Filomat 40:8 (2026), 2895–2906
https://doi.org/10.2298/FIL2608895B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some notes on strongly topological gyrogroups

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Abstract. A topological gyrogroup is a gyrogroup endowed with a topology such that the binary operation is jointly continuous and the inverse mapping is also continuous. In this paper, we find an error of the proof in [6, Lemma 3.13], then we revise the construction of metric and reprove the result that a strongly topological gyrogroup G is feathered iff it contains a compact L-subgyrogroup P such that the quotient space G/P is metrizable. Then, combining generalized metric properties, some characterizations of metrizability on quotient spaces of gyrogroups with respect to strong subgyrogroups are researched.

1. Introduction

The gyrogroup was firstly posed by A.A. Ungar when he studied the c-ball of relativistically admissible velocities with Einstein velocity addition [25]. The Einstein velocity addition $\oplus_E$ in the c-ball is given by the following equation

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1+\frac{\langle \mathbf{u},\mathbf{v}\rangle}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}\langle \mathbf{u},\mathbf{v}\rangle \mathbf{u} \right\},$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 : \| \mathbf{v} \| < c\}$ and $\gamma_{\mathbf{u}}$ is the Lorentz factor given by

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{u}\|^2}{c^2}}}.$$

The system $(\mathbb{R}^3_c, \oplus_E)$ does not form a group since $\oplus_E$ is neither associative nor commutative. A gyrogroup is a relaxation of a group such that the associativity condition has been replaced by a weaker one. Indeed, a gyrogroup is the most natural extension of a group into the regime of the non-associative algebra that we need for extending analytic Euclidean geometry into analytic hyperbolic, see [22–25]. In 2017, W.

2020 *Mathematics Subject Classification*. Primary 22A22, Secondary 54A20, 20N05, 18A32.
Keywords. Topological gyrogroups, metrizable, feathered.
Received: 14 March 2025; Revised: 25 January 2026; Accepted: 27 January 2026
Communicated by Ljubiša D. R. Kočinac
Research supported by the National Natural Science Foundation of China (Nos. 12471070), the Project of Suqian Sci&Tech Program (Grant No. K202441), 2024 Research Project of Beijing Polytechnic College (Nos. BGY2024KY-42Z), the Foundation of PhD start-up of Suqian University (Nos. 2024XRC003), the Qing Lan Project, and the interdisciplinary Integration and Innovation Project of Suqian university (2025XKTD02).
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Atiponrat [3] introduced the concept of topological gyrogroups as a generalization of topological groups and investigated some properties of them. A topological gyrogroup is a gyrogroup endowed with a topology such that the binary operation is jointly continuous and the inverse mapping is also continuous. It is obvious that every topological group is a topological gyrogroup. However, every topological gyrogroup whose gyrations are not identically equal to the identity is not a topological group, such as the classical Möbius gyrogroup and Einstein gyrogroup.

By further study on Möbius gyrogroups, Bao and Lin [6] introduced the concept of strongly topological gyrogroups. A topological gyrogroup G is called a strongly topological gyrogroup if there exists a neighborhood base $\mathscr{U}$ of 0 such that, for every $U \in \mathscr{U}$, $\text{gyr}[x, y](U) = U$ for any $x, y \in G$. Clearly, every topological group is a strongly topological gyrogroup. For more about strongly topological gyrogroups, see [7–10, 18, 26, 27]. In [6], the authors investigated feathered strongly topological gyrogroups and showed that a strongly topological gyrogroup G is feathered if and only if it contains a compact L-subgyrogroup H such that the quotient space G/H is metrizable, which deduces that every feathered strongly topological gyrogroup is a paracompact space, see [6, Lemma 3.13]. However, there is a gap in the proof of the metric ϱ generating the quotient topology of the quotient space. In Section 3, we revise the proof and show that a strongly topological gyrogroup G is feathered if and only if it contains a compact L-subgyrogroup P such that the quotient space G/P is metrizable. Then, we introduce the concept of topological N-gyrogroups and show that every feathered topological N-gyrogroup is paracompact.

Then, we are continuous to investigate the generalized metric property of quotient space G/H, where G is a strongly topological gyrogroup and H is a closed strong subgyrogroup of G. Assume that if G is a strongly topological gyrogroup, H is a closed strong subgyrogroup of G and H is inner neutral. It is shown that if the quotient space G/H is an snf-countable space, then G/H is an sof-countable space; if the space G/H is a csf-countable and sequential α_7-space, then G/H is first-countable, which deduces that the quotient space G/H is metrizable if and only if the quotient space G/H is first-countable if and only if G/H is a bisequential space if and only if G/H is a weakly first-countable space if and only if G/H is a csf-countable and sequential α_7-space.

2. Preliminary

Throughout this paper, all topological spaces are assumed to be Hausdorff, unless otherwise is explicitly stated. Let $\mathbb{N}$ be the set of all positive integers and ω the first infinite ordinal. The readers may consult [2, 12, 17, 25] for notation and terminology not explicitly given here. Next we recall some definitions and facts.

Definition 2.1. ([3]) Let G be a nonempty set, and let $\oplus : G \times G \to G$ be a binary operation on G. Then the pair $(G, \oplus)$ is called a *magma or groupoid*. A function f from a groupoid $(G_1, \oplus_1)$ to a groupoid $(G_2, \oplus_2)$ is called a *groupoid homomorphism* if $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$ for any elements $x, y \in G_1$. Furthermore, a bijective groupoid homomorphism from a groupoid $(G, \oplus)$ to itself will be called a *groupoid automorphism*. We write $\text{Aut}(G, \oplus)$ for the set of all automorphisms of a groupoid $(G, \oplus)$.

Definition 2.2. ([25]) Let $(G, \oplus)$ be a groupoid. The system $(G, \oplus)$ is called a *gyrogroup*, if its binary operation satisfies the following conditions:

($G1$) There exists a unique identity element $0 \in G$ such that $0 \oplus a = a = a \oplus 0$ for all $a \in G$.

