\Gamma(3.9)}+\frac{1.3}{\Gamma(4.9)}\right)2^{3.9}\right. \\ &\left.+\frac{(37.93357+172.6498)}{\Gamma(2.9)}2^{1.9}\right)+\frac{0.03}{\Gamma(3.8)}2^{2.8}+\frac{0.03}{\Gamma(3.9)}2^{2.9}+0.05+0.05 \\ &\simeq 0.749<1,\end{aligned}$$

we conclude that all the assumptions required by Theorems 3.5, 3.7 and 3.9 are fulfilled. Therefore, the coupled system (37) with conditions (38) possesses a unique mild solution $(\varphi,\psi)\in C(I,E)\times C(I,E)$ *that continuously depends on the given functions* f_1, f_2, g_1 *and* g_2, *and satisfies the Ulam-Hyers stability.*

4. Conclusion

In this study, we investigated a nonlinear coupled system of Caputo fractional differential equations with integral boundary conditions in the setting of Banach spaces. By employing the Kuratowski measure of noncompactness in conjunction with the Generalized Darbo's theorem involving a nondecreasing control function, we established the existence of mild solutions. The uniqueness of these solutions was obtained through the Banach contraction principle. Moreover, we demonstrated that the solutions depend continuously on the given data, ensuring the model's stability under small perturbations. The Ulam-Hyers stability of the system was also proven, confirming the robustness of the solutions. To illustrate the theoretical results, a concrete example was provided. Overall, this study contributes to the growing body of research on fractional differential systems and offers a foundation for future investigations involving more generalized boundary conditions or applications in the applied sciences.

5. Declarations

Conflict of interest. The authors declare that they have no conflict of interest.
Availability of data. No data were used.
Funding. No financial support was received for this work.
Author Contributions. All authors contributed equally in this work.

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