Filomat 40:8 (2026), 2871–2881
https://doi.org/10.2298/FIL2608871L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On p-statistical convergence in fuzzy metric spaces

Changqing Li[a,*], Yangli Hu[a]

[a]*School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, Fujian 363000, China*

Abstract. The idea of statistical convergence was independently presented by Fast and Steinhaus in 1951 and has now become an area of active research in the field of mathematics. Recently, it has been successfully used in the realm of fuzzy metrics, in the sense of George and Veeramani, by several authors. The appearance of a parameter t in the definition of fuzzy metric space allows for the introduction of new particular definitions in fuzzy metric context. In view of this, we introduce the concept of p-statistical convergence in fuzzy metric spaces and explore some useful results. Also, we give several equivalent characterizations those are relative to the concept of principal fuzzy metric space that Gregori et al. proposed in [16]. Moreover, we provide some illustrative examples.

1. Introduction

Several versions of fuzzy metric spaces have been studied in different ways [9, 15, 28, 30]. Specifically, Kramosil and Michalek [30] introduced the concept of KM-fuzzy metric space, which could be considered an extension of Menger space [35] in fuzzy setting. To make the topology induced by a fuzzy metric to be Hausdorff, George and Veeramani in [15] gave a slight modification of KM-fuzzy metric and obtained the concept of GV-fuzzy metric. In [25], Gregori and Romaguera proved that the topological spaces generated by GV-fuzzy metrics are metrizable. Afterwards, It was found that some well-known results in classical metric spaces are applicable to the realm of GV-fuzzy metric spaces.

In this paper we deal with the concept of GV-fuzzy metric. From this time on, for simplicity, fuzzy metric we refer to is GV-fuzzy metric.

It is widely known that the completeness of fuzzy metric spaces is very different from that of classical metric spaces. In fact, as it can be see in [26, 27] that there exist non-completable fuzzy metric spaces. In view of this, many authors became interesting to the completeness of fuzzy metric spaces and gave some contributions to this study in the literature (see [14, 20, 22–24, 33, 42]). Another significant distinction between a fuzzy metric and a classical metric is that the former one includes the parameter t in its definition.

2020 *Mathematics Subject Classification*. Primary 40A05; Secondary 54A40, 54D35, 54E50.

Keywords. Fuzzy metric, p-statistically convergent sequence, p-statistically Cauchy sequence, p-statistical completeness; statistical convergence.

Received: 16 August 2025; Revised: 27 December 2025; Accepted: 17 January 2025

Communicated by Ljubiša D. R. Kočinac

Research supported by Nation Natural Science Foundation of China(No. 11701258), Fujian Provincial Natural Science Foundation of China (No. 2024J01803, No. 2022J05168, No. 2020J01801), Fujian Key Laboratory of Granular Computing and Applications, the Institute of Meteorological Big Data-Digital Fujian and Fujian Key Laboratory of Data Science and Statistics.

* Corresponding author: Changqing Li

Email addresses: helen_smile0320@163.com (Changqing Li), 805429925@qq.com (Yangli Hu)

ORCID iDs: https://orcid.org/0009-0005-9475-3162 (Changqing Li), https://orcid.org/0009-0007-6326-6235 (Yangli Hu)

So some novel fuzzy metric concepts can be defined in fuzzy metric spaces, however those concepts are invalid in classical case. This fact has been successfully used in convergence in fuzzy metric spaces, as *p*-convergence [37], *s*-convergence [19], *st*-convergence [18], *std*-convergence [38] and *sts*-convergence [31]. In addition, we can find other more references devoted to this topic (see [5, 16, 17, 21, 32]).

In 1951, the concept of statistical convergence in real number space was initially introduced by Fast [10] and Steinhaus [41], respectively. Later, the concept attracted a great deal of attention, and some authors provided various applications in different fields of mathematics [1–4, 6, 7, 11–13, 29, 31, 34, 36, 39]. In particular, Li et al. in [31] first introduced the concept of the statistical convergence in fuzzy metric spaces and explored several aspects relative to the concept. Since the theory of fuzzy metric spaces is richer than that of classical case, it is a natural problem to modify the definition of statistical convergence in fuzzy metric spaces and get a more general concept with the help of the parameter t. Here we do it. After introduce the concept of *p*-statistically convergent sequence in fuzzy metric spaces, we will study *p*-statistical convergence, *p*-statistical Cauchyness, *p*-statistical completeness and *s*-*p*-statistical completeness. We show that statistical convergence (statistical Cauchyness, resp.) implies *p*-statistical convergence (*p*-statistical Cauchyness, resp.) in fuzzy metric spaces, but the converse is not true. At the same time we give an equivalent characterization that a fuzzy metric space is principle (*s*-*p*-statistically complete, resp.). Also, we provide a necessary and sufficient condition that a sequence is *p*-statistically convergent in a fuzzy metric space. In addition, we give several examples.

The paper is organized as follows. In Section 2 preliminaries are given. In Section 3 the concept of *p*-statistical convergence in fuzzy metric spaces is introduced, and also some aspects relative to this concept are investigated. In Section 4 *p*-statistical Cauchyness and *p*-statistical completeness in fuzzy metric spaces are studied. At the end, conclusions are provided in Section 5.

2. Preliminaries

From now on, $\mathbb{N}$ and $\mathbb{R}$ shall denote the set of all positive integer numbers and the set of all real numbers, respectively. Our basic reference for general topology is [8].

Definition 2.1. ([40]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if it satisfies the following conditions:

(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, and $a, b, c, d \in [0,1]$.

Notice that $a * b = a \cdot b$ and $a * b = \min\{a, b\}$ are two common examples of continuous t-norms. Clearly, for any $\varepsilon \in (0,1)$, there exist $\varepsilon_1, \varepsilon_2 \in (\varepsilon, 1)$ such that $\varepsilon_1 * \varepsilon_2 > \varepsilon$.