($G2$) For each $x \in G$, there exists a unique inverse element $\ominus x \in G$ such that $\ominus x \oplus x = 0 = x \oplus (\ominus x)$.

($G3$) For all $x, y \in G$, there exists $\text{gyr}[x, y] \in \text{Aut}(G, \oplus)$ with the property that $x \oplus (y \oplus z) = (x \oplus y) \oplus \text{gyr}[x, y](z)$ for all $z \in G$.

($G4$) For any $x, y \in G$, $\text{gyr}[x \oplus y, y] = \text{gyr}[x, y]$.

Clearly, every group is a gyrogroup, where the groupoid automorphism is the identity function.

Definition 2.3. ([21]) Let $(G, \oplus)$ be a gyrogroup. A nonempty subset H of G is called a *subgyrogroup*, denoted by $H \leq G$, if H forms a gyrogroup under the operation inherited from G and the restriction of $\mathrm{gyr}[a, b]$ to H is an automorphism of H for all $a, b \in H$.

Furthermore, a subgyrogroup H of G is said to be an *L-subgyrogroup*, denoted by $H \leq_L G$, if $\mathrm{gyr}[a, h](H) = H$ for all $a \in G$ and $h \in H$.

Definition 2.4. ([9])A subgyrogroup H of a topological gyrogroup G is called *strong subgyrogroup* if for any $x, y \in G$, we have $gyr[x, y](H) = H$.

Obviously, every strong subgyrogroup is an L-subgyrogroup.

Lemma 2.5. ([25]). *Let $(G, \oplus)$ be a gyrogroup. Then for any $x, y, z \in G$, we obtain the following:*

1. $(\ominus x) \oplus (x \oplus y) = y$;
2. $(x \oplus (\ominus y)) \oplus gyr[x, \ominus y](y) = x$;
3. $(x \oplus gyr[x, y](\ominus y)) \oplus y = x$;
4. $gyr[x, y](z) = \ominus(x \oplus y) \oplus (x \oplus (y \oplus z))$;
5. $(\ominus x \oplus y) \oplus gyr[\ominus x, y](\ominus y \oplus z) = \ominus x \oplus z$;
6. $(x \oplus y) \oplus z = x \oplus (y \oplus gyr[y, x](z))$;
7. $Agyr[x, y] = gyr[Ax, Ay]A$ *for any automorphism* A *on* G;
8. $gyr[\ominus x, \ominus y] = gyr[x, y]$;
9. $gyr^{-1}[x, y] = gyr[y, x]$;
10. $\ominus(x \oplus y) = gyr[x, y](\ominus y \ominus x)$.

Proposition 2.6. ([21]) *Let G be a gyrogroup and let $X \subseteq G$. Then the following are equivalent:*
(1) $gyr[x, y](X) \subseteq X$ *for all* $x, y \in G$;
(2) $gyr[x, y](X) = X$ *for all* $x, y \in G$;

Definition 2.7. ([3])A triple $(G, \tau, \oplus)$ is called a *topological gyrogroup* if the following statements hold:

(1) (G, τ) is a topological space.

(2) $(G, \oplus)$ is a gyrogroup.

(3) The binary operation $\oplus : G \times G \to G$ is jointly continuous while $G \times G$ is endowed with the product topology, and the operation of taking the inverse $\ominus(\cdot) : G \to G$, i.e. $x \to \ominus x$, is also continuous.

It is clear that every topological group is a topological gyrogroup. The following example is a topological gyrogroup, but not a topological group.

Example 2.8. ([3, Example 3])The Einstein gyrogroup with the standard topology is a topological gyrogroup but not a topological group.

By further research on topological gyrogroups, Bao and Lin in [6] introduced the concept of strongly topological gyrogroups.

Definition 2.9. ([6])Let G be a topological gyrogroup. We say that G is a *strongly topological gyrogroup* if there exists a neighborhood base $\mathscr{U}$ of 0 such that, for every $U \in \mathscr{U}$, $\mathrm{gyr}[x, y](U) = U$ for any $x, y \in G$. For convenience, we say that G is a strongly topological gyrogroup with neighborhood base $\mathscr{U}$ of 0.

The following example is a strongly topological gyrogroup and it is not a topological group.

Example 2.10. The Möbius gyrogroup with the standard topology is a strongly topological gyrogroup but not a topological group.

Let $\mathbb{D}$ be the complex open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. We consider $\mathbb{D}$ with the standard topology. Define a Möbius addition $\oplus_M : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ to be a function such that

$$a \oplus_M b = \frac{a+b}{1+\bar{a}b} \text{ for all } a, b \in \mathbb{D}.$$

Firstly, we show that $\oplus_M$ is a binary operation on $\mathbb{D}$. Let $a, b \in \mathbb{D}$. Then $|a| < 1$ and $|b| < 1$, and so $1 + \bar{a}b \neq 0$; otherwise, we would have $\bar{a}b = -1$ and hence $|a||b| = |\bar{a}b| = 1$, a contradiction. Note that

$$0 < (1 - |a|^2)(1 - |b|^2) = 1 - (|a|^2 + |b|^2) + |a|^2|b|^2.$$

Hence, $|a|^2 + |b|^2 < 1 + |a|^2|b|^2$. So,

$$\begin{aligned} |a+b|^2 &= (a+b)\overline{a+b} \\ &= (a+b)(\bar{a}+\bar{b}) \\ &= a\bar{a} + \bar{a}b + a\bar{b} + b\bar{b} \\ &= |a|^2 + \bar{a}b + a\bar{b} + |b|^2 \\ &< 1 + |a|^2|b|^2 + \bar{a}b + a\bar{b} \\ &= (1+\bar{a}b)(1+a\bar{b}) \\ &= (1+\bar{a}b)\overline{(1+\bar{a}b)} \\ &= (1+\bar{a}b)^2, \end{aligned}$$

which deduces that

$$|a \oplus_M b|^2 = |\frac{a+b}{1+\bar{a}b}|^2 < 1.$$

Secondly, for any $a \in \mathbb{D}$,

$$0 \oplus_M a = \frac{0+a}{1} = a = a \oplus_M 0 = \frac{a+0}{1} \text{ and } (-a) \oplus_M a = 0 = a \oplus_M (-a).$$

