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Topological manifestations of group-theoretic structures

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Abstract. Inspired by approximation operators in rough set theory and utilizing group theoretical concepts, approximation operators on groups have been defined in this study. These operators have been used to construct various topological structures. Key topologies introduced on groups include the cn-topology derived from cyclic subgroups, the Cn-topology from centralizers, the Nn-topology from normalizers, the Ln-topology and Rn-topology from left and right cosets of a subgroup, and the on-topology from group actions. It has been determined that isomorphic groups yield homeomorphic spaces, revealing a fundamental connection between group structures and topological properties. Furthermore, it has been observed that for Abelian groups, the cn-topology, Ln-topology, and Rn-topology form topological groups.

1. Introduction and background

In mathematics, some of the most profound insights emerge from the interplay between seemingly distinct disciplines. While group theory focuses on the analysis of algebraic structures and symmetries, topology offers a framework for understanding spatial and continuity-related properties. This study aims to establish a conceptual bridge between these two fields by investigating how abstract algebraic structures can give rise to meaningful topological frameworks. In particular, we explore the adaptation of approximation operators—originally developed in the context of rough set theory to manage uncertainty—into the setting of group theory. The resulting topologies, derived from group-theoretic notions such as cyclic subgroups, centralizers, and group actions, not only provide new perspectives on the structural features of groups but also reveal deep connections between isomorphic groups and homeomorphic topological spaces. This synthesis opens a novel pathway for analyzing algebraic symmetry through topological lenses, offering both theoretical richness and potential for further applications.

Rough set theory [16], introduced by Zdzisław Pawlak in the early 1980s, is a mathematical framework for dealing with vagueness and uncertainty in data analysis. Unlike traditional set theory, where an element either belongs to a set or it doesn't, rough set theory provides tools for representing uncertainty about set membership. This theory is particularly useful in situations where incomplete or imprecise information is available, making it a powerful tool for data mining, pattern recognition, and decision analysis.

Central to rough set theory are the concepts of lower and upper approximation operators. These operators are used to approximate a target set by using the available information. The lower approximation of a set includes all elements that are definitively classified as members of the set based on the available

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data, while the upper approximation consists of elements that possibly belong to the set. The difference between the upper and lower approximations forms the boundary region, which represents the uncertainty or ambiguity in classifying elements.

The strength of rough set theory lies in its ability to work without additional information, such as probability distributions or membership functions, which are often required in other theories like fuzzy set theory. By relying solely on the granularity of the data, rough sets provide a simple yet effective way to manage uncertainty, particularly in the field of information systems, where data may be incomplete or noisy.

In practical applications, rough set theory has been used in fields such as medical diagnosis, image processing, and feature selection, where it helps to identify essential data patterns and support decision-making processes. The approximation operators play a key role in identifying which data points are core to a classification decision and which may require further investigation or refinement.

This theoretical framework continues to evolve, offering insights into new ways to handle imprecision and making it an indispensable tool in modern data-driven research.

As originally introduced by Pawlak [16, 17], rough sets are grounded in a universal set and an equivalence relation defined upon it. Let U denote a non-empty set referred to as the universe, and let R be an equivalence relation on U. The pair (U, R) is then termed an approximation space. For each $x \in U$, the equivalence class of x is called an elementary set of (U, R). Clearly, the family of all elementary sets is the quotient set $U/_R$. The equivalence relation R partitions the universe into equivalence classes, thereby defining the granularity of information available about the elements of U. This granularity is reflected in the lower and upper approximations, which are central to rough set theory. Therefore, using the equivalence classes, denoted by $[x]_R$, of a point x in the approximation space (U, R); the lower approximation of X is defined by

$$R^{\downarrow}(X) = \{x \in U \mid [x]_R \subseteq X\}$$

and the upper approximation of X is defined by

$$R^{\uparrow}(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}$$

for each $X \subseteq U$. The pair $(R^{\downarrow}(X), R^{\uparrow}(X))$ is defined as the rough set of X. The rough set $(R^{\downarrow}(X), R^{\uparrow}(X))$ is an approximation of X based on the current knowledge about R, which allows for the classification of U. Numerous studies have explored extensions of rough set theory by substituting the equivalence relation in the approximation space with a more general relation. These generalizations have enhanced the theory's applicability to a wide range of domains. Now, assume that R be an ordinary relation on U. Then, (U, R) is referred as an approximation space. Here, the definition of approximation operators is based on the concept of neighborhood of a point in the universe defined with respect to the ordinary relation R. According to the relation R, the neighborhood of a point x is defined as $n(x) = \{y \in U \mid (x, y) \in R\}$. Clearly, according to this definition, n is an operator of the form $n : U \to \mathcal{P}(U)$, which is called the neighborhood operator. With this generalization, the lower and upper approximations of a subset $X \subseteq U$ with respect to R are given as follows:

$$R^{\downarrow}(X) = \{x \in U \mid n(x) \subseteq X\} \text{ and} \tag{1}$$

$$R^{\uparrow}(X) = \{x \in U \mid n(x) \cap X \neq \emptyset\} \tag{2}$$

Due to this generalization, the pair $(R^{\downarrow}(X), R^{\uparrow}(X))$ is called a generalized rough set. The difference between $R^{\uparrow}(X)$ and $R^{\downarrow}(X)$ is called the R-boundary of X and is denoted by $BN_R(X)$, i.e. $BN_R(X) = R^{\uparrow}(X) - R^{\downarrow}(X)$.

In [13], in a generalized approximation space (U, R), a point x is called a solitary element if $n(x) = \emptyset$, and the set of all solitary element in (U, R) is denoted by $S = \{x \in U \mid n(x) = \emptyset\}$. However, as we recall, if

$n(x) \neq \emptyset$ for every $x \in U$, then the relation R on U is called a serial relation, and also, n is called the serial neighborhood operator. We can clearly see that; n is serial neighborhood if and only if $S = \emptyset$.

Considering an arbitrary universal set U, a relation R defined on it, and a set S of solitary elements, the following properties hold:

Proposition 1.1. ([13])

(a) $R^{\downarrow}(\emptyset) = S$, $R^{\uparrow}(\emptyset) = \emptyset$, $R^{\downarrow}(U) = U$ *and* $R^{\uparrow}(U) = S^c$, *where* S^c *is the complement of* S.

(b) $S \subseteq R^{\downarrow}(X)$ *and* $R^{\uparrow}(X) \subseteq S^c$ *for all* $X \in \mathcal{P}(U)$.

(c) $R^{\downarrow}(X) - S \subseteq R^{\uparrow}(X)$ *for each* $X \in \mathcal{P}(U)$.

(d) $R^{\downarrow}(X) = U$ *iff* $\bigcup_{x\in U} n(x) \subseteq X$ *and* $R^{\uparrow}(X) = \emptyset$ *iff* $X \subseteq (\bigcup_{x\in U} n(x))^c$.

(e) *If* $S \neq \emptyset$, *then* $R^{\downarrow}(X) \neq R^{\uparrow}(X)$ *for each* $X \in \mathcal{P}(U)$.

(f) *Let* I *be an arbitrary index set. Then,* $R^{\downarrow}(\bigcap_{i\in I} X_i) = \bigcap_{i\in I} R^{\downarrow}(X_i)$ *and* $R^{\uparrow}(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} R^{\uparrow}(X_i)$.

(g) *For each* $X, Y \subseteq U$, *if* $X \subseteq Y$ *then* $R^{\downarrow}(X) \subseteq R^{\downarrow}(Y)$ *and* $R^{\uparrow}(X) \subseteq R^{\uparrow}(Y)$.

(h) $R^{\downarrow}(X) \cup R^{\downarrow}(Y) \subseteq R^{\downarrow}(X \cup Y)$ *and* $R^{\uparrow}(X \cap Y) \subseteq R^{\uparrow}(X) \cap R^{\uparrow}(Y)$.

(i) $(R^{\downarrow}(X))^c = R^{\uparrow}(X^c)$ *and* $(R^{\uparrow}(X))^c = R^{\downarrow}(X^c)$.

(j) *There is some* $X \in \mathcal{P}(U)$ *such that* $R^{\downarrow}(X) = R^{\uparrow}(X)$ *iff* R *is serial.*

This rough set introduction is primarily intended to highlight the role of approximation operators. In essence, the concept of a rough set is grounded in these approximation operators. One of the foundational concepts of this study is the notion of lower approximation operators.

One of the key concepts we will employ in this study is the interior operator. An interior operator is generated by its dual concept, known as the closure operator. Initially defined as a method for generating topologies, these concepts appear in many areas of mathematics. A closure operator on a set U is a function $\mathbf{cl} : \mathcal{P}(U) \to \mathcal{P}(U)$ that fulfills axioms;

(a) $X \subseteq \mathbf{cl}(X)$ for any $X \subseteq U$.

(b) If $X \subseteq Y$ for each $X, Y \subseteq U$, then $\mathbf{cl}(X) \subseteq \mathbf{cl}(Y)$.

(c) $\mathbf{cl}(\mathbf{cl}(X)) = \mathbf{cl}(X)$ for any $X \subseteq U$.

As the dual of this concept, the interior operator is defined as a mapping $\mathbf{int} : \mathcal{P}(U) \to \mathcal{P}(U)$ that satisfies conditions

(a) $\mathbf{int}(X) \subseteq X$ for any $X \subseteq U$.

(b) If $X \subseteq Y$ for each $X, Y \subseteq U$, then $\mathbf{int}(X) \subseteq \mathbf{int}(Y)$.

(c) $\mathbf{int}(\mathbf{int}(X)) = \mathbf{int}(X)$ for any $X \subseteq U$.

There exists a relationship

$$\mathbf{int}(X) = U - \mathbf{cl}(U - X) \tag{3}$$

or

$$\mathbf{cl}(X) = U - \mathbf{int}(U - X) \tag{4}$$

between the interior operator and the closure operator.

We shall now discuss a significant area of mathematics. Topology is a major branch of mathematics that studies the invariant properties of spaces under continuous deformations. Theoretically, given a set U and a collection $\mathcal{T}$ of subsets of U, if the intersection of any finite subcollection of $\mathcal{T}$ and the union of any arbitrary subcollection of $\mathcal{T}$ are also elements of $\mathcal{T}$, then $\mathcal{T}$ is called a topology on U, and the pair $(U,\mathcal{T})$ is called a topological space. All elements of the topology (i.e., all subsets of U that belong to $\mathcal{T}$) are called open sets, and sets whose complements are open are called closed sets. Let $(U,\mathcal{T})$ and $(V,\mathcal{T}')$ be two topological spaces and $f : U \to V$ be a function. If the preimage of every open set in $(V,\mathcal{T}')$ under f is an open set in $(U,\mathcal{T})$, then f is called a continuous function. In a topological space $(U,\mathcal{T})$, a collection $\mathcal{B}$ of subsets of U is called a basis for the topology $\mathcal{T}$ if every open set in $\mathcal{T}$ can be expressed as a union of elements of $\mathcal{B}$ [14]. An Alexandroff topology (or Alexandroff space) is a topological space where arbitrary intersections of open sets are still open. This property contrasts with general topological spaces, where typically only finite intersections of open sets are required to be open [3].

It is a well-known fact that the collection of subsets of a set U, denoted by

$$\mathcal{T} = \{X \subseteq U | \mathbf{int}(X) = X\}, \tag{5}$$

defined using the interior operator, forms a unique topology on U [14].

The following lemma provides a characterization of minimal bases in Alexandroff spaces.

Lemma 1.2. ([6]) *A collection $\mathcal{M}$ of open sets in an Alexandroff space $(U,\mathcal{T})$ is a minimal base for the topology if and only if*

(a) $\mathcal{B}$ covers U;

(b) If $X, Y \in \mathcal{M}$, there exists a subcollection $\{M_i \mid i \in I\}$ of $\mathcal{M}$ such that $X \cap Y = \bigcup_{i\in I} M_i$;

(c) If a subcollection $\{M_i \mid i \in I\}$ of $\mathcal{M}$ satisfies $\bigcup_{i\in I} M_i \in \mathcal{M}$, then there exists $i_0 \in I$ such that $\bigcup_{i\in I} M_i = M_{i_0}$.

