



Advances in rough ideal convergence within L -fuzzy normed spaces

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Abstract. This study introduces and develops the concept of rough ideal convergence for sequences in \mathcal{L} -fuzzy normed spaces. Within this framework, it defines the notion of the rough \mathcal{I} -limit set and investigates its topological and geometrical properties. The concept of \mathcal{I} -boundedness for sequences is also introduced, and its relationship with the rough \mathcal{I} -limit set is thoroughly examined. Furthermore, the study defines the rough \mathcal{I} -cluster point of a sequence and derives several notable results concerning closed balls and their connection to the rough \mathcal{I} -limit set in the context of \mathcal{L} -fuzzy normed spaces.

1. Introduction and preliminaries

A brief historical overview provides valuable insight into the character and motivation behind this article. Since its introduction by Zadeh [60] in 1965, the theory of fuzzy sets has undergone remarkable evolution, influencing a wide range of disciplines. Its applications span artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic, management science, operations research, pattern recognition, and robotics. Over the decades, the mathematical foundations