Note that

$$a \oplus_M b = \frac{a \oplus_M b}{b \oplus_M a}(b \oplus_M a) \text{ and } \frac{a \oplus_M b}{b \oplus_M a} = \frac{\frac{a+b}{1+\bar{a}b}}{\frac{b+a}{1+\bar{b}a}} = \frac{1+a\bar{b}}{1+\bar{a}b}.$$

Therefore, define

$$gyr[a,b] = \frac{1+a\bar{b}}{1+\bar{a}b} \in S,$$

where $S = \{z \in \mathbb{C} : |z| = 1\}$. Since

$$a \oplus_M b = \frac{a+b}{1+\bar{a}b},$$

we have that

$$a \oplus_M (b \oplus_M c) = \frac{a + \frac{b+c}{1+\bar{b}c}}{1 + \bar{a}\frac{b+c}{1+\bar{b}c}} = \frac{a+b+c+a\bar{b}c}{1+\bar{a}b+\bar{b}c+\bar{a}c}$$

and

$$\begin{aligned} (a \oplus_M b) \oplus_M gyr[a,b]c &= \frac{(a+b+c+a\bar{b}c)(1+a\bar{b})(1+\bar{a}b)}{(1+\bar{a}b)(1+a\bar{b}+\bar{a}b+a^2b^2+\bar{a}c+a^2\bar{b}c+\bar{b}c+ab^2c)} \\ &= \frac{(a+b+c+a\bar{b}c)(1+a\bar{b})(1+\bar{a}b)}{(1+\bar{a}b)(1+a\bar{b})(1+\bar{a}b+\bar{a}c+\bar{b}c)} = \frac{a+b+c+a\bar{b}c}{1+\bar{a}b+\bar{b}c+\bar{a}c}. \end{aligned}$$

Therefore, $a \oplus_M (b \oplus_M c) = (a \oplus_M b) \oplus_M gyr[a,b]c$ for any $a, b, c \in \mathbb{D}$.

Thirdly, from

$$gyr[a,b] = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

it follows that

$$\begin{aligned} gyr[a \oplus_M b, b] &= \frac{1 + (a \oplus_M b)\overline{b}}{1 + \overline{(a \oplus_M b)}b} \\ &= \frac{1 + \frac{a+b}{1+\overline{a}b}\overline{b}}{1 + \frac{\overline{a}+\overline{b}}{1+a\overline{b}}b} \\ &= \frac{1 + a\overline{b}}{1 + \overline{a}b} \\ &= gyr[a,b]. \end{aligned}$$

Therefore, we conclude that $(\mathbb{D}, \oplus_M)$ is a gyrogroup, and

$$\text{gyr}[a,b](c) = \frac{1 + a\bar{b}}{1 + \bar{a}b}c \text{ , for any } a, b, c \in \mathbb{D}.$$

However, if we take

$$a = \frac{1}{2}, b = \frac{i}{2}, c = -\frac{1}{2},$$

we have

$$(a \oplus_M b) \oplus_M c = \frac{8 + 15i}{34} \text{ and } a \oplus_M (b \oplus_M c) = \frac{10 + 15i}{26}.$$

Thus, we obtain that $(\mathbb{D}, \oplus_M)$ is not a group.

Finally, we show that $(\mathbb{D}, \oplus_M)$ is a strongly topological gyrogroup with the standard topology. Indeed, for any $n \in \omega$, let $U_n = \{x \in \mathbb{D} : |x| < \frac{1}{n}\}$. Then, $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ is a neighborhood base of 0. Moreover, we observe that $|\frac{1+a\bar{b}}{1+\bar{a}b}| = 1$. Therefore, we obtain that $\text{gyr}[x, y](U) \subseteq U$, for any $x, y \in \mathbb{D}$ and each $U \in \mathscr{U}$, then it follows that $\text{gyr}[x, y](U) = U$ by Proposition 2.6. Hence, $(\mathbb{D}, \oplus_M)$ is a strongly topological gyrogroup with the standard topology.

We recall the concept of the coset space of a topological gyrogroup.

Let $(G, \tau, \oplus)$ be a topological gyrogroup and H an L-subgyrogroup of G. Then $G/H = \{a \oplus H : a \in G\}$ is a coset space which defines a partition of G. We denote by π the mapping $a \mapsto a \oplus H$ from G onto G/H. Denote by $\tau(G)$ the topology of G, the quotient topology on G/H is as follows:

$$\tau(G/H) = \{O \subseteq G/H : \pi^{-1}(O) \in \tau(G)\}.$$

3. On feathered (strongly) topological gyrogroups

In [6], Bao and Lin posed the following questions.

Question 3.1. *If a topological gyrogroup is a p-space, is it paracompact? What if the topological gyrogroup is a strongly topological gyrogroup?*

Question 3.2. *If a topological gyrogroup is a p-space, is it a D-space? What if the topological gyrogroup is a strongly topological gyrogroup?*

Recall that a topological gyrogroup G is *feathered* if it contains a non-empty compact set K of countable character in G. It is well known that each p-space is feathered. They showed that a strongly topological gyrogroup G is feathered if and only if it contains a compact L-subgyrogroup H such that the quotient space G/H is metrizable [6, Lemma 3.13], which deduces that every feathered strongly topological gyrogroup is paracompact, which gives an affirmative answer to Question 3.1 when the topological gyrogroup is a strongly topological gyrogroup.

