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On bi-controlled fuzzy φ-normed linear spaces

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Abstract. In this paper, from modifying the notion of the Bag-Samanta's type fuzzy norm, we introduce the concepts of bi-controlled φ-norms and bi-controlled fuzzy φ-normed linear spaces, by utilizing a special function φ which initiated by Goleţ, and two non-comparable functions ω and μ which initiated by N. Saleem et al., respectively. Furthermore, we establish some basic results of bi-controlled fuzzy φ-normed linear spaces. Finally, we introduce the concept of $l^{\varphi}_{[\omega,\mu]}$-fuzzy convergent sequence, and investigate the completeness in finite dimensional bi-controlled fuzzy φ-normed linear spaces.

1. Introduction and preliminaries

Since Zadeh [25] initiated the notion of fuzzy sets in 1965, many researchers have investigated numerous results for the theory of fuzzy sets. Such as Kaleva and Seikkala [13], Kramosil and Michálek [17], George and Veeramani [11] have obtained some results in fuzzy metric spaces in various directions. Furthermore, Katsaras [15], Felbin [10], Cheng and Mordeson [5], Bag and Samanta [2] have proposed several notions of fuzzy normed linear spaces by different approaches. Additionally, since Bakhtin [4] and Czerwik [6] introduced the generalization of metric space by replacing $d(x,z) \leq d(x,y) + d(y,z)$ with $d(x,z) \leq b[d(x,y) + d(y,z)]$ for some number $b \geq 1$, respectively, and Nădăban [19] proposed the idea of fuzzy b-metric spaces, which extended fuzzy metric spaces in 2016, by replacing $M(x,z,t+s) \leq M(x,y,t)*M(y,z,s)$ with $M(x,z,b(t+s)) \leq M(x,y,t)*M(y,z,s)$ for some number $b \geq 1$. Many authors presented various types of generalizations of fuzzy b-metric spaces. Kamran et al.[14] gave a generalization of b-metric spaces, and Mehmood et al.[18] established the definition of extended fuzzy b-metric. In 2018, Abdeljawad [1] initiated the concept of double controlled metric, which is a new type of generalization of b-metric, by using two non-comparable functions. Following the work of Abdeljawad [1], Saleem [23] and Sezen [24] introduced the concepts of fuzzy double controlled metric spaces and controlled fuzzy metric spaces in 2021, respectively.

Furthermore, in 2003, Bag and Samanta [2] established another definition of fuzzy norm, which was different from Felbin's type fuzzy norm [10] and Katsaras's type fuzzy norm [15]. Later, Goleţ [12] introduced the nation of generalized fuzzy norms by a special function φ. In 2023, Das, Bag and Chatterjee [9] proposed a notion of fuzzy strong ϕ-b-normed spaces in general t-morm settings, and obtained some basic results on finite dimensional fuzzy strong ϕ-b-normed space. Recently, Das and Bag [7, 8] established the completeness related to l-fuzzy convergent sequence on fuzzy strong ϕ-b-normed linear spaces, and developed some results for fuzzy bounded linear operators.

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Motivated by the works of Saleem [23] and Das, Bag et al. [9] on fuzzy normed linear spaces, we are able to proceed further. In this paper, the concept of bi-controlled fuzzy φ-b-norm is introduced, which generalizes the condition ($bN4$) of Definition 3.2 (see [21]), by replacing $M(x,z,t+bs) \le M(x,y,t) * M(y,z,s)$ with $M^{\varphi}_{[\omega,\mu]}(x,z,t+s) \le M^{\varphi}_{[\omega,\mu]}(x,y,\frac{t}{\omega(x,y)}) * M^{\varphi}_{[\omega,\mu]}(y,z,\frac{s}{\mu(y,z)})$ for two non-comparable functions. In Section 2, we illustrate several examples by using a non-negative real valued function φ, which satisfies some properties (Φ1)-(Φ5). Moreover, we establish some basic results of finite dimensional bi-controlled fuzzy φ-b-normed linear spaces including some decomposition theorems for bi-controlled fuzzy φ-b-norms into a family of pseudo-φ-norms. In Section 3, we introduce the concept of $I^{\varphi}_{[\omega,\mu]}$-fuzzy convergence and investigate the completeness in finite dimensional bi-controlled fuzzy φ-normed linear spaces.

Throughout this paper, X is always a nonempty set, the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}^+$ always denote the set of real numbers, of positive real numbers and of positive integers, respectively. Firstly, we briefly recall some definitions and facts which are used in the following sections (see more details in [3, 9, 11, 12, 22–24]).

Definition 1.1. ([17]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a *continuous triangular norm* (briefly t-norm) if the following conditions are satisfied for all $a,b,c,d \in [0,1]$:
(T1) $*$ is associative and commutative;
(T2) $*$ is continuous;
(T3) $a * 1 = a$;
(T4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$.

Examples of such t-norm functions are $a *_M b = \min\{a,b\}$ and $a *_P b = a \cdot b$, $\forall a,b \in [0,1]$.

Additionally, following the non-negative real valued function φ which discovered by Goleţ [12], we assume that φ is a function defined on the field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) to $\mathbb{R}$, with the following properties:
(Φ1) $\varphi(-\lambda) = \varphi(\lambda)$ for all $\lambda \in \mathbb{K}$;
(Φ2) $\varphi(0) = 0$;
(Φ3) $\varphi(\lambda_1 \cdot \lambda_2) = \varphi(\lambda_1) \cdot \varphi(\lambda_2)$ for all $\lambda_1, \lambda_2 \in \mathbb{K}$;
(Φ4) φ is strictly increasing on $[0,+\infty)$ and $\varphi(\lambda) \le \lambda$ on $[0,1]$;
(Φ5) φ is continuous on $[0,+\infty)$ and $\lim_{\lambda\to+\infty} \varphi(\lambda) = +\infty$.
From the above properties (Φ3) and (Φ4), taking $\lambda_1 = \lambda_2 = 1$, we get that $\varphi(1) = 1$.

Remark 1.2. (1) Examples of such functions are: $\varphi(\lambda) = |\lambda|$ and $\varphi(\lambda) = |\lambda|^p$, $\forall \lambda \in \mathbb{K}$, $p \in (0,+\infty)$.
(2) By (Φ4), it is clear that $\varphi(1) = 1$;
(3) From (Φ3) and (Φ4), it follows that $\varphi(\lambda) = 0$ if and only if $\lambda = 0$.

2. Bi-controlled fuzzy φ-normed linear spaces

In this section, we introduce the notions of the extended φ-norm and bi-controlled fuzzy φ-norm, and give several examples of bi-controlled fuzzy φ-normed linear spaces in general t-norm setting.

Definition 2.1. Let X be a linear space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\|^{\varphi}_{\alpha} : X \to [0,+\infty)$ is called an *extended φ-norm*, if for each $x,y \in X$, $\lambda \in \mathbb{K}$, where $\alpha \in (0,1)$, the following conditions hold:
($E\Phi N1$) $\|x\|^{\varphi}_{\alpha} \ge 0$;
($E\Phi N2$) $\|x\|^{\varphi}_{\alpha} = 0$ if and only if $x = \theta$, where θ is the null element of X;
($E\Phi N3$) $\|\lambda x\|^{\varphi}_{\alpha} = \varphi(\lambda)\|x\|^{\varphi}_{\alpha}$, where φ is a function satisfying (Φ1)-(Φ5);
($E\Phi N4$) $\|x+y\|^{\varphi}_{\alpha} \le \|x\|^{\varphi}_{\alpha} + \|y\|^{\varphi}_{\alpha}$.
A pair $(X, \|\cdot\|^{\varphi}_{\alpha})$ is called an *extended φ-normed space* if and only if $\|\cdot\|^{\varphi}_{\alpha}$ is an extended φ-norm on X.

