φ-normal criteria for meromorphic functions and its applications

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Abstract. In 2011, Aulaskari and Rättyä (J. Michigan. Math. Vol. 60) introduced the concept of φ-normal meromorphic functions on the unit disc D, where the function $\varphi(r) : [0, 1) \to (0, \infty)$ admits a sufficient regularity near 1 and exceeds $\frac{1}{1-r^2}$ in growth. They examined the class of meromorphic functions f on D satisfying $f^{\#}(z) = O(\varphi(|z|))$, as $|z| \to 1^-$. In this paper, using a result of Pang and Zalcman (Bull. London. Math. Soc. **32**(2000), 325-331), we prove some Zalcman's type result for φ-normal function. Using that results and Nevanlinna theory, we give some φ-normal criteria for meromorphic functions. In our best knowledge, they are first criteria for φ-normal functions sharing sets. Finally, we investigate the φ-normal function which is a solution of algebraic differential equations. Theorem 1.9 shows that the solution is φ-normal without any comparision between nk and the degree of $P[w](z)$ as previous results.

1. Introduction

An increasing function $\varphi : [0, 1) \to (0, \infty)$ is called *smoothly increasing* if

$$\varphi(r)(1-r) \to \infty \quad \text{as} \quad r \to 1^-, \tag{1.1}$$

and

$$\mathcal{R}_a(z) := \frac{\varphi(|a + z/\varphi(|a|)|)}{\varphi(|a|)} \to 1, \quad \text{as} \quad |a| \to 1^- \tag{1.2}$$

uniformly on compact subsets of $\mathbb{C}$.

In 2011, Aulaskari and Rättyä [1] introduced the concept of φ-normal meromorphic functions as follows: For a smoothly increasing function φ, we say that the meromorphic function f in the unit disc D is φ-normal if

$$\|f\|_{N^\varphi} := \sup_{z \in D} \frac{f^{\#}(z)}{\varphi(|z|)} < \infty, \tag{1.3}$$

where $f^{\#}(z) := \frac{|f'(z)|}{(1+|f(z)|^2)}$ is the spherical derivative of f.

Denote by $\mathcal{N}^\varphi$ the set of all φ-normal meromorphic functions on D. For each smooth increasing function φ, set $\varphi^*(r) := \varphi(r) + (1-r)^{-1}$, then φ^* is also smoothly increasing and $\mathcal{N}^{\varphi^*} = \mathcal{N}^\varphi$. Furthermore, the function φ^* satisfies the condition: $\varphi^*(r)(1-r) \geq 1$ for all $r \in [0,1)$. Therefore, in [1], the authors considered that the smoothly increasing φ satisfies the added condition $\varphi(r)(1-r) \geq 1$ for all $r \in [0,1)$. We refer readers to [1] for comments on the concept of φ-normal meromorphic functions.

In 2017, Tan and Thin [10] considered a family of functions φ : in the above definition of smoothly increasing functions mentioned above, condition (1.1) is replaced by the following condition

$$\varphi(r)(1-r) \geq 1 \text{ for all } r \in [0,1). \tag{1.4}$$

It means that the class of function φ is larger than the class of smoothly increasing due to Aulaskari and Rättyä [1]. If we take $\varphi_0 = \frac{1}{1-r}$, then the concept φ_0-normal functions coincides with the concept of normal functions (note that φ_0 does not satisfy condition (1.1), but it satisfies conditions (1.4) and (1.2)). They also proved a Zalcman's type result for φ-normal meromorphic functions and gave some Lappan's type theorem with 3 and 4 points. After that, many authors extended that results in other situations. I recommend the readers to references [7, 11, 12] to see more details on this topic.

The first main suppose of this paper is to establish the types of Lohwater-Pommerenke-Zalcman's criterion [6, 13] for φ-normal functions.