During the proof of [6, Lemma 3.13], the authors constructed a pseudometric by defining a function d from $G \times G$ to $\mathbb{R}$ by $d(x, y) = |N(x) - N(y)|$ for all $x, y \in G$. Then, they defined a function ϱ on $G/P \times G/P$ by

$$\varrho(\pi_P(x), \pi_P(y)) = d(\ominus x \oplus y, 0) + d(\ominus y \oplus x, 0)$$

for any $x, y \in G$ and showed that ϱ is a metric on G/P and ϱ generates the quotient topology of the space G/P. **However, there is a gap** in the proof of the metric ϱ generating the quotient topology of G/P. Here, we revise the proof and show that a strongly topological gyrogroup G is feathered if and only if it contains a compact L-subgyrogroup P such that the quotient space G/P is metrizable.

Indeed, the construction of the pseudometric d by $d(x, y) = N(\ominus x \oplus y)$ for all $x, y \in G$ and the metric ϱ generating the quotient topology of the quotient space G/P are similar to [7, Theorem 3.3]. Therefore, we omit some details about the proof that d defined by $d(x, y) = N(\ominus x \oplus y)$ is a pseudometric, ϱ defined by $\varrho(\pi_P(x), \pi_P(y)) = d(x, y)$ is a metric and ϱ generates the quotient topology of the space G/P. For more details, see [7, Theorem 3.3].

Theorem 3.3. *A strongly topological gyrogroup G is feathered if and only if it contains a compact L-subgyrogroup P such that the quotient space G/P is metrizable.*

Proof. The sufficiency is claimed in [6, Theorem 3.14], and it suffices to claim the necessity.

Let G be a feathered strongly topological gyrogroup with the symmetric neighborhood base $\mathscr{U}$ of 0. Since G is feathered, it follows from [6, Theorem 3.11] that it contains a compact L-subgyrogroup H of countable character in G. Let $\{U_n : n \in \mathbb{N}\}$ be a countable base for G at H. We define by induction a sequence $\{V_n : n \in \mathbb{N}\}$ of open symmetric neighborhoods of the identity 0 in G such that $V_n \in \mathscr{U}$ and $V_{n+1} \oplus V_{n+1} \subseteq V_n \cap U_n$ for each $n \in \mathbb{N}$. Put $P = \bigcap_{n\in\mathbb{N}} V_n$. Then, by [6, Lemma 3.10], P is a compact L-subgyrogroup of G, $P \subseteq H$, and $\{V_n : n \in \mathbb{N}\}$ is a countable base of G at P. Moreover, P is a strong subgyrogroup of G, since

$$\mathrm{gyr}[x, y](P) = gyr[x, y](\bigcap_{n\in\mathbb{N}} V_n) \subseteq \bigcap_{n\in\mathbb{N}} \mathrm{gyr}[x, y](V_n) = \bigcap_{n\in\mathbb{N}} V_n = P,$$

for any $x, y \in G$, which implies that $gyr[x, y](P) = P$ by Proposition 2.6. Let $\pi_P : G \to G/P$ be the quotient mapping of G onto the left coset space G/P.

Applying [6, Lemma 3.12], there exists a continuous prenorm N on G which satisfies

$$N(gyr[x, y](z)) = N(z)$$

for any $x, y, z \in G$ and

$$\{x \in G : N(x) < 1/2^n\} \subseteq V_n \subseteq \{x \in G : N(x) \leq 2/2^n\},$$

for each integer $n \geq 0$.

It is clear that $N(x) = 0$ if and only if $x \in P$ and $N(x \oplus p) = N(x)$ for every $x \in G$ and $p \in P$.

Define a function d from $G \times G$ to $\mathbb{R}$ by $d(x, y) = N(\ominus x \oplus y)$ for all $x, y \in G$. It is clear that d is continuous and d is a pseudometric.

Then we define a function ϱ on $G/P \times G/P$ by

$$\varrho(\pi_P(x), \pi_P(y)) = d(x, y)$$

for any $x, y \in G$. Then ϱ generates the quotient topology of the space G/P. Therefore, the quotient space G/P is metrizable. □

A mapping $\varphi : G \to H$ between gyrogroups is called *gyrogroup homomorphism* if $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$ for all $a, b \in G$. Then we discuss the class $\mathscr{N}$ of topological gyrogroups such that each $G \in \mathscr{N}$ satisfies the following condition ($\star$):

($\star$) each subgyrogroup N of G is normal, denoted by $N \trianglelefteq G$, if it is the kernel of a gyrogroup homomorphism of G.

If $G \in \mathscr{N}$, we say that G is a topological N-gyrogroup.

Example 3.4. Let G_8 be the gyrogroup $\{0, 1, 2, 3, 4, 5, 6, 7\}$, whose gyroaddition and gyration tables are presented in Table 1. For each $x \in G_8$, we use $\langle x\rangle$ denote the smallest subgyrogroup of G_8 which contains x. Then it is easy to see that all subgyrogroups are normal in G_8. Indeed, we have that $\langle 1\rangle$, $\langle 2\rangle$, $\langle 5\rangle$ are all normal subgyrogroups of G_8, $\langle 7\rangle$ is a normal subgyrogroup of $\langle 1\rangle$ (resp., $\langle 2\rangle$, $\langle 5\rangle$) and $\langle 0\rangle$ is a normal subgyrogroup of $\langle 7\rangle$.