Definition 2.2. Let X be a linear space over $\mathbb{K}$, $\omega,\mu : X \to [1,+\infty)$ be given non-comparable bounded functions, and $*$ be a continuous t-norm. A fuzzy set $N^{\varphi}_{[\omega,\mu]}$ on $X \times \mathbb{R}$ is called a *bi-controlled fuzzy φ-norm* on X.
($BCF\Phi N1$) $N^{\varphi}_{[\omega,\mu]}(x,t) = 0$ for all $t \in \mathbb{R}$ with $t \le 0$;
($BCF\Phi N2$) $N^{\varphi}_{[\omega,\mu]}(x,t) = 1$ for all $t \in \mathbb{R}^+$ if $x = \theta$, where θ is the zero element of X;

($BCF\Phi N3$) $N^{\varphi}_{[\omega,\mu]}(\lambda x, t) = N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\varphi(\lambda)})$ for all $t \in \mathbb{R}$ and $\lambda \neq 0$;
($BCF\Phi N4$) $N^{\varphi}_{[\omega,\mu]}(x + y, t + s) \geq N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\omega(x)}) * N^{\varphi}_{[\omega,\mu]}(y, \frac{s}{\mu(y)})$ for all $x, y \in X, s, t \in \mathbb{R}$;
($BCF\Phi N5$) $N^{\varphi}_{[\omega,\mu]}(x, \cdot)$ is a non-decreasing function of t and $\lim_{t\to+\infty} N^{\varphi}_{[\omega,\mu]}(x, t) = 1$;
($BCF\Phi N6$) If there exists $\alpha \in (0, 1)$ such that $N^{\varphi}_{[\omega,\mu]}(x, t) > \alpha$ for all $t \in \mathbb{R}^+$ implies $x = \theta$;
($BCF\Phi N7$) $N^{\varphi}_{[\omega,\mu]}(x, \cdot)$ is left continuous on $\mathbb{R}$ for all $x \in X$.
The triple $(X, N^{\varphi}_{[\omega,\mu]}, *)$ is called a *bi-controlled fuzzy φ-normed linear space*.

Remark 2.3. (1) If we have $\omega(x) = \mu(y) = b$ for all $x, y \in X$, then $(X, N^{\varphi}_{[\omega,\mu]}, *)$ is a fuzzy strong ϕ-b-normed space in the sense of Bag-Samanta type [7];

(2) If we have $\omega(x) = \mu(y) = 1$ and $\varphi(\lambda) = |\lambda|$ for all $x, y \in X$, then $(X, N^{\varphi}_{[\omega,\mu]}, *)$ is a fuzzy normed linear space in the sense of Nădăban-Dzitac type [21].

Example 2.4. Let X be a linear space over $\mathbb{K}$ and $\|\cdot\|^{\varphi}_{\alpha}$ be an extended φ-norm. Let $\omega, \mu : X \to [1, +\infty)$ be given non-comparable bounded functions. Define a fuzzy set $N^{\varphi}_{[\omega,\mu]}$: $X \times \mathbb{R} \to [0, 1]$ by

$$N^{\varphi}_{[\omega,\mu]}(x, t) = \begin{cases} \frac{t}{t+\|x\|^{\varphi}_{\alpha}}, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

for all $x \in X$. Then $(X, N^{\varphi}_{[\omega,\mu]}, *_M)$ is a bi-controlled fuzzy φ-normed space.

It is trivial to verify ($BCF\Phi N1$), ($BCF\Phi N2$), ($BCF\Phi N5$) and ($BCF\Phi N7$).

We need to verify the conditions ($BCF\Phi N3$), ($BCF\Phi N4$) and ($BCF\Phi N6$), respectively.

($BCF\Phi N3$): We will distinguish the following cases:

Case 1: Suppose that $t \leq 0$. It implies that $N^{\varphi}_{[\omega,\mu]}(\lambda x, t) = N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\varphi(\lambda)}) = 0$ if $\lambda \neq 0$.

Case 2: For all $\lambda \in \mathbb{K}$ with $\lambda \neq 0$, by ($E\Phi N3$), we have $N^{\varphi}_{[\omega,\mu]}(\lambda x, t) = \frac{t}{t+\|\lambda x\|^{\varphi}_{\alpha}} = \frac{t}{t+\varphi(\lambda)\|x\|^{\varphi}_{\alpha}} = \frac{\frac{t}{\varphi(\lambda)}}{\frac{t}{\varphi(\lambda)}+\|x\|^{\varphi}_{\alpha}} = N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\varphi(\lambda)})$ for all $x \in X$.

($BCF\Phi N4$): We will distinguish the following cases:

Case 1: Suppose that $t \leq 0$ or $s \leq 0$. It follows that $N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\omega(x)}) = 0$ or $N^{\varphi}_{[\omega,\mu]}(y, \frac{s}{\mu(y)}) = 0$ for all $x, y \in X$, then $N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\omega(x)}) *_M N^{\varphi}_{[\omega,\mu]}(y, \frac{s}{\mu(y)}) = 0$. Hence, $N^{\varphi}_{[\omega,\mu]}(x + y, t + s) \geq N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\omega(x)}) *_M N^{\varphi}_{[\omega,\mu]}(y, \frac{s}{\mu(y)})$.

Case 2: For any $x, y \in X, t, s > 0$, since $N^{\varphi}_{[\omega,\mu]}(x + y, t + s) = \frac{s+t}{s+t+\|x+y\|^{\varphi}_{\alpha}}$, and $N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\omega(x)}) *_M N^{\varphi}_{[\omega,\mu]}(y, \frac{s}{\mu(y)}) = \min\{N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\omega(x)}), N^{\varphi}_{[\omega,\mu]}(y, \frac{s}{\mu(y)})\}$.

Without loss of generality, suppose that $N^{\varphi}_{[\omega,\mu]}(y, \frac{s}{\mu(y)}) \geq N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\omega(x)})$. Namely, $\frac{\frac{s}{\mu(y)}}{\frac{s}{\mu(y)}+\|y\|^{\varphi}_{\alpha}} \geq \frac{\frac{t}{\omega(x)}}{\frac{t}{\omega(x)}+\|x\|^{\varphi}_{\alpha}}$, which implies that $s\omega(x)\|x\|^{\varphi}_{\alpha} \geq t\mu(y)\|y\|^{\varphi}_{\alpha}$. Then we claim that $\frac{t+s}{t+s+\|x+y\|^{\varphi}_{\alpha}} \geq \frac{t}{t+\omega(x)\|x\|^{\varphi}_{\alpha}}$.

In fact, by ($E\Phi N4$), we have

$$\begin{aligned} &\frac{t + s}{t + s + \|x + y\|^{\varphi}_{\alpha}} - \frac{t}{t + \omega(x)\|x\|^{\varphi}_{\alpha}} \\ \geq &\frac{t + s}{t + s + \|x\|^{\varphi}_{\alpha} + \|y\|^{\varphi}_{\alpha}} - \frac{t}{t + \omega(x)\|x\|^{\varphi}_{\alpha}} \\ = &\frac{(t + s)(t + \omega(x)\|x\|^{\varphi}_{\alpha}) - t(t + s + \|x\|^{\varphi}_{\alpha} + \|y\|^{\varphi}_{\alpha})}{(t + s + \|x\|^{\varphi}_{\alpha} + \|y\|^{\varphi}_{\alpha})(t + \omega(x)\|x\|^{\varphi}_{\alpha})} \\ = &\frac{(\omega(x) - 1)\|x\|^{\varphi}_{\alpha} + s\omega(x)\|x\|^{\varphi}_{\alpha} - t\|y\|^{\varphi}_{\alpha})}{(t + s + \|x\|^{\varphi}_{\alpha} + \|y\|^{\varphi}_{\alpha})(t + \omega(x)\|x\|^{\varphi}_{\alpha}}. \end{aligned}$$

Since $s\omega(x)\|x\|^{\varphi}_{\alpha} \geq t\mu(y)\|y\|^{\varphi}_{\alpha}$ and $\mu(y) \geq 1$, it is clear that $s\omega(x)\|x\|^{\varphi}_{\alpha} \geq t\|y\|^{\varphi}_{\alpha}$. Then $N^{\varphi}_{[\omega,\mu]}(x + y, t + s) \geq N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\omega(x)})$. Thus, $N^{\varphi}_{[\omega,\mu]}(x + y, t + s) \geq N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\omega(x)}) *_M N^{\varphi}_{[\omega,\mu]}(y, \frac{s}{\mu(y)})$.