Theorem 1.1. *Let $\varphi : [0,1) \to (0,\infty)$ be a smoothly increasing function, and let f be a meromorphic function on the unit disc D. Assume that all zeros and all poles of f have multiplicity at least p, q, respectively. Let β be a real number satisfying $-p < \beta < q$. If f is not φ-normal then, there exist*
(i) a sequence $\{a_n\} \subset D$, $|a_n| \to 1$;
(ii) a sequence $\{z_n\} \subset D$ with $z_n \to z_* \in D$ and $w_n := a_n + \dfrac{z_n}{\varphi(|a_n|)} \in D$;
(iii) a sequence $\{\rho_n\}$, $\rho_n \to 0^+$
such that $g_n(\xi) := \rho_n^\beta f(w_n + \dfrac{\rho_n}{\varphi(|a_n|)}\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros and poles have multiplicity at least p, q respectively.

We note that in the case where $\beta = 0$, the above theorem is similar to Theorem 6 in [1].

Theorem 1.2. *Let $\varphi : [0,1) \to (0,\infty)$ be a smoothly increasing function, and let f be a meromorphic function on the unit disc D which has all zeros with multiplicity at least k. Assume that there exists a constant $A \geq 1$ such that $|f^{(k)}(z)| \leq A\varphi^k(|z|)$ whenever $f(z) = 0$. If f is not φ-normal, then for each $0 \leq \alpha \leq k$, there exist,*
(i) a sequence $\{a_n\} \subset D$, $|a_n| \to 1$;
(ii) a number $0 < r < 1$;
(iii) points $z_n, |z_n| < r$ satisfying $w_n := a_n + \dfrac{z_n}{\varphi(|a_n|)} \in D$;
(iv) a sequence $\{\rho_n\}$, $\rho_n \to 0^+$
such that $g_n(\xi) = \dfrac{f(w_n + \dfrac{\rho_n}{\varphi(|a_n|)}\xi)}{\rho_n^\alpha} \to g(\xi)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, $g^\#(\xi) \leq g^\#(0) = \dfrac{3}{2}kA + 1$.

Remark 1.3. *Since $\varphi(|z|) \geq 1$, for all $z \in D$, the above theorem remains valid if the condition $|f^{(k)}(z)| \leq A\varphi^k(|z|)$ whenever $f(z) = 0$ is replaced by the condition $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$.*

By using the above types of Lohwater-Pommerenke-Zalcman's lemma, now we establish some φ-normal criteria.

Theorem 1.4. *Let f be a meromorphic function on the unit disc D, all of whose zeros and poles are multiplicity at least m and n respectively, where m, n are positive integers satisfying $\dfrac{1}{m} + \dfrac{1}{n} < 1$. Let α_1, α_2 be two distinct nonzero complex numbers and the set $S = \{\alpha_1, \alpha_2\}$, and let M be positive number. If $|f'(z)| \leq M\varphi(|z|)$ whenever $f(z) \in S$, then f is a φ-normal function.*

Take $\varphi := \frac{1}{1-|z|}$, and $M := \max\{|\alpha_1|, |\alpha_2|\}$, we get the following corollary of Theorem 1.4.

Corollary 1.5. *Let f be meromorphic function on the unit disc D, all of whose zeros and poles are multiplicity at least m and n respectively, where m, n are positive integers satisfying $\frac{1}{m} + \frac{1}{n} < 1$. Let α_1, α_2 be two distinct nonzero complex numbers and the set $S = \{\alpha_1, \alpha_2\}$. If $f'(z)$ and $f(z)$ share S-IM (in the sense of that $\{z : f(z) \in S\} = \{z : f'(z) \in S\}$), then f is a normal function.*

Theorem 1.6. *Let f be an entire function in a domain D, all of whose zeros are multiplicity at least 3. let α be a nonzero complex number and M be positive number. If $|f'(z)| \leq M\varphi(|z|)$ whenever $f(z) = \alpha$, then f is a φ-normal function.*

Take $\varphi := \frac{1}{1-|z|}$, and $M := |\alpha|$, we get the following corollary of Theorem 1.6.