Furthermore, group theory, a branch of mathematics that studies algebraic structures, is a fundamental field with significant applications in various disciplines. A group is defined as a set equipped with an operation satisfying four fundamental properties: closure, associativity, identity, and inverse. Formally, let G be nonempty set, and $* : G \times G \to G$ be a mapping. Then the pair $(G, *)$ is called a group such that

Closedness For any $x, y \in G$, $x * y \in G$.

Associavity $x * (y * z) = (x * y) * z$ for every $x, y, z \in G$.

Identity There exists $e \in G$ such that $x * e = e * x = x$ for each $x \in G$.

Inverse For any $x \in G$, there exists a $y \in G$ such that $x * y = y * x = e$, and it is denoted by $y = x^{-1}$.

Such groups are typically referred to as multiplicative groups, and the group operation is usually denoted by juxtaposition, i.e., $x * y$ is simply written as xy and then $x^n = \underbrace{xx\cdots x}_{n}$. This notation will be adopted throughout this work.

A subgroup is a subset of a group that is itself a group under the same operation. More technically, for $S \subseteq G$, S is a subgroup of G if and only if for all $x, y \in S$, $xy^{-1} \in S$.

Let G_1 and G_2 be two groups, and let $f : G_1 \to G_2$ be a function. If for all $x, y \in G_1$, $f(xy) = f(x)f(y)$, then f is called a homomorphism. A homomorphism that is injective is called a monomorphism, and a homomorphism that is surjective is called an epimorphism. A homomorphism that is both injective and surjective is called an isomorphism.

In [12], group-theoretic concepts such as the cyclic subgroup generated by an element, the centralizer of an element, and the normalizer of an element were employed to define approximation operators on groups, and some of their fundamental properties were investigated. To recapitulate these concepts:

Let G be a group and S be a subgroup of G.

Cyclic Subgroup [10] The cyclic subgroup of G generated by an element $x \in G$ is defined to be

$$\langle x \rangle = \{x^n \mid n \in \mathbb{Z}\},$$

where $\mathbb{Z}$ is the set of integers.

Centralizer [15] The centralizer of an element $x \in G$ is defined to be

$$C(x) = \{y \in G \mid xy = yx\}.$$

Normalizer [15] The normalizer of an element $x \in G$ is defined to be

$$N(x) = \{y \in G \mid y^{-1}xy \in \langle x \rangle\}.$$

Left - Right Coset [10] The left coset of S in G is defined to be

$$xS = \{xy \mid y \in S\},$$

and the right coset of S in G is defined to be

$$Sx = \{yx \mid y \in S\},$$

where $x \in G$.

It should be noted that there is a relationship between the cyclic subgroup, the centralizer, and the normalizer, such that

$$\langle x \rangle \subseteq C(x) \subseteq N(x). \tag{6}$$

The following general information is available for cyclic subgroups.

Lemma 1.3. ([6, 10]) *Let G be a group and $x \in G$.*

(a) $\langle x \rangle$ is a subgroup of G.

(b) $x \in \langle x \rangle$.

(c) Let H be a subgroup of G, and let x be an element of H. Then, $\langle x \rangle$, is also subgroup of H. Moreover, $\langle x \rangle$ is the smallest subgroup of G containing x.

(d) $\langle x \rangle$ is abelian.

Definition 1.4. ([10]) Let G be a group and U be a non-empty set. We say that $\cdot : G \times U \to U$ is a group action of G on U that satisfies the following two axioms;

(a) $e \cdot x = x$, for all $x \in U$ and $e \in G$ is an identity element of G.

(b) $(gh) \cdot x = g \cdot (h \cdot x)$, for all $g, h \in G$ and for all $x \in U$.

We say that U is G-set if G is acting on U.

Definition 1.5. ([10]) Let G be a group acting on a set U. The orbit of an element $x \in U$ is defined as

$$\mathbf{Orb}(x) = \{y \in U \mid \exists g \in G, y = g \cdot x\}.$$

That is $\mathbf{Orb}(x) = G \cdot x$.

At the same time, for given $x \in U$, its stabilizer is defined as

$$\mathbf{Stab}(x) = \{g \in G \mid g \cdot x = x\}$$

which is subset of G.

Note that, for each $x \in U$, **Stab**(x) is a subgroup of G. It is called a stabilizer subgroup of G or isotropy subgroup of x.

Definition 1.6. ([10]) Let U and V be two G-sets, and $f : U \to V$ be a function. We call that f is a morphism of G-sets or G-function, if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in U$. If f is a bijective G-function then we call f is an isomorphism.

Definition 1.7. ([10]) Let G be a group. For any $g \in G$, the element aga^{-1} is called conjugate of g with respect to a. The automorphism $f : G \to G$ such that $f(x) = axa^{-1}$ is called conjugation.

Let G be a group. If we define the operation $\cdot : G \times G \to G$ such that $g \cdot x = gxg^{-1}$, then we have an action $\cdot$, and G acts on itself. We call that $\cdot$ is conjugation action.

Recent neighborhood-driven rough set models have been actively developed with an eye toward application-oriented approximation mechanisms: for instance, [9] introduces initial-neighborhood systems and demonstrates their usefulness on medical diagnosis tasks (Covid-19 variants), while [11] proposes a primal approximation framework built from κ-neighborhoods and illustrates its applicability on representative decision problems. From a methodological standpoint, the β-basic rough-set formalism in [8] further enriches neighborhood-based approximations with medically motivated case studies, and a topological reduction pipeline for generalized rough sets is explored for lung-cancer prediction in [7]. On the purely topological side, the neighborhood-to-topology principle appears in relation-generated constructions: [4] develops general mechanisms for generating topologies from relations, the dual-topology viewpoint is advanced in [2], and a systematic comparison of rough approximations induced by different topologies is provided in [1]. The present paper builds on this line by selecting relations canonically from group-theoretic data (cosets, centralizers, normalizers, and actions), thereby producing structured neighborhood systems whose induced topologies admit additional algebraic interpretations (cf. Figure 1).

This paper introduces approximation operators on groups, defined in terms of the aforementioned group-theoretic notions. Fundamental properties of these operators will be established, demonstrating that they are inner operators. Subsequently, we will utilize these concepts to introduce and investigate topologies on the group G, such as the cn-topology, Cn-topology, Nn-topology, and Ln-topology and Rn-topology with respect to a subgroup exploring the interconnections between these topological structures. Subsequently, employing the same approach, we will obtain a topology on a set, called the on-topology, arising from the group action of a group on the set. The fundamental properties of this topology will be investigated.

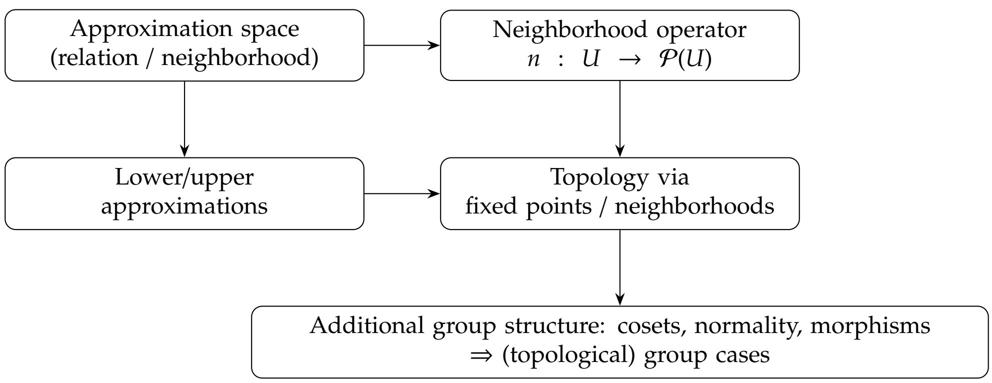

Figure 1: From rough approximations to topologies and group-topological structures in this paper.

Symbol	Meaning
G	A group with identity element e.
e	The identity element of G.
$S \leq G$	A fixed subgroup of G (used in coset-based neighborhoods).
$\mathcal{P}(G)$	The power set of G.
$X \subseteq G$	An arbitrary subset of G.
$n : G \to \mathcal{P}(G)$	A generic neighborhood operator.
$L_n : G \to \mathcal{P}(G),\ L_n(x) = xS$	Left-coset neighborhood of x with respect to S.
$R_n : G \to \mathcal{P}(G),\ R_n(x) = Sx$	Right-coset neighborhood of x with respect to S.
$c_n : G \to \mathcal{P}(G)$	Neighborhood operator induced by cyclic subgroups (as defined in the text).
$C_n : G \to \mathcal{P}(G)$	Neighborhood operator induced by centralizers (as defined in the text).
$N_n : G \to \mathcal{P}(G)$	Neighborhood operator induced by normalizers (as defined in the text).
$\mathbf{apr}_n(X)$	Approximation operator associated with n.
$\underline{\mathbf{apr}}_n(X) = \{x \in G : n(x) \subseteq X\}$	Lower approximation.
$\overline{\mathbf{apr}}_n(X) = \{x \in G : n(x) \cap X \neq \emptyset\}$	Upper approximation.
$\mathcal{T}_n = \{X \subseteq G : \underline{\mathbf{apr}}_n(X) = X\}$	Topology induced by n via fixed points of $\underline{\mathbf{apr}}_n$.
$\mathcal{T}_{L_n}$	The topology induced by L_n.
$\mathcal{T}_{R_n}$	The topology induced by R_n.
$\mathcal{T}_{c_n}$	The topology induced by c_n.
$\mathcal{T}_{C_n}$	The topology induced by C_n.
$\mathcal{T}_{N_n}$	The topology induced by N_n.
$(U, \varphi),\ \varphi : G \times U \to U$	A G-set (a left group action on U).
$G \cdot u = \{\varphi(g, u) : g \in G\}$	The orbit of $u \in U$.
$G_u = \{g \in G : \varphi(g, u) = u\}$	The stabilizer subgroup of $u \in U$.
$o_n : U \to \mathcal{P}(U)$	Orbit-based neighborhood operator (as defined for the on-topology).
$\mathcal{T}_{on}$	The on-topology on U induced by o_n.

Table 1: Main symbols and notation used throughout the paper.

2. Topologies with respect to group concepts

2.1. The cn-topology, Cn-topology and Nn-topology

The group structural concepts presented in the preceding section allow for a direct definition of neighborhood operators on the group.

Definition 2.1. ([13]) Assume that G is a group. Define set valued mappings as follows:

(1) The mapping $cn : G \to \mathcal{P}(G)$ such that $cn(x) = \langle x \rangle$ for each $x \in G$ is called the cyclical neighborhood operation on G.

(2) The mapping $Cn : G \to \mathcal{P}(G)$ such that $Cn(x) = C(x)$ for each $x \in G$ is called the centralizing neighborhood operator on G.

(3) The mapping $Nn : G \to \mathcal{P}(G)$ such that $Nn(x) = N(x)$ for each $x \in G$ is called the normalizing neighborhood operator on G.

Based on Yao's [19] definitions of lower and upper approximation operators for a set , which are formulated using the notion of neighborhood operators, the following definition can be presented.

Definition 2.2. ([12]) Let G be a group, and X be non-empty subset of G.

(1) The lower approximation of X with respect to the cyclical neighborhood operator is defined as $\underline{\mathbf{apr}}_{cn}(X) = \{x \in U \mid cn(x) \subseteq X\}$, and the upper approximation of X with respect to the cyclical neighborhood operator is defined as $\overline{\mathbf{apr}}_{cn}(X) = \{x \in U \mid cn(x) \cap X \neq \emptyset\}$.