Similarly, if $N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\omega(x)}) \geq N^{\varphi}_{[\omega,\mu]}(y, \frac{s}{\mu(y)})$, then we have $N^{\varphi}_{[\omega,\mu]}(x+y, t+s) \geq N^{\varphi}_{[\omega,\mu]}(y, \frac{s}{\mu(y)})$. Hence, ($BCF\Phi N4$) holds.

($BCF\Phi N6$): We suppose that there exists $\lambda_0 \in (0,1)$, such that $N^{\varphi}_{[\omega,\mu]}(x,t) > \lambda_0$ for all $t > 0$. Then $\frac{t}{t+\|x\|^{\varphi}_{\alpha}} > \lambda_0$. It implies that $t > \frac{\lambda_0}{1-\lambda_0}\|x\|^{\varphi}_{\alpha}$ for all $t > 0$. Thus $\|x\|^{\varphi}_{\alpha} = 0$. By ($E\Phi N2$), we have $x = \theta$.

Example 2.5. Let X be a linear space over $\mathbb{K}$ and $\|\cdot\|^{\varphi}_{\alpha}$ be an extended φ-norm. Let $\omega, \mu : X \to [1, +\infty)$ be given non-comparable bounded functions. Define a fuzzy set $N^{\varphi}_{[\omega,\mu]}$: $X \times \mathbb{R} \to [0,1]$ by

$$N^{\varphi}_{[\omega,\mu]}(x,t) = \begin{cases} e^{-\frac{\|x\|^{\varphi}_{\alpha}}{t}}, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

for all $x \in X$. Then $(X, N^{\varphi}_{[\omega,\mu]}, *_P)$ is a bi-controlled fuzzy φ-normed space.

It is trivial to verify ($BCF\Phi N1$), ($BCF\Phi N2$), ($BCF\Phi N5$) and ($BCF\Phi N7$).

We need to verify the conditions ($BCF\Phi N3$), ($BCF\Phi N4$) and ($BCF\Phi N6$), respectively.

($BCF\Phi N3$): We will distinguish the following cases:

Case 1: Suppose that $t \leq 0$. It implies that $N^{\varphi}_{[\omega,\mu]}(\lambda x, t) = N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\varphi(\lambda)}) = 0$ if $\lambda \neq 0$.

Case 2: For all $\lambda \in \mathbb{K}$ with $\lambda \neq 0$, by ($E\Phi N3$), we have $N^{\varphi}_{[\omega,\mu]}(\lambda x, t) = e^{-\frac{\|\lambda x\|^{\varphi}_{\alpha}}{t}} = e^{-\frac{\varphi(\lambda)\|x\|^{\varphi}_{\alpha}}{t}} = e^{-\frac{\|x\|^{\varphi}_{\alpha}}{\frac{t}{\varphi(\lambda)}}} = N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\varphi(\lambda)})$ for all $x \in X$.

($BCF\Phi N4$): Let $t > 0, s > 0$, then $N^{\varphi}_{[\omega,\mu]}(x+y, t+s) = e^{-\frac{\|x+y\|^{\varphi}_{\alpha}}{t+s}}$, and $N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\omega(x)}) *_P N^{\varphi}_{[\omega,\mu]}(y, \frac{s}{\mu(y)}) = e^{-\frac{\|x\|^{\varphi}_{\alpha}}{\frac{t}{\omega(x)}}} \cdot e^{-\frac{\|y\|^{\varphi}_{\alpha}}{\frac{s}{\mu(y)}}} = e^{-\frac{\omega(x)\|x\|^{\varphi}_{\alpha}}{t} - \frac{\mu(y)\|y\|^{\varphi}_{\alpha}}{s}}$. By ($E\Phi N4$), we have $-\frac{\|x+y\|^{\varphi}_{\alpha}}{t+s} \geq -\frac{\|x\|^{\varphi}_{\alpha}+\|y\|^{\varphi}_{\alpha}}{t+s}$. Since $\omega(x), \mu(y) \geq 1$ for all $x, y \in X$, it implies that $-\frac{\|x+y\|^{\varphi}_{\alpha}}{t+s} \geq -\frac{\omega(x)\|x\|^{\varphi}_{\alpha}+\mu(y)\|y\|^{\varphi}_{\alpha}}{t+s} \geq -\frac{\omega(x)\|x\|^{\varphi}_{\alpha}}{t} - \frac{\mu(y)\|y\|^{\varphi}_{\alpha}}{s}$. Thus $e^{-\frac{\|x+y\|^{\varphi}_{\alpha}}{t+s}} \geq e^{-\frac{\omega(x)\|x\|^{\varphi}_{\alpha}}{t} - \frac{\mu(y)\|y\|^{\varphi}_{\alpha}}{s}} = e^{-\frac{\|x\|^{\varphi}_{\alpha}}{\frac{t}{\omega(x)}}} \cdot e^{-\frac{\|y\|^{\varphi}_{\alpha}}{\frac{s}{\mu(y)}}}$. Namely, $N^{\varphi}_{[\omega,\mu]}(x+y, t+s) \geq N^{\varphi}_{[\omega,\mu]}(x, \frac{t}{\omega(x)}) *_M N^{\varphi}_{[\omega,\mu]}(y, \frac{s}{\mu(y)})$.

($BCF\Phi N6$): We suppose that there exists $\lambda_0 \in (0,1)$, such that $N^{\varphi}_{[\omega,\mu]}(x,t) > \lambda_0$ for all $t > 0$. Then $e^{-\frac{\|x\|^{\varphi}_{\alpha}}{t}} > \lambda_0$. It implies that $t > -\frac{1}{\ln \lambda_0}\|x\|^{\varphi}_{\alpha}$ for all $t > 0$. Thus $\|x\|^{\varphi}_{\alpha} = 0$. By ($E\Phi N2$), we have $x = \theta$.

Following [20], we have the following proposition:

Proposition 2.6. *$(X, N^{\varphi}_{[\omega,\mu]}, *)$ is a bi-controlled fuzzy φ-normed space. Define a mapping $\|\cdot\|^{\varphi}_{\alpha} : X \to [0, +\infty)$ as follows: $\|x\|^{\varphi}_{\alpha} = \bigwedge\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x,t) > \alpha\}$ for all $x \in X$, where $\alpha \in (0,1)$. Then the following statements hold: for all $x \in X$ and $\alpha \in (0,1)$*

(1) $\{\|x\|^{\varphi}_{\alpha} : \alpha \in (0,1)\}$ is non-decreasing with respect to α;

(2) $\{\|x\|^{\varphi}_{\alpha}\}$ satisfies ($E\Phi N1$)-($E\Phi N3$);

(3) $N^{\varphi}_{[\omega,\mu]}(x, \|x\|^{\varphi}_{\alpha}) \leq \alpha$;

(4) For each $s > 0$, we have that $\|x\|^{\varphi}_{\alpha} < s$ if and only if $N^{\varphi}_{[\omega,\mu]}(x,s) > \alpha$.

Proof. (1) Case 1: Suppose that $x = \theta$, it is evident.

Case 2: Let $x \neq \theta$, for all $\alpha, \beta \in (0,1)$, $\alpha < \beta$, we have $\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x,t) > \beta\} \subset \{t > 0 : N^{\varphi}_{[\omega,\mu]}(x,t) > \alpha\}$. Thus $\bigwedge\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x,t) > \beta\} \geq \bigwedge\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x,t) > \alpha\}$. Namely, $\|x\|^{\varphi}_{\beta} \geq \|x\|^{\varphi}_{\alpha}$. Hence, $\{\|x\|^{\varphi}_{\alpha} : \alpha \in (0,1)\}$ is non-decreasing.