Definition 2.2. ([15]) A 3-tuple $(X, M, *)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t \in (0,\infty)$:

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;
(v) the function $M(x, y, \cdot) : (0,\infty) \to [0,1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, then we will call $(M, *)$, or simply M, a *fuzzy metric on* X.

Definition 2.3. ([15]) Let $(X, M, *)$ be a fuzzy metric space and let $\varepsilon \in (0,1)$, $t > 0$ and $x \in X$. The set

$$B_M(x, \varepsilon, t) = \{y \in X | M(x, y, t) > 1 - \varepsilon\}$$

is called *the open ball with center x and radius ε with respect to t*.

George and Veeramani [15] proved that $\{B_M(x, \varepsilon, t)|x \in X, t > 0, \varepsilon \in (0, 1)\}$ forms a base of a topology τ_M in X, and for every $x \in X$ the family $\{B_M(x, \frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at x.

Definition 2.4. ([15]) Let (X, d) be a metric space. Define $a * b = a \cdot b$ for all $a, b \in [0, 1]$, and let M_d be the real value mapping on $X \times X \times (0, +\infty)$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space and $(M_d, \cdot)$ is called *the standard fuzzy metric induced by d*.

Proposition 2.5. ([15]) *Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X converges to $x_0 \in X$ if and only if $\lim_{n\to\infty} M(x_n, x_0, t) = 1$ for all $t > 0$.*

Definition 2.6. ([37]) Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is called p-convergent to $x_0 \in X$ for t_0 (or, simply, $\{x_n\}$ is p-convergent) if $\lim_n M(x_n, x_0, t_0) = 1$.

Definition 2.7. ([16]) We say that the fuzzy metric space $(X, M, *)$ is *principal* (or simply, M is principal) if $\{B_M(x, r, t) : r \in (0, 1)\}$ is a local base at $x \in X$, for each $x \in X$ and each $t > 0$.

Proposition 2.8. ([16]*) The fuzzy metric space $(X, M, *)$ is principal if and only if all p-convergent sequences in X are convergent.*

In the following, for any $B \subseteq \mathbb{N}$, $|B|$ will denote the cardinality of B.

Definition 2.9. ([10, 41]) Let $A \subseteq \mathbb{N}$. For any $n \in \mathbb{N}$, put $A(n) = \{k \leq n | k \in A\}$. Define $\delta(A)$ by

$$\delta(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}$$

and call it an *asymptotic (or natural) density* of A. Obviously, if $\delta(A)$ exists, then $\delta(A) \in [0, 1]$ and $\delta(\mathbb{N} - A) = 1 - \delta(A)$. A is said to be *statistically dense* provided that $\delta(A) = 1$.

Definition 2.10. *[31]* Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be *statistically convergent* to $x_0 \in X$ if $\delta(\{n \in \mathbb{N} | M(x_n, x_0, t) > 1 - \varepsilon\}) = 1$ for all $\varepsilon \in (0, 1)$ and $t > 0$.

Definition 2.11. ([31]) Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is called a *statistically Cauchy sequence*, if for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $N_0 \in \mathbb{N}$ such that $\delta(\{n \in \mathbb{N} | M(x_n, x_{N_0}, t) > 1 - \varepsilon\}) = 1$.

Definition 2.12. ([31]) The fuzzy metric space $(X, M, *)$ is called *statistically complete* if every statistically Cauchy sequence in X is statistically convergent.

3. *p*-statistical convergence

In this section we introduce the concept of p-statistical convergence in fuzzy metric spaces and explore some aspects relative to this concept.

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be *p-statistically convergent* to $x_0 \in X$, if for some $t_0 > 0$, $\delta(\{n \in \mathbb{N} | M(x_n, x_0, t_0) > 1 - \varepsilon\}) = 1$ whenever $\varepsilon \in (0, 1)$.

In such a case we say that the sequence $\{x_n\}$ is p-statistically convergent to x_0 for t_0 (or, simply, $\{x_n\}$ is p-statistically convergent). It is evident that

$$\delta(\{n \in \mathbb{N} | M(x_n, x_0, t_0) > 1 - \varepsilon\}) = 1 \Leftrightarrow \lim_{n\to\infty} \frac{|\{k \leq n | M(x_k, x_0, t_0) > 1 - \varepsilon\}|}{n} = 1.$$

Remark 3.2. It is easy see that a sequence $\{x_n\}$ is statistically convergent to x_0 if and only if it is p-statistically convergent to x_0 for all $t > 0$.

It is worth to mention that if a sequence $\{x_n\}$ in the fuzzy metric space $(X, M, *)$ is p-statistically convergent to x_0 for $t_0 > 0$, then any subsequence $\{x_{n_i}\}$ of $\{x_n\}$ need not be p-statistically convergent to x_0 for t_0. This fact is illustrated in the next example.

Example 3.3. Let $X = [1, 3]$. Denote $a * b = a \cdot b$ for all $a, b \in [0, 1]$. Define M by $M(x, y, t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space (see [15]). Now, choose the sequence $\{x_n\}$ in X, where

$$x_n = \begin{cases} 2, & n = \frac{m(m+3)}{2}, m \in \mathbb{N}, \\ 1, & \text{otherwise}. \end{cases}$$

Then by Example 3.3 of [31], we get that $\{x_n\}$ is statistically convergent to 1, which implies from Remark 3.2 that $\{x_n\}$ is p-statistically convergent to 1 for all $t > 0$. Now, take the subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $n_i = \frac{i(i+3)}{2}$ $(i \in \mathbb{N})$. Then $x_{n_i} = 2$ for all $i \in \mathbb{N}$. Obviously, $\{x_{n_i}\}$ is p-statistically convergent to 2 for all $t > 0$.