Corollary 1.7. *Let f be an entire function on the unit disc D, all of whose zeros have multiplicity at least 3, let α be a nonzero complex number. If $f'(z)$ and $f(z)$ share α-IM (in the sense of that $f(z) = \alpha$ iff $f'(z) = \alpha$), then f is a normal function.*

Thanks for the reviewer's comments. We think that Corollary 1.7 can be extended for the case $f^{(k)}(z)$ ($k \geq 2$) and $f(z)$ share α-IM and adding an assumption: $f(z) = \alpha$ implies that $|f^{(j)}(z) \leq M\varphi^j(z)$, $j = 1, \ldots, k-1$, for some $M > 0$. This result is left to the reader. It is very interesting to consider the extension of Corollary 1.7 if we only assume that $f^{(k)}(z)$ ($k \geq 2$) and $f(z)$ share α-IM. We will consider this question in the future.

Next, we study the solution of a differential algebraic equation. We consider the following differential algebraic equation on the unit disc D.

$$\left(\frac{d^k w}{dz^k}\right)^n = \sum_{j=1}^{m} P_j(z, w) D_j[w]. \tag{1.5}$$

Here, k, n are positive integers; $P_j(z, w) := \sum_{i=0}^{m_j} a_{ij}(z) w^i$, where $a_{ij}(z)$ are holomorphic functions, m_j is a positive integer for all $j = 1, \ldots, m$; and $D_j[w]$ is a differential monomial in w of the form

$$D_j[w] = \left(\frac{dw}{dz}\right)^{j_1} \left(\frac{d^2 w}{dz^2}\right)^{j_2} \ldots \left(\frac{d^l w}{dz^l}\right)^{j_l}, \qquad j_1, \ldots, j_l \in \mathbb{N} \cup \{0\}.$$

We define the weight of D_j by $v_{D_j} := j_1 + 2j_2 + \cdots + lj_l$ and the weight of $P[w](z) := \sum_{j=1}^{m} P_j(z, w) D_j[w]$ by $v(P) := \max_{j=1,\ldots,m}\{v_{D_j}\}$.

Theorem 1.8. *[9] Asumme that $nk > v(P)$ and f is a meromorphic solution of (1.5) such that all zeros of f having multiplicity at least k. If the coefficients $\{a_{ij}\}$ satisfy the condition*

$$\lim_{|z| \to 1} \left(\frac{1}{\varphi(|z|)}\right)^{nk-v(P)} \max_{1 \leq j \leq m} \sum_{i=0}^{m_j} |a_{ij}(z)| < +\infty, \tag{1.6}$$

then f is $\varphi-$ normal.

We note that for the case where $k = 1$ the above theorem is similar to the one obtained by Aulaskari and Wulan [6] for strongly normal function. In the following theorem, we examine equation (1.5) in the case where kn can be smaller than $v(P)$. Namely, we prove that any solution of (1.5) under some suitable conditions is φ-normal without any comparision on $v(P)$ and nk.

Set $\Gamma_{D_j} = j_1 + \cdots + j_l$. We assume that $1 \leq l \leq k$, then $k\Gamma_{D_j} \geq v_{D_j}, j = 1, \ldots, m$. Consider the **Condition** C on $P[w](z)$ as follows:

Condition C. $a_{0j} = 0$ for all $j = 1, \ldots, m$, $1 \leq l \leq k$, $\Gamma_{D_j} \geq 1$.

Theorem 1.9. *Assume that the* **Condition C** *holds and*

$$\lim_{|z|\to 1}(\varphi(|z|))^{\nu_{D_j}-nk} \max_{1\le j\le m} \sum_{i=1}^{m_j} |a_{ij}(z)| < +\infty, \ j = 1, \ldots, m. \tag{1.7}$$

If f is a meromorphic solution of (1.5) such that of zeros of f having multiple at least k, then f is a φ-normal.

We note that Theorem 1.6, Theorem 1.8 and Theorem 1.9 are independent results. Finally, we give the following estimate for the order of φ-normal functions.

Theorem 1.10. *Let f be a φ-normal meromorphic function. Then, the order $\rho(f)$ of f satisfies*

$$\rho(f) := \limsup_{r\to 1^-} \frac{\log T_0(r, f)}{-\log(1-r)} \le 2\chi,$$

where χ is the order of φ, defined by

$$\chi = \limsup_{r\to 1^-} \frac{\log \varphi(r)}{-\log(1-r)},$$

and $T_0(r, f)$ is the Ahlfors-Shimizu characteristic function of f.