(2) The lower approximation of X with respect to the centralizing neighborhood operator is defined as $\underline{\mathbf{apr}}_{Cn}(X) = \{x \in U \mid Cn(x) \subseteq X\}$, and the upper approximation of X with respect to the centralizing neighborhood operator is defined as $\overline{\mathbf{apr}}_{Cn}(X) = \{x \in U \mid Cn(x) \cap X \neq \emptyset\}$.

(3) The lower approximation of X with respect to the normalizing neighborhood operator is defined as $\underline{\mathbf{apr}}_{Nn}(X) = \{x \in U \mid Nn(x) \subseteq X\}$, and the upper approximation of X with respect to the normalizing neighborhood operator is defined as $\overline{\mathbf{apr}}_{Nn}(X) = \{x \in U \mid Nn(x) \cap X \neq \emptyset\}$.

Proposition 2.3. ([12]) *Let G be a group and $S \subseteq G$. $\underline{\mathbf{apr}}_{cn}(S) \supseteq \underline{\mathbf{apr}}_{Cn}(S) \supseteq \underline{\mathbf{apr}}_{Nn}(S)$.*

Proof. Assume that $x \in \underline{\mathbf{apr}}_{Nn}(S)$. So, we have $Nn(x) \subseteq S$. From Equation 6, since $Cn(x) \subseteq Nn(x)$, it follows that $Cn(x) \subseteq S$, and therefore $x \in \underline{\mathbf{apr}}_{Cn}(S)$. Hence, we obtain that $\underline{\mathbf{apr}}_{Cn}(S) \supseteq \underline{\mathbf{apr}}_{Nn}(S)$.

In a similar manner, it can be demonstrated that $\underline{\mathbf{apr}}_{cn}(S) \supseteq \underline{\mathbf{apr}}_{Cn}(S)$ holds. As a result, the desired outcome is achieved. □

Proposition 2.4. *In a group G, for any $x \in G$, we have $\underline{\mathbf{apr}}_{cn}(\langle x \rangle) = \langle x \rangle$.*

Proof. Suppose $y \in \underline{\mathbf{apr}}_{cn}(\langle x \rangle)$. Then, by definition, $\langle y \rangle \subseteq \langle x \rangle$. Clearly, since $y \in \langle y \rangle$, we have $y \in \langle x \rangle$. Conversely, suppose $y \in \langle x \rangle$. Thus, there exists a $k \in \mathbb{Z}$ such that $y = x^k$. From this, since $\langle y \rangle = \langle x^k \rangle$ is a subgroup of $\langle x \rangle$, the condition $\langle y \rangle \subseteq \langle x \rangle$ holds. Therefore, we obtain $y \in \underline{\mathbf{apr}}_{cn}(\langle x \rangle)$. □

Proposition 2.5. *Let G be a group, $S_1, S_2 \subseteq G$ and $S_i \subseteq G$ for each $i \in I$.*

(a) $\underline{\mathbf{apr}}_{cn}(G) = G$, $\underline{\mathbf{apr}}_{cn}(\emptyset) = \emptyset$.

(b) $\underline{\mathbf{apr}}_{cn}(S) \subseteq S$.

(c) $\underline{\mathbf{apr}}_{cn}(\underline{\mathbf{apr}}_{cn}(S)) = \underline{\mathbf{apr}}_{cn}(S)$.

(d) *If* $S_1 \subseteq S_2$, *then* $\underline{\mathbf{apr}}_{cn}(S_1) \subseteq \underline{\mathbf{apr}}_{cn}(S_2)$.

(e) $\underline{\mathbf{apr}}_{cn}(\bigcap_{i\in I} S_i) = \bigcap_{i\in I} \underline{\mathbf{apr}}_{cn}(S_i)$.

(f) $\underline{\mathbf{apr}}_{cn}(\bigcup_{i\in I} S_i) = \bigcup_{i\in I} \underline{\mathbf{apr}}_{cn}(S_i)$.

Proof. (a) Clearly, assuming $x \in G$, we have $\langle x \rangle \subseteq G$, and hence $x \in \underline{\mathbf{apr}}_{cn}(G)$. Moreover, it is evident that $\underline{\mathbf{apr}}_{cn}(\emptyset) = \emptyset$.

(b) Let $x \in \underline{\mathbf{apr}}_{cn}(S)$. Then $\langle x \rangle \subseteq S$. Since $x \in \langle x \rangle$, we have $x \in S$.

(c) It is evident from (b) that $\underline{\mathbf{apr}}_{cn}(\underline{\mathbf{apr}}_{cn}(S)) \subseteq \underline{\mathbf{apr}}_{cn}(S)$. On the other hand, let x be an element of $\underline{\mathbf{apr}}_{cn}(S)$. Then, $\langle x \rangle$ is a subset of S. From (d), we have $\underline{\mathbf{apr}}_{cn}(\langle x \rangle)$ is a subset of $\underline{\mathbf{apr}}_{cn}(S)$. Since $\underline{\mathbf{apr}}_{cn}(\langle x \rangle)$ is equal to $\langle x \rangle$ by Proposition 2.4, it follows that $\langle x \rangle$ is clearly a subset of $\underline{\mathbf{apr}}_{cn}(S)$. Hence, $\underline{\mathbf{apr}}_{cn}(S) \subseteq \underline{\mathbf{apr}}_{cn}(\underline{\mathbf{apr}}_{cn}(S))$. This is the desired result.

(d) Let $x \in \underline{\mathbf{apr}}_{cn}(S_1)$. Then, $\langle x \rangle \subseteq S_1$. Since S_1 is a subset of S_2 by hypothesis, it follows that $\langle x \rangle$ is also a subset of S_2. Consequently, we have $x \in \underline{\mathbf{apr}}_{cn}(S_2)$.

(e) Suppose $x \in \underline{\mathbf{apr}}_{cn}(\bigcap_{i\in I} S_i)$. By definition, $\langle x \rangle \subseteq \bigcap_{i\in I} S_i$. Then, for all $i \in I$, $\langle x \rangle \subseteq S_i$. This clearly implies that, for all $i \in I$, $x \in \underline{\mathbf{apr}}_{cn}(S_i)$. Therefore, $x \in \bigcap_{i\in I} \underline{\mathbf{apr}}_{cn}(S_i)$. The reverse inclusion can be established in a similar manner, thus showing the equality of the sets.

(f) The proof is analogous to that of (e). □

As Proposition 2.5 guarantees that $\underline{\mathbf{apr}}_{cn} : P(G) \to P(G)$ defines an interior operator, the following result is immediate.

Proposition 2.6. *Let G be any group. The family defined*

$$\mathcal{T}_{cn} = \{S \subseteq G \mid \underline{\mathbf{apr}}_{cn}(S) = S\} \tag{7}$$

is a topology on G.

The topology on G obtained in Proposition 2.6 is called the cyclic topology or, briefly, the cn-topology, and the space $(G, \mathcal{T}_{cn})$ is called a cn-space.

It follows directly from Propositions 2.4 and 2.6 that the following is true.

Corollary 2.7. *In the cn-space $(G, \mathcal{T}_{cn})$, every cyclic subgroup $\langle x \rangle$, where $x \in G$, is an open set.*

It can be demonstrated now that for an arbitrary subset S of the cn-space $(G, \mathcal{T}_{cn})$, the interior of S, $\mathring{S}$, coincides with the $\underline{\mathbf{apr}}_{cn}(S)$. Let $(G, \mathcal{T}_{cn})$ be a cn-space and S be a subset of G. Consider an arbitrary element $x \in \mathring{S}$. Then, there exists an open set O in $\mathcal{T}_{cn}$ such that $x \in O \subseteq S$. By the definition of the cn-topology, $\underline{\mathbf{apr}}_{cn}(O) = O$, implying $x \in \underline{\mathbf{apr}}_{cn}(O)$ and thus $\langle x \rangle \subseteq O$. Consequently, $\langle x \rangle \subseteq S$, which yields $x \in \underline{\mathbf{apr}}_{cn}(S)$. On the other hand, suppose $x \in \underline{\mathbf{apr}}_{cn}(S)$. Then, clearly, $x \in \langle x \rangle \subseteq S$. By Proposition 2.4, the cyclic subgroup generated by any element x in G belongs to the cn-topology. Therefore, x is an interior point of S. Consequently, the interior of S is equal to the $\underline{\mathbf{apr}}_{cn}(S)$.

In addition, let $\underline{\mathbf{apr}}_{cn}$ denote the interior operator associated with an arbitrary topological space $(G, \mathcal{T})$. It is straightforward to verify that a subset S of G is open precisely when the $\underline{\mathbf{apr}}_{cn}(S) = S$. This characterization implies that the cn-topology $\mathcal{T}_{cn}$ is identical to the given topology $\mathcal{T}$. Hence, $\mathcal{T}_{cn}$ is unique.

Example 2.8. Let the dihedral group

$$D_3 = \langle r, s \mid r^3 = s^2 = e, rs = sr^{-1} \rangle = \{e, r, r^2, s, rs, r^2s\},$$

whose operation table is given below in Table 2, act on itself by conjugation.

	e	r	r^2	s	rs	r^2s
e	e	r	r^2	s	rs	r^2s
r	r	r^2	e	rs	r^2s	s
r^2	r^2	e	r	r^2s	s	rs
s	s	r^2s	rs	e	r^2	r
rs	rs	s	r^2s	r	e	r^2
r^2s	r^2s	rs	s	r^2	r	e

Table 2: Operation table of the dihedral group D_3

As is conventional, the symbol r signifies a rotation, and s signifies a reflection.

Computing the cyclic subgroups generated by each element, we obtain the following: $\langle e \rangle = \{e\}$, $\langle r \rangle = \langle r^2 \rangle = \{e, r, r^2\}$, $\langle s \rangle = \{e, s\}$, $\langle rs \rangle = \{e, rs\}$, and $\langle r^2s \rangle = \{e, r^2s\}$.

It is evident that the power set of D_3, $\mathcal{P}(D_3)$, contains $2^6 = 64$ elements. Upon computing the lower approximations of all subsets using the cyclical neighborhood, the cn-topology is determined together with the subsets S fulfilling the condition $\underline{\mathbf{apr}}_{cn}(S) = S$, which are given below:

$$\begin{aligned}\mathcal{T}_{cn} = \{&\emptyset, D_3, \{e\}, \{e,s\}, \{e,rs\}, \{e,r^2s\}, \{e,r,r^2\}, \{e,s,rs\}, \{e,s,r^2s\}\\ &\{e,rs,r^2s\}, \{e,r,r^2,s\}, \{e,r,r^2,rs\}, \{e,r,r^2,r^2s\}, \{e,s,rs,r^2s\},\\ &\{e,r,r^2,s,rs\}, \{e,r,r^2,s,r^2s\}, \{e,r,r^2,rs,r^2s\}\}.\end{aligned}$$

Hence, $(D_3, \mathcal{T}_{cn})$ is a cn-space. Upon closer examination, this space is revealed to be a T_0-space as well.

By Proposition 2.5, we have the following.

Proposition 2.9. *The cn-space $(G, \mathcal{T}_{cn})$ obtained from an arbitrary group G is an Alexandroff space.*

Since $(G, \mathcal{T}_{cn})$ is an Alexandroff topology, it has a minimal base, which is expressed as follows:

Proposition 2.10. *The family $\mathcal{M} = \{\langle x\rangle \mid x \in G\}$ is a minimal base for the cn-space $(G, \mathcal{T}_{cn})$.*

Proof. To complete the proof, it remains to verify that the conditions of Lemma 1.2 are satisfied.

Since $x \in \langle x\rangle$ for all $x \in G$ by Lemma 1.3, it follows that $\bigcup\{\langle x\rangle | x \in G\} = G$.