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	7	6	0	5	2	4	3
2	2	5	7	6	0	3	1	4
3	3	0	5	7	6	4	2	1
4	4	6	0	5	7	1	3	2
5	5	2	3	4	1	7	0	6
6	6	4	1	2	3	0	7	5
7	7	3	4	1	2	6	5	0

gyr	0	1	2	3	4	5	6	7
0	I	I	I	I	I	I	I	I
1	I	I	A	I	A	A	A	I
2	I	A	I	A	I	A	A	I
3	I	I	A	I	A	A	A	I
4	I	A	I	A	I	A	A	I
5	I	A	A	A	A	I	I	I
6	I	A	A	A	A	I	I	I
7	I	I	I	I	I	I	I	I

Table1. The gyroaddition and gyration tables for the gyrogroup G_8, where I is the identity automorphism and $A = (1\ 3)(2\ 4)$

Motivated by Questions 3.1 and 3.2, it is natural to pose the following two questions.

Question 3.5. *If a topological N-gyrogroup is a p-space, is it paracompact?*

Question 3.6. *If a topological N-gyrogroup is a p-space, is it a D-space?*

Let G be a topological gyrogroup and $H \trianglelefteq G$. We define a binary operation on the set G/H in the following natural way:

$$(a \oplus H) \oplus (b \oplus H) = (a \oplus b) \oplus H,$$

for any $a, b \in G$. It follows from [21, Theorem 5.5] that G/H with the above operation is a gyrogroup.

Theorem 3.7. *Let G be a topological gyrogroup and $H \trianglelefteq G$. Then the gyrogroup G/H equipped with the quotient topology $\tau(G/H)$ is a topological gyrogroup.*

Proof. Let $\tau(G) = \tau$ and $\tau(G/H) = \iota$. We show that $(G/H, \iota)$ is a topological space. For every family $\mathcal{O}$ of subsets of G, we have $\pi^{-1}(\bigcup \mathcal{O}) = \bigcup\{\pi^{-1}(O) : O \in \mathcal{O}\}$ and $\pi^{-1}(\bigcap \mathcal{O}) = \bigcap\{\pi^{-1}(O) : O \in \mathcal{O}\}$. Then ι is a topology of G/H.

Next we show that $\pi(U) \in \iota$ for every $U \in \tau$. By the definition of topology ι, we have $\pi(U) \in \iota$ when $\pi^{-1}(\pi(U)) \in \tau$. For every $g \in G$, we have $\pi^{-1}(\pi(g)) = g \oplus H$. As a consequence, $\pi^{-1}(\pi(U)) = \bigcup_{g\in U}(g \oplus H) = U \oplus H$. So, $U \oplus H \in \tau$. Hence, we have $\pi(U) \in \iota$ for every $U \in \tau$.

We show that the joint mapping $h : (G/H, \iota) \times (G/H, \iota) \to (G/H, \iota)$ with $h(a, b) = a \oplus b$ for any $a, b \in G/H$ is continuous. Take arbitrary $O \in \iota$ and $a, b \in G/H$ such that $a \oplus b \in O$. Then there exist $x, y \in G$ such that $\pi(x) = a$ and $\pi(y) = b$. Then $\pi(x \oplus y) = \pi(x) \oplus \pi(y) = a \oplus b \in O$, and thus $x \oplus y \in \pi^{-1}(O)$. Since $O \in \iota$, we have $\pi^{-1}(O) \in \tau$. Since τ is a gyrogroup topology and $x \oplus y \in \pi^{-1}(O) \in \tau$, there exist $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \oplus V \subseteq \pi^{-1}(O)$. Since π is a homomorphism, we have $\pi(U \oplus V) = \pi(U) \oplus \pi(V) \subseteq O$. Since $\pi(U) \in \iota$ and $\pi(V) \in \iota$, it follows that the mapping h is continuous at point (a, b). By arbitrary of choice a, b, we obtain that h is continuous.

Finally, we prove that the inverse mapping $g : (G/H, \iota) \to (G/H, \iota)$ with $g(a) = \ominus a$ is continuous. We note first that $\ominus(\pi^{-1}(A)) = \pi^{-1}(\ominus A)$ for every $A \subseteq G/H$. Indeed, because for every $x \in G$, we have $\ominus\pi(x) = \pi(\ominus x)$, it follows that $x \in \ominus(\pi^{-1}(A)) \Leftrightarrow \ominus x \in \pi^{1}(A) \Leftrightarrow \pi(\ominus x) \in A \Leftrightarrow \pi(x) \in \ominus A \Leftrightarrow x \in \pi^{-1}(\ominus A)$. Since we have $\ominus U \in \tau$ for every $U \in \tau$, it follows from $\ominus(\pi^{-1}(A)) = \pi^{-1}(\ominus A)$ for every $A \subseteq G/H$ that $\ominus O \in \iota$ for every $O \in \iota$. This means that g is continuous. □

Theorem 3.8. *Let G be a topological gyrogroup and $H \trianglelefteq G$. If H is a compact subgyrogroup with countable character, then the quotient space G/H is metrizable.*

Proof. Since H has countable character in G, we can fix a family $\{U_n\}_{n\in\mathbb{N}}$ of open sets such that $H \subseteq \bigcap U_n$ and for every open subset U with $H \subseteq U$, there exists $n \in \mathbb{N}$ such that $H \subseteq U_n \subseteq U$. Then we claim that $\{\pi(U_n) : n \in \mathbb{N}\}$ is a neighborhood base at the identity $\pi(0)$ in G/H.

Indeed, for every open neighborhood V of $\pi(0)$ in G/H, we have $H \subseteq \pi^{-1}(V)$. Obviously, $\pi^{-1}(V)$ is open in G, then there exists $n \in \mathbb{N}$ such that $H \subseteq U_n \subseteq \pi^{-1}(V)$. So, $\pi(U_n) \subseteq V$ and we obtain that $\{\pi(U_n) : n \in \mathbb{N}\}$ is a neighborhood base at the identity $\pi(0)$ in G/H.

Therefore, the gyrogroup G/H is first-countable. It is claimed in [11] that every first-countable topological gyrogroup is metrizable, we conclude that G/H is metrizable. □

The following theorem gives a characterization for a topological N-gyrogroup being a feathered space.

