On separable idempotent uninorms on a bounded chain and the structure of their demarcating functions

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Abstract. As not all idempotent uninorms on a bounded (but not complete) chain can be characterized by decreasing unary functions on the chain as the demarcating functions, we introduce separable idempotent uninorms on a bounded chain. The separability of idempotent uninorms on a bounded chain is characterized. The structure of the demarcating functions of separable idempotent uninorms on a bounded chain and that of the demarcating functions of unilaterally continuous idempotent uninorms on a complete chain are also investigated.

1. Introduction

Recently, the structure of idempotent uninorms on a bounded chain attracted moderate attention. We [5] characterized idempotent uninorms on a complete chain in terms of decreasing unary functions with a symmetry-related property and separated unilaterally continuous idempotent uninorms on a complete chain by their demarcating functions. Nevertheless, on a proper (non-complete) bounded chain, there may be idempotent uninorms that cannot be separated by decreasing unary functions from the chain to itself, as exemplified in [6]. Making use of the Dedekind-MacNeille completion of a bounded chain, Su et al. [6] separated idempotent uninorms on a bounded chain by decreasing unary functions defined on the Dedekind-MacNeille completion of the bounded chain.

It is noteworthy that there are indeed numerous idempotent uninorms on a proper bounded chain that can be separated more directly by decreasing unary functions from the chain to itself and it is easier to visualize decreasing unary functions on a proper bounded chain than on its Dedekind-MacNeille completion as the structure of the Dedekind-MacNeille completion is more abstract than that of the underlying chain. It is then natural and interesting to wonder whether there exists a proper bounded chain on which all idempotent uninorms are separable by decreasing unary functions from the chain to itself. The answer to this question will prove to be negative, thus motivating the classification of idempotent uninorms on a proper bounded chain into two classes: the class of idempotent uninorms that can be separated by decreasing

unary functions from the chain to itself (we call them separable idempotent uninorms) and the class that cannot be. Naturally, the first aim of this paper is to identify separable idempotent uninorms on a proper bounded chain by presenting several easily verifiable characterizations.

Due to the characterization theorems for idempotent uninorms on a complete chain [5] and on a bounded chain [6], constructing idempotent uninorms amounts to constructing decreasing unary functions with a symmetry-related property on a complete chain. Thus, as a supplement to the characterization theorems, the second aim of this paper is to investigate the structure of such decreasing unary functions. Inspired by the work [2] on finite chains showing that such decreasing unary functions are decomposable into two interrelated decreasing unary functions, we shall consider the extendability of a general decreasing unary function to a decreasing unary function with the symmetry-related property on a bounded chain containing that on a complete chain as a special case.

The remainder of this paper is organized as follows. We recall some basic notions and results related to posets and uninorms on a bounded lattice in Section 2. Section 3 is devoted to the characterization of separable idempotent uninorms on a bounded chain. The structure of the demarcating functions of separable idempotent uninorms on a bounded chain is studied in Section 4, while Section 5 is concerned with the structure of the demarcating functions of unilaterally continuous idempotent uninorms on a complete chain. We end with some conclusions in Section 6.

2. Preliminaries

In this section, we recall some basic notions and results related to posets (for more information, see, e.g., [1]) and uninorms on a bounded lattice. A poset $(P, \leq)$ is a nonempty set P equipped with an order relation $\leq$ (i.e., a reflexive, antisymmetric and transitive binary relation). For $a, b \in P$, the notation $a < b$ means that $a \leq b$ and $a \neq b$. P is said to be a chain if for any $a, b \in P$, either $a \leq b$ or $b \leq a$. P is called bounded if it has a greatest (also known as top) element and a smallest (also known as bottom) element. For $a, b \in P$ with $a \leq b$, the subintervals $[a, b]$, $]a, b]$, $[a, b[$ and $]a, b[$ of P are defined as usual.

A subset $A \subseteq P$ is called an upper set (resp. lower set) w.r.t. $(P, \leq)$ if, for any $x, y \in P$, $x \in A$ and $y \geq x$ (resp. $y \leq x$) imply $y \in A$ [3]. For any $x \in P$, the subset $\uparrow x = \{y \in P \mid y \geq x\}$ (resp. $\downarrow x = \{y \in P \mid y \leq x\}$) is called the principal filter (resp. principal ideal) generated by x. For any $A \subseteq P$, define $\downarrow A = \bigcup_{a \in A} (\downarrow a)$. A^u (resp. A^l) denotes the set of all upper (resp. lower) bounds of A, i.e.,

$$A^u = \{a \in P \mid (\forall b \in A)(b \leq a)\}; \quad A^l = \{a \in P \mid (\forall b \in A)(a \leq b)\}.$$

Note that $\emptyset^u = \emptyset^l = P$. A lattice is a poset $(L, \leq)$ such that any two elements x and y have a greatest lower bound (called meet or infimum), denoted by $x \wedge y$, as well as a smallest upper bound (called join or supremum), denoted by $x \vee y$. It is clear that any chain is a lattice and in this case, we often use min (resp. max) instead of $\wedge$ (resp. $\vee$). A lattice $(L, \leq)$ is said to be complete if, for any $A \subseteq L$, the greatest lower bound $\bigwedge A$ and the smallest upper bound $\bigvee A$ of A exist. Obviously, any finite lattice is complete and any complete lattice is bounded.