(2) It is trivial to verify ($E\Phi N1$) and ($E\Phi N2$), then we only check ($E\Phi N3$) as follows.

($E\Phi N3$): For any $\lambda \in \mathbb{K}$ and $\lambda \neq 0$, by ($BCF\Phi N3$), we have

$$\begin{aligned}\|\lambda x\|_\alpha^\varphi &= \bigwedge\{t>0 : N^\varphi_{[\omega,\mu]}(\lambda x,t)>\alpha\}\\ &= \bigwedge\{t>0 : N^\varphi_{[\omega,\mu]}(x,\frac{t}{\varphi(\lambda)})>\alpha\}\\ &= \bigwedge\{\varphi(\lambda)t>0 : N^\varphi_{[\omega,\mu]}(x,t)>\alpha\}\\ &= \varphi(\lambda)\|x\|_\alpha^\varphi.\end{aligned}$$

(3) Since $\|x\|_\alpha^\varphi = \bigwedge\{t>0 : N^\varphi_{[\omega,\mu]}(x,t)>\alpha\}$, we get that $N^\varphi_{[\omega,\mu]}(x,t)>\alpha$ implies $\|x\|_\alpha^\varphi \le t$. Otherwise, if $\|x\|_\alpha^\varphi > t$, then $N^\varphi_{[\omega,\mu]}(x,t)\le\alpha$. Taking $t\to\|x\|_\alpha^\varphi$, where $t<\|x\|_\alpha^\varphi$, then $\lim_{t\to\|x\|_\alpha^\varphi, t<\|x\|_\alpha^\varphi} N^\varphi_{[\omega,\mu]}(x,t)\le\alpha$. By ($BCF\Phi N7$), we have $\lim_{t\to\|x\|_\alpha^\varphi, t<\|x\|_\alpha^\varphi} N^\varphi_{[\omega,\mu]}(x,t) = N^\varphi_{[\omega,\mu]}(x,\|x\|_\alpha^\varphi)$. Thus $N^\varphi_{[\omega,\mu]}(x,\|x\|_\alpha^\varphi)\le\alpha$. which is a contradiction.

(4)Now we prove its sufficiency and necessity as follows.

($\Leftarrow$) For each $s>0$, we suppose that $\|x\|_\alpha^\varphi \ge s$. From ($BCF\Phi N5$), we have $N^\varphi_{[\omega,\mu]}(x,\|x\|_\alpha^\varphi)\ge N^\varphi_{[\omega,\mu]}(x,s)$. Since $N^\varphi_{[\omega,\mu]}(x,s)>\alpha$, we get $N^\varphi_{[\omega,\mu]}(x,\|x\|_\alpha^\varphi)>\alpha$. From above Proposition 2.6 (3), which is a contradiction.

($\Rightarrow$) Suppose $\|x\|_\alpha^\varphi < s$. By assumption , we have $\bigwedge\{t>0 : N^\varphi_{[\omega,\mu]}(x,t)>\alpha\}<s$. It follows that there exists $t_0\in\{t>0 : N^\varphi_{[\omega,\mu]}(x,t)>\alpha\}$, such that $t_0<s$. From ($BCF\Phi N5$), we have that $N^\varphi_{[\omega,\mu]}(x,t_0)\le N^\varphi_{[\omega,\mu]}(x,s)$. Thus $N^\varphi_{[\omega,\mu]}(x,s)>\alpha$. □

Following [23], we introduce a concept of fuzzy bi-controlled metric spaces. Furthermore, we present a characterized theorem of fuzzy double controlled metrics in terms of bi-controlled fuzzy φ-norms.

Definition 2.7. Let X be a nonempty set, $\psi,\nu : X\times X\to[1,+\infty)$ be given non-comparable functions, and $*$ be a continuous t-norm. A fuzzy set $M_{[\psi,\nu]}$ on $X\times X\times[0,+\infty)$ is called a *fuzzy bi-controlled metric* on X, if for all $x,y,z\in X$, the following conditions hold:

($FBCM1$) $M_{[\psi,\nu]}(x,y,0)=0$;

($FBCM2$) $M_{[\psi,\nu]}(x,x,t)=1$ for all $t>0$;

($FBCM3$) $M_{[\psi,\nu]}(x,y,t)=M(y,x,t)$;

($FBCM4$) $M_{[\psi,\nu]}(x,z,t+s)\ge M_{[\psi,\nu]}(x,y,\frac{t}{\psi(x,y)}) * M_{[\psi,\nu]}(y,z,\frac{s}{\nu(y,z)})$;

($FBCM5$) The function $M_{[\psi,\nu]}(x,y,\cdot):[0,+\infty)\to[0,1]$ is left-continuous;

($FBCM6$) $\lim_{t\to+\infty} M_{[\psi,\nu]}(x,y,t)=1$.

A triple $(X,M_{[\psi,\nu]},*)$ is called a *fuzzy bi-controlled metric space*.

Example 2.8. Let $X=\{1,2,3\}$ and $\psi,\nu : X\times X\to[1,+\infty)$ be given by $\psi(x,y)=x+y+1$ and $\nu(y,z)=y^2+z^2-1$ for all $x,y,z\in X$. Define a fuzzy set $M_{[\psi,\nu]} : X\times X\times\mathbb{R}\to[0,1]$ by

$$M_{[\psi,\nu]}(x,y,t)=\begin{cases}\frac{x\wedge y+t}{x\vee y+t}, & t>0;\\ 0, & t\le 0.\end{cases}$$

for all $x,y\in X$. Then $(X,M_{[\psi,\nu]},*_P)$ is a fuzzy bi-controlled metric space.

It is trivial to verify ($FBCM1$)-($FBCM6$).

Remark 2.9. (1) If we have $\psi(x,y)=\nu(y,z)=1$ for all $x,y,z\in X$, then $(X,M_{[\psi,\nu]},*)$ becomes a fuzzy metric space in the sense of Kramosil-Michálek type [17];

(2) Furthermore, we claim that a fuzzy bi-controlled metric space is not Hausdorff. In fact, we define open ball $B(x,y,t)$ with center $x\in X$ and radius r as $B(x,r,t)=\{y\in X : M_{[\psi,\nu]}(x,y,t)>1-r\}$ for all $t>0$ in fuzzy bi-controlled metric spaces, where $0<r<1$. From the above Example 2.8, we note that $B(1,0.4,5)=\{2,3\}$ and $B(2,0.6,10)=\{1,3\}$. So $B(1,0.4,5)\cap B(2,0.6,10)\neq\emptyset$.

Theorem 2.10. *Let* $(X, N^{\varphi}_{[\omega,\mu]}, *)$ *be a bi-controlled fuzzy* φ*-normed linear space. Define a mapping* $M_{[\omega,\mu]}$*:* $X \times X \times [0, +\infty) \to [0, 1]$ *by* $M_{[\omega,\mu]}(x, y, t) = N^{\varphi}_{[\omega,\mu]}(x - y, t)$ *for all* $x, y \in X$ *and* $t \geq 0$*. Then* $(X, M_{[\omega,\mu]}, *)$ *is a fuzzy bi-controlled metric space.*

Proof. It is trivial to prove that $(X, M_{[\omega,\mu]}, *)$ satisfies ($FBCM1$), ($FBCM2$), ($FBCM5$), ($FBCM6$). We verify conditions ($FBCM3$) and ($FBCM4$) in the following.

($FBCM3$): By ($BCF\Phi N3$) and ($\Phi 1$), we have $M_{[\omega,\mu]}(x, y, t) = N^{\varphi}_{[\omega,\mu]}(x - y, t) = N^{\varphi}_{[\omega,\mu]}(y - x, \frac{t}{\varphi(-1)}) = N^{\varphi}_{[\omega,\mu]}(y - x, t) = M_{[\omega,\mu]}(y, x, t)$.