Theorem 3.9. *A topological N-gyrogroup G is feathered if and only if it contains a compact subgyrogroup H such that the quotient space G/H is metrizable.*

Proof. If the topological N-gyrogroup G is feathered, it contains a non-empty compact set F with $\chi(F, G) \leq \omega$. By the homogeneity of G, we can assume that F contains the identity element 0 of G. Let $\{U_n : n \in \mathbb{N}\}$ be a countable base for G at F. We define by induction a sequence $\{V_n : n \in \omega\}$ of symmetric open neighborhood of 0 in G satisfying the following conditions:

(i) $V_0 \subset O$.

(ii) $V_{n+1} \oplus V_{n+1} \subset U_n \bigcap V_n$ for each $n \in \mathbb{N}$.

Put $H = \bigcap_{n\in\mathbb{N}} V_n$. Then $H \subset O$ by (i). Furthermore, it follows from [6, Theorem 3.10] that H is a compact subgyrogroup of G and $\{V_n : n \in \mathbb{N}\}$ is a base for G at H. Hence, it follows from Theorem 3.8 that the quotient space G/H is metrizable.

Conversely, assume that G contains a compact subgyrogroup H such that the quotient space G/H is metrizable. It follows from [6, Theorem 3.8] the mapping $\pi : G \to G/H$ is perfect. We claim that the subgyrogroup $H \subseteq G$ has countable character in G. Indeed, let $\{U_n : n \in \omega\}$ be a countable base for G/H at the point $\pi(0)$. Then the family $\{V_n : n \in \omega\}$ is a base for G at H, where $V_n = \pi^{-1}(U_n)$ for each $n \in \omega$. Indeed, for every open neighborhood O of H in G, $F = G\backslash O$ is closed in G and the image $\pi(F)$ is a closed subset of G/H which does not contain $\pi(0)$. Hence, $U_n \bigcap \pi(F) = \emptyset$ for some $n \in \omega$, which in turn implies that $V_n \bigcap F = \emptyset$. Therefore, $H \subseteq V_n \subseteq O$. Therefore, $\chi(H, G) \leq \omega$. Since H is compact, the gyrogroup G is feathered. □

The following two corollaries give answers to Questions 3.5 and 3.6.

Corollary 3.10. *Every feathered topological N-gyrogroup is paracompact.*

Corollary 3.11. *Every feathered topological N-gyrogroup is a D-space.*

4. Quotient spaces of strongly topological gyrogroups

Then we recall some weakly first-countable concepts which are important in the following researches.

Definition 4.1. Let X be a topological space.

(1) X is called a *weakly first-countable space* or *gf-countable space* [1] if for each point $x \in X$ it is possible to assign a sequence $\{B(n,x) : n \in \mathbb{N}\}$ of subsets of X containing x in such a way that $B(n+1,x) \subseteq B(n,x)$ and so that a set U is open if, and only if, for each $x \in U$ there exists $n \in \mathbb{N}$ such that $B(n,x) \subseteq U$.

(2) X is called a *sequential space* [14] if for each non-closed subset $A \subseteq X$, there are a point $x \in X \setminus A$ and a sequence in A converging to x in X.

(3) X is called a *Fréchet-Urysohn space* [14] if for any subset $A \subseteq X$ and $x \in \overline{A}$, there is a sequence in A converging to x in X.

(4) X is called a *strongly Fréchet-Urysohn space* [20] if the following condition is satisfied:

(SFU) For each $x \in X$ and every sequence $\xi = \{A_n : n \in \mathbb{N}\}$ of subsets of X such that $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, there exists a sequence $\eta = \{b_n : n \in \mathbb{N}\}$ in X converging to x and intersecting infinitely many members of ξ.

(5) X is called an *α_4-space* [19], if for every point $x \in X$ and each sheaf $\{S_n : n \in \mathbb{N}\}$ with the vertex x, there exists a sequence converging to x which meets infinitely many sequences S_n.

(6) X is called an *α_7-space* [4], if for every point $x \in X$ and each sheaf $\{S_n : n \in \omega\}$ with the vertex x, there exists a sequence converging to some point $y \in X$ which meets infinitely many sequences S_n.

Definition 4.2. Let $\mathscr{P}$ be a family of subsets of a space X with $x \in \bigcap \mathscr{P}$.

(1) The family $\mathscr{P}$ is called a *network at x* [12] if for each neighborhood U of x there exists $P \in \mathscr{P}$ such that $P \subseteq U$.

(2) The family $\mathscr{P}$ is called a *cs-network at x* [16] if for any sequence L converging to x and a neighborhood U of x, there exists $P \in \mathscr{P}$ such that L is eventually in P and $P \subseteq U$.

(3) The family $\mathscr{P}$ is called an *sn-network at x* [15] if $\mathscr{P}$ is a network at x and each element of $\mathscr{P}$ is a sequential neighborhood of x.

(4) The family $\mathscr{P}$ is called an *so-network at x* [15] if $\mathscr{P}$ is a network at x and each element of $\mathscr{P}$ is a sequential open subset of X.

(5) A space X is called *csf-countable* (resp., *snf-countable*, *sof-countable*) [15] if for each $x \in X$, there is a countable *cs*-network (resp., *sn*-network, *so*-network) at x.

Note that in [4, 5], csf-countable spaces and snf-countable spaces are called *spaces with countable cs*-character* and *spaces with countable sb-character*, respectively.

Lemma 4.3. ([6]) *Let G be a topological gyrogroup and H an L-subgyrogroup of G. Then the natural homomorphism π from a topological gyrogroup G to its quotient topology on G/H is an open and continuous mapping.*

Lemma 4.4. ([9]) *Let G be a strongly topological gyrogroup and H a closed strong subgyrogroup of G. Then the family $\{\pi(x \oplus V) : V \in \tau, 0 \in U\}$ is a local base of the space G/H at the point $x \oplus H \in G/H$, and G/H is a homogeneous T_1-space.*

A topological space X is called a coset space if X is homeomorphic to G/H, for some closed subgroup H of a topological group G. It is well-known that every first-countable topological group is metrizable by the Birkhoff-Kakutani theorem. However, the following example shows that it does not hold in coset spaces.