The following proposition will be useful hereinafter.

Proposition 2.1. *Let $(L, \leq)$ be a chain, $A \subseteq L$ be a lower set and $B \subseteq L$ be an upper set with $A = L \setminus B$. Then $\bigvee A$ exists if and only if $\bigwedge B$ exists.*

Proof. Suppose that $\bigvee A$ exists. We consider two cases:

(i) $B \cap B^l \neq \emptyset$.

In this case, it is clear that $B \cap B^l$ is a singleton with the element being $\bigwedge B$.

(ii) $B \cap B^l = \emptyset$.

In this case, it is easy to see that $B^l = A$. Hence, we have $\bigwedge B = \bigvee A$.

Similarly, the existence of $\bigwedge B$ implies that of $\bigvee A$. $\square$

A unary function f on a chain $(L, \leq)$ is said to have property **(S)** [5] provided that

$$(\forall(x, y) \in L^2)(y < f(x) \Longrightarrow x \leq f(y)). \tag{S}$$

Note that **(S)** is equivalent to

$$(\forall(x, y) \in L^2)(y > f(x) \Longrightarrow x \geq f(y)).$$

Property **(S)** can also be equivalently stated as follows.

Proposition 2.2. *Let f be a unary function on a chain $(L, \leq)$. Then the following statements are mutually equivalent:*

  (i) *f has property* **(S)**;

  (ii) *$f(x) \in \{y \in L \mid f(y) > x\}^u$ for all $x \in L$;*

  (iii) *$f(x) \in \{y \in L \mid f(y) < x\}^l$ for all $x \in L$.*

Aggregation functions such as uninorms can be straightforwardly defined on a bounded lattice. Note that an order relation $\leq$ on a set P naturally induces an order relation on P^2, also denoted by $\leq$ for convenience: $(x, y) \leq (u, v)$ if and only if $x \leq u$ and $y \leq v$.

Definition 2.3. ([4, 7]) Let $(L, \leq, 0, 1)$ be a bounded lattice with top element 1 and bottom element 0. A binary operation $U: L^2 \to L$ is said to be a uninorm on L with neutral element $e \in L$ if, for any $x, y, z, w \in L$, the following conditions are fulfilled:

  (i) $U(x, y) = U(y, x)$ (commutativity);

  (ii) If $(x, z) \leq (y, w)$, then $U(x, z) \leq U(y, w)$ (increasingness);

  (iii) $U(U(x, y), z) = U(x, U(y, z))$ (associativity);

  (iv) $U(e, x) = U(x, e) = x$ (neutral element).

Due to the commutativity, the increasingness of a uninorm is equivalent to that w.r.t. one variable. A uninorm on L is precisely an abelian ordered monoid on L.

A binary operation F on a nonempty set X is said to be idempotent if $F(x, x) = x$ for all $x \in X$.

3. Characterizations of separability of idempotent uninorms on a bounded chain

Let $(L, \leq, 0, 1)$ be a bounded chain with top element 1 and bottom element 0. We denote by $\mathcal{U}_{\mathrm{Id}}(L, e)$ the set of all idempotent uninorms on L with neutral element $e \in L \setminus \{0, 1\}$ and by $\mathcal{G}(L, e, \mathbf{S})$ the set of all decreasing unary functions on L with fixed point e and property **(S)**.

We say that $U \in \mathcal{U}_{\mathrm{Id}}(L, e)$ is *separable* if there exists $g \in \mathcal{G}(L, e, \mathbf{S})$ such that $U(x, y) = \min\{x, y\}$ for all $(x, y) \in A_g$ and $U(x, y) = \max\{x, y\}$ for all $(x, y) \in B_g$, where $A_g, B_g \subseteq L^2$ are defined as follows:

$$A_g = \{(x, y) \in L^2 \mid y < g(x) \text{ or } x < g(y)\};$$
$$B_g = \{(x, y) \in L^2 \mid y > g(x) \text{ or } x > g(y)\}.$$

It is known from Theorem 3.15 of [5] that every $U \in \mathcal{U}_{\mathrm{Id}}(L, e)$ is separable when $(L, \leq, 0, 1)$ is complete, while this is not the case when $(L, \leq, 0, 1)$ is not complete (see Example 1(i) of [6]). However, this does not mean that there exists no separable idempotent uninorm on a proper bounded chain. Indeed, given $e \in L \setminus \{0, 1\}$, the greatest ($U_{\mathrm{g}}$) and the smallest ($U_{\mathrm{s}}$) idempotent uninorms on L with neutral element e are both separable, where U_{g} is given by

$$U_{\mathrm{g}}(x, y) = \begin{cases} \min\{x, y\} & \text{if } (x, y) \in [0, e]^2 \\ \max\{x, y\} & \text{otherwise,} \end{cases}$$

and U_s is given by

$$U_s(x, y) = \begin{cases} \max\{x, y\} & \text{if } (x, y) \in [e, 1]^2 \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

Now, we characterize the separability of idempotent uninorms on a proper bounded chain. Let $U \in \mathcal{U}_{\text{Id}}(L, e)$. For any $x \in L$, define the sets $A_x, B_x \subseteq L$ as follows:

$$A_x = \left\{ y \in L \mid (\forall z \in \downarrow y)(U(x, z) = \min\{x, z\}) \right\};$$

$$B_x = \left\{ y \in L \mid (\forall z \in \uparrow y)(U(x, z) = \max\{x, z\}) \right\}.$$