Let X and Y be arbitrary elements of $\mathcal{M}$. Then, there exist x, y in G such that $X = \langle x\rangle$ and $Y = \langle y\rangle$. From this, we have

$$X \cap Y = \langle x\rangle \cap \langle y\rangle = \begin{cases} \langle a^k\rangle, & (\exists k, l \in F)(a^k = b^l) \\ \{e\}, & \text{otherwise} \end{cases}$$

and therefore $X \cap Y \in \mathcal{M}$. Thus, the desired result follows directly. □

Proposition 2.11. ([12]) *Let G be a group and S be a subgroup of G. Then, $\underline{\mathbf{apr}}_{cn}(S) = S$.*

Proof. Obviously, $\underline{\mathbf{apr}}_{cn}(S) \subseteq S$ from Proposition 2.5.

Conversely, let x be an arbitrary element of S. Since S is a subgroup of G, it is itself a group, and for any $x \in S$, the cyclic subgroup generated by x,$\langle x\rangle$, is a subgroup of S. Thus, $\langle x\rangle \subseteq S$. This implies that $x \in \underline{\mathbf{apr}}_{cn}(S)$, and consequently, $S \subseteq \underline{\mathbf{apr}}_{cn}(S)$. Therefore, $\underline{\mathbf{apr}}_{cn}(S) = S$. □

The converse of the Proposition 2.11 is false. For instance, consider Example 2.8. The $\underline{\mathbf{apr}}_{cn}(\{e,r,rs\}) = \{e,r,rs\}$, but $\{e,r,rs\}$ is not a subgroup of D_3 since

$$r(rs) = r^2 \notin \{e,r,rs\}.$$

Lemma 2.12. *Let G_1 and G_2 be two groups and $f : G_1 \to G_2$ a homomorphism.*

(a) For any $x \in G_1$, $f(\langle x\rangle) = \langle f(x)\rangle$.

(b) If $S_1 \subseteq G_1$, then $f(\underline{\mathbf{apr}}_{cn}(S_1)) \subseteq \underline{\mathbf{apr}}_{cn}(f(S_1))$.

(c) If $S_2 \subseteq G_2$, then $f^{-1}(\underline{\mathbf{apr}}_{cn}(S_2)) = \underline{\mathbf{apr}}_{cn}(f^{-1}(S_2))$.

Proof. (a) It is straightforward.

(b) Suppose that $y \in f(\underline{\mathbf{apr}}_{cn}(S_1))$. Then, there exists $x \in \underline{\mathbf{apr}}_{cn}(S_1)$ such that $f(x) = y$. By definition, $\langle x\rangle \subseteq S_1$, and applying f to both sides, we have $f(\langle x\rangle) \subseteq f(S_1)$. From part (a), we obtain $\langle f(x)\rangle \subseteq f(S_1)$. Since $f(x) = y$, we have $\langle y\rangle \subseteq f(S_1)$, and hence $y \in \underline{\mathbf{apr}}_{cn}(f(S_1))$.

(c) Let $x \in f^{-1}(\underline{\mathbf{apr}}_{cn}(S_2))$ be arbitrary. Then, $f(x) \in \underline{\mathbf{apr}}_{cn}(S_2)$, and hence $\langle f(x)\rangle \subseteq S_2$. From part (a), we have $f(\langle x\rangle) \subseteq S_2$. Taking the preimage of both sides under f, we obtain $\langle x\rangle \subseteq f^{-1}(f(\langle x\rangle)) \subseteq$

$f^{-1}(S_2)$. Thus, $\langle x\rangle \subseteq f^{-1})(S_2)$, and consequently, $x \in \underline{\mathbf{apr}}_{cn}(f^{-1}(S_2))$. On the other hand, suppose $x \in \underline{\mathbf{apr}}_{cn}(f^{-1}(S_2))$.*Then*, $\langle x\rangle \subseteq f^{-1}(S_2)$. Applying f to both sides of this inclusion gives $f(\langle x\rangle) \subseteq f(f^{-1}(S_2))$. Since $f(f^{-1}(S_2)) \subseteq S_2$, it follows that $f(\langle x\rangle) \subseteq S_2$. By part (a), $\langle f(x)\rangle \subseteq S_2$ and hence, $f(x) \in \underline{\mathbf{apr}}_{cn}(S_2)$. From this, it is clear that $x \in f^{-1}(\underline{\mathbf{apr}}_{cn}(S_2))$. □

Proposition 2.13. *If f is a homomorphism from a group G_1 to any group G_2, then f is a continuous map from the cn-space $(G_1, \mathcal{T}_{cn})$ to the cn-space $(G_2, \mathcal{T}'_{cn})$.*

Proof. Let O be an arbitrary element of $\mathcal{T}'_{cn}$. By definition, $O = \underline{\mathbf{apr}}_{cn}(O)$. Taking the preimage of both sides under f, we have $f^{-1}(O) = f^{-1}(\underline{\mathbf{apr}}_{cn}(O))$. By Lemma 2.12(c), $f^{-1}(O) = \underline{\mathbf{apr}}_{cn}(f^{-1}(O))$. Thus, $f^{-1}(O) \in \mathcal{T}_{cn}$, and so f is continuous. □

Considering part (b) of Lemma 2.12, the condition $\underline{\mathbf{apr}}_{cn}(f(S_1)) \subseteq f(\underline{\mathbf{apr}}_{cn}(S_1))$ does not generally hold. However, if f is an injective homomorphism, then the relation holds. Indeed, suppose $y \in \underline{\mathbf{apr}}_{cn}(f(S_1))$. Then, $\langle y\rangle \subseteq f(S_1)$. Since $y \in \langle y\rangle$, it follows that $y \in f(S_1)$ and thus there exists $x \in S_1$ such that $f(x) = y$. From this, we can write $\langle f(x)\rangle \subseteq f(S_1)$. Taking the preimage of both sides under f, and using the injectivity of f and part (a) of Lemma 2.12, we obtain $\langle x\rangle \subseteq S_1$. Hence, $x \in \underline{\mathbf{apr}}_{cn}(S_1)$ and consequently, $f(x) = y \in f(\underline{\mathbf{apr}}_{cn}(S_1))$.

Therefore, the following conclusion can be drawn.

Corollary 2.14. *Isomorphic groups give rise to homeomorphic cn-spaces.*

Considering the relations in Equations 3 and 4, for an arbitrary subset $S \subseteq G$ in a cn-space, the equality

$$\overline{\mathbf{apr}}_{cn}(S) = G - \underline{\mathbf{apr}}_{cn}(G - S) \tag{8}$$

holds. Indeed, if we arbitrarily choose an element $x \in G - \underline{\mathbf{apr}}_{cn}(G-S)$, it is clear that $x \in G$ and $x \notin \underline{\mathbf{apr}}_{cn}(G-S)$. From this, we conclude that $\langle x\rangle \nsubseteq G - S$ and hence $\langle x\rangle \cap S \neq \emptyset$. On the other hand, if we arbitrarily choose an element $x \in \overline{\mathbf{apr}}_{cn}(S)$, by definition, $\langle x\rangle \cap S \neq \emptyset$. From this, it is easy to see that $\langle x\rangle \nsubseteq G - S$. Consequently, the equality in Equation 8 holds.

Using the concepts of centralizing and normalizing neighborhood operators defined above, two unique topologies can be obtained on a group G. These topologies are called the centralizing topology (or Cn-topology for short) and the normalizing topology (or Nn-topology for short), respectively.

Indeed, if we define

$$\mathcal{T}_{Cn} = \{S \subseteq G | \underline{\mathbf{apr}}_{Cn}(S) = S\} \tag{9}$$

and

$$\mathcal{T}_{Nn} = \{S \subseteq G | \underline{\mathbf{apr}}_{Nn}(S) = S\}, \tag{10}$$

then both sets form topologies on G.

Proposition 2.15. *Let G be a group equipped with the cn-topology $\mathcal{T}_{cn}$, the Cn-topology $\mathcal{T}_{Cn}$, and the Nn-topology $\mathcal{T}_{Nn}$. Then, $\mathcal{T}_{cn}$ is finer than $\mathcal{T}_{Cn}$, and $\mathcal{T}_{Cn}$ is finer than $\mathcal{T}_{Nn}$, i.e.,*

$$\mathcal{T}_{cn} \supseteq \mathcal{T}_{Cn} \supseteq \mathcal{T}_{Nn}.$$

Proof. Suppose $S \in \mathcal{T}_{Cn}$. By definition, $\underline{\mathbf{apr}}_{Cn}(S) = S$. Since $\underline{\mathbf{apr}}_{cn}(S) \supseteq \underline{\mathbf{apr}}_{Cn}(S) = S$ by Proposition 2.3, it follows immediately that $\underline{\mathbf{apr}}_{cn}(S) = S$. Thus, $S \in \mathcal{T}_{cn}$. Similarly, it can be easily shown that $\mathcal{T}_{Cn} \supseteq \mathcal{T}_{Nn}$. □

If G is an abelian group, then for any $x, y \in G$, we have $\langle xy \rangle = \langle x \rangle \langle y \rangle$. Indeed, suppose $z \in \langle xy \rangle$. Then $z = (xy)^n$ for some integer n. Since G is abelian, $(xy)^n = x^n y^n$. However, $x^n \in \langle x \rangle$ and $y^n \in \langle y \rangle$, so $z = x^n y^n \in \langle x \rangle \langle y \rangle$. Conversely, suppose $z \in \langle x \rangle \langle y \rangle$. Then $z = x^n y^m$ for some integers n and m. Since G is abelian, $x^n y^n = (xy)^n$ and $x^n y^m = (xy)^n (xy)^m$, so $z = (xy)^{n+m}$, and hence $z \in \langle xy \rangle$. Moreover, for any $x \in G$, we have $\langle x^{-1} \rangle = (\langle x \rangle)^{-1}$. Indeed,

$$\begin{aligned}(\langle x \rangle)^{-1} &= \{g^{-1} | g \in \langle x \rangle\} \\ &= \{(x^n)^{-1} | n \in \mathbb{Z}\} \\ &= \{x^{-n} | n \in \mathbb{Z}\} \\ &= \{(x^{-1})^n | n \in \mathbb{Z}\} = \langle x^{-1} \rangle.\end{aligned}$$

Topological groups serve as a crucial link between group theory and topology, providing a framework for studying the interplay of algebraic and topological structures. The concept of a topological group is defined by the compatibility of a group operation and a topology defined on a set. This compatibility is ensured by the continuity of the group operation and the inversion map. More formally; if G is a group and $\mathcal{T}$ is a topology on G, such that the group operation is a continuous map from $G \times G$ to G and the inversion map is a continuous map from G to G, then the resulting mathematical structure is called a topological group. To examine the relevant continuities, the following conditions are considered. The product map is continuous if and only if for any $x, y \in G$ and any neighborhood Z of xy in G, there exist neighborhoods X of x and Y of y in G such that $XY \subseteq Z$, where $XY = \{xy | x \in X, y \in Y\}$. The inversion map is continuous if and only if for any $x \in G$ and any neighborhood Y of x^{-1} in G, there exists a neighborhood X of x in G such that $X^{-1} \subseteq Y$, where $X^{-1} = \{x^{-1} | x \in X\}$. In light of the preceding discussion, the following proposition concerning cn-spaces is proposed.