Theorem 3.4. *Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$, $a, b \in X$ and $t_1, t_2 \in (0, +\infty)$. If $\{x_n\}$ is p-statistically convergent to a for t_1 and it is also p-statistically convergent to b for t_2, then $a = b$.*

Proof. Suppose that $a \neq b$. Then $M(a, b, t_1 + t_2) \in (0, 1)$. Choose $\varepsilon_0 \in (0, 1 - M(a, b, t_1 + t_2))$. Then there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that $(1 - \varepsilon_1) * (1 - \varepsilon_1) \geq 1 - \varepsilon_0$. Since

$$\delta(\{n \in \mathbb{N} | M(x_n, a, t_1) > 1 - \varepsilon_1\}) = 1$$

and

$$\delta(\{n \in \mathbb{N} | M(x_n, b, t_2) > 1 - \varepsilon_1\}) = 1.$$

It follows that

$$\{n \in \mathbb{N} | M(x_n, a, t_1) > 1 - \varepsilon_1\} \cap \{n \in \mathbb{N} | M(x_n, b, t_2) > 1 - \varepsilon_1\} \neq \emptyset.$$

Take

$$m_0 \in \{n \in \mathbb{N} | M(x_n, a, t_1) > 1 - \varepsilon_1\} \cap \{n \in \mathbb{N} | M(x_n, b, t_2) > 1 - \varepsilon_1\}.$$

Then

$$M(a, b, t_1 + t_2) \geq M(a, x_{m_0}, t_1)) * M(x_{m_0}, b, t_2)) \geq (1 - \varepsilon_1) * (1 - \varepsilon_1) \geq 1 - \varepsilon_0 > M(a, b, t_1 + t_2),$$

which is a contradiction. This completes the proof. □

By virtue of Remark 3.2 and Theorem 3.4, we get the next corollary immediately.

Corollary 3.5. *Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$, $x_0 \in X$ and $t_0 > 0$. If $\{x_n\}$ is p-statistically convergent to x_0 for t_0 and it is statistically convergent, then $\{x_n\}$ statistically converges to x_0.*

As shown in the following example, a p-statistically convergent sequence need not be statistically convergent in general.

Example 3.6. Let $X = \{x_1, x_2, \cdots\} \cup \{1\}$, where $\{x_n\}$ is a strictly increasing sequence in (0,1) with $\{x_n\}$ converges to 1 in the usual topology of $\mathbb{R}$. Denote $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Consider the function $M : X \times X \times (0, +\infty) \to (0, 1]$ given by

$$M(x, x, t) = 1 \text{ for each } x \in X, t > 0,$$

$$M(x_n, x_m, t) = \min\{x_n, x_m\} \text{ for all } n, m \in \mathbb{N} \text{ with } n \neq m, t > 0,$$

$$M(x_n, 1, t) = M(1, x_n, t) = \min\{x_n, t\} \text{ for all } n \in \mathbb{N}, t > 0.$$

Then, by Example 2.2 of [37], we obtain that $(X, M, *)$ is a fuzzy metric space. Let $\varepsilon \in (0, 1)$. Since $\{x_n\}$ is strictly increasing in (0,1) with $\{x_n\}$ converges to 1 in the usual topology of $\mathbb{R}$, we can find $N_0 \in \mathbb{N}$ such that $x_n > 1 - \varepsilon$ for all $n > N_0$. Hence $M(x_n, 1, 1) = x_n > 1 - \varepsilon$ for all $n > N_0$. Therefore

$$|\{k \leq n | M(x_k, 1, 1) > 1 - \varepsilon\}| \geq n - N_0$$

for all $n > N_0$, which follows that

$$\lim_{n\to\infty} \frac{|\{k \leq n | M(x_k, 1, 1) > 1 - \varepsilon\}|}{n} \geq \lim_{n\to\infty} \frac{n - N_0}{n} = 1.$$

So $\{x_n\}$ is p-statistically convergent to 1 for $t = 1$, i.e., $\{x_n\}$ is p-statistically convergent.

On the other hand, since $M(x_n, 1, \frac{1}{2}) \leq \frac{1}{2}$ for all $n \in \mathbb{N}$, we deduce that

$$|\{k \leq n | M(x_k, 1, \frac{1}{2}) > \frac{1}{2}\}| = 0$$

for all $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} \frac{|\{k \leq n | M(x_k, 1, \frac{1}{2}) > \frac{1}{2}\}|}{n} = 0.$$

Consequently, $\{x_n\}$ is not p-statistically convergent to 1 for $t = \frac{1}{2}$. According to Remark 3.2, we get that $\{x_n\}$ is not statistically convergent to 1. With Theorem 3.4 and Corollary 3.5, we conclude that $\{x_n\}$ is not statistically convergent to any point in X, that is, $\{x_n\}$ is not statistically convergent.

Theorem 3.7. *Let $(X, M, *)$ be a fuzzy metric space. Then M is principal if and only if all p-statistically convergent sequences in X are statistically convergent.*

Proof. Suppose that all p-statistically convergent sequences in X are statistically convergent. We assert that M is principal. If not, then there exist $x_0 \in X$ and $t_0 > 0$ such that $\{B_M(x_0, \frac{1}{n}, t_0) | n \in \mathbb{N}\}$ is not a local base at x_0. Then we can choose $\varepsilon_1 \in (0, 1)$ and $t_1 > 0$ such that $B_M(x_0, \frac{1}{n}, t_0) \nsubseteq B_M(x_0, \varepsilon_1, t_1)$ for all $n \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$, we choose $x_n \in B_M(x_0, \frac{1}{n}, t_0) \backslash B_M(x_0, \varepsilon_1, t_1)$. Consider the obtained sequence $\{x_n\}$ by induction. Let $\varepsilon \in (0, 1)$. Then there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$. Therefore

$$M(x_n, x_0, t_0) > 1 - \frac{1}{n} > 1 - \frac{1}{n_0} > 1 - \varepsilon$$

for all $n \geq n_0$. Thus

$$|\{k \leq n | M(x_k, x_0, t_0) > 1 - \varepsilon\}| \geq n - n_0.$$

It follows that

$$\lim_{n\to\infty} \frac{|\{k \leq n | M(x_k, x_0, t_0) > 1 - \varepsilon\}|}{n} \geq \lim_{n\to\infty} \frac{n - n_0}{n} = 1.$$

As a consequence, $\{x_n\}$ is p-statistically convergent to x_0 for t_0. Notice that $x_n \notin B_M(x_0, \varepsilon_1, t_1)$, we have that

$$\{n \in \mathbb{N} | M(x_n, x_0, t_1) > 1 - \varepsilon_1\} = \emptyset.$$

Hence

$$\lim_{n\to\infty} \frac{|\{k \leq n | M(x_k, x_0, t_1) > 1 - \varepsilon_1\}|}{n} = 0.$$

So $\{x_n\}$ is not p-statistically convergent to x_0 for t_1, which means that $\{x_n\}$ is not statistically convergent to x_0. By Theorem 3.4 and Corollary 3.5, we obtain that $\{x_n\}$ is not statistically convergent to any point in X. A contradiction occurs.