Theorem 3.1. *Let $U \in \mathcal{U}_{\text{Id}}(L, e)$. Then the following statements are mutually equivalent:*

(i) *U is separable;*

(ii) *For every $x \in L \setminus \{e\}$, there exists $y \in L$ such that $U(x, z) = \min\{x, z\}$ for all $z \in (\downarrow y) \setminus \{y\}$ and $U(x, z) = \max\{x, z\}$ for all $z \in (\uparrow y) \setminus \{y\}$;*

(iii) *$\bigvee A_x$ exists for all $x \in L \setminus \{e\}$;*

(iv) *$\bigwedge B_x$ exists for all $x \in L \setminus \{e\}$.*

Proof. For all $x \in L \setminus \{e\}$, it is easy to see that A_x is a lower set and B_x is an upper set. Similarly to Proposition 3.3 of [5], we have $A_x = L \setminus B_x$. Hence, it follows from Proposition 2.1 that $\bigvee A_x$ exists if and only if $\bigwedge B_x$ exists, which implies that (iii) $\Longleftrightarrow$ (iv).

(i) $\Longrightarrow$ (ii) Let U be separable, i.e., there exists $g \in \mathcal{G}(L, e, \mathbf{S})$ such that $U(x, y) = \min\{x, y\}$ for all $(x, y) \in A_g$ and $U(x, y) = \max\{x, y\}$ for all $(x, y) \in B_g$. Hence, for every $x \in L \setminus \{e\}$, we have $U(x, z) = \min\{x, z\}$ for all $z \in (\downarrow g(x)) \setminus \{g(x)\}$ and $U(x, z) = \max\{x, z\}$ for all $z \in (\uparrow g(x)) \setminus \{g(x)\}$.

(ii) $\Longrightarrow$ (iii) Let $x \in L \setminus \{e\}$. Then there exists $y \in L$ such that $U(x, z) = \min\{x, z\}$ for all $z \in (\downarrow y) \setminus \{y\}$ and $U(x, z) = \max\{x, z\}$ for all $z \in (\uparrow y) \setminus \{y\}$. We consider two cases:

(a) $U(x, y) = \min\{x, y\}$.

In this case, we have $A_x = [0, y]$. Hence, $\bigvee A_x = y$.

(b) $U(x, y) = \max\{x, y\}$.

In this case, we have $B_x = [y, 1]$. Hence, $\bigwedge B_x = y$, and so $\bigvee A_x$ exists as proved at the beginning.

(iii) $\Longrightarrow$ (i) Suppose that $\bigvee A_x$ exists for all $x \in L \setminus \{e\}$. Define a unary function $g \colon L \to L$ as

$$g(x) = \begin{cases} \bigvee A_x & \text{if } x \in L \setminus \{e\} \\ e & \text{if } x = e. \end{cases}$$

Similarly to Propositions 3.5 and 3.6 of [5], we have $g \in \mathcal{G}(L, e, \mathbf{S})$. Hence, similarly to Proposition 3.8 of [5], it holds that $U(x, y) = \min\{x, y\}$ for all $(x, y) \in A_g$ and $U(x, y) = \max\{x, y\}$ for all $(x, y) \in B_g$. Therefore, U is separable. $\square$

Due to Theorem 3.1, there are ample separable idempotent uninorms on a proper bounded chain.

Example 3.2. Let $(L, \leq, 0, 1)$ be a bounded chain with $e \in L \setminus \{0, 1\}$. For any $(a, b) \in [0, e[\times]e, 1]$, $U_{a,b}$ and $U^{a,b}$ are separable idempotent uninorms on L with neutral element e, where $U_{a,b}$ is given by

$$U_{a,b}(x, y) = \begin{cases} \min\{x, y\} & \text{if } (x, y) \in [0, e]^2 \cup ([0, a] \times]e, b]) \cup (]e, b] \times [0, a]) \\ \max\{x, y\} & \text{otherwise,} \end{cases}$$

and $U^{a,b}$ is given by

$$U^{a,b}(x, y) = \begin{cases} \max\{x, y\} & \text{if } (x, y) \in [e, 1]^2 \cup ([a, e[\times [b, 1]) \cup ([b, 1] \times [a, e[) \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

However, on a proper bounded chain, not all idempotent uninorms are separable as completeness of the underlying chain is a necessary and sufficient condition for all idempotent uninorms to be separable.

Theorem 3.3. *Let $(L, \leq, 0, 1)$ be a bounded chain. Then the following statements are mutually equivalent:*

(i) *L is complete;*

(ii) *For all $e \in L \setminus \{0, 1\}$, every $U \in \mathcal{U}_{\mathrm{Id}}(L, e)$ is separable;*

(iii) *There exists $e_0 \in L \setminus \{0, 1\}$ such that every $U \in \mathcal{U}_{\mathrm{Id}}(L, e_0)$ is separable.*

Proof. (i) $\Longrightarrow$ (ii) It follows from Theorem 3.15 of [5].

(ii) $\Longrightarrow$ (iii) Trivial.