Proposition 2.16. *An abelian group G, equipped with the cn-topology, becomes a topological group.*

Proof. Let $x, y \in G$ be arbitrary and suppose $xy \in W \in \mathcal{T}_{cn}$. Since $W = \underline{\mathbf{apr}}_{cn}(W)$, $xy \in \underline{\mathbf{apr}}_{cn}(W)$, and hence $\langle xy \rangle \subseteq W$. As G is abelian, $\langle xy \rangle = \langle x \rangle \langle y \rangle$, so $\langle x \rangle \langle y \rangle \subseteq W$. Since $x \in \langle x \rangle$ and $y \in \langle y \rangle$, it follows that $xy \in \langle x \rangle \langle y \rangle$, and since $\langle x \rangle, \langle y \rangle \in \mathcal{T}_{cn}$, the multiplication is continuous. Similarly, let $x \in G$ and suppose $x^{-1} \in V \in \mathcal{T}_{cn}$. Then $x^{-1} \in \underline{\mathbf{apr}}_{cn}(V)$, so $\langle x^{-1} \rangle \subseteq V$. Since $\langle x^{-1} \rangle = (\langle x \rangle)^{-1}$, we have $(\langle x \rangle)^{-1} \subseteq V$, and since $\langle x \rangle \in \mathcal{T}_{cn}$, the inversion is continuous. Therefore, G is a topological group with respect to the cn-topology. □

2.2. Ln-topology and Rn-topology

It has been previously established that given a group G and its subgroup S, the left coset generated by an element x in G is defined as the set $xS = \{xy \mid y \in S\}$. Likewise, the right coset generated by x is defined as the set $Sx = \{yx \mid y \in S\}$.

The following well-known proposition about cosets is frequently used in abstract algebra.

Proposition 2.17. ([6, 10]) *Let G be a group and S be subgroup of G. Then;*

(a) G is the union of all left (right) cosets of S.

(b) For all x, y in G, either $xS = yS$ or $xS \cap yS = \emptyset$ (or, equivalently, either $Sx = Sy$ or $Sx \cap Sy = \emptyset$).

(c) For all x, y in G, $xS = yS$ if and only if $x^{-1}y \in S$. An analogous result holds for right cosets.

(d) The number of left cosets of S is equal to the number of right cosets of S.

Building upon the concepts introduced in [12], the operators defined below are constructed as follows.

Definition 2.18. Let G be a group and S be a subgroup of G.

(1) The mapping $Ln : G \to \mathcal{P}(G)$ such that $Ln(x) = xS$ for each $x \in G$ is called the left cosetial neighborhood operator on G.

(2) The mapping $Rn : G \to \mathcal{P}(G)$ such that $Rn(x) = Sx$ for each $x \in G$ is called the right cosetial neighborhood operator on G.

Based on this definition, the approximation operators are given as follows.

Definition 2.19. Let G be a group, S be a subgroup of G and X be subset of G.

(1) The lower approximation of X with respect to the left cosetial neighborhood operator is defined as $\underline{\mathbf{apr}}_{Ln}(X) = \{x \in U \mid Ln(x) \subseteq X\}$, and the upper approximation of X with respect to the left cosetial neighborhood operator is defined as $\overline{\mathbf{apr}}_{Ln}(X) = \{x \in U \mid Ln(x) \cap X \neq \emptyset\}$.

(2) The lower approximation of X with respect to the right cosetial neighborhood operator is defined as $\underline{\mathbf{apr}}_{Rn}(X) = \{x \in U \mid Rn(x) \subseteq X\}$, and the upper approximation of X with respect to the right cosetial neighborhood operator is defined as $\overline{\mathbf{apr}}_{Rn}(X) = \{x \in U \mid Rn(x) \cap X \neq \emptyset\}$.

Using these definitions, the following results can be obtained.

Proposition 2.20. *Let G be a group and S be a subgroup of G. Then, $\underline{\mathbf{apr}}_{Ln}(xS) = xS$.*

Proof. Let y be in $\underline{\mathbf{apr}}_{Ln}(xS)$. Then, yS is a subset of xS. By Proposition 2.17(a), it follows that $yS = xS$. Hence, y is in xS. Conversely, if y is in xS, then there exists an $s \in S$ such that $y = xs$. From this, we have $x^{-1}y = s$, and since $x^{-1}y \in S$, by Proposition 2.17(c), $xS = yS$. This clearly implies that yS is a subset of xS. Therefore, y is in $\underline{\mathbf{apr}}_{Ln}(xS)$. Consequently, $\underline{\mathbf{apr}}_{Ln}(xS) = xS$. □

It should be noted that the above proposition also holds for right cosets.

Proposition 2.21. *Let G be a group, S be a subgroup of G and X_i be subsets of G for each $i \in I$ and $X \subseteq G$.*

(a) $\underline{\mathbf{apr}}_{Ln}(G) = G$ *and* $\underline{\mathbf{apr}}_{Ln}(\emptyset) = \emptyset$.

(b) $\underline{\mathbf{apr}}_{Ln}(X) \subseteq X$.

(c) $\underline{\mathbf{apr}}_{Ln}(\underline{\mathbf{apr}}_{Ln}(X)) = \underline{\mathbf{apr}}_{Ln}(X)$.

(d) If $X_1 \subseteq X_2$, *then* $\underline{\mathbf{apr}}_{Ln}(X_1) \subseteq \underline{\mathbf{apr}}_{Ln}(X_2)$.

(e) $\underline{\mathbf{apr}}_{Ln}(\bigcap_{i\in I} X_i) = \bigcap_{i\in I} \underline{\mathbf{apr}}_{Ln}(X_i)$.

(f) $\underline{\mathbf{apr}}_{Ln}(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} \underline{\mathbf{apr}}_{Ln}(X_i)$.

Proof. Parts (a) and (b) are obvious.

(c) It is clear from part (b) that $\underline{\mathbf{apr}}_{Ln}(\underline{\mathbf{apr}}_{Ln}(X))$ is a subset of $\underline{\mathbf{apr}}_{Ln}(X)$. Now, let x be an arbitrary element of $\underline{\mathbf{apr}}_{Ln}(X)$. By definition, xS is a subset of X. Then, by part (d), $\underline{\mathbf{apr}}_{Ln}(xS)$ is a subset of $\underline{\mathbf{apr}}_{Ln}(X)$. Since $\underline{\mathbf{apr}}_{Ln}(xS)$ equals xS by Proposition 2.20, we have xS is a subset of $\underline{\mathbf{apr}}_{Ln}(X)$. Thus, x is an element of $\underline{\mathbf{apr}}_{Ln}(\underline{\mathbf{apr}}_{Ln}(X))$.

The proofs of parts (d), (e), and (f) are straightforward. □

In view of Proposition 2.21, the operation $\underline{\mathbf{apr}}_{Ln}$ defines an internal binary operation on the power set of G, $\mathcal{P}(G)$. Therefore, the following proposition is straightforward.

Hence $\underline{\mathbf{apr}}_{Ln}$ satisfies the Kuratowski interior axioms $\underline{\mathbf{apr}}_{Ln}(G) = G$, $\underline{\mathbf{apr}}_{Ln}(X) \subseteq X$, $\underline{\mathbf{apr}}_{Ln}(\underline{\mathbf{apr}}_{Ln}(X)) = \underline{\mathbf{apr}}_{Ln}(X)$, and monotonicity; therefore its fixed points form a topology.

Proposition 2.22. *For any group G and its subgroup S, the family*

$$\mathcal{T}_{Ln} = \{X \subseteq G \mid \underline{\mathbf{apr}}_{Ln}(X) = X\} \tag{11}$$

defines a topology on G.

The topology obtained in Proposition 2.22 is called the left cosetial topology of G with respect to S, or simply the Ln-topology.

The operator $\underline{\mathbf{apr}}_{Rn} : \mathcal{P}(G) \to \mathcal{P}(G)$ is shown to satisfy all the conditions stipulated in Proposition 2.21, establishing it as an interior operator. As a result, the collection

$$\mathcal{T}_{Rn} = \{X \subseteq G | \underline{\mathbf{apr}}_{Rn}(X) = X\} \tag{12}$$

constitutes a topology on the group G. This specific topology is conventionally termed the right cosetial topology relative to S, or more succinctly, the Rn-topology.

Consider the Ln-space $(G, \mathcal{T}_{Ln})$ generated by an arbitrary subgroup S of G. For any subset X of G, the interior of X, $\mathring{X}$, is equal to $\underline{\mathbf{apr}}_{Ln}(X)$. Indeed, let x be an arbitrary element in the interior of X. Then, there exists an open set O in $\mathcal{T}_{Ln}$ such that x is contained in O and O is a subset of X. Since O is equal to $\underline{\mathbf{apr}}_{Ln}(O)$, x belongs to $\underline{\mathbf{apr}}_{Ln}(O)$, which is a subset of X. Therefore, $xS \subseteq O$, which is a subset of X. Hence, xS is a subset of X. Conversely, suppose x belongs to $\underline{\mathbf{apr}}_{Ln}(X)$. Then, xS is a subset of X. By Proposition 2.20, every coset xS is an open set in $\mathcal{T}_{Ln}$. Consequently, x is in the interior of X, and we conclude that $\mathring{X} = \underline{\mathbf{apr}}_{Ln}(X)$.

Given a topological space $(G, \mathcal{T})$ with the interior operator $\underline{\mathbf{apr}}_{Ln}$, it follows directly that a subset X of G is open precisely when $\underline{\mathbf{apr}}_{Ln}(X) = X$. Consequently, the topologies $\mathcal{T}$ and $\mathcal{T}_{Ln}$ coincide, implying the uniqueness of $\mathcal{T}_{Ln}$.

It should be noted that the same situation holds for the Rn-topology $\mathcal{T}_{Rn}$.

Remark 2.23. Moreover, it should be noted that for an arbitrary subset $X \subseteq G$ in the Ln-space $(G, \mathcal{T}_{Ln})$, the equality

$$\overline{\mathbf{apr}}_{Ln}(X) = G - \underline{\mathbf{apr}}_{Ln}(G - X) \tag{13}$$

clearly holds. Consequently, the upper approximation operator also possesses closure operator properties in this space.

Example 2.24. Consider the symmetric group S_3 and its subgroup $S = \{(1), (12)\}$. The left and right cosets of this subgroup are as follows.

$$\begin{aligned} (1)S = (12)S &= \{(1), (12)\} \\ (13)S = (123)S &= \{(13), (123)\} \\ (23)S = (132)S &= \{(23), (132)\} \end{aligned} \tag{14}$$

and

$$\begin{aligned} S(1) = S(12) &= \{(1), (12)\} \\ S(13) = S(132) &= \{(13), (132)\} \\ S(23) = S(132) &= \{(23), (123)\} \end{aligned} \tag{15}$$

Thus, the Ln-topology is obtained as

$$\begin{aligned} \mathcal{T}_{Ln} = \{&\emptyset, S_3, \{(1), (12)\}, \{(13), (123)\}, \{(23), (132)\}, \{(1), (12), (13), (123)\} \\ &\{(1), (12), (23), (132)\}, \{(13), (123), (23), (132)\}\}. \end{aligned}$$

However, the Rn-topology is also obtained as

$$\mathcal{T}_{Rn} = \{\emptyset, S_3, \{(1),(12)\}, \{(13),(132)\}, \{(23),(123)\}, \{(1),(12),(13),(132)\}$$
$$\{(1),(12),(23),(123)\}, \{(13),(132),(23),(123)\}\}.$$

It is important to note that these two topologies are not identical. Consequently, $\mathcal{T}_{Ln} \neq \mathcal{T}_{Rn}$ in this non-abelian example.

Within the framework of group theory, a subgroup S of a group G is classified as a normal subgroup if, for every element x belonging to G and every element s belonging to S, the conjugate xsx^{-1} is also an element of S. This definition directly implies that a necessary and sufficient criterion for S to be a normal subgroup is the equality of its left and right cosets, explicitly expressed as $xS = Sx$ for all $x \in G$. Using these arguments, the following proposition can be established.

Proposition 2.25. *If S is a normal subgroup of an arbitrary group G, then the Ln-topology $\mathcal{T}_{Ln}$ coincides with the Rn-topology $\mathcal{T}_{Rn}$.*

Nevertheless, given that every subgroup of an abelian group is a normal subgroup, it is evident that:

Corollary 2.26. *If G is an abelian group, then $\mathcal{T}_{Ln} = \mathcal{T}_{Rn}$.*

Example 2.27. Given the additive group $\mathbb{Z}_4$ and its subgroup $S = \{0, 2\}$, the abelian nature of $\mathbb{Z}_4$ ensures that S is normal. This implies that for any $x \in \mathbb{Z}_4$, the left coset $xS = x + S$ equals the right coset $Sx = S + x$. As a result, we have

$$\mathcal{T}_{Ln} = \{\emptyset, \mathbb{Z}_4, \{0, 2\}, \{1, 3\}\} = \mathcal{T}_{Rn}.$$

As clearly shown in Example 2.27, an arbitrary topological space $(G, \mathcal{T}_{Ln})$ is not T_0, and consequently, it is neither T_1 nor T_2.