2. Some Lemmas

In order to prove our theorems, we need the following lemmas.

Lemma 2.1 (Zalcman's Lemma, see [13]). *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc D whose all zeros and poles have multiplicity at least p, q, respectively. Let α be a real number satisfying $-p < \alpha < q$. Then, $\mathcal{F}$ is not normal at z_0 if and only if there exist*
(i) a number $r, 0 < r < 1$;
(ii) points z_n with $|z_n| < r, z_n \to z_0$;
(iii) functions $f_n \in \mathcal{F}$;
(iv) positive numbers $\rho_n \to 0^+$;
such that $g_n(\xi) = \rho_n^\alpha f_n(z_n + \rho_n \xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros and poles have multiplicity at least p, q respectively. Moreover, g has order at most 2.

Lemma 2.2 (see [10]). *Let $\varphi : [0, 1) \to (0, \infty)$ be a smoothly increasing function, and let f is a meromorphic function on D. Then $f \in \mathcal{N}^\varphi$ if and only if the family $\{f_a(z) := f(a + \frac{z}{\varphi(|a|)})\}_{a\in D}$ is normal in D.*

Lemma 2.3 (see [8]). *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \ge 1$ such that $|f^{(k)}(z)| \le A$ whenever $f(z) = 0$. If $\mathcal{F}$ is not normal, then there exist, for each $0 \le \alpha \le k$,*
(i) a number $r, 0 < r < 1$,
(ii) points $z_n, |z_n| < r$,
(ii) functions $f_n \in \mathcal{F}$, and
(iv) the sequence $\{\rho_n\} \to 0^+$
such that $g_n(\xi) = \dfrac{f(z_n + \rho_n \xi)}{\rho_n^\alpha} \to g(\xi)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$ such that $g^\#(\xi) \le g^\#(0) = kA + 1$.

3. Proof of Theorems

Proof. [Proof of Theorem 1.1] Suppose that f is not φ-normal. Then by Lemma 2.2, the family $\mathcal{F} = \{f_a\}_{a \in D}$ is not normal (see the definition of functions $\{f_a\}$ in Lemma 2.2). Hence, there exists the sequence $a_m \in D$ such that

$$\mathcal{F}_{a_m} = \left\{ f_{a_m} : f_{a_m}(z) = f(a_m + \frac{z}{\varphi(|a_m|)}) \right\}$$

is not normal at some point z_*. It is clear that there exists a subsequence, denote again $\{a_j\}$ such that $|a_j| \to 1^-$. By Lemma 2.1, for all $-p < \beta < q$, there exist
(i) points $\{z_n\} \subset D$ with, $z_n \to z_*$;
(ii) functions $f_{a_{t_n}} \in \mathcal{F}_{a_m}$;
(iii) positive numbers $\rho_n \to 0^+$;
such that $g_n(\xi) = \rho_n^\beta f_{a_{t_n}}(z_n + \rho_n \xi)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros and poles have multiplicity at least p, q respectively.

We have $f_{a_{t_n}}(z_n + \rho_n \xi) = f(a_{t_n} + \frac{z_n + \rho_n \xi}{\varphi(|a_{t_n}|)})$. Take $w_n = a_{t_n} + \frac{z_n}{\varphi(|a_{t_n}|)}$, we get the conclusion of Theorem 1.1. $\square$