(iii) $\Longrightarrow$ (i) Suppose that there exists $e_0 \in L \setminus \{0, 1\}$ such that every $U \in \mathcal{U}_{\mathrm{Id}}(L, e_0)$ is separable. Since L is bounded, it is complete if and only if it is join-complete, i.e., $\bigvee A$ exists for all nonempty $A \subseteq L$. We consider three cases:

(a) There exists $a_0 \in A$ with $a_0 > e_0$.

In this case, set $A_0 = \{a \in A \mid a > e_0\}$. Consider $U \in \mathcal{U}_{\mathrm{Id}}(L, e_0)$ defined by

$$U(x, y) = \begin{cases} \min\{x, y\} & \text{if } (x, y) \in [0, e_0]^2 \cup ([0, e_0[\times \bigcup_{a \in A_0}]e_0, a]) \cup (\bigcup_{a \in A_0}]e_0, a] \times [0, e_0[) \\ \max\{x, y\} & \text{otherwise.} \end{cases}$$

For all $x \in [0, e_0[$, we have $A_x = \downarrow A$. It follows from Theorem 3.1 that $\bigvee(\downarrow A)$ exists. Hence, $\bigvee A = \bigvee(\downarrow A)$.

(b) $A \subseteq [0, e_0]$ and $e_0 \in A$.

In this case, it is clear that $\bigvee A = e_0$.

(c) $A \subseteq [0, e_0[$.

In this case, consider $U \in \mathcal{U}_{\mathrm{Id}}(L, e_0)$ defined by

$$U(x, y) = \begin{cases} \min\{x, y\} & \text{if } (x, y) \in [0, e_0]^2 \cup (]e_0, 1] \times \bigcup_{a \in A} [0, a]) \cup (\bigcup_{a \in A} [0, a] \times]e_0, 1]) \\ \max\{x, y\} & \text{otherwise.} \end{cases}$$

For all $x \in]e_0, 1]$, we have $A_x = \downarrow A$. It follows from Theorem 3.1 that $\bigvee(\downarrow A)$ exists. Hence, $\bigvee A = \bigvee(\downarrow A)$.

We conclude that L is complete. $\quad \square$

4. Structure of the demarcating functions of separable idempotent uninorms on a bounded chain

Let $(L, \leq, 0, 1)$ be a bounded chain with $e \in L \setminus \{0, 1\}$. According to Remark 3.22 of [5], the set of all separable idempotent uninorms on L with neutral element e coincides with that of all idempotent uninorms on L with neutral element e demarcated by the members of $\mathcal{G}(L, e, \mathbf{S})$ as in Theorem 3.21 of [5]. In this section, we study the structure of the members of $\mathcal{G}(L, e, \mathbf{S})$.

As every $g \in \mathcal{G}(L, e, \mathbf{S})$ can be viewed as the combination of two decreasing unary functions: $f_1 : [0, e] \to [e, 1]$ and $f_2 : [e, 1] \to [0, e]$, it is natural to ask whether either of them can be extended to the member of $\mathcal{G}(L, e, \mathbf{S})$. We only consider the extendability of the first function as similar results hold for the second function.

Theorem 4.1. *Let $(L, \leq, 0, 1)$ be a bounded chain with $e \in L \setminus \{0, 1\}$ and $f : [0, e] \to [e, 1]$ be decreasing with $f(e) = e$. Then the following statements are mutually equivalent:*

(i) *There exists $g \in \mathcal{G}(L, e, \mathbf{S})$ such that $g|_{[0,e]} = f$;*

(ii) $\{a \in [0, e] \mid f(a) > x\}^u \cap \{a \in [0, e] \mid f(a) < x\}^l \neq \emptyset$ *for all $x > e$;*

(iii) $\bigvee \{a \in [0, e] \mid f(a) > x\}$ *exists for all $x \in]e, 1] \setminus \operatorname{Ran}(f)$, where $\operatorname{Ran}(f)$ is the range of f;*

(iv) $\bigwedge \{a \in [0, e] \mid f(a) < x\}$ *exists for all $x \in]e, 1] \setminus \operatorname{Ran}(f)$.*

Proof. For all $x \in]e, 1] \setminus \operatorname{Ran}(f)$, it is clear that $\{a \in [0, e] \mid f(a) > x\}$ is a lower set, $\{a \in [0, e] \mid f(a) < x\}$ is an upper set, and $\{a \in [0, e] \mid f(a) > x\} = [0, e] \setminus \{a \in [0, e] \mid f(a) < x\}$. Hence, it follows from Proposition 2.1 that $\bigvee \{a \in [0, e] \mid f(a) > x\}$ exists if and only if $\bigwedge \{a \in [0, e] \mid f(a) < x\}$ exists, which implies that (iii) $\Longleftrightarrow$ (iv).

(i) $\Longrightarrow$ (ii) Since g has property (**S**), it follows from Proposition 2.2 that $g(x) \in \{a \in L \mid g(a) > x\}^u \cap \{a \in L \mid g(a) < x\}^l$ for all $x > e$. Note that $\{a \in L \mid g(a) > x\} = \{a \in [0, e] \mid f(a) > x\}$ and $\{a \in L \mid g(a) < x\}^l = \{a \in [0, e] \mid f(a) < x\}^l$ for all $x > e$. Hence, $g(x) \in \{a \in [0, e] \mid f(a) > x\}^u \cap \{a \in [0, e] \mid f(a) < x\}^l$ for all $x > e$.