($FBCM4$): By ($BCF\Phi N4$), we have $M_{[\omega,\mu]}(x, z, t + s) = N^{\varphi}_{[\omega,\mu]}(x - z, t + s) = N^{\varphi}_{[\omega,\mu]}(x - y + y - z, t + s) \geq N^{\varphi}_{[\omega,\mu]}(x - y, \frac{t}{\omega(x-y)}) * N^{\varphi}_{[\omega,\mu]}(y - z, \frac{s}{\mu(y-z)})$. Taking $\psi(x, y) = \omega(x - y)$ and $\nu(y, z) = \mu(y - z)$. Then $M_{[\omega,\mu]}(x, z, t + s) \geq N^{\varphi}_{[\omega,\mu]}(x - y, \frac{t}{\psi(x,y)}) * N^{\varphi}_{[\omega,\mu]}(y - z, \frac{s}{\nu(y,z)}) = M_{[\omega,\mu]}(x, y, \frac{t}{\psi(x,y)}) * M_{[\omega,\mu]}(y, z, \frac{s}{\nu(y,z)})$. □

Corollary 2.11. *Let* $(X, N^{\varphi}_{[\omega,\mu]}, *)$ *be a bi-controlled fuzzy* φ*-normed linear space. Define a family of subsets of* X *by* $\mathcal{T}_{N^{\varphi}_{[\omega,\mu]}} = \{V \subset X : x \in V$ if and only if there exist $t > 0, r \in (0, 1)$ such that $B(x, r, t) \subset V\}$*. Then the following statements hold:*

(1) $\mathcal{T}_{N^{\varphi}_{[\omega,\mu]}}$ *is a topology on* X*.*

(2) $(X, \mathcal{T}_{N^{\varphi}_{[\omega,\mu]}})$ *is Hausdorff if* $*$ *satisfies* (T5)*:* $\bigvee_{a\in(0,1)} a * a = 1$.

Proof. (1) From Theorem 2.10, we can deduce that every fuzzy bi-controlled metric induces a topology (see [12]).

(2) Let $x, y \in X, x \neq y$. Then there exists $t_0 > 0$, such that $N^{\varphi}_{[\omega,\mu]}(x - y, t_0) < 1$. Otherwise, suppose that $N^{\varphi}_{[\omega,\mu]}(x - y, t) = 1$ for all $t > 0$. By ($BCF\Phi N6$), we have $x - y = \theta$, namely $x = y$, which is a contradiction. Set $r = N^{\varphi}_{[\omega,\mu]}(x - y, t_0)$. By (T5), there is $r_0 \in (0, 1)$, such that $r_0 * r_0 > r$. Then we claim that $B(x, 1 - r_0, \frac{t_0}{2}) \cap B(y, 1 - r_0, \frac{t_0}{2}) = \emptyset$. Otherwise, suppose that $B(x, 1 - r_0, \frac{t_0}{2}) \cap B(y, 1 - r_0, \frac{t_0}{2}) \neq \emptyset$. It follows that $B(x, 1 - r_0, \frac{t_0}{2\omega(x-z)}) \cap B(y, 1 - r_0, \frac{t_0}{2\mu(z-y)}) \neq \emptyset$. Then there exists $z \in B(x, 1 - r_0, \frac{t_0}{2\omega(x-z)}) \cap B(y, 1 - r_0, \frac{t_0}{2\mu(z-y)})$, that is, $z \in B(x, 1 - r_0, \frac{t_0}{2\omega(x-z)})$ and $z \in B(y, 1 - r_0, \frac{t_0}{2\mu(z-y)})$, which implies that $N^{\varphi}_{[\omega,\mu]}(x - z, \frac{t_0}{2\omega(x-z)}) > r_0$ and $N^{\varphi}_{[\omega,\mu]}(y - z, \frac{t_0}{2\mu(z-y)}) > r_0$. By ($BCF\Phi N4$), we have $N^{\varphi}_{[\omega,\mu]}(x - y, t_0) \geq N^{\varphi}_{[\omega,\mu]}(x - z, \frac{t_0}{2\omega(x-z)}) * N^{\varphi}_{[\omega,\mu]}(y - z, \frac{t_0}{2\mu(z-y)}) > r_0 * r_0 > r$, which is a contradiction. □

3. Some results in finite dimensional bi-controlled fuzzy φ-normed linear spaces

In this section, the concept of l-fuzzy convergent sequence in fuzzy normed linear spaces, which was introduced by Das, Bag and Chatterjee [9], is extended to bi-controlled fuzzy φ-normed linear spaces, and some results in finite dimensional bi-controlled fuzzy φ-normed linear space are established.

First, we recall some concepts and results which will be used in the following.

Definition 3.1. ([2]) Let (X, N) be a fuzzy normed linear space and $\{x_n\}$ a sequence in X.

(1) A sequence $\{x_n\}$ is said to be convergent if there exists some point $x \in X$, such that $\lim_{n\to+\infty} N(x_n - x, t) = 1$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote $\lim_{n\to+\infty} x_n = x$.

(2) A sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{n\to+\infty} N(x_{n+p} - x_n, t) = 1$ for all $p \in \mathbb{N}$ and $t > 0$.

(3) (X, N) is said to be complete if every Cauchy sequence $\{x_n\}$ in X is convergent.

Definition 3.2. Let $(X, N^{\varphi}_{[\omega,\mu]}, *)$ be a bi-controlled fuzzy φ-normed linear space and $\{x_n\}$ a sequence in X.

(1) A sequence $\{x_n\}$ is said to be $\alpha^{\varphi}_{[\omega,\mu]}$-fuzzy convergent if there exists some point $x \in X$, such that $\lim_{n\to+\infty} \bigwedge\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x_n - x, t) > 1 - \alpha\} = 0$, where $\alpha \in (0, 1)$.

If $\{x_n\}$ is an $\alpha^{\varphi}_{[\omega,\mu]}$-fuzzy convergent for all $\alpha \in (0, 1)$, then $\{x_n\}$ is called $l^{\varphi}_{[\omega,\mu]}$-fuzzy convergent.

(2) A sequence $\{x_n\}$ is said to be an $\alpha^{\varphi}_{[\omega,\mu]}$-fuzzy Cauchy sequence if there exists some point $x \in X$, such that $\lim_{n,m\to+\infty} \bigwedge\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x_n - x_m, t) > 1 - \alpha\} = 0$, where $\alpha \in (0, 1)$.

If $\{x_n\}$ is $\alpha^{\varphi}_{[\omega,\mu]}$-fuzzy Cauchy sequence for all $\alpha \in (0, 1)$, then $\{x_n\}$ is called $l^{\varphi}_{[\omega,\mu]}$-fuzzy Cauchy sequence.

(3) F is said to be $\alpha^{\varphi}_{[\omega,\mu]}$-fuzzy complete if every $\alpha^{\varphi}_{[\omega,\mu]}$-fuzzy Cauchy sequence converges to some point in F, where $F \subset X$.

If F is $\alpha^{\varphi}_{[\omega,\mu]}$-fuzzy complete for all $\alpha \in (0, 1)$, then F is called $l^{\varphi}_{[\omega,\mu]}$-fuzzy complete.

Theorem 3.3. *Let $(X, N^{\varphi}_{[\omega,\mu]}, *)$ be a bi-controlled fuzzy φ-normed linear space and $\{x_n\}$ a sequence in X. Then the following statements hold:*

(1) If $\{x_n\}$ is a convergent sequence, then $\{x_n\}$ is $l^{\varphi}_{[\omega,\mu]}$-fuzzy convergent.