The additive group $\mathbb{Z}_p$ is a cyclic group of order p for any prime number p. In a cyclic group of order n, the subgroups are completely determined by the divisors of n. As p is a prime number, its only divisors are 1 and p. Consequently, the only subgroups of $\mathbb{Z}_p$ are the trivial subgroup $\{0\}$ and the group $\mathbb{Z}_p$ itself. Considering the trivial subgroup $S = \{0\}$, the left cosets are $\{0\}, \{1\}, \{2\}, \ldots, \{p-1\}$. By Proposition 2.20, all cosets are open in the Ln-topology, hence $\mathcal{T}_{Ln} = \mathcal{P}(\mathbb{Z}_p)$, which is the discrete topology. However, if we choose $S = \mathbb{Z}_p$, then there is only one coset, namely $\mathbb{Z}_p$ itself. Consequently, the resulting Ln-topology is $\mathcal{T}_{Ln} = \{\emptyset, \mathbb{Z}_p\}$, which is the indiscrete topology.

In the more general case of an arbitrary group G, if we take the trivial subgroup $S = \{e_G\}$, then for each $x \in G$, the left coset xS is simply the singleton set $\{x\}$. Thus, the Ln-topology is the discrete topology, consisting of all subsets of G. On the other hand, if we choose $S = G$, then there is only one left coset, namely G itself, and hence the Ln-topology is the indiscrete topology, consisting only of the empty set and the entire group.

Remark 2.28. It is important to observe that the collection

$$\mathcal{B}_{Ln} = \{xS | x \in G\}$$

forms a basis for the Ln-topology $\mathcal{T}_{Ln}$. The proof of this fact is straightforward.

It is a well-known fact that if $f : G_1 \to G_2$ is a homomorphism between groups G_1 and G_2, and S_1 is a subgroup of G_1, then the image $f(S_1)$ of S_1 under f is also a subgroup of G_2. Say $f(S_1) = S_2$. Under this condition, it is clear that $f^{-1}(S_2) = S_1$. Indeed, suppose $x \in S_1$. Then, $f(x) \in f(S_1)$. Since $f(S_1) = S_2$, $f(x) \in S_2$ and thus $x \in f^{-1}(S_2)$. The other inclusion can be shown similarly. Based on this, for any $x \in G_1$, the image $f(xS_1)$ of the coset xS_1 under f is a coset in G_2 and is defined by $f(xS_1) = f(x)S_2$. If f is surjective, then clearly for any $y \in G_2$, the preimage of the coset yS_2 is also a coset in G_1 and $xS_1 \subseteq f^{-1}(yS_2)$ if $f(x) = y$. Moreover, for any $x \in G_1$, $xf^{-1}(S_2) \subseteq f^{-1}(f(x)S_2)$. If f is bijective, then $xS_1 = xf^{-1}(S_2) = f^{-1}(f(x)S_2)$. Based on these arguments, the following conclusions can be drawn.

Lemma 2.29. *Let G_1 and G_2 be two groups, $f : G_1 \to G_2$ homomorphism and S_1 be a subgroup of G_1. Then,*

(a) $f(\underline{\mathbf{apr}}_{Ln}(X)) \subseteq \underline{\mathbf{apr}}_{Ln}(f(X))$ *for any* $X \subseteq G_1$.

(b) If f is bijective, then $f(\underline{\mathbf{apr}}_{Ln}(X)) = \underline{\mathbf{apr}}_{Ln}(f(X))$.

(c) $f^{-1}(\underline{\mathbf{apr}}_{Ln}(Y)) = \underline{\mathbf{apr}}_{Ln}(f^{-1}(Y))$ *for any* $Y \subseteq G_2$.

Proof. (a) Let X be an arbitrary subset of G_1, and suppose y is an element of $f(\underline{\mathbf{apr}}_{Ln}(X))$. Then, there exists an element $x \in \underline{\mathbf{apr}}_{Ln}(X)$ such that $f(x) = y$. From this, it follows that $xS_1 \subseteq X$. Applying f to both sides of this inclusion yields $f(xS_1) \subseteq f(X)$, and consequently, $f(x)S_2 \subseteq f(X)$. Since $f(x) = y$, we have $yS_2 \subseteq f(X)$, and thus $y \in \underline{\mathbf{apr}}_{Ln}(f(X))$.

(b) The first inclusion follows from (a). Now, assume $y \in \underline{\mathbf{apr}}_{Ln}(f(X))$. Consequently, $yS_2 \subseteq f(X)$. Given the definition $f(S_1) = S_2$, we have $yf(S_1) \subseteq f(X)$. Taking the preimage of both sides under f, we obtain $f^{-1}(yf(S_1)) \subseteq f^{-1}(f(X))$, which implies $f^{-1}(y)f^{-1}(f(S_1)) \subseteq f^{-1}(f(X))$. Since f is bijective, we can denote $f(x) = y$. Then, $xS_1 \subseteq X$. Hence, $x \in \underline{\mathbf{apr}}_{Ln}(X)$, and consequently, $f(x) = y \in f(\underline{\mathbf{apr}}_{Ln}(X))$.

(c) Let Y be an arbitrary subset of G_2 and suppose $x \in f^{-1}(\underline{\mathbf{apr}}_{Ln}(Y))$. Then, $f(x) \in \underline{\mathbf{apr}}_{Ln}(Y)$. Consequently, $f(x)S_2 \subseteq Y$. Taking the preimage of both sides under f, we obtain $f^{-1}(f(x)S_2) \subseteq f^{-1}(Y)$. Since $xf^{-1}(S_2) = xS_1 \subseteq f^{-1}(f(x)S_2)$, it follows that $xS_1 \subseteq f^{-1}(Y)$. Hence, $x \in \underline{\mathbf{apr}}_{Ln}(f^{-1}(Y))$. Conversely, suppose $x \in \underline{\mathbf{apr}}_{Ln}(f^{-1}(Y))$. Then, $xS_1 \subseteq f^{-1}(Y)$. Applying f to both sides of this inclusion, we obtain $f(xS_1) \subseteq f(f^{-1}(Y))$. Since $f(f^{-1}(Y)) \subseteq Y$, it follows that $f(xS_1) \subseteq Y$. As $f(xS_1) = f(x)S_2$, we have $f(x)S_2 \subseteq Y$, which implies $f(x) \in \underline{\mathbf{apr}}_{Ln}(Y)$. Therefore, $x \in f^{-1}(\underline{\mathbf{apr}}_{Ln}(Y))$. □

Let G_1 and G_2 be groups, and let $f : G_1 \to G_2$ be a homomorphism. Suppose S_1 is a subgroup of G_1. Consequently, $f(S_1) = S_2$ is a subgroup of G_2. Given that $(G_1, \mathcal{T}_{Ln})$ is an Ln-space with respect to S_1 and $(G_2, \mathcal{T}'_{Ln})$ is an Ln-space with respect to $f(S_1) = S_2$, the following proposition can be stated for f:

Proposition 2.30. $f : (G_1, \mathcal{T}_{Ln}) \to (G_2, \mathcal{T}'_{Ln})$ *is continuous.*

Proof. Let O be an arbitrary open set in $\mathcal{T}'_{Ln}$. To show that $f^{-1}(O) \in \mathcal{T}_{Ln}$, it suffices to prove that $f^{-1}(O) = \underline{\mathbf{apr}}_{Ln}(f^{-1}(O))$. The inclusion $\underline{\mathbf{apr}}_{Ln}(f^{-1}(O)) \subseteq f^{-1}(O)$ follows from Proposition 2.21 (b). Now, let x be an arbitrary element of $f^{-1}(O)$. Then, $f(x) \in O$, and since $O = \underline{\mathbf{apr}}_{Ln}(O)$, we have $f(x)S_2 \subseteq O$. Taking the preimage of both sides under f, we obtain $f^{-1}(f(x)S_2) \subseteq f^{-1}(O)$, and hence $xf^{-1}(S_2) \subseteq f^{-1}(O)$. As $f^{-1}(S_2) = S_1$, we have $xS_1 \subseteq f^{-1}(O)$, which implies $x \in \underline{\mathbf{apr}}_{Ln}(f^{-1}(O))$. Consequently, $f^{-1}(O) \in \mathcal{T}_{Ln}$, and therefore f is continuous. □

Corollary 2.31. *If two groups are isomorphic, then the induced Ln-spaces are homeomorphic.*

Proposition 2.16 demonstrated that an abelian group G becomes a topological group under the cn-topology. Analogously, it is possible to prove that a group endowed with the Ln-topology associated with a normal subgroup is a topological group. Consequently, we propose the following proposition.

Proposition 2.32. *Let G be a group and S be its normal subgroup. The Ln-topology, $\mathcal{T}_{Ln}$, is compatible with the group operation. In other words, the Ln-space is a topological group.*

Proof. The aforementioned conditions are sufficient for the proof. Suppose $x, y \in G$ and $xy \in Z \in \mathcal{T}_{Ln}$. Then, $xy \in \underline{\mathbf{apr}}_{Ln}(Z) = Z$, and hence $(xy)S \subseteq Z$. Since S is a normal subgroup, $(xy)S = (xS)(yS)$. By Proposition 2.20, for all $x, y \in G$, $xS, yS \in \mathcal{T}_{Ln}$. Thus, $x \in xS$, $y \in yS$, and $(xS)(yS) \subseteq Z$. Consequently, the group multiplication is continuous.

Suppose $x \in G$ and $x^{-1} \in V \in \mathcal{T}_{Ln}$. Then, $x^{-1} \in \underline{\mathbf{apr}}_{Ln}(V) = V$, and hence $x^{-1}S \subseteq V$. Since S is a normal subgroup, $x^{-1}S = (xS)^{-1}$. Thus, $x \in (xS)^{-1} \subseteq V$. By Proposition 2.20, $xS \in \mathcal{T}_{Ln}$. Consequently, the inversion map is continuous.

Therefore, G is a topological group. □

Note that, similarly, The Rn-space is also a topological group.

Given that every subgroup of an abelian group is normal, the following conclusion is immediate.

Corollary 2.33. *An abelian group equipped with a Ln-topology (or Rn-topology) induced by its subgroups is a topological group.*

2.2.1. Connection with relation-generated (neighborhood) topologies

The construction of $\mathcal{T}_{Ln}$ and $\mathcal{T}_{Rn}$ fits into the standard scheme of generating topologies from binary relations (see, e.g., [4] and the dual-topology approach in [1, 2]).

Fix a group G and a subgroup $S \leq G$. Define relations R_L^S and R_R^S on G by

$$(x, y) \in R_L^S \Leftrightarrow y \in xS \Leftrightarrow x^{-1}y \in S,$$

$$(x, y) \in R_R^S \Leftrightarrow y \in Sx \Leftrightarrow yx^{-1} \in S.$$

Let $n_R(x) = \{y \in G : (x, y) \in R\}$ be the (right) neighborhood induced by a relation R. Then $n_{R_L^S}(x) = xS = Ln(x)$ and $n_{R_R^S}(x) = Sx = Rn(x)$.

Consequently, the neighborhood-generated topology

$$\tau(R) = \{A \subseteq G : \forall x \in A,\ n_R(x) \subseteq A\}$$

coincides with $\mathcal{T}_{Ln}$ when $R = R_L^S$, and with $\mathcal{T}_{Rn}$ when $R = R_R^S$. Indeed, since $x \in xS$ (resp. $x \in Sx$), the operator $\underline{\mathbf{apr}}_{Ln}(A) = \{x \mid xS \subseteq A\}$ is an interior operator and its family of fixed points equals $\tau(R_L^S)$; similarly for $\underline{\mathbf{apr}}_{Rn}$.