(ii) $\Longrightarrow$ (iii) For all $x \in]e, 1] \setminus \operatorname{Ran}(f)$, let $b \in \{a \in [0, e] \mid f(a) > x\}^u \cap \{a \in [0, e] \mid f(a) < x\}^l$. If $f(b) > x$, then $\bigvee \{a \in [0, e] \mid f(a) > x\} = b$; If $f(b) < x$, then $\bigwedge \{a \in [0, e] \mid f(a) < x\} = b$, and so $\bigvee \{a \in [0, e] \mid f(a) > x\}$ exists as proved at the beginning.

(iii) $\Longrightarrow$ (i) For all $x \in \operatorname{Ran}(f)$, set $f^{-1}(x) = \{a \in [0, e] \mid f(a) = x\}$. By the Axiom of Choice, there exists a function $f^* : \operatorname{Ran}(f) \to [0, e]$ with $f^*(x) \in f^{-1}(x)$. Define a function $g : L \to L$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in [0, e] \\ \bigvee \{a \in [0, e] \mid f(a) > x\} & \text{if } x \in]e, 1] \setminus \operatorname{Ran}(f) \\ f^*(x) & \text{if } x \in \operatorname{Ran}(f) \setminus \{e\}. \end{cases}$$

It is clear that $g|_{[0,e]} = f$, and $g(x) \leq e$ for all $x > e$. It remains to show that g is decreasing on $]e, 1]$ and has property (**S**). For the former, let $p, q \in]e, 1]$ with $p < q$, we consider four cases to prove that $g(q) \leq g(p)$:

(a) $p, q \in]e, 1] \setminus \operatorname{Ran}(f)$.

In this case, since $\{a \in [0, e] \mid f(a) > q\} \subseteq \{a \in [0, e] \mid f(a) > p\}$, it follows that

$$g(q) = \bigvee \{a \in [0, e] \mid f(a) > q\} \leq \bigvee \{a \in [0, e] \mid f(a) > p\} = g(p).$$

(b) $p, q \in \operatorname{Ran}(f) \setminus \{e\}$.

In this case, since $f(g(p)) = p < q = f(g(q))$ and f is decreasing, it follows that $g(q) < g(p)$.

(c) $p \in]e, 1] \setminus \operatorname{Ran}(f)$ and $q \in \operatorname{Ran}(f) \setminus \{e\}$.

In this case, since $f(g(q)) = q > p$, it follows that

$$g(q) \leq \bigvee \{a \in [0, e] \mid f(a) > p\} = g(p).$$

(d) $p \in \operatorname{Ran}(f) \setminus \{e\}$ and $q \in]e, 1] \setminus \operatorname{Ran}(f)$.

In this case, for all $a \in [0, e]$ with $f(a) > q$, since $f(g(p)) = p < f(a)$, it follows that $a < g(p)$. Hence,

$$g(q) = \bigvee \{a \in [0, e] \mid f(a) > q\} \leq g(p).$$

For the latter, let $(x, y) \in L^2$ with $y < g(x)$, we consider three cases to prove that $x \leq g(y)$:

(a) $x \in [0, e]$.

If $y \in [0, e]$, then $x \leq e \leq g(y)$;

If $y \in]e, 1] \setminus \operatorname{Ran}(f)$, then $x \leq \bigvee \{a \in [0, e] \mid f(a) > y\} = g(y)$;

If $y \in \operatorname{Ran}(f) \setminus \{e\}$, then it follows from $f(g(y)) = y < g(x) = f(x)$ that $x < g(y)$.

(b) $x \in]e, 1] \setminus \mathrm{Ran}(f)$.

In this case, we have $y < e$ and $g(x) = \bigvee\{a \in [0, e] \mid f(a) > x\}$. Since $y < g(x)$, it follows that there exists $a_0 \in [0, e]$ such that $y < a_0$ and $f(a_0) > x$. Hence, $x < f(y) = g(y)$.

(c) $x \in \mathrm{Ran}(f) \setminus \{e\}$.

In this case, we have $y < e$ and $g(x) = f^*(x)$. Since $y < f^*(x)$, it follows that $x = f(f^*(x)) \le f(y) = g(y)$.

We conclude that $g \in \mathcal{G}(L, e, \mathbf{S})$ and $g|_{[0,e]} = f$. $\square$

As a corollary of Theorem 4.1, every surjective and decreasing function $f : [0, e] \to [e, 1]$ with $f(e) = e$ is extendable.

Corollary 4.2. *Let $(L, \le, 0, 1)$ be a bounded chain with $e \in L \setminus \{0, 1\}$ and $f : [0, e] \to [e, 1]$ be surjective and decreasing with $f(e) = e$. Then there exists $g \in \mathcal{G}(L, e, \mathbf{S})$ such that $g|_{[0,e]} = f$.*

Example 4.3. Let $(L, \le, 0, 1)$ be a bounded chain with $e \in L \setminus \{0, 1\}$, $\{a_i\}_{i=0}^{n} \subseteq [0, e]$ with $0 = a_0 < a_1 < a_2 < \cdots < a_n \le e$ (n is a positive integer) and $\{b_i\}_{i=1}^{n} \subseteq [e, 1]$ with $1 \ge b_1 > b_2 > \cdots > b_n \ge e$. Define a decreasing function $f : [0, e] \to [e, 1]$ with $f(e) = e$ by

$$f(x) = \begin{cases} b_i & \text{if } x \in [a_{i-1}, a_i[\text{ and } i \in \{1, 2, \cdots, n\} \\ e & \text{if } x \in [a_n, e]. \end{cases}$$

Then there exists $g \in \mathcal{G}(L, e, \mathbf{S})$ such that $g|_{[0,e]} = f$.