(2) Furthermore, for every $l^{\varphi}_{[\omega,\mu]}$-fuzzy convergent sequence $\{x_n\}$, the limit of $\{x_n\}$ is independent on α if $*$ satisfies (T6)*: $a * a > 0$, $\forall a \in (0, 1)$, and $N^{\varphi}_{[\omega,\mu]}$ satisfies ($BCF\Phi N6^*$): If $N^{\varphi}_{[\omega,\mu]}(x, t) > 0$ for all $t \in \mathbb{R}^+$ implies $x = \theta$.*

Proof. (1) From Definition 3.1 and Definition 3.2, we have $\lim_{n\to+\infty} N^{\varphi}_{[\omega,\mu]}(x_n - x, t) = 1$ for all $t > 0$, where converges to $x \in X$. Taking $\alpha_0 \in (0, 1)$, we have $\lim_{n\to+\infty} N^{\varphi}_{[\omega,\mu]}(x_n - x, t) > 1 - \alpha_0$ for all $t > 0$. It follows that there exists $n(t_0) \in \mathbb{N}^+$, such that $N^{\varphi}_{[\omega,\mu]}(x_n - x, t_0) > 1 - \alpha_0$ for each $t_0 > 0$ and all $n > n(t_0)$. Thus, $\bigwedge\{s > 0 : N^{\varphi}_{[\omega,\mu]}(x_n - x, s) > 1 - \alpha_0\} < t_0$ for all $n > n(t_0)$. Taking the limit as $n \to +\infty$ on both sides, we have $\lim_{n\to+\infty} \bigwedge\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x_n - x, t) > 1 - \alpha_0\} \leq t_0$. By the arbitrariness of t_0, which implies that $\lim_{n\to+\infty} \bigwedge\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x_n - x, t) > 1 - \alpha_0\} = 0$. Hence $\{x_n\}$ is $\alpha^{\varphi}_{[\omega,\mu]}$-fuzzy convergent. Since α_0 is arbitrary, so $\{x_n\}$ is $l^{\varphi}_{[\omega,\mu]}$-fuzzy convergent.

(2) Now we prove that the limit of $\{x_n\}$ is independent on α in the following.

Let $\{x_n\}$ be an $l^{\varphi}_{[\omega,\mu]}$-fuzzy convergent sequence. From Definition 3.2, we have $\lim_{n\to+\infty} \bigwedge\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x_n - x, t) > 1 - \alpha\} = 0$ for each $\alpha \in (0, 1)$. We suppose that there exist $\alpha_1, \alpha_2 \in (0, 1)$ with $\alpha_1 \neq \alpha_2$, such that $\lim_{n\to+\infty} \bigwedge\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x_n - x_1, t) > 1 - \alpha_1\} = 0$ and $\lim_{n\to+\infty} \bigwedge\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x_n - x_2, t) > 1 - \alpha_2\} = 0$ for $x_1, x_2 \in X$, respectively. Then for a given $\varepsilon > 0$, there exist $n(\alpha_1), n(\alpha_2) \in \mathbb{N}^+$, such that

$$\bigwedge\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x_n - x_1, t) > 1 - \alpha_1\} < \tfrac{\varepsilon}{2\omega(x_n - x_1)}, \forall n > n(\alpha_1),$$

and

$$\bigwedge\{t > 0 : N^{\varphi}_{[\omega,\mu]}(x_n - x_2, t) > 1 - \alpha_2\} < \tfrac{\varepsilon}{2\mu(x_n - x_2)}, \forall n > n(\alpha_2).$$

Then $N^{\varphi}_{[\omega,\mu]}(x_n - x_1, \frac{\varepsilon}{2\omega(x_n - x_1)}) > 1 - \alpha_1$ and $N^{\varphi}_{[\omega,\mu]}(x_n - x_2, \frac{\varepsilon}{2\mu(x_n - x_2)}) > 1 - \alpha_2$.

Since $\alpha_1 \neq \alpha_2$, without loss of generality, suppose $\alpha_1 > \alpha_2$. Then we have $N^{\varphi}_{[\omega,\mu]}(x_n - x_1, \frac{\varepsilon}{2\omega(x_n - x_1)}) > 1 - \alpha_1$ and $N^{\varphi}_{[\omega,\mu]}(x_n - x_2, \frac{\varepsilon}{2\mu(x_n - x_2)}) > 1 - \alpha_2 > 1 - \alpha_1$, for all $n > n_0$, where $n_0 = \max\{n(\alpha_1), n(\alpha_2)\}$. By ($BCF\Phi N4$) and ($T6$), we have

$$\begin{aligned}
& N^{\varphi}_{[\omega,\mu]}(x_1 - x_2, \varepsilon) \\
\geq\; & N^{\varphi}_{[\omega,\mu]}\left(x_n - x_1, \frac{\varepsilon}{2\omega(x_n - x_1)}\right) * N^{\varphi}_{[\omega,\mu]}\left(x_n - x_2, \frac{\varepsilon}{2\mu(x_n - x_2)}\right) \\
>\; & (1 - \alpha_1) * (1 - \alpha_1) > 0
\end{aligned}$$

Taking $t = \varepsilon$. By the arbitrariness of ε, then we get $N^{\varphi}_{[\omega,\mu]}(x_1 - x_2, t) > 0$ for all $t > 0$. From ($BCF\Phi N6^*$), we have $x_1 - x_2 = \theta$. So $x_1 = x_2$. □

Lemma 3.4. *Let $(X, N^{\varphi}_{[\omega,\mu]}, *)$ be a bi-controlled fuzzy φ-normed linearly space with the underlying t-norm $*$, which is continuous at $(1,1)$, and $\{x_i\}_{i=1}^{n}$ be a linear independent set of vectors in X. Then there exist $c > 0$ and $\delta \in (0,1)$, such that $N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\alpha_i x_i, \frac{\varphi(\sum_{i=1}^{n}|\alpha_i|)c}{\omega(\sum_{i=1}^{n}\alpha_i x_i)\mu(\sum_{i=1}^{n}\alpha_i x_i)}\right) < 1-\delta$, for all $\alpha_i \in \mathbb{R}, i = 1,2,\ldots,n$, where $\sum_{i=1}^{n}|\alpha_i| \neq 0$.*

Proof. Firstly, it is clear that $N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\alpha_i x_i, \frac{\varphi(\sum_{i=1}^{n}|\alpha_i|)c}{\omega(\sum_{i=1}^{n}\alpha_i x_i)\mu(\sum_{i=1}^{n}\alpha_i x_i)}\right) < 1-\delta$ is equivalent to

$$N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\beta_i x_i, \frac{c}{\omega(\sum_{i=1}^{n}\beta_i x_i)\mu(\sum_{i=1}^{n}\beta_i x_i)}\right) < 1-\delta,$$

for some $c > 0$ and $\delta \in (0,1)$, where $\beta_i \in \mathbb{R}, i = 1,2,\ldots,n$ and $\sum_{i=1}^{n}|\beta_i| = 1$. Now we prove that $N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\beta_i x_i, \frac{c}{\omega(\sum_{i=1}^{n}\beta_i x_i)\mu(\sum_{i=1}^{n}\beta_i x_i)}\right) < 1-\delta$ holds in the following steps.