Conversely, suppose that M is principal and $\{x_n\}$ is a sequence in X which is p-statistically convergent to $x_0 \in X$ for some $t_0 > 0$. Let $r \in (0, 1)$ and $t > 0$. Since M is principal, we get that $\{B_M(x_0, \varepsilon, t_0) | \varepsilon \in (0, 1)\}$ is a local base at x_0. Hence there exists $\varepsilon_1 \in (0, 1)$ such that $B_M(x_0, \varepsilon_1, t_0) \subseteq B_M(x_0, r, t)$. It follows that

$$\{n \in \mathbb{N} | x_n \in B_M(x_0, \varepsilon_1, t_0)\} \subseteq \{n \in \mathbb{N} | x_n \in B_M(x_0, r, t)\}.$$

Since $\{x_n\}$ is p-statistically convergent to x_0 for t_0, we obtain that

$$\lim_{n\to\infty}\frac{|\{k\le n|x_k\in B_M(x_0,r,t)\}|}{n}\ge\lim_{n\to\infty}\frac{|\{k\le n|x_k\in B_M(x_0,\varepsilon_1,t_0)\}|}{n}=1.$$

So $\{x_n\}$ is statistically convergent to x_0. We are done. □

From Proposition 2.8 and Theorem 3.7 we immediately deduce the next corollary.

Corollary 3.8. *In the fuzzy metric space $(X, M, *)$, the following are equivalent.*

(i) *M is principal.*
(ii) *All p-convergent sequences in X are convergent.*
(iii) *All p-statistically convergent sequences in X are statistically convergent.*

4. p–statistical Cauchyness and p-statistical completeness

We start this section with the following definition.

Definition 4.1. Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be *p-statistically Cauchy*, if one can find $t_0 > 0$ such that for every $\varepsilon \in (0, 1)$, there exists $N_0 \in \mathbb{N}$ such that $\delta(\{n \in \mathbb{N}|M(x_n, x_{N_0}, t_0) > 1-\varepsilon\}) = 1$.

In such a case we say that the sequence $\{x_n\}$ is p-statistically Cauchy for t_0 (or, simply, $\{x_n\}$ is p-statistically Cauchy).

Remark 4.2. It is obvious that a sequence $\{x_n\}$ is statistically Cauchy if and only if it is p-statistically Cauchy for all $t > 0$.

In the next example, we shall show that a p-statistically Cauchy sequence in X need not be statistically Cauchy.

Example 4.3. Let $X = \{x_1, x_2, \cdots\} \cup \{1\}$, where $x_n = 1 - \frac{1}{n+1}$ $(n \in \mathbb{N})$. Denote $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Define on $X \times X \times (0, +\infty)$ the function M by

$$M(x,x,t)=1 \text{ for each } x\in X, t>0,$$

$$M(x_n,x_m,t)=\min\{x_n,x_m,\frac{t}{2}\} \text{ for all } n,m\in\mathbb{N} \text{ with } n\ne m, t>0,$$

$$M(x_n,1,t)=M(1,x_n,t)=\min\{x_n,t\} \text{ for all } n\in\mathbb{N}, t>0.$$

Then $(X, M, *)$ is a fuzzy metric space (see Example 3.2 of [32]). Fix $t_0 = 2$. Let $\varepsilon \in (0, 1)$. Then there exists $N_0 \in \mathbb{N}$ such that $x_{N_0} = 1 - \frac{1}{N_0+1} > 1-\varepsilon$. Hence

$$M(x_n,x_{N_0},t_0)=x_{N_0}=1-\frac{1}{N_0+1}>1-\varepsilon$$

for all $n > N_0$. Therefore

$$|\{k\le n|M(x_k,x_{N_0},t_0)>1-\varepsilon\}|\ge n-N_0.$$

It follows that

$$\lim_{n\to\infty}\frac{|\{k\le n|M(x_k,x_{N_0},t_0)>1-\varepsilon\}|}{n}\ge\lim_{n\to\infty}\frac{n-N_0}{n}=1.$$

So the sequence $\{x_n\}$ is p-statistically Cauchy for t_0. On the other hand, put $t_1 = 1$ and $\varepsilon_1 = \frac{1}{3}$. Notice that for any $n, N_1 \in \mathbb{N}$,

$$M(x_n,x_{N_1},t_1)=\min\{1-\frac{1}{n+1},1-\frac{1}{N_1+1},\frac{t_1}{2}\}=\frac{t_1}{2}=\frac{1}{2}.$$

We deduce that $\{n \in \mathbb{N} | M(x_n, x_{N_1}, t_1) > 1 - \varepsilon_1\} = \emptyset$ for all $N_1 \in \mathbb{N}$. Hence

$$\delta(\{n \in \mathbb{N} | M(x_n, x_{N_1}, t_1) > 1 - \varepsilon_1\}) = 0$$

for all $N_1 \in \mathbb{N}$. Thus $\{x_n\}$ is not p-statistically Cauchy for t_1. It follows from Remark 4.2 that $\{x_n\}$ is not statistically Cauchy.