Proof. [Proof of Theorem 1.2] Suppose that f is not φ-normal. Then by Lemma 2.2, we have that the family $\{f(a + \frac{z}{\varphi(|a|)}) : a \in D\}$ is not normal in D. Thus, there exists the sequence $a_m \to 1^-$ such that

$$\mathcal{F}_{a_m} = \left\{ f_{a_m} : f_{a_m}(z) = f(a_m + \frac{z}{\varphi(|a_m|)}) \right\}$$

is not normal in D. Then, without loss of the generality, we may assume that $|a_m| \to 1^-$. Since

$$\lim_{m \to \infty} \frac{\varphi^k(|a_m + \frac{z}{\varphi(|a_m|)}|)}{\varphi^k(|a_m|)} = 1,$$

uniformly on compact subsets of $\mathbb{C}$, there exists a positive integer M such that

$$\frac{\varphi^k(|a_m + \frac{z}{\varphi(|a_m|)}|)}{\varphi^k(|a_m|)} \le \frac{3}{2} \tag{3.1}$$

for all $z \in D$, and $m \ge M$. For any $m \ge M$ and for any $z_* \in D$ satisfying $f_{a_m}(z_*) = 0$, we have also $f(a_m + \frac{z_*}{\varphi(|a_m|)}) = 0$. Therefore, by the assumption, we have

$$|f^{(k)}(a_m + \frac{z_*}{\varphi(|a_m|)})| \le A \varphi^k(|a_m + \frac{z_*}{\varphi(|a_m|)}|).$$

Then, by (3.1), we have

$$|f_{a_m}^{(k)}(z_*)| = \frac{1}{\varphi^k(|a_m|)} |f^{(k)}(a_m + \frac{z_*}{\varphi(|a_m|)})| \le A \frac{\varphi^k(|a_m + \frac{z_*}{\varphi(|a_m|)}|)}{\varphi^k(|a_m|)} \le \frac{3}{2} A.$$

Then, by Lemma 2.3 (for the family $\{f_{a_m}\}_{m \ge M}$), for all $0 \le \alpha \le k$, there exist:
(i) a number $0 < r < 1$,
(ii) points $z_n, |z_n| < r$,
(iii) a sub-sequence, denote again by $\{f_{a_n}\}$, and
(iv) a sequence $\rho_n \to 0^+$

such that $g_n(\xi) = \dfrac{f_{a_n}(z_n + \rho_n\xi)}{\rho_n^\alpha} \to g(\xi)$ locally uniformly with respect to the spherical metric, where g is a

nonconstant meromorphic function on $\mathbb{C}$ such that $g^\#(\xi) \leq g^\#(0) = \dfrac{3}{2}kA + 1$.

Since $f_{a_n}(z_n + \rho_n\xi) = f(a_n + \dfrac{z_n + \rho_n\xi}{\varphi(|a_n|)})$, by taking $w_n = a_n + \dfrac{z_n}{\varphi(|a_n|)}$, we complete the proof of Theorem 1.2. $\square$

Proof. [Proof of Theorem 1.4]

Suppose that f is not $\varphi-$ normal. Then, by Theorem 1.1 (with $\beta = 0$), there exist:

(*i*) a sequence $\{a_n\} \subset D$, $|a_n| \to 1$;

(*ii*) a sequence $\{z_n\} \subset D$: $z_n \to z_* \in D$, and $w_n = a_n + \dfrac{z_n}{\varphi(|a_n|)}$;

(*iii*) the sequence $\rho_n \to 0^+$

such that $g_n(\xi) = f(w_n + \dfrac{\rho_n}{\varphi(|a_n|)}\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where g is

a nonconstant meromorphic function on $\mathbb{C}$, all of zeros and poles of g have multiplicity at least m and n, respectively.

Claim 1: *All zeros of $g - \alpha_i$ ($i = 1, 2$) are multiple.*

Indeed, for any zero point ξ_0 of $g - \alpha_i$ (for some $i \in \{1, 2\}$), by Hurwitz's theorem, there exists a sequence $\{\xi_n\} \to \xi_0$ such that

$$f(w_n + \dfrac{\rho_n}{\varphi(|a_n|)}\xi_n) = g_n(\xi_n) = \alpha_i.$$

Then, by the assumption, we have

$$\left| f'(w_n + \dfrac{\rho_n}{\varphi(|a_n|)}\xi_n) \right| \leq M\varphi(|w_n + \dfrac{\rho_n}{\varphi(|a_n|)}\xi_n|).$$