Step 1: We suppose that there exists $\{\beta_i\}_{i=1}^{n}$ with $\sum_{i=1}^{n}|\beta_i| = 1$, such that $N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\beta_i x_i, \frac{c}{\omega(\sum_{i=1}^{n}\beta_i x_i)\mu(\sum_{i=1}^{n}\beta_i x_i)}\right) \geq 1-\delta$ for all $c > 0$ and $\delta \in (0,1)$. Taking $c_m = \frac{1}{m}$ and $\delta_m = \frac{1}{m}$. Then we have

$$N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\beta_{[i,m]}x_i, \frac{1}{m\omega(\sum_{i=1}^{n}\beta_{[i,m]}x_i)\mu(\sum_{i=1}^{n}\beta_{[i,m]}x_i)}\right) \geq 1-\frac{1}{m},$$

for some sequence $\{\beta_{[i,m]}\}_{i=1}^{n}$ with $\sum_{i=1}^{n}|\beta_{[i,m]}| = 1$, where $\{\beta_{[i,m]}\}_{i=1}^{n}$ is the subsequence of $\{\beta_i\}_{i=1}^{n}$ with $\sum_{i=1}^{n}|\beta_{[i,m]}| = 1$ and $m = 1,2,\ldots$. On the other hand, since $\sum_{i=1}^{n}|\beta_{[i,m]}| = 1$, it follows that $0 \leq |\beta_{[i,m]}| \leq 1$ for all $i = 1,2,\ldots,n$. Namely, the sequence $\{\beta_{[i,m]}\}_{i=1}^{n}$ is bounded for each fixed i. It implies that sequence $\{\beta_{[i,m]}\}_{i=1}^{n}$ has a convergent subsequence $\{\gamma_{[i,m]}\}_{i=1}^{n}$ with $\sum_{i=1}^{n}|\gamma_{[i,m]}| = 1$ for each fixed i, where $m = 1,2,\ldots$. For each i, we denote $\lim_{m\to+\infty}\gamma_{[i,m]} = \beta_i$, where $i = 1,2,\ldots,n$. By $(BCF\Phi N3)$ and $(BCF\Phi N4)$, we have

$$\begin{aligned}
&N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}(\gamma_{[i,m]}-\beta_i)x_i, t\right)\\
&= N^{\varphi}_{[\omega,\mu]}\left((\gamma_{[1,m]}-\beta_1)x_1 + \sum_{i=2}^{n}(\gamma_{[i,m]}-\beta_i)x_i, \frac{1}{n}t + \left(1-\frac{1}{n}\right)t\right)\\
&\geq N^{\varphi}_{[\omega,\mu]}\left((\gamma_{[1,m]}-\beta_1)x_1, \frac{\frac{1}{n}t}{\omega\left((\gamma_{[1,m]}-\beta_1)x_1\right)}\right) * N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=2}^{n}(\gamma_{[i,m]}-\beta_i)x_i, \frac{\left(1-\frac{1}{n}\right)t}{\mu\left(\sum_{i=2}^{n}(\gamma_{[i,m]}-\beta_i)x_i\right)}\right)\\
&\geq N^{\varphi}_{[\omega,\mu]}\left(x_1, \frac{t}{n\omega\left((\gamma_{[1,m]}-\beta_1)x_1\right)\varphi(\gamma_{[1,m]}-\beta_1)}\right) * N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=2}^{n}(\gamma_{[i,m]}-\beta_i)x_i, \frac{\left(1-\frac{1}{n}\right)t}{\mu\left(\sum_{i=2}^{n}(\gamma_{[i,m]}-\beta_i)x_i\right)}\right)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=2}^{n}(\gamma_{[i,m]}-\beta_i)x_i, \frac{\left(1-\frac{1}{n}\right)t}{\mu\left(\sum_{i=2}^{n}(\gamma_{[i,m]}-\beta_i)x_i\right)}\right)\\
&= N^{\varphi}_{[\omega,\mu]}\left((\gamma_{[2,m]}-\beta_2)x_2 + \sum_{i=3}^{n}(\gamma_{[i,m]}-\beta_i)x_i, \frac{\frac{1}{n}t + \left(1-\frac{2}{n}\right)t}{\mu\left(\sum_{i=2}^{n}(\gamma_{[i,m]}-\beta_i)x_i\right)}\right)\\
&\geq N^{\varphi}_{[\omega,\mu]}\left(x_2, \frac{t}{n\omega\left((\gamma_{[2,m]}-\beta_1)x_2\right)\varphi(\gamma_{[2,m]}-\beta_2)\mu\left(\sum_{i=2}^{n}(\gamma_{[i,m]}-\beta_i)x_i\right)}\right)\\
&* N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=3}^{n}(\gamma_{[i,m]}-\beta_i)x_i, \frac{\left(1-\frac{2}{n}\right)t}{\mu\left(\sum_{i=2}^{n}(\gamma_{[i,m]}-\beta_i)x_i\right)\mu\left(\sum_{i=3}^{n}(\gamma_{[i,m]}-\beta_i)x_i\right)}\right)
\end{aligned}$$

Continuing this way, we have

$$
\begin{aligned}
& N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}(\gamma_{[i,m]}-\beta_i)x_i, t\right) \\
& \geq N^{\varphi}_{[\omega,\mu]}\left(x_1, \frac{t}{n\omega\,(\gamma_{[1,m]}-\beta_1)x_1)\,\varphi(\gamma_{[1,m]}-\beta_1)}\right) \\
& * N^{\varphi}_{[\omega,\mu]}\left(x_2, \frac{t}{n\omega\,(\gamma_{[2,m]}-\beta_1)x_2)\,\varphi(\gamma_{[2,m]}-\beta_2)\mu\,(\sum_{i=2}^{n}(\gamma_{[i,m]}-\beta_i)x_i)}\right) * \cdots \\
& * N^{\varphi}_{[\omega,\mu]}\left(x_n, \frac{t}{n\varphi(\gamma_{[n,m]}-\beta_n)\prod_{j=2}^{n}\mu\left(\sum_{i=j}^{n}(\gamma_{[i,m]}-\beta_i)x_i\right)}\right)
\end{aligned}
$$

Taking limit as $m \to +\infty$ on both sides. By $(BCF\Phi N5)$ and $\lim_{m\to+\infty}\gamma_{[i,m]} = \beta_i$ for each i, it implies that

$$\lim_{m\to+\infty} N^{\varphi}_{[\omega,\mu]}\,(\textstyle\sum_{i=1}^{n}(\gamma_{[i,m]}-\beta_i)x_i, t) = 1 \text{ for all } t > 0.$$

Step 2: By $(BCF\Phi N3)$ and $(BCF\Phi N4)$, we have

$$
\begin{aligned}
& N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\beta_i x_i, 2t\right) \\
& \geq N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}(\beta_i-\gamma_{[i,m]})x_i + \sum_{i=1}^{n}\gamma_{[i,m]}x_i, 2t\right) \\
& \geq N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}(\beta_i-\gamma_{[i,m]})x_i, \frac{t}{\omega(\sum_{i=1}^{n}(\beta_i-\gamma_{[i,m]})x_i)}\right) * N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\gamma_{[i,m]}x_i, \frac{t}{\mu(\sum_{i=1}^{n}\gamma_{[i,m]}x_i)}\right)
\end{aligned}
$$

Additionally, we note that there exist some m such that $mt > 1$ for all $t > 0$. By $(T3)$, $(BCF\Phi N2)$ and $(BCF\Phi N3)$, we have

$$
\begin{aligned}
& N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\gamma_{[i,m]}x_i, \frac{t}{\mu(\sum_{i=1}^{n}\gamma_{[i,m]}x_i)}\right) \\
& \geq N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\gamma_{[i,m]}x_i, \frac{t}{m\omega(\sum_{i=1}^{n}\gamma_{[i,m]}x_i)\mu(\sum_{i=1}^{n}\gamma_{[i,m]}x_i)}\right) * N^{\varphi}_{[\omega,\mu]}\left(\theta, \frac{mt-1}{m\mu(\theta)\mu(\sum_{i=1}^{n}\gamma_{[i,m]}x_i)}\right) \\
& \geq N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\gamma_{[i,m]}x_i, \frac{1}{m\omega(\sum_{i=1}^{n}\gamma_{[i,m]}x_i)\mu(\sum_{i=1}^{n}\gamma_{[i,m]}x_i)}\right)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
& N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\beta_i x_i, 2t\right) \\
& \geq N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}(\beta_i-\gamma_{[i,m]})x_i, \frac{t}{\omega(\sum_{i=1}^{n}(\beta_i-\gamma_{[i,m]})x_i)}\right) * N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{n}\gamma_{[i,m]}x_i, \frac{1}{m\omega(\sum_{i=1}^{n}\gamma_{[i,m]}x_i)\mu(\sum_{i=1}^{n}\gamma_{[i,m]}x_i)}\right)
\end{aligned}
$$