When $([0, e], \le)$ is complete (in particular, $(L, \le, 0, 1)$ is complete), every decreasing function $f : [0, e] \to [e, 1]$ with $f(e) = e$ is extendable. In this case, the proof does not rely on the Axiom of Choice.

Theorem 4.4. *Let $(L, \le, 0, 1)$ be a bounded chain with $e \in L \setminus \{0, 1\}$, $([0, e], \le)$ be complete and $f : [0, e] \to [e, 1]$ be decreasing with $f(e) = e$. Then $\underline{f}, \overline{f} \in \mathcal{G}(L, e, \mathbf{S})$, where $\underline{f}$ and $\overline{f}$ are defined by*

$$\underline{f}(x) = \begin{cases} f(x) & \text{if } x \in [0, e] \\ \bigvee\{a \in [0, e] \mid f(a) > x\} & \text{if } x \in]e, 1]; \end{cases}$$

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in [0, e] \\ \bigwedge\{a \in [0, e] \mid f(a) < x\} & \text{if } x \in]e, 1]. \end{cases}$$

Proof. The proof is similar to that of (iii) $\implies$ (i) of Theorem 4.1, but the Axiom of Choice is not needed. $\square$

Remark 4.5. Given a decreasing function $f : [0, e] \to [e, 1]$ with $f(e) = e$,

(i) it is easy to see that $\underline{f}$ (resp. $\overline{f}$) is the smallest (resp. greatest) in $\mathcal{G}(L, e, \mathbf{S})$ whose restriction to $[0, e]$ is f, i.e., for all $g \in \mathcal{G}(L, e, \mathbf{S})$ with $g|_{[0,e]} = f$, it holds that $\underline{f}(x) \le g(x) \le \overline{f}(x)$ for all $x \in L$;

(ii) if $f^{-1}(x_0)$ contains more than one element for some $x_0 \in]e, 1]$, then $\underline{f} \ne \overline{f}$ as $\underline{f}(x_0) < \overline{f}(x_0)$;

(iii) if $([0, e], \le)$ is a finite chain, then $\underline{f} \ne \overline{f}$ as $\underline{f}(x) < \overline{f}(x)$ for all $x \in]e, 1]$.

When $]e, 1[\ne \emptyset$, the completeness of $([0, e], \le)$ is a necessary and sufficient condition for the extendability of all decreasing functions $f : [0, e] \to [e, 1]$ with $f(e) = e$.

Theorem 4.6. *Let $(L, \le, 0, 1)$ be a bounded chain with $e \in L \setminus \{0, 1\}$. Then the following statements are mutually equivalent:*

(i) $([0, e], \le)$ *is complete or* $]e, 1[= \emptyset$;

(ii) *For every decreasing function $f : [0, e] \to [e, 1]$ with $f(e) = e$, there exists $g \in \mathcal{G}(L, e, \mathbf{S})$ such that $g|_{[0,e]} = f$;*

(iii) *$\{a \in [0, e] \mid f(a) > x\}^u \cap \{a \in [0, e] \mid f(a) < x\}^l \neq \emptyset$ for all decreasing functions $f : [0, e] \to [e, 1]$ with $f(e) = e$ and all $x > e$;*

(iv) *$\bigvee\{a \in [0, e] \mid f(a) > x\}$ exists for all decreasing functions $f : [0, e] \to [e, 1]$ with $f(e) = e$ and all $x \in \,]e, 1] \setminus \mathrm{Ran}(f)$;*

(v) *$\bigwedge\{a \in [0, e] \mid f(a) < x\}$ exists for all decreasing functions $f : [0, e] \to [e, 1]$ with $f(e) = e$ and all $x \in \,]e, 1] \setminus \mathrm{Ran}(f)$.*

Proof. It follows from Theorem 4.1 that (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v). It is easy to see that (i) $\Longrightarrow$ (iv).

(iv) $\Longrightarrow$ (i) Suppose that $]e, 1[\,\neq \emptyset$. Let $b \in \,]e, 1[$. For any nonempty $A \subseteq [0, e[$, define $f_A : [0, e] \to [e, 1]$ by

$$f_A(x) = \begin{cases} 1 & \text{if } x \in (\downarrow A) \\ e & \text{if } x \in [0, e] \setminus (\downarrow A). \end{cases}$$

It is clear that f_A is decreasing with $f_A(e) = e$ and $b \notin \mathrm{Ran}(f_A)$. Hence, $\bigvee\{a \in [0, e] \mid f_A(a) > b\} = \bigvee(\downarrow A)$ exists, and so $\bigvee A = \bigvee(\downarrow A)$. Therefore, $([0, e], \leq)$ is complete. $\square$