Hence,

$$|g_n'(\xi_n)| = \dfrac{\rho_n}{\varphi(|a_n|)}\left| f'(w_n + \dfrac{\rho_n}{\varphi(|a_n|)}\xi_n) \right| \leq M\rho_n \dfrac{\varphi(|a_n + \dfrac{z_n + \rho_n\xi_n}{\varphi(|a_n|)}|)}{\varphi(|a_n|)}. \tag{3.2}$$

From (3.2) and (1.2), we get

$$g'(\xi_0) = \lim_{n\to\infty} g_n'(\xi_n) = 0.$$

On the other hand, since g is non-constant and all zeros of g have multiplicity at least $m(\geq 2)$, we get that $g'(\xi) \not\equiv 0$. Hence, ξ_0 is a multiple zero of $g - \alpha_i$. Then, we get Claim 1.

For each meromorphic function u, denote by $T_u(r)$ the Nevanlinna characteristic function of u (in the disc $\{z : |z| < r\}$), and denote by $N(r, \dfrac{1}{u})$ the counting funtion (counted multiplicity) of zeros (in the disc $\{z : |z| < r\}$) of u, (and by $\overline{N}(r, \dfrac{1}{u})$) for the case of regardless multiplicity). For more details on Nevanlinna theory for meromorphic functions, we recommend the readers to the books [4, 5].

By the First and the Second Main Theorems (in Nevanlinna theory), we have

$$2T_g(r) \leq \overline{N}(r, g) + \overline{N}(r, \dfrac{1}{g}) + \overline{N}(r, \dfrac{1}{g - \alpha_1}) + \overline{N}(r, \dfrac{1}{g - \alpha_2}) + o(T_g(r))$$

$$\leq \dfrac{1}{n}N(r, g) + \dfrac{1}{m}N(r, \dfrac{1}{g}) + \dfrac{1}{2}N(r, \dfrac{1}{g - \alpha_1}) + \dfrac{1}{2}N(r, \dfrac{1}{f - \alpha_2}) + o(T_g(r))$$

$$\leq (\dfrac{1}{m} + \dfrac{1}{n} + 1)T_g(r) + o(T_g(r)). \tag{3.3}$$

This contradicts to the condition $\dfrac{1}{m} + \dfrac{1}{n} < 1$. We have completed the proof of Theorem 1.4. $\square$

Proof. [Proof of Theorem 1.6] We can get the proof of Theorem 1.6 by an argument similar to the proof of Theorem 1.4 with the following remark:

• Claim 1 is replaced by the claim: all zeros of $g - \alpha$ are multiple.

• Inequality (3.3) is replaced by the following estimate:

$$T_g(r) \le \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{g-\alpha}) + o(T_g(r)) \le \frac{1}{3}N(r, \frac{1}{g}) + \frac{1}{2}N(r, \frac{1}{g-\alpha}) + o(T_g(r))$$

$$\le \frac{5}{6}T_g(r) + o(T_g(r)).$$

This is a contradiction. □

Proof. [Proof of Theorem 1.9] It is clear that $\sum_{j=1}^{m} P_j(z, f)D_j[f] = 0$ on $\{z : f(z) = 0\}$ (note that $a_{0j} \equiv 0$ for $j = 1, \ldots, m$). Thus for a solution f of (1.5), we have $|f^{(k)}(z)| = 0 < 1$ for all z such that $f(z) = 0$. Hence, if f is not φ-normal. Then, by Theorem 1.2 (for $\alpha = k$) and Remark 1.3, there exist
(i) a sequence $\{a_v\} \subset D, |a_v| \to 1$;
(ii) a number $0 < r < 1$;
(iii) points $z_v, |z_v| < r$, and $w_v = a_v + \frac{z_v}{\varphi(|a_v|)}$;
(iv) a sequence $\rho_v \to 0^+$

such that $g_v(\xi) = \dfrac{f(w_v + \frac{\rho_v}{\varphi(|a_v|)}\xi)}{\rho_v^k} \to g(\xi)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k.