Example 4.5. ([13]) The two arrows space is a compact coset space which is first-countable, but not submetrizable.

Since every topological group is a strongly topological gyrogroup and each subgroup is a strong subgyrogroup, the Example 4.5 shows that the axioms of first-countability is not equivalent with metrizability in the quotient space G/H, where G is a strongly topological gyrogroup and H is a closed strong subgyrogroup of G. However, if G is a strongly topological gyrogroup with neighborhood base $\mathscr{U}$ of 0 and P is an admissible subgyrogroup generated from $\mathscr{U}$, if the quotient space G/P is first-countable, then it is metrizable.

Theorem 4.6. ([8]) *Let G be a strongly topological gyrogroup with a symmetric neighborhood base $\mathscr{U}$ at 0 and P an admissible subgyrogroup generated from $\mathscr{U}$. If the quotient space G/P is first-countable, then it is metrizable.*

Definition 4.7. ([9]) A subgyrogroup H of a topological gyrogroup G is called *inner (outer) neutral* if for every open neighborhood U of 0 in G, there exists an open neighborhood V of 0 such that $H \oplus V \subseteq U \oplus H$ ($V \oplus H \subseteq H \oplus U$). H is called *neutral* if it is not only inner neutral but also outer neutral.

According to [9, Proposition 3.16], it was proved that if G is a strongly topological gyrogroup, then every compact strong subgyrogroup H of G is outer neutral, that is, for every open neighborhood U of 0 in G, there exists an open neighborhood V of 0 such that $V \oplus H \subseteq H \oplus U$. It is not difficult to see that every compact strong subgyrogroup H of G is inner neutral by the similar proof. Therefore, every compact strong subgyrogroup of a strongly topological gyrogroup is neutral.

Theorem 4.8. *Suppose that G is a strongly topological gyrogroup with neighborhood base $\mathscr{U}$ of 0, H is a closed strong subgyrogroup of G and H is inner neutral, if the space G/H is bisequential, then G/H is metrizable.*

Proof. Since the space G/H is regular by [10, Theorem 5.8] and is also bisequential, we can find a countable open prefilter ξ on G/H converging to $\pi(0)$ by [2, Lemma 4.7.11]. Let $Q_P = \pi^{-1}(P)$ for each $P \in \xi$. Put $\gamma = \{\pi((\ominus Q_P) \oplus Q_P) : P \in \xi\}$. Then γ is a base at $\pi(0)$ in G/H. Indeed, all elements of γ are open in G/H and contain $\pi(0)$. Let O be an open neighborhood of $\pi(0)$. Take open symmetric neighborhoods $U, V \in \mathscr{U}$ in G such that $V \oplus V \subseteq U$, $H \oplus (V \oplus V) \subseteq U \oplus H$ and $\pi(U) \subseteq O$. Since ξ converges to $\pi(0)$, there exists $P \in \xi$ such that $P \subseteq \pi(V)$. It follows that $Q_P \subseteq V \oplus H$. Then

$$\begin{aligned}
\ominus Q_P &\subseteq \ominus(V \oplus H) \\
&= \bigcup_{v\in V, h\in H} \{\ominus(v \oplus h)\} \\
&= \bigcup_{v\in V, h\in H} \{gyr[v, h](\ominus h \ominus v)\} \\
&= \bigcup_{v\in V, h\in H} \{gyr[v, h](\ominus h) \oplus gyr[v, h](\ominus v)\} \\
&= H \oplus V.
\end{aligned}$$

Therefore, $0 \in (\ominus Q_P) \oplus Q_P \subseteq (H \oplus V) \oplus (V \oplus H)$ and

$$\begin{aligned}
(H \oplus V) \oplus (V \oplus H) &= \bigcup_{v_1,v_2\in V, h_1,h_2\in H} \{(h_1 \oplus v_1) \oplus (v_2 \oplus h_2)\} \\
&= \bigcup_{v_1,v_2\in V, h_1,h_2\in H} \{((h_1 \oplus v_1) \oplus v_2) \oplus gyr[h_1 \oplus v_1, v_2](h_2)\} \\
&= \bigcup_{v_1,v_2\in V, h_1,h_2\in H} \{(h_1 \oplus (v_1 \oplus gyr[v_1, h_1](v_2))) \oplus gyr[h_1 \oplus v_1, v_2](h_2)\} \\
&= (H \oplus (V \oplus V)) \oplus H \\
&\subseteq (U \oplus H) \oplus H,
\end{aligned}$$

hence, $\pi(0) \in \pi((\ominus Q_P) \oplus Q_P) \subseteq \pi((U \oplus H) \oplus H) = \pi(U) \subseteq O$. Hence, G/H is first-countable. Then it was claimed in [18, Theorem 4.4] that if H is a closed neutral strong subgyrogroup of a strongly topological gyrogroup G, then G/H is metrizable iff G/H is first-countable, we conclude that the quotient space G/H is metrizable. □

According to the process of the proof of [2, Lemma 4.7.1] and [2, Proposition 4.7.2], we can easily obtain the following two lemmas:

Lemma 4.9. *Let $\{V_n(x) : n \in \omega, x \in X\}$ and $\{W_n(x) : n \in \omega, x \in X\}$ be two sn-networks on a Hausdorff space X. Then, for each $x \in X$ and each $n \in \omega$, there is $m \in \omega$ such that $W_m(x) \subseteq V_n(x)$.*

Lemma 4.10. *Let $\{V_n(x) : n \in \omega, x \in X\}$ be an sn-network on a homogeneous Hausdorff space X, and let b be an element of X. Suppose further that, for each $x \in X$, f_x is a homeomorphism of X onto X such that $f_x(b) = x$. Put $W_n(x) = f_x(V_n(b))$. Then $\{W_n(x) : n \in \omega, x \in X\}$ is an sn-network on X.*

Theorem 4.11. *Suppose that G is a strongly topological gyrogroup with neighborhood base $\mathscr{U}$ of 0, H is a closed strong subgyrogroup of G and H is inner neutral, if the space G/H is an snf-countable space, then G/H is an sof-countable space.*

Proof. For each $x \in G/H$, fix a point $a_x \in G$ such that $\pi(a_x) = x$, where $a_{\pi(0)} = 0$. For each subset K of G/H, put $T_K = \pi^{-1}(K)$.