5. Structure of the demarcating functions of unilaterally continuous idempotent uninorms on a complete chain

Let $(L, \leq, 0, 1)$ be a complete chain with top element 1 and bottom element 0. A unary function f on L is said to have property $(\tilde{\mathbf{S}})$ provided that

$$(\forall(x, y) \in L^2)(0 < y \leq f(x) \Longrightarrow x \leq f(y)). \tag{$\tilde{\mathbf{S}}$}$$

A unary function f on L is said to have property $(\hat{\mathbf{S}})$ provided that

$$(\forall(x, y) \in L^2)(1 > y \geq f(x) \Longrightarrow x \geq f(y)). \tag{$\hat{\mathbf{S}}$}$$

Let $e \in L \setminus \{0, 1\}$. Denote by $\mathcal{G}(L, e, \tilde{\mathbf{S}})$ (resp. $\mathcal{G}(L, e, \hat{\mathbf{S}})$) the set of all decreasing unary functions on L with fixed point e and property $(\tilde{\mathbf{S}})$ (resp. $(\hat{\mathbf{S}})$). It is known from Section 5 of [5] that left-continuous (resp. right-continuous) idempotent uninorms on L with neutral element e can be characterized by members of $\mathcal{G}(L, e, \tilde{\mathbf{S}})$ (resp. $\mathcal{G}(L, e, \hat{\mathbf{S}})$) as the demarcating functions. In this section, we study the structure of members of $\mathcal{G}(L, e, \tilde{\mathbf{S}})$ as well as that of $\mathcal{G}(L, e, \hat{\mathbf{S}})$.

Every $g \in \mathcal{G}(L, e, \tilde{\mathbf{S}})$, as the combination of two decreasing unary functions: $f_1 : [0, e] \to [e, 1]$ and $f_2 : [e, 1] \to [0, e]$, is uniquely determined by the first function, but not necessarily uniquely by the second one.

Theorem 5.1. *Let $(L, \leq, 0, 1)$ be a complete chain with $e \in L \setminus \{0, 1\}$ and g be a unary function on L. Then the following statements are mutually equivalent:*

(i) *$g \in \mathcal{G}(L, e, \tilde{\mathbf{S}})$;*

(ii) *there exists a decreasing function $f : [0, e] \to [e, 1]$ with $f(e) = e$ and $f(\bigvee_{i \in I} a_i) = \bigwedge_{i \in I} f(a_i)$ for any nonempty subset $\{a_i\}_{i \in I}$ of $[0, e]$ such that*

$$g(x) = \begin{cases} f(x) & \text{if } x \in [0, e] \\ \bigvee\{a \in [0, e] \mid f(a) \geq x\} & \text{if } x \in \,]e, 1]; \end{cases}$$

(iii) *there exists a decreasing function $f : [e, 1] \to [0, e]$ satisfying: $f(x) = e$ if and only if $x = e$, and $f(\bigvee_{i \in I} a_i) = \bigwedge_{i \in I} f(a_i)$ for any nonempty subset $\{a_i\}_{i \in I}$ of $[e, 1]$, such that*

$$g(x) = \begin{cases} f(x) & \text{if } x \in [e, 1] \\ \bigvee\{a \in [e, 1] \mid f(a) \geq x\} & \text{if } x \in]0, e[\\ \alpha & \text{if } x = 0, \end{cases}$$

where $\alpha \in [\bigvee_{x \in]0, e]} g(x), 1]$.

Proof. According to Proposition 5.6 of [5], a decreasing unary function g on L has property $(\tilde{\mathbf{S}})$ if and only if $g(g(x)) \geq x$ for all $x \in [0, g(0)]$, and $g(x) = 0$ for all $x > g(0)$.

(i) $\Longrightarrow$ (ii) Set $f = g|_{[0, e]}$. It is clear that $f : [0, e] \to [e, 1]$ is decreasing with $f(e) = e$. Hence, $f(\bigvee_{i \in I} a_i) \leq \bigwedge_{i \in I} f(a_i)$ for any nonempty subset $\{a_i\}_{i \in I}$ of $[0, e]$. The equality $f(\bigvee_{i \in I} a_i) = \bigwedge_{i \in I} f(a_i)$ trivially holds if $\bigvee_{i \in I} a_i = 0$. We suppose that $\bigvee_{i \in I} a_i > 0$. For any $i \in I$, it follows from $0 < \bigwedge_{i \in I} f(a_i) \leq f(a_i)$ that $a_i \leq g(\bigwedge_{i \in I} f(a_i))$, and so $0 < \bigvee_{i \in I} a_i \leq g(\bigwedge_{i \in I} f(a_i))$. Therefore, $\bigwedge_{i \in I} f(a_i) \leq f(\bigvee_{i \in I} a_i)$. It remains to show that $g(x) = \bigvee\{a \in [0, e] \mid f(a) \geq x\}$ for all $x > e$. If $x > g(0) = f(0)$, then $g(x) = 0 = \bigvee \emptyset = \bigvee\{a \in [0, e] \mid f(a) \geq x\}$; If $e < x \leq g(0)$, then $f(g(x)) = g(g(x)) \geq x$, and so $g(x) \leq \bigvee\{a \in [0, e] \mid f(a) \geq x\}$. For all $a \in [0, e]$ with $f(a) \geq x$, we have $a \leq g(x)$, and so $\bigvee\{a \in [0, e] \mid f(a) \geq x\} \leq g(x)$. Hence, $g(x) = \bigvee\{a \in [0, e] \mid f(a) \geq x\}$.