Since all zeros of f have multiplicity at least k, and by Hurwitz's theorem, all zeros of g also have multiplicity at least k. Hence, there exist $\xi_0 \in \mathbb{C}$ such that

$$g^{(k)}(\xi_0) \neq 0. \tag{3.4}$$

We have

$$\rho_v^{k-j}\varphi^j(|a_v|)g_v^{(j)}(\xi) = f^{(j)}(w_v + \frac{\rho_v}{\varphi(|a_v|)}\xi),$$

for all $j \in \mathbb{N}^*$. Therefore, since f is a solution of (1.5), we have

$$|\varphi(|a_v|)^{nk}(g_v^{(k)})^n(\xi_0)| \le \sum_{j=1}^{m} \left| P_j(w_v + \frac{\rho_v}{\varphi(|a_v|)}\xi_0, \rho_v^k g_v(\xi_0)) \right| |D_j[g_v](\xi_0)|\varphi(|a_v|)^{v_{D_j}}\rho_v^{k\Gamma_{D_j}-v_{D_j}}.$$

Thus, there exist a constant $M > 0$ such that

$$|(g_v^{(k)})^n(\xi_0)| \le M \sum_{j=1}^{m} \rho_v^{1+k\Gamma_{D_j}-v_{D_j}}(\varphi(|a_v|))^{v_{D_j}-nk} \cdot \max_{1 \le j \le m} \sum_{i=1}^{m_j} \left| a_{ij}(w_v + \frac{\rho_v}{\varphi(|a_v|)}\xi_0) \right|. \tag{3.5}$$

This implies

$$|(g_v^{(k)})^n(\xi_0)| \le M \sum_{j=1}^{m} \rho_v^{1+k\Gamma_{D_j}-v_{D_j}} \left(\frac{\varphi(|a_n|)}{\varphi\left(\left|a_n + \frac{z_n+\rho_n\xi_0}{\varphi(|a_n|)}\right|\right)} \right)^{v_{D_j}-nk} \tag{3.6}$$

$$\cdot \max_{1 \le j \le m} \sum_{i=1}^{m_j} \left| a_{ij}(w_n + \frac{\rho_n}{\varphi(|a_n|)}\xi_0) \right| \left(\varphi\left(\left|w_n + \frac{\rho_n}{\varphi(|a_n|)}\xi_0\right|\right) \right)^{v_{D_j}-nk}.$$

On the other hand, by the definition of φ (and note that $z_v + \rho_v \xi_0 \in D$ for $v >> 0$), we have

$$\lim_{v \to \infty} \frac{\varphi(|a_v|)}{\varphi\left(\left|a_v + \dfrac{z_v + \rho_v \xi_0}{\varphi(|a_v|)}\right|\right)} = 1$$

without the any comparision on nk and v_{D_j}, $j = 1, \ldots, m$. Hence, by taking limits both sides in (3.6), we get $g^{(k)}(\xi_0) = \lim_{v \to \infty} g_v^{(k)}(\xi_0) = 0$, which contradicts to (3.4). We have completed the proof of Theorem 1.9. $\square$

Proof. [Proof of Theorem 1.10] Since f is φ-normal, there exist a constant $M > 0$ such that

$$f^{\#}(z) \leq M\varphi(|z|), \text{ for all } z \in D$$

Then, we have

$$S(t) = \frac{1}{\pi} \iint_{|z| \leq t} |f^{\#}(z)|^2 dx dy \leq \frac{M^2}{\pi} \iint_{|z| \leq t} \varphi^2(|z|) dx dy.$$

On the other hand, φ is a increasing function. Hence, we have

$$S(t) \leq \iint_{|z| \leq t} \varphi^2(t) dx dy = M^2 \varphi^2(t) t^2.$$

By the definition of the Ahlfors-Shimizu characteristic function, we have

$$T_0(r, f) = \int_0^r \frac{S(t)}{t} dt \leq M^2 \int_0^r \varphi^2(t) t dt = O\left(r^2 \varphi^2(r)\right).$$

Hence,

$$\rho(f) = \limsup_{r \to 1^-} \frac{\log T_0(r, f)}{-\log(1 - r)} \leq \limsup_{r \to 1^-} \frac{\log \varphi^2(r)}{-\log(1 - r)} = 2\chi.$$

$\square$

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