Taking limit as $m \to +\infty$ on both sides. By Step 1, then we have

$$N^{\varphi}_{[\omega,\mu]}\,(\textstyle\sum_{i=1}^{n}\beta_i x_i, 2t) \geq \lim_{m\to+\infty}\left(1-\frac{1}{m}\right) * 1 = 1.$$

So $N^{\varphi}_{[\omega,\mu]}\,(\sum_{i=1}^{n}\beta_i x_i, 2t) = 1$ for all $t > 0$. By $(BCF\Phi N2)$, hence $\sum_{i=1}^{n}\beta_i x_i = \theta$, which is a contradiction. □

Theorem 3.5. *If $(X, N^{\varphi}_{[\omega,\mu]}, *)$ is a finite dimensional bi-controlled fuzzy φ-normed linear space with the underlying t-norm $*$, which is continuous at $(1,1)$, then $(X, N^{\varphi}_{[\omega,\mu]}, *)$ is complete.*

Proof. By assumption, the dimension of $(X, N^{\varphi}_{[\omega,\mu]}, *)$ is finite. Set $\dim X = r$ and let $\{e_i\}_{i=1}^{r}$ be a basis for X. Let $\{x_n\}$ be a Cauchy sequence in X. Then there exists suitable scalars $\{\beta_{[i,n]}\}_{i=1}^{r}$ such that $x_n = \sum_{i=1}^{r} \beta_{[i,n]} e_i$. Firstly, we note that $\{x_n\}$ is not necessarily a constant sequence, otherwise, $\sum_{i=1}^{r} |\beta_{[i,n]} - \beta_{[i,m]}| = 0$ for all $m, n \in \mathbb{N}^+$. Since $\{x_n\}$ be a Cauchy sequence in X, we have $\lim_{m,n\to+\infty} N^{\varphi}_{[\omega,\mu]}(x_m - x_n, t) = 1$, where $x_m = \sum_{i=1}^{r} \beta_{[i,m]} e_i$. Namely, $\lim_{m,n\to+\infty} N^{\varphi}_{[\omega,\mu]}(\sum_{i=1}^{r}(\beta_{[i,m]} - \beta_{[i,n]})e_i, t) = 1$. From Lemma 3.4, there exist some $c > 0$ and $\delta \in (0,1)$, such that

$$N^{\varphi}_{[\omega,\mu]}(\textstyle\sum_{i=1}^{r}(\beta_{[i,m]} - \beta_{[i,n]})e_i, \frac{\varphi(\sum_{i=1}^{r}|\beta_{[i,m]}-\beta_{[i,n]}|)c}{\omega(\sum_{i=1}^{r}(\beta_{[i,m]}-\beta_{[i,n]})e_i)\mu(\sum_{i=1}^{r}(\beta_{[i,m]}-\beta_{[i,n]})e_i)}) < 1 - \delta.$$

Furthermore, for the above δ, there exists $n_0 \in \mathbb{N}^+$ such that $N^{\varphi}_{[\omega,\mu]}(\sum_{i=1}^{r}(\beta_{[i,m]} - \beta_{[i,n]})e_i, t) > 1 - \delta$ for all $n, m > n_0$. Thus, we have

$$N^{\varphi}_{[\omega,\mu]}(\sum_{i=1}^{r}(\beta_{[i,m]} - \beta_{[i,n]})e_i, \frac{\varphi(\sum_{i=1}^{r}|\beta_{[i,m]} - \beta_{[i,n]}|)c}{\omega(\sum_{i=1}^{r}(\beta_{[i,m]} - \beta_{[i,n]})e_i)\mu(\sum_{i=1}^{r}(\beta_{[i,m]} - \beta_{[i,n]})e_i)}) < N^{\varphi}_{[\omega,\mu]}(\sum_{i=1}^{r}(\beta_{[i,m]} - \beta_{[i,n]})e_i, t).$$

By $(BCF\Phi N5)$, we have $\frac{\varphi(\sum_{i=1}^{r}|\beta_{[i,m]}-\beta_{[i,n]}|)c}{\omega(\sum_{i=1}^{r}(\beta_{[i,m]}-\beta_{[i,n]})e_i)\mu(\sum_{i=1}^{r}(\beta_{[i,m]}-\beta_{[i,n]})e_i)} < t$ for all $t > 0$. Hence,

$$\lim_{m,n\to+\infty} \frac{\varphi(\sum_{i=1}^{r}|\beta_{[i,m]}-\beta_{[i,n]}|)}{\omega(\sum_{i=1}^{r}(\beta_{[i,m]}-\beta_{[i,n]})e_i)\mu(\sum_{i=1}^{r}(\beta_{[i,m]}-\beta_{[i,n]})e_i)} = 0.$$

Since ω and μ are bounded, it follows that $\lim_{m,n\to+\infty} \varphi(\sum_{i=1}^{r}|\beta_{[i,m]} - \beta_{[i,n]}|) = 0$. By $(\Phi 2)$ and $(\Phi 5)$, we have $|\beta_{[i,m]} - \beta_{[i,n]}| = 0$. Thus, $\{\beta_{[i,m]}\}$ is a Cauchy sequence for each $i = 1, 2, \ldots, r$. Set $\lim_{m,n\to+\infty} \beta_{[i,m]} = \beta_i$ for each $i = 1, 2, \ldots r$, and denote $\sum_{i=1}^{r} \beta_i e_i = x \in X$. We claim that $\lim_{m\to+\infty} x_m = x$. Indeed, for all $t > 0$ and each $i = 1, 2, \ldots, r$, we have

$$\begin{aligned}
& N^{\varphi}_{[\omega,\mu]}(x_m - x, t)) \\
&= N^{\varphi}_{[\omega,\mu]}\left(\sum_{i=1}^{r}(\beta_{[i,m]} - \beta_i)e_i, t\right) \\
&\geq N^{\varphi}_{[\omega,\mu]}\left(e_1, \frac{t}{r\omega\left((\beta_{[1,m]} - \beta_1)e_1\right)\varphi(\beta_{[1,m]} - \beta_1)}\right) \\
&* N^{\varphi}_{[\omega,\mu]}\left(e_2, \frac{t}{r\omega\left((\beta_{[2,m]} - \beta_2)e_2\right)\varphi(\beta_{[2,m]} - \beta_2)\mu\left(\sum_{i=2}^{r}(\beta_{[i,m]} - \beta_i)e_i\right)}\right) * \cdots \\
&* N^{\varphi}_{[\omega,\mu]}\left(e_m, \frac{t}{r\varphi(\beta_{[r,m]} - \beta_r)\prod_{j=2}^{r}\mu\left(\sum_{i=j}^{r}(\beta_{[i,m]} - \beta_i)e_i\right)}\right)
\end{aligned}$$

Taking limit as $m \to +\infty$ on both sides. For each $i = 1, 2, \cdots, r$, by $(BCF\Phi N5)$, we have

$$\lim_{m\to+\infty} N^{\varphi}_{[\omega,\mu]}(x_m - x, t) = 1, \forall t > 0.$$

By $(BCF\Phi N2)$, then $\lim_{m\to+\infty}(x_m - x) = \theta$, that is $\lim_{m\to+\infty} x_m = x$. Hence, X is complete. □

4. Conclusions

The main objective of this paper was to present some basic results on bi-controlled fuzzy φ-normed linear spaces. In Section 2, based on a dilation function φ, by modifying the notion of Bag-Samanta's type fuzzy norms, we introduce the concept of bi-controlled fuzzy φ-norm for two non-comparable functions, and illustrate several examples. Additionally, we establish the relationship for bi-controlled fuzzy φ-norms into a family of pseudo-φ-norms. In Section 3, we introduce the concept of $I^{\varphi}_{[\omega,\mu]}$-fuzzy convergence and investigate the completeness of finite dimensional bi-controlled fuzzy φ-normed linear spaces. In the future, we think that it may be interesting to present some new theorems and results in three-variable generalizations of fuzzy metric spaces from functional and topological points of view, as well as to make a similar investigation in the sense of Kočinac and Rashid [16] in generalized fuzzy 2-normed spaces and related structures.

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