Structural enhancement. While $\tau(R)$ can be defined for an arbitrary relation R, the group-theoretic choice R_L^S, R_R^S yields additional structure not available in the purely relational setting:

(i) cosets form canonical minimal open neighborhoods;

(ii) $\mathcal{T}_{Ln}$ and $\mathcal{T}_{Rn}$ encode left/right asymmetry and coincide exactly when S is normal;

(iii) algebraic morphisms (homomorphisms/isomorphisms) induce continuous maps/homeomorphisms between the resulting spaces.

These features are exploited in Sections 2.2–2.3 and in the on-topology section.

2.3. The on-topology

In [12], lower and upper approximation operators on a set were obtained by using the action of a group.

Let G be a group acting on a set U, i.e. U be an G-set. Then, from the definition of the orbit of an element $x \in U$, a partition of U is induced, denoted by

$$\mathcal{P} = \{G \cdot x | x \in U\},$$

and an equivalence relation is naturally defined on U using this partition, given by

$$x \sim y :\Leftrightarrow G \cdot x = G \cdot y.$$

Equivalence relations induced by group actions on sets naturally give rise to approximation operators in the sense of Pawlak. In addition, if the operator $n_{on} : U \to \mathcal{P}(U)$ is defined as $n_{on}(x) = G \cdot x$, then $n_{on}(x)$ is called the orbital neighborhood of the element x. Thus, using Equations 1 and 2, lower and upper approximation sets are obtained for arbitrary subsets of U. These approximation sets are defined, for $X \subseteq U$, as

$$\underline{\mathbf{apr}}_{on}(X) = \{x \in U | n_{on}(x) \subseteq X\} \tag{16}$$

and

$$\overline{\mathbf{apr}}_{on}(X) = \{x \in U | n_{on}(x) \cap X \neq \emptyset\}, \tag{17}$$

and are called the orbital lower approximation set and orbital upper approximation set of X, respectively. The orbital boundary of a set, denoted by $\mathbf{bnd}_{on}(X)$, can be defined using the orbital approximations of the set as follows:

$$\mathbf{bnd}_{on}(X) = \overline{\mathbf{apr}}_{on}(X) - \underline{\mathbf{apr}}_{on}(X). \tag{18}$$

The $\underline{\mathbf{apr}}_{on}$ operator possesses the following characteristics.

Proposition 2.34. *Let U be a G-set, $X, Y \subseteq U$ and $X_i \subseteq U$ for each $i \in I$.*

(a) $\underline{\mathbf{apr}}_{on}(U) = U$.

(b) $\underline{\mathbf{apr}}_{on}(X) \subseteq X$.

(c) $\underline{\mathbf{apr}}_{on}(\underline{\mathbf{apr}}_{on}(X)) = \underline{\mathbf{apr}}_{on}(X)$.

(d) If $X \subseteq Y$, *then* $\underline{\mathbf{apr}}_{on}(X) \subseteq \underline{\mathbf{apr}}_{on}(Y)$.

(e) $\underline{\mathbf{apr}}_{on}(\bigcap_{i\in I} X_i) = \bigcap_{i\in I} \underline{\mathbf{apr}}_{on}(X_i)$.

(f) $\bigcup_{i\in I} \underline{\mathbf{apr}}_{on}(X_i) = \underline{\mathbf{apr}}_{on}(\bigcup_{i\in I} X_i)$.

Proof. (a) Suppose that $x \in U$. Then, we have $n_{on}(x) = G \cdot x \subseteq U$. Explicitly, $x \in \underline{\mathbf{apr}}_{on}(U)$.

(b) Assuming $x \in \underline{\mathbf{apr}}_{on}(X)$, by definition, $G \cdot x \subseteq X$. Since $e \in G$ and $e \cdot x = x$ for all $x \in U$, it follows that $x \in G \cdot x$ and hence $x \in X$.

(c) Suppose $x \in \underline{\mathbf{apr}}_{on}(\underline{\mathbf{apr}}_{on}(X))$. Then, by definition, $G \cdot x \subseteq \underline{\mathbf{apr}}_{on}(X)$. Since $\underline{\mathbf{apr}}_{on}(X) \subseteq X$, it follows that $G \cdot x \subseteq X$ and hence $x \in \underline{\mathbf{apr}}_{on}(X)$. On the other hand, based on the fact that $G \cdot (G \cdot x) = G \cdot x$, it is readily seen that $\underline{\mathbf{apr}}_{on}(X) \subseteq \underline{\mathbf{apr}}_{on}(\underline{\mathbf{apr}}_{on}(X))$. Consequently, the desired result is obtained.

(d) Assuming $x \in \underline{\mathbf{apr}}_{on}(X)$, we have $G \cdot x \subseteq X$ by definition. Given that $X \subseteq Y$, it follows that $G \cdot x \subseteq Y$, implying $x \in \underline{\mathbf{apr}}_{on}(Y)$.

(e) Suppose $x \in \underline{\mathbf{apr}}_{on}(\bigcap_{i\in I} X_i)$. By definition, $G \cdot x \subseteq \bigcap_{i\in I} X_i$. It follows that for each $i \in I$, $G \cdot x \subseteq X_i$. Consequently, for all $i \in I$, $x \in \underline{\mathbf{apr}}_{on}(X_i)$. Therefore, $x \in \bigcap_{i\in I} \underline{\mathbf{apr}}(X_i)$. The reverse implication can be proved analogously. Hence, the desired equality is obtained.

(f) Suppose $x \in \bigcup_{i\in I} \underline{\mathbf{apr}}_{on}(X_i)$. Then, there exists $i_0 \in I$ such that $x \in \underline{\mathbf{apr}}_{on}(X_{i_0})$. By definition, $G \cdot x \subseteq X_{i_0}$. Hence, $G \cdot x \subseteq \bigcup_{i\in I} X_i$. Consequently, $x \in \underline{\mathbf{apr}}_{on}(\bigcup_{i\in I} X_i)$. The other direction of the proof can be shown similarly to that in proof (e). Therefore, the required equality follows immediately. □

As a consequence of Proposition 2.34, it is clear that the operator $\underline{\mathbf{apr}}_{on}$ acts as an interior operator from $\mathcal{P}(U)$ to $\mathcal{P}(U)$. Consequently, the following proposition holds.

Proposition 2.35. *Let U be a G-set. The family defined as*

$$\mathcal{T}_{on} = \{X \subseteq U \mid \underline{\mathbf{apr}}_{on}(X) = X\} \tag{19}$$

is a topology on U.

Proof. Proposition 2.34 immediately implies the proof. □

Hence, $(U, \mathcal{T}_{on})$ is a topological space, which is called the orbital space or, in short, on-space. It should be noted that this should not be confused with the orbit space in group action theory. In the classical setting, the orbit space is customarily constructed as a quotient of a topological space under a canonical mapping. In contrast, the topological space presented here is generated by the lower approximation operator based on the orbital neighborhood.

Proposition 2.36. *$G \cdot x$ is an open set in the on-space $(U, \mathcal{T}_{on})$ for any $x \in U$.*

Proof. It is obvious that for any x in U, $\underline{\mathbf{apr}}_{on}(G \cdot x)$ is a subset of $G \cdot x$. Conversely, let us take $x' \in G \cdot x$ for any $x \in U$. Clearly, $x' \in G \cdot x'$. Consider an arbitrary element y belonging to $G \cdot x'$. By definition, there exists $g_1 \in G$ such that $y = g_1 \cdot x$. Additionally, as x' is an element of $G \cdot x$, we can find $g_2 \in G$ satisfying $x' = g_2 \cdot x$. Consequently, y can be expressed as

$$y = g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x = g_3 \cdot x$$

for some $g_3 \in G$. Hence, y is also in $G \cdot x$. We conclude that $G \cdot x' \subseteq G \cdot x$.

Since $x' \in G \cdot x' \subseteq G \cdot x$, it follows that $x' \in \underline{\mathbf{apr}}_{on}(G \cdot x)$.

Therefore, $\underline{\mathbf{apr}}_{on}(G \cdot x) = G \cdot x$. Consequently, $G \cdot x$ is an element of the topology $\mathcal{T}_{on}$. □

Let $(U, \mathcal{T}_{on})$ be a on-space and let X be a subset of U. Then, for any x in the interior of X, $x \in \mathring{X}$, by the definition of interior in topological spaces, there exists an open set O in $\mathcal{T}_{on}$ such that x is in O and O is a subset of X. By the definition of $\mathcal{T}_{on}$ and the conditions of Proposition 2.34, it follows that $x \in \underline{\mathbf{apr}}_{on}(O)$ and $\underline{\mathbf{apr}}_{on}(O)$ is a subset of $\underline{\mathbf{apr}}_{on}(X)$. Thus, $x \in \underline{\mathbf{apr}}_{on}(X)$. Hence, the interior of X, $\mathring{X}$, is a subset of $\underline{\mathbf{apr}}_{on}(X)$. Conversely, let x be an arbitrary element of $\underline{\mathbf{apr}}_{on}(X)$. Then, $x \in G \cdot x$ and $G \cdot x$ is in $\mathcal{T}_{on}$ by Proposition 2.36, implying that x is in the interior of X. Therefore, $\underline{\mathbf{apr}}_{on}(X)$ is a subset of the interior of X. Consequently, $\underline{\mathbf{apr}}_{on}(X) = \mathring{X}$.

Furthermore, suppose $(U, \mathcal{T})$ is a topological space with $\underline{\mathbf{apr}}_{on}$ as its interior operator. It is a straightforward consequence of the definition that X is an open set if and only if $\underline{\mathbf{apr}}_{on}(X) = X$. Therefore, the topology $\mathcal{T}$ is identical to the topology $\mathcal{T}_{on}$. In other words, $\mathcal{T}_{on}$ is uniquely determined.

By virtue of Proposition 2.34(e), we can deduce the following proposition.

Proposition 2.37. *The on-space $(U, \mathcal{T}_{on})$ is an Alexandroff space.*

It should be noted that, by Propositions 2.36 and 2.37, $\mathbf{Orb}(x) = G \cdot x$ is the minimal neighborhood of x, that is, $\mathbf{Orb}(\mathbf{x}) = G \cdot x = \bigcap_{i \in I} O_i$, where $x \in O_i \in \mathcal{T}_{on}$.

Example 2.38. Consider a subgroup

$$G = \{(1), (123), (132), (45), (123)(45), (132)(45)\}$$

of the symmetric group S_5 acting on the set $U = \{1, 2, 3, 4, 5\}$ via the natural permutation action.

Based on Equation 16, and given that the orbit of x under the action of G is denoted by $n_{on}(x) = G \cdot x$, we observe that the orbits of 1, 2, and 3 under G are identical and equal to$\{1, 2, 3\}$, while the orbits of 4 and 5 under G are identical and equal to $\{4, 5\}$.

Therefore, by these arguments, if the lower approximations of all subsets of U are calculated, then the lower approximation of any subset of X containing $\{1, 2, 3\}$ is $\underline{\mathbf{apr}}_{on}(X) = \{1, 2, 3\}$, the lower approximation of any subset of X containing $\{4, 5\}$ is $\underline{\mathbf{apr}}_{on}(X) = \{4, 5\}$, and the lower approximation of any other subset of X is $\underline{\mathbf{apr}}_{on}(X) = \varnothing$. Of course, $\underline{\mathbf{apr}}_{on}(U) = U$.

Among these subsets, the ones satisfying the condition $\underline{\mathbf{apr}}_{on}(X) = X$ are the empty set, the entire set U, and the subsets $\{1, 2, 3\}$ and $\{4, 5\}$.