(ii) $\Longrightarrow$ (i) It is clear that g is decreasing with $g(e) = e$. For all $x > g(0) = f(0)$, $g(x) = \bigvee \emptyset = 0$. It remains to show that $g(g(x)) \geq x$ for all $x \in [0, g(0)]$. If $x \in [0, e]$, then $g(g(x)) = g(f(x)) = \bigvee\{a \in [0, e] \mid f(a) \geq f(x)\} \geq x$; If $x \in]e, g(0)]$, then $g(g(x)) = f(\bigvee_{f(a) \geq x} a) = \bigwedge_{f(a) \geq x} f(a) \geq x$.

The proof of (i) $\Longleftrightarrow$ (iii) is similar to that of (i) $\Longleftrightarrow$ (ii). $\square$

Remark 5.2. For every decreasing function $f : [e, 1] \to [0, e]$ satisfying: $f(x) = e$ if and only if $x = e$, and $f(\bigvee_{i \in I} a_i) = \bigwedge_{i \in I} f(a_i)$ for any nonempty subset $\{a_i\}_{i \in I}$ of $[e, 1]$,

(i) if $f(1) > 0$, then there exists a unique $g \in \mathcal{G}(L, e, \tilde{\mathbf{S}})$ with $g|_{[e, 1]} = f$ as $g(f(1)) = 1$ implies $g(0) = 1$;

(ii) if there exists $a_0 \in [e, 1[$ such that $f(x) = 0$ for all $x \in]a_0, 1]$, then there exist more than one $g \in \mathcal{G}(L, e, \tilde{\mathbf{S}})$ with $g|_{[e, 1]} = f$ as $g(0)$ can be arbitrarily chosen from $[a_0, 1]$. We give an example to show this observation.

Example 5.3. Let L be the real unit interval $[0, 1]$. Consider $e = \frac{1}{3}$ and the decreasing function $f : [\frac{1}{3}, 1] \to [0, \frac{1}{3}]$ defined by

$$f(x) = \begin{cases} \frac{1}{2}(1 - x) & \text{if } x \in [\frac{1}{3}, \frac{2}{3}] \\ 0 & \text{if } x \in]\frac{2}{3}, 1]. \end{cases}$$

For any $\alpha \in [\frac{2}{3}, 1]$, we have $g_\alpha \in \mathcal{G}(L, e, \tilde{\mathbf{S}})$ with $g_\alpha|_{[e, 1]} = f$, where g_α is given by

$$g_\alpha(x) = \begin{cases} f(x) & \text{if } x \in [\frac{1}{3}, 1] \\ 1 - 2x & \text{if } x \in]\frac{1}{6}, \frac{1}{3}[\\ \frac{2}{3} & \text{if } x \in]0, \frac{1}{6}] \\ \alpha & \text{if } x = 0. \end{cases}$$

Similarly, every $g \in \mathcal{G}(L, e, \hat{\mathbf{S}})$, as the combination of two decreasing unary functions: $f_1 : [0, e] \to [e, 1]$ and $f_2 : [e, 1] \to [0, e]$, is uniquely determined by the second function, but not necessarily uniquely by the first one.

Theorem 5.4. *Let $(L, \leq, 0, 1)$ be a complete chain with $e \in L \setminus \{0, 1\}$ and g be a unary function on L. Then the following statements are mutually equivalent:*

(i) $g \in \mathcal{G}(L, e, \hat{\mathbf{S}})$;

(ii) *there exists a decreasing function $f : [e, 1] \to [0, e]$ with $f(e) = e$ and $f(\bigwedge_{i \in I} a_i) = \bigvee_{i \in I} f(a_i)$ for any nonempty subset $\{a_i\}_{i \in I}$ of $[e, 1]$ such that*

$$g(x) = \begin{cases} f(x) & \text{if } x \in [e, 1] \\ \bigwedge \{a \in [e, 1] \mid f(a) \leq x\} & \text{if } x \in [0, e[\; ; \end{cases}$$

(iii) *there exists a decreasing function $f : [0, e] \to [e, 1]$ satisfying: $f(x) = e$ if and only if $x = e$, and $f(\bigwedge_{i \in I} a_i) = \bigvee_{i \in I} f(a_i)$ for any nonempty subset $\{a_i\}_{i \in I}$ of $[0, e]$, such that*

$$g(x) = \begin{cases} f(x) & \text{if } x \in [0, e] \\ \bigwedge \{a \in [0, e] \mid f(a) \leq x\} & \text{if } x \in]e, 1[\\ \alpha & \text{if } x = 1, \end{cases}$$

where $\alpha \in [0, \bigwedge_{x \in [e, 1[} g(x)]$.

6. Concluding remarks

We have introduced and characterized the separability of idempotent uninorms on a bounded chain. In addition, we have investigated the structure of the demarcating functions of separable idempotent uninorms on a bounded chain and that of the demarcating functions of unilaterally continuous idempotent uninorms on a complete chain. These results as well as that of [5, 6] are beneficial to deeper comprehension of the structure of idempotent uninorms on a bounded chain. We hope these results would inspire the research into full characterization of idempotent uninorms on more general structures, e.g., complete lattices and bounded lattices further.

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