Theorem 4.4. *Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. If $\{x_n\}$ is p-statistically convergent, then it is p-statistically Cauchy.*

Proof. Assume that $\{x_n\}$ is p-statistically convergent to $x_0 \in X$ for $t_0 > 0$. Let $\varepsilon \in (0, 1)$. Then there exists $\varepsilon_1 \in (0, \varepsilon)$ such that $(1 - \varepsilon_1) * (1 - \varepsilon_1) > 1 - \varepsilon$. Note that $\delta(\{n \in \mathbb{N} | M(x_n, x_0, t_0) > 1 - \varepsilon_1\}) = 1$. We pick $N_0 \in \{n \in \mathbb{N} | M(x_n, x_0, t_0) > 1 - \varepsilon_1\}$. Then $M(x_{N_0}, x_0, t_0) > 1 - \varepsilon_1$. Let $m \in \{n \in \mathbb{N} | M(x_n, x_0, t_0) > 1 - \varepsilon_1\}$. Then we have that

$$M(x_m, x_{N_0}, 2t_0) \geq M(x_m, x_0, t_0) * M(x_0, x_{N_0}, t_0) \geq (1 - \varepsilon_1) * (1 - \varepsilon_1) > 1 - \varepsilon.$$

It follows that $m \in \{n \in \mathbb{N} | M(x_n, x_{N_0}, 2t_0) > 1 - \varepsilon\}$. Thus

$$\{n \in \mathbb{N} | M(x_n, x_0, t_0) > 1 - \varepsilon_1\} \subseteq \{n \in \mathbb{N} | M(x_n, x_{N_0}, 2t_0) > 1 - \varepsilon\},$$

which implies that

$$\lim_{n\to\infty} \frac{|\{k \leq n | M(x_k, x_{N_0}, 2t_0) > 1 - \varepsilon\}|}{n} \geq \lim_{n\to\infty} \frac{|\{k \leq n | M(x_k, x_0, t_0) > 1 - \varepsilon_1\}|}{n} = 1.$$

So $\{x_n\}$ is p-statistically Cauchy for $2t_0$, that is, $\{x_n\}$ is p-statistically Cauchy. □

Remark 4.5. As it is shown in the above demonstration, one can see that a p-statistically convergent sequence for $t_0 > 0$ is p-statistically Cauchy for $2t_0$. However, in general, a p-statistically convergent sequence for $t_0 > 0$ need not be p-statistically Cauchy for t_0. Indeed, consider Example 4.3, it is straightforward to show that the sequence $\{x_n\}$ is p-statistically convergent to $x = 1$ for $t = 1$, but it is not p-statistically Cauchy for $t = 1$.

It is worth to mention that the converse of Theorem 4.4 is not true. We illustrate this fact with the following example.

Example 4.6. Let $X = (0, 1)$. Denote $a * b = a \cdot b$ for all $a, b \in [0, 1]$. Define the function $M : X \times X \times (0, +\infty) \to (0, 1]$ by the formula

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}, \; x, y \in X, t > 0.$$

Then $(X, M, *)$ is a fuzzy metric space (see [43]). Now, take the sequence $\{x_n\}$ in X, where $x_n = \frac{1}{n}$ $(n \in \mathbb{N})$. Fix $t_0 = 1$. Let $\varepsilon \in (0, 1)$. Then there exists $N_0 = [\frac{1-\varepsilon}{\varepsilon}] + 1$ such that

$$M(x_n, x_{N_0}, t_0) = \frac{\frac{1}{n} + 1}{\frac{1}{N_0} + 1} > \frac{1}{\frac{1}{N_0} + 1} > 1 - \varepsilon$$

for all $n > N_0$. Hence

$$|\{k \leq n | M(x_k, x_{N_0}, t_0) > 1 - \varepsilon\}| \geq n - N_0$$

for all $n > N_0$. It follows that

$$\lim_{n\to\infty} \frac{|\{k \leq n | M(x_k, x_{N_0}, t_0) > 1 - \varepsilon\}|}{n} \geq \lim_{n\to\infty} \frac{n - N_0}{n} = 1.$$

Thus $\{x_n\}$ is p-statistically Cauchy for t_0. Let $x \in X$ and $t > 0$. We put $\varepsilon_0 = \frac{x}{2(x+t)}$. Since $\lim_{n\to\infty} x_n = 0$, then we can find large enough $N_1 \in \mathbb{N}$ such that $\frac{1}{n} < \frac{1}{2}x$ for all $n > N_1$. Hence

$$M(x_n, x, t) = \frac{\min\{x_n, x\} + t}{\max\{x_n, x\} + t} = \frac{\frac{1}{n} + t}{x + t} < \frac{\frac{1}{2}x + t}{x + t} = 1 - \frac{x}{2(x+t)} = 1 - \varepsilon_0$$

for all $n > N_1$. Therefore

$$|\{k \leq n | M(x_k, x, t) > 1 - \varepsilon_0\}| \leq N_1$$

for all $n > N_1$. It follows that

$$\lim_{n\to\infty} \frac{|\{k \leq n | M(x_k, x, t) > 1 - \varepsilon_0\}|}{n} \leq \lim_{n\to\infty} \frac{N_1}{n} = 0.$$

Thus $\{x_n\}$ is not p-statistically convergent to x for t, which implies that $\{x_n\}$ is not p-statistically convergent.

Theorem 4.7. *Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. Then $\{x_n\}$ is p-statistically convergent if and only if it is p-statistically Cauchy, also there exist $x_0 \in X$ and $t_0 > 0$ such that $\delta(\{n \in \mathbb{N} | M(x_n, x_0, t_0) > 1 - r\}) > 0$ whenever $r \in (0, 1)$.*

Proof. By Theorem 4.4 and the concept of p-statistically convergent sequence, it is straightforward to show that the "only if" part hold.