Consequently, based on the definition of on-space, we obtain

$$\mathcal{T}_{on} = \{X \subseteq U \mid \underline{\mathbf{apr}}_{on}(X) = X\} = \{\varnothing, U, \{1, 2, 3\}, \{4, 5\}\}.$$

Example 2.38 demonstrates that an on-space need not be a T_0 space. Consequently, it cannot be a T_1 or T_2 space, as these axioms are strictly stronger than the T_0 axiom.

Example 2.39. Consider the action of the orthogonal group $O(2)$ on $\mathbb{R}^2$ via rotations and reflections. The on-topology $\mathcal{T}_{on}$ is characterized by sets that are invariant under the action of $O(2)$. These sets commonly exhibit the following properties:

- Circular, disk, and annular regions centered at the origin.
- Lines passing through the origin and angular segments of disks.
- Regular polygons enclosing the origin and surfaces with particular symmetry elements.

The following specific instances exemplify this:

The empty set, the origin $\{(0,0)\}$, all circles centered at the origin, i.e. $X = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}$, all disks centered at the origin i.e. $X = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}$, all annuli centered at the origin, i.e. $X = \{(x,y) \in \mathbb{R}^2 \mid r_1^2 \leq x^2 + y^2 \leq r_2^2\}$, lines fixed under reflection, i.e. lines passing through the origin as $X = \{(x,0) \mid x \in \mathbb{R}\}$, and the entire plane $\mathbb{R}^2$.

It should be noted that there is no direct relation between the usual Euclidean topology on $\mathbb{R}^2$ and the on-topology $\mathcal{T}_{on}$. However, they do share some common elements.

Let U be a G-set. If for any two elements $x, y \in U$, there exists a group element $g \in G$ such that $g \cdot x = y$, then this action is called a transitive group action. In other words, for any two elements in the set, there is always a group element that maps one to the other. Let G act transitively on a set U. Then, for any $x \in U$, the orbit of x under the action of G is equal to the entire set U, i.e., $\mathbf{Orb}(x) = G \cdot x = U$. To see this, note that $\mathbf{Orb}(x)$ is always a subset of U. Moreover, since the action is transitive, for any $y \in U$, there exists $g \in G$ such that $g \cdot x = y$, implying that y is in the orbit of x. Hence, $U \subseteq \mathbf{Orb}(x)$, and thus, $\mathbf{Orb}(x) = U$. It follows that, under a transitive group action, the orbit lower approximation set of any non-empty proper subset of U is trivial, i.e., the empty set. Given that $\underline{\mathbf{apr}}_{on}(\emptyset) = \emptyset$ and $\underline{\mathbf{apr}}_{on}(U) = U$, it is evident that the on-topology $\mathcal{T}_{on}$ consists only of the empty set and the entire set U, rendering it the indiscrete topology. Consequently, the following proposition can be stated.

Proposition 2.40. *If the group action of G on U is transitive, then the on-space $(U, \mathcal{T}_{on})$ is an indiscrete space.*

Proposition 2.40 immediately yields the following corollary.

Corollary 2.41. *If the group action of G on U is transitive, then the on-space $(U, \mathcal{T}_{on})$ is a connected space.*

As is well-known, every group acts on itself by group multiplication. Under this action, the orbit of each group element is naturally the entire group. When considering this particular group action, the orbital topology on any group is the indiscrete topology. It follows that the on-space of a group with respect to its natural action is connected.

Example 2.42. Let G be the group of fourth roots of unity under complex number multiplication, where $G = \{1, -1, i, -i\}$. The group operation table is given below Table 3.

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Table 3: Operation table of G

Clearly, this group is abelian. Now, suppose that the group acts on itself by conjugation. Since $g \cdot x = gxg^{-1} = gg^{-1}x = x$ for all $x, g \in G$, it is clear that $\mathbf{Orb}(x) = G \cdot x = \{x\}$ for every $x \in G$. Accordingly, for any $X \subseteq G$, we have $\underline{\mathbf{apr}}_{on}(X) = X$. That is, all subsets of G satisfy the condition $\underline{\mathbf{apr}}_{on}(X) = X$. Consequently, the topology $\mathcal{T}_{on} = \mathcal{P}(G)$ is the discrete topology.

It is evident that the conclusion reached in Example 2.42 can be extended to the more general case of abelian groups. To formalize this generalization, we present the following proposition.

Proposition 2.43. *The conjugation action of an abelian group G on itself induces the discrete topology.*

Proof. It is straightforward. □

The relationship between two G-sets is established by a transformation called a G-morphism. Formally, given two G-sets U and V, a G-morphism is a function $\phi : U \to V$ such that $\phi(g \cdot x) = g \cdot \phi(x)$ for all $g \in G$ and $x \in U$.

Lemma 2.44. *Let U and V be two G-sets and $\phi : U \to V$ be a G-morphism.*

(a) For any $x \in U$, $\phi(\mathbf{Orb}(\mathbf{x})) = \mathbf{Orb}(\phi(x))$.

(b) For any $y \in V$, if ϕ is a bijection, then $\phi^{-1}(\mathbf{Orb}(y)) = \mathbf{Orb}(\phi^{-1}(y))$.

(c) For any $X \subseteq U$, $\phi(\underline{\mathbf{apr}}_{on}(X)) = \underline{\mathbf{apr}}_{on}(\phi(X))$.

(d) For any $Y \subseteq V$, $\phi^{-1}(\underline{\mathbf{apr}}_{on}(Y)) = \underline{\mathbf{apr}}_{on}(\phi^{-1}(Y))$.

Proof. (a) Suppose that $y' \in \phi(\mathbf{Orb}(x))$ for any $x \in U$. Then, there exists $x' \in \mathbf{Orb}(x) = G \cdot x$ such that $\phi(x') = y'$. Consequently, there exists $g \in G$ such that $x' = g \cdot x$. Since ϕ is a G-morphism, we have

$$\phi(x') = \phi(g \cdot x) = g \cdot \phi(x).$$

Thus, we conclude that $y' \in \mathbf{Orb}(\phi(x))$.

Conversely, let $y' \in G \cdot \phi(x)$ for some arbitrary $x \in U$. By the definition of the orbit, there exists $g \in G$ such that $y' = g \cdot \phi(x)$. Since ϕ is a G-morphism, we have $y' = \phi(g \cdot x)$. Clearly, $g \cdot x \in G \cdot x$, so $y' \in \phi(G \cdot x)$.

(b) Suppose that for an arbitrary $y \in V$, $x' \in \phi^{-1}(\mathbf{Orb}(y))$. Then, $\phi(x') \in G \cdot y$. By the definition of an orbit, there exists $g \in G$ such that $\phi(x') = g \cdot y$. Since ϕ is a bijection, there exists $x \in U$ such that $\phi(x) = y$. Thus, we have $\phi(x') = g \cdot \phi(x)$. As ϕ is a G-morphism, $\phi(x') = \phi(g \cdot x)$, and since ϕ is injective, $x' = g \cdot x$. Hence, $x' \in G \cdot x = \mathbf{Orb}(x) = \mathbf{Orb}(\phi^{-1}(y))$. Consequently, $\phi^{-1}(\mathbf{Orb}(y)) \subseteq \mathbf{Orb}(\phi^{-1}(y))$.

The other direction of the proof can be shown similarly.

The demonstrations for parts (c) and (d) are likewise straightforward. □

Proposition 2.45. *Let U and V be two G-sets, $\phi : U \to V$ be a G-morphism and $(U, \mathcal{T}_{on}$ and $(V, \mathcal{T}'_{on})$ be the on-spaces.*

(a) ϕ is a continuous function from the on-space $(U, \mathcal{T}_{on})$ to $(V, \mathcal{T}'_{on})$.

(b) If ϕ is a G-isomorphism, then the on-space $(U, \mathcal{T}_{on})$ homeomorphic to $(V, \mathcal{T}'_{on})$.

Proof. (a) To establish the continuity of ϕ, we must show that for any open set $O \in \mathcal{T}'_{on}$, the preimage $\phi^{-1}(O)$ is also open in the on-spaces $(U, \mathcal{T}_{on}$. To this end, it suffices to demonstrate that $\underline{\mathbf{apr}}_{on}(\phi^{-1}(O)) = \phi^{-1}(O)$. From Proposition 2.34, it is clear that $\underline{\mathbf{apr}}_{on}(\phi^{-1}(O)) \subseteq \phi^{-1}(O)$. Now, let x be an arbitrary element in $\phi^{-1}(O)$. Consequently, $\phi(x) \in O$. Since $O \in \mathcal{T}'_{on}$, $\phi(x) \in \underline{\mathbf{apr}}_{on}(O)$, which implies, by definition, that $G \cdot \phi(x) \subseteq O$. By Lemma 2.44(a), $\phi(G \cdot x) \subseteq O$. Taking the preimage of both sides under ϕ, we obtain $\phi^{-1}[\phi(G \cdot x)] \subseteq \phi^{-1}(O)$. As $G \cdot x \subseteq \phi^{-1}[\phi(G \cdot x)]$, it follows that $G \cdot x \subseteq \phi^{-1}(O)$. By the definition of the orbital lower approximation, $x \in \underline{\mathbf{apr}}_{on}(\phi^{-1}(O))$. Therefore, ϕ is continuous.

(b) It is obvious from (a). □

Given the previous results, that is Lemma 2.44 and Proposition 2.45, the following theorem can be deduced, clearly.

Theorem 2.46. *Every G-isomorphism gives rise to a homeomorphism on the corresponding on-spaces.*

3. Conclusion

Pawlak [16, 17] introduced a method for approximating a set in rough set theory and defined the roughness of a set based on these approximations. Liu and Zhu [13] generalized rough set approximations by defining the neighborhood operator concept. The neighborhood operator is given as a set-valued mapping according to an arbitrary relation. In this study, neighborhood operators are defined using set-valued functions with the help of group theoretical concepts such as cyclic subgroup, centralizer, normalizer, coset, and group action. Using these operators, lower and upper approximation operators for subsets of a group are defined. In topological spaces, the interior concept exhibits the behavior of an approximation operator. Moreover, it is a well-known fact that a unique topology can be obtained on a set through a transformation called the interior operator, which is defined on the power set of a set and has certain properties. Based on this argument, in this study, the interior behavior of the approximation operators obtained from group theoretical concepts is examined, and various topologies on the group are constructed. The basic properties of these topologies are investigated. As a result of these studies, the concepts of cn-space obtained from the cyclic subgroup approximation of each element in a group, Cn-space obtained from the centralizer approximation, Nn-space obtained from the normalizer approximation, Ln-space and Rn-space obtained from the left and right coset approximations with respect to a subgroup, and on-space obtained from the orbit approximation of the action of a group on a set are introduced. Using these spaces, it is concluded that when two groups are isomorphic, the spaces obtained from these groups and mentioned above are homeomorphic. In this sense, the fundamental structures of topological spaces obtained directly from algebraic structures are constructed and presented to provide a perspective for researchers in this field. Furthermore, it is shown that when an abelian group is considered together with cn-space and Ln(Rn)-space structures, these structures are topological groups.
Relation-based perspective and future directions. The Ln- and Rn-topologies introduced here can also be viewed as instances of relation-generated neighborhood topologies in the sense of Allam–Bakeir–Abo-Tabl and related dual-topology constructions. The novelty of our approach lies in deriving the underlying relations canonically from group-theoretic data (cosets, centralizers, normalizers and actions), which yields translation-invariant neighborhood systems and permits structural results such as the characterization of $\mathcal{T}_{L_n} = \mathcal{T}_{R_n}$ via normality and the compatibility with group operations under natural algebraic hypotheses.

As future work, the basic properties of these topological groups will be investigated, and their relations with other structures will be examined. Moreover, since the concept of topological groups finds applications in many fields such as physics and computer science, it is hoped that this study will attract the attention of researchers in this field.

We hope that the present framework will stimulate further research at the interface of algebraic and rough topological structures.

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