Let $\{O_n(x) : n \in \omega, x \in G/H\}$ be an *sn*-network on G/H. Put $Q_n(x) = \pi(a_x \oplus (T_{O_n(\pi(0))} \oplus T_{O_n(\pi(0))}))$ for each $x \in G/H$ and $n \in \omega$.

Claim 1: $\{Q_n(x) : n \in \omega, x \in G/H\}$ is a network on G/H.

We can assume that $O_n(x) = \pi(a_x \oplus T_{O_n(\pi(0))})$ for each $x \in G/H$ and $n \in \omega$. Indeed, this follows from Lemma 4.10 since h_a on G/H given by the formula $h_a(b\oplus H) = (a\oplus b)\oplus H$, for each $a \in G$, is a homeomorphism of G/H onto itself. Next, we only need prove that $\{Q_n(x) : n \in \omega, x \in G/H\}$ is a network of G/H. Indeed, let V be a neighborhood of a point $x \in G/H$. Since G is a strongly topological gyrogroup and H is inner neutral, there exists $W \in \mathscr{U}$ such that $\pi(a_x \oplus ((W \oplus H) \oplus (W \oplus H))) \subseteq V$. It follows from $\pi(0) \in \pi(W \oplus H)$ that there exists $n \in \omega$ such that $O_n(\pi(0)) \subseteq \pi(W \oplus H)$. So $x \in Q_n(x) = \pi(a_x \oplus (T_{O_n(\pi(0))} \oplus T_{O_n(\pi(0))})) \subseteq V$, whence $\{Q_n(x) : n \in \omega, x \in G/H\}$ is a network of G/H.

Claim 2: $O_n(\pi(0))$ contains a sequentially open neighbourhood of $\pi(0)$ for each $n \in \omega$.

Let B_n be the set of all points $x \in O_n(\pi(0))$ such that $\pi(a_x \oplus T_{O_k(\pi(0))}) \subseteq O_n(\pi(0))$, for some $k \in \omega$. Clearly, $\pi(0) \in B_n \subseteq O_n(\pi(0))$. We claim that the set B_n is sequentially open in G/H. Indeed, take any $y \in B_n$. Then $\pi(a_y \oplus T_{O_k(\pi(0))}) \subseteq O_n(\pi(0))$, for some $k \in \omega$. By Claim 1 and Lemma 4.9, there is $m \in \omega$ such that $\pi(T_{O_m(\pi(0))} \oplus T_{O_m(\pi(0))}) \subseteq O_k(\pi(0))$. Then $\pi(a_y \oplus (T_{O_m(\pi(0))} \oplus T_{O_m(\pi(0))})) \subseteq \pi(a_y \oplus T_{O_k(\pi(0))}) \subseteq O_n(\pi(0))$, which implies that $O_m(y) = \pi(a_y \oplus T_{O_m(\pi(0))}) \subseteq B_n$. Since y was an arbitrary point of B_n, it follows that B_n is sequentially open in G/H.

Now it is clear that $\{B_n : n \in \omega\}$ is a countable *so*-network of G/H at $\pi(0)$. Hence G/H is an *sof*-countable space. □

Corollary 4.12. *Suppose that G is a strongly topological gyrogroup, H is a closed strong subgyrogroup of G and H is inner neutral, if the space G/H is weakly first-countable, then G/H is a first-countable space.*

In [4], T. Banakh and T. Zdomskyy introduced the concept of a sequence tree, and showed that if a sequential csf-countable topological group G is an α_7-space, then G is metrizable. By the similar method, it is easy to show that if the space G/H is a *csf*-countable and sequential α_7-space, then G/H is first-countable, where G is a strongly topological gyrogroup, H is an inner neutral and closed strong subgyrogroup of G. Therefore, we omit the proof of the following result.

Theorem 4.13. *Let G be a strongly topological gyrogroup, H an inner neutral and closed strong subgyrogroup of G. If the space G/H is a csf-countable and sequential α_7-space, then G/H is first-countable.*

It was claimed in [18, Theorem 4.4] and [9, Corollary 3.22] that if H is a closed neutral strong subgyrogroup of a strongly topological gyrogroup G, then G/H is metrizable iff G/H is first-countable, and G/H is first-countable if and only if G/H is Fréchet-Urysohn with an ω^ω-base, respectively. Therefore, we conclude the following result.

Corollary 4.14. *Suppose that G is a strongly topological gyrogroup, H is a closed strong subgyrogroup of G and H is inner neutral, then the followings are equivalent.*

1. *G/H is metrizable;*
2. *G/H is first-countable;*

3. *G/H is a weakly first-countable space;*
4. *G/H is a bisequential space;*
5. *G/H is Fréchet-Urysohn with an ω^ω-base;*
6. *G/H is a csf-countable and sequential α_7-space.*

Acknowledgements

The authors are thankful to Professor T. Suksumran for communicating the examples of gyrogroups in this article.

The authors are thankful to the anonymous referees for valuable remarks and corrections and all other sort of help related to the content of this article.

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