Next, we will demonstrate the "if" part. Let $\varepsilon \in (0, 1)$. Then there exists $\varepsilon_0 \in (0, \varepsilon)$ such that $(1 - \varepsilon_0) * (1 - \varepsilon_0) * (1 - \varepsilon_0) > 1 - \varepsilon$. Assume that the sequence $\{x_n\}$ is p-statistically Cauchy for some $t_1 > 0$. Then there exists $N_0 \in \mathbb{N}$ such that $\delta(\{n \in \mathbb{N} | M(x_n, x_{N_0}, t_1) > 1 - \varepsilon_0\}) = 1$. Since there exist $x_0 \in X$ and $t_0 > 0$ such that $\delta(\{n \in \mathbb{N} | M(x_n, x_0, t_0) > 1 - r\}) > 0$ whenever $r \in (0, 1)$. Therefore $\delta(\{n \in \mathbb{N} | M(x_n, x_0, t_0) > 1 - \varepsilon_0\}) > 0$. It follows that

$$\{n \in \mathbb{N} | M(x_n, x_{N_0}, t_1) > 1 - \varepsilon_0\} \cap \{n \in \mathbb{N} | M(x_n, x_0, t_0) > 1 - \varepsilon_0\} \neq \emptyset.$$

Take

$$m_0 \in \{n \in \mathbb{N} | M(x_n, x_{N_0}, t_1) > 1 - \varepsilon_0\} \cap \{n \in \mathbb{N} | M(x_n, x_0, t_0) > 1 - \varepsilon_0\}.$$

Then, for any $m \in \{n \in \mathbb{N} | M(x_n, x_{N_0}, t_1) > 1 - \varepsilon_0\}$, we get that

$$\begin{aligned} M(x_m, x_0, 2t_1 + t_0) &\geq M(x_m, x_{N_0}, t_1) * M(x_{N_0}, x_{m_0}, t_1) * M(x_{m_0}, x_0, t_0) \\ &> (1 - \varepsilon_0) * (1 - \varepsilon_0) * (1 - \varepsilon_0) > 1 - \varepsilon. \end{aligned}$$

Hence

$$\{n \in \mathbb{N} | M(x_n, x_{N_0}, t_1) > 1 - \varepsilon_0\} \subseteq \{n \in \mathbb{N} | M(x_n, x_0, 2t_1 + t_0) > 1 - \varepsilon\}.$$

which means that

$$\lim_{n\to\infty} \frac{|\{k \leq n | M(x_k, x_0, 2t_1 + t_0) > 1 - \varepsilon\}|}{n} \geq \lim_{n\to\infty} \frac{|\{k \leq n | M(x_k, x_{N_0}, t_1) > 1 - \varepsilon_0\}|}{n} = 1.$$

As a consequence, $\{x_n\}$ is p-statistically convergent to x_0 for $(2t_1 + t_0)$, i.e., $\{x_n\}$ is p-statistically convergent. □

Definition 4.8. Let $(X, M, *)$ be a fuzzy metric space. We say that $(X, M, *)$ (or simply, X) is *p-statistically complete* if every p-statistically Cauchy sequence in X is p-statistically convergent.

Theorem 4.9. *Let $(X, M, *)$ be a principal fuzzy metric space. If X is p-statistically complete, then it is statistically complete.*

Proof. Suppose that $\{x_n\}$ is a statistically Cauchy sequence in X. Then, by Remark 4.2, $\{x_n\}$ is p-statistically Cauchy. Since X is p-statistically complete, we deduce that $\{x_n\}$ is p-statistically convergent. Note that M is principal. It follows from Theorem 3.7 that $\{x_n\}$ is statistically convergent. So X is statistically complete. □

As it is shown in the next example, the assumption that M is principal cannot be omitted in the above theorem in general.

Example 4.10. Consider Example 3.6. Due to Theorem 3.7 and the demonstration of Example 3.6, it is immediate to see that the fuzzy metric space $(X, M, *)$ is not principal. Choose the sequence $\{y_n\}$ in X, where

$$y_n = \begin{cases} x_1, & n = m^2, m \in \mathbb{N}, \\ x_n, & \text{otherwise}. \end{cases}$$

Let $r \in (0, 1)$ and $t > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} > 1 - r$. Then

$$|\{k \leq n | M(y_k, y_{n_0}, t) > 1 - r\}| \geq n - n_0 - \sqrt{n} - 1 > 0$$

for all sufficiently large $n > n_0$. Hence

$$\lim_{n\to\infty} \frac{|\{k \leq n | M(y_k, y_{n_0}, t) > 1 - r\}|}{n} \geq \lim_{n\to\infty} \frac{n - n_0 - \sqrt{n} - 1}{n} = 1.$$

Therefore $\{y_n\}$ is statistically Cauchy. Take $t_0 = \frac{1}{4}x_1$ and $\varepsilon_0 = 1 - \frac{1}{2}x_1$. Observe that

$$M(y_n, 1, t_0) = M(y_n, 1, \frac{1}{4}x_1) = \min\{y_n, \frac{1}{4}x_1\} = \frac{1}{4}x_1 = t_0 < 1 - \varepsilon_0 = \frac{1}{2}x_1$$

for all $n \in \mathbb{N}$. we get that

$$\{n \in \mathbb{N} | M(y_n, 1, t_0) > 1 - \varepsilon_0\} = \emptyset,$$

which means that

$$\delta(\{n \in \mathbb{N} | M(y_n, 1, t_0) > 1 - \varepsilon_0\}) = 0.$$

Thus $\{y_n\}$ is not p-statistically convergent to 1 for t_0. Take $t_1 = 2$. Let $\varepsilon \in (0, 1)$. Then we can find $n_1 \in \mathbb{N}$ such that $y_{n_1} > 1 - \varepsilon$. Since $M(y_n, 1, t_1) = y_n$ for all $n \in \mathbb{N}$, we obtain that

$$|\{k \leq n | M(y_k, 1, t_1) > 1 - \varepsilon\}| \geq n - n_1 - \sqrt{n} - 1 > 0$$

for all sufficiently large $n > N_1$. Thus

$$\lim_{n\to\infty} \frac{|\{k \leq n | M(y_k, 1, t_1) > 1 - \varepsilon\}|}{n} \geq \lim_{n\to\infty} \frac{n - n_1 - \sqrt{n} - 1}{n} = 1,$$

which implies that $\{y_n\}$ is p-statistically convergent to 1 for t_1. It follows from Remark 3.2 and Theorem 3.4 that $\{y_n\}$ is not statistically convergent to any point in X. So X is not statistically complete.

On the other hand, let $\{z_n\}$ be a p-statistically Cauchy sequence in X. Then it is not difficult to see that there exist $x_0 \in X$ and $t_0 \geq 1$ such that $\delta(\{n \in \mathbb{N} | M(z_n, x_0, t_0) > 1 - \varepsilon\}) > 0$ whenever $\varepsilon \in (0, 1)$. According to Theorem 4.7, we conclude that $\{z_n\}$ is p-statistically convergent. Consequently, X is p-statistically complete.

Now, we give a example to illustrate that the converse of Theorem 4.9 is not true.

Example 4.11. Let $X = (0, 1)$. Denote $a * b = a \cdot b$ for all $a, b \in [0, 1]$. Define the function $M : X \times X \times (0, +\infty) \to (0, 1]$ as follows: for any $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \begin{cases} 1, & x = y, \\ xyt, & x \neq y, t \leq 1, \\ xy, & x \neq y, t > 1. \end{cases}$$

Then, by Example 19 of [16], we have that $(X, M, *)$ is a principal fuzzy metric space. Let $\{x_n\}$ be a statistically Cauchy sequence. Then it is easy to see that there exists a constant $a \in X$ such that $\delta(\{n \in \mathbb{N} | x_n = a\}) = 1$. Therefore $\delta(\{n \in \mathbb{N} | M(x_n, a, t) > 1 - \varepsilon\}) = 1$ for all $\varepsilon \in (0, 1)$ and $t > 0$, which means that $\{x_n\}$ is statistically convergent to a. So X is statistically complete. Take $y_n = 1 - \frac{1}{n+1}$ $(n \in \mathbb{N})$. Fix $t_0 = 2$. Let $\varepsilon \in (0, 1)$. Then there exists $N_0 \in \mathbb{N}$ such that $(1 - \frac{1}{N_0+1})^2 > 1 - \varepsilon$. Hence

$$(1 - \frac{1}{n+1})(1 - \frac{1}{N_0+1}) > (1 - \frac{1}{N_0+1})^2 > 1 - \varepsilon$$

for all $n > N_0$. Therefore

$$|\{k \le n | M(y_k, y_{N_0}, t) > 1 - \varepsilon\}| \ge n - N_0$$

for all $n > N_0$. It follows that

$$\lim_{n\to\infty} \frac{|\{k \le n | M(y_k, y_{N_0}, t) > 1 - \varepsilon\}|}{n} \ge \lim_{n\to\infty} \frac{n - N_0}{n} = 1.$$

Thus the sequence $\{y_n\}$ in X is p-statistically Cauchy for t_0. Let $b \in X$. In case $t > 1$. Put $\varepsilon_0 = 1 - b$. Then $M(y_n, b, t) = y_n \cdot b < b = 1 - \varepsilon_0$ for all $n \in \mathbb{N}$. Hence $\delta(\{n \in \mathbb{N} | M(y_n, b, t) > 1 - \varepsilon_0) = 0$. We deduce that $\{y_n\}$ is not p-statistically convergent to b for t. In case $t \in (0, 1]$, a similar argument as above shows that $\{y_n\}$ is not p-statistically convergent to b for t. So $\{y_n\}$ is not p-statistically convergent to any point b in X for $t > 0$. As a consequence, X is not p-statistically complete.

Definition 4.12. Let $(X, M, *)$ be a fuzzy metric space. We say that $(X, M, *)$ (or simply, X) is *s-p-statistically complete* if every p-statistically Cauchy sequence in X is statistically convergent.

Theorem 4.13. *Let $(X, M, *)$ be a fuzzy metric space. Then X is s-p-statistically complete if and only if M is principal and X is p-statistically complete.*

Proof. Suppose that X is s-p-statistically complete. Then, according to Remark 3.2, we get that X is p-statistically complete. Let $\{x_n\}$ be a p-statistically convergent sequence in X. Then, by Theorem 4.4, we have that $\{x_n\}$ is p-statistically Cauchy. Since X is s-p-statistically complete, we obtain that $\{x_n\}$ is statistically convergent. It follows from Theorem 3.7 that M is principal.

Conversely, suppose that M is principal and X is p-statistically complete. Let $\{y_n\}$ be a p-statistically Cauchy sequence in X. Since X is p-statistically complete, we get that $\{y_n\}$ is p-statistically convergent. Observe that M is principal, it follows from Theorem 3.7 that $\{y_n\}$ is statistically convergent. Consequently, X is s-p-statistically complete. □

5. Conclusions

We have modified the definition of statistical convergence that due to Li et al. [31] in fuzzy metric spaces, and obtained a more general concept that we call p-statistical convergence. We have investigated the relationship between p-statistical convergence and statistical convergence in fuzzy metric spaces. Furthermore, we have provided several necessary and sufficient conditions those are relative to p-statistical convergence in fuzzy metric spaces.

As we can see that statistical Cauchyness implies p-statistical Cauchyness in fuzzy metric spaces, but the inverse is not. In a further work we will continue to explore when it is satisfied that p-statistical Cauchyness agrees with statistical Cauchyness in fuzzy metric spaces.

References

[1] B. Bilalov, F. Guliyeva, *On the basis properties of the systems in the intuitionistic fuzzy metric space*, Azerbaijan J. Math. **4** (2014), 136–149.

[2] B. Bilalov, T. Nazarova, *On statistical convergence in metric spaces*, J. Math. Res. **7** (2015), 37–43.

[3] B. Bilalov, T. Nazarova, *Statistical convergence of functional sequences* , Rocky Mt. J. Math. **45** (2015), 1413–1423.

[4] B. Bilalov, T. Nazarova, *On statistical type convergence in uniform spaces* , B. Iran. Math. Soc. **42** (2016), 975–986.

[5] K. Cho, C. Lee, *Some results on convergences in fuzzy metric spaces and fuzzy normed spaces* , Commun. Korean Math. S. **35** (2020), 185-199.

[6] J. S. Connor, *The statistical and strong p-Cesaro convergence of sequences* , Analysis **8** (1988), 47–63.
[7] G. Di Maio, Lj.D. R. Kočinac, *Statistical convergence in topology* , Topol Appl. **156** (2008), 28–45.
[8] R. Engelking, *General Topology* , PWN-Polish Science Publishers, warsaw, 1977.
[9] M. A. Erceg, *Metric spaces in fuzzy set theory* , J. Math. Anal. Appl. **69** (1979), 205–230.
[10] H. Fast, *Sur la convergence statistique* , Colloq. Math. **2** (1951), 241–244.
[11] J. A. Fridy, *On statistical convergence* , Analysis **5** (1985), 301–313.
[12] J. A. Fridy, M. K. Khan, *Tauberian theorems via statistical convergence* , J. Math. Anal. Appl. **228** (1998) 73–95.
[13] A. D. Gadjiev, C. Orhan, *Some approximation theorems via statistical convergence* , Rocky Mt. J. Math. **32** (2002), 129–138.
[14] J. G. García, S. Romaguera, M. Sanchis, *An identification theorem for the completion of the Hausdorff fuzzy metric* , Fuzzy Set Syst. **227** (2013), 96–106.
[15] A. George, P. Veeramani, *On some resules in fuzzy metric spaces* , Fuzzy Set Syst. **64** (1994), 395–399.
[16] V. Gregori, A. López-Crevillén, S. Morillas, A. Sapena, *On convergence in fuzzy metric spaces* , Topol Appl. **156** (2009), 3002–3006.
[17] V. Gregori, J. J. Miñana, *std-convergence in fuzzy metric spaces*, Fuzzy Set Syst. **267** (2015), 140–143.
[18] V. Gregori, J. J. Miñana, *Strong convergence in fuzzy metric spaces* , Filomat **31** (2017), 1619–1625.
[19] V. Gregori, J. J. Miñana, S. Morillas, *A note on convergence in fuzzy metric spaces* , Iran. J. Fuzzy Syst. **11** (2014), 75–85.
[20] V. Gregori, J. J. Miñana, S. Morillas, A. Sapena, *Characterizing a class of completable fuzzy metric spaces* , Topol Appl. **203** (2016), 3–11.
[21] V. Gregori, J. J. Miñana, B. Roig, A. Sapena, *A characterization of p-complete fuzzy metric spaces* , Fuzzy Set Syst. **444** (2022), 144–155.
[22] V. Gregori, J. J. Miñana, B. Roig, A. Sapena, *On completeness and fixed point theorems in fuzzy metric spaces* , Mathematics **12** (2024), Art. ID 287.
[23] V. Gregori, J. J. Miñana, B. Roig, A. Sapena, *On the axiomatic of GV-fuzzy metric spaces and its completion* , Axioms **14** (2025), Art. ID 89.
[24] V. Gregori, S. Morillas, A. Sapena, *On a class of completable fuzzy metric spaces* , Fuzzy Set Syst. **161** (2010), 2193–2205.
[25] V. Gregori, S. Romaguera, *Some properties of fuzzy metric spaces* , Fuzzy Set Syst. **115** (2000), 485–489.
[26] V. Gregori, S. Romaguera, *On completion of fuzzy metric spaces* , Fuzzy Set Syst. **130** (2002), 399–404.
[27] V. Gregori, S. Romaguera, *Characterizing completable fuzzy metric spaces* , Fuzzy Set Syst. **144** (2004), 411–420.
[28] O. Kaleva, S. Seikkala, *On fuzzy metric spaces* , Fuzzy Set Syst. **12** (1984), 215–229.
[29] P. Kostyrko, M. Mačaj, T. Šalát, O. Strauch, *On statistical limit points* , P. Am. Math. Soc. **129** (2000), 2647–2654.
[30] I. Kramosil, J. Michalek, *Fuzzy metric and statistical metric spaces* , Kybernetika **11** (1975) 326–334.
[31] C. Q. Li, Y. L. Zhang, J. Zhang, *On statistical convergence in fuzzy metric spaces* , J. Intell. Fuzzy Syst. **39** (2020), 3987–3993.
[32] C. Q. Li, Y. L. Zhang, *On p-convergent sequences and p-Cauchy sequences in fuzzy metric spaces* , Fuzzy Set Syst. **466** (2023), Art. ID 108464.
[33] C. Q. Li, Y. L. Zhang, *On completion of principal fuzzy metric spaces* , Fuzzy Set Syst. **505** (2025), Art. ID 109261.
[34] K. D. Li, S. Lin, Y. Ge, *On statistical convergence in cone metricspaces* , Topol. Appl. **196** (2015), 641–651.
[35] K. Menger, *Statistical metrics* , Proc. Natl. Acad. Sci. USA, **28** (1942), 535–537.
[36] H. I. Miller, *A measure theoretical subsequence characterization of statistical convergence* , Trans. Amer. Math. Soc. **347** (1995), 1811–1819.
[37] D. Mihet, *On fuzzy contractive mapping in fuzzy metric spaces* , Fuzzy Set Syst. **158** (2007), 915–921.
[38] S. Morillas, A. Sapena, *On standard Cauchy sequences in fuzzy metric spaces* , in: Proceedings of the Conference in Applied Topology, WiAT'13, 2013, pp. 101–108.
[39] S. Pehlivan, A. M. Mamedov, *Statistical cluster points and turnpike* , Optimization **48** (2000), 93–106.
[40] B. Schweizer, A. Sklar, *Statistical metric spaces* , Pac. J. Math. **10** (1960), 313–334.
[41] H. Steinhaus, *Sur la convergence ordinarieet la convergence asymptotique* , Colloq. Math **2** (1951), 73–74.
[42] P. Tirado, *On compactness and G-completeness in fuzzy metrics spaces* , Iran. J. Fuzzy Syst. **9** (2012), 151–158.
[43] P. Veeramani, *Best approximation in fuzzy metric spaces* , J. Fuzzy Math. **9** (2001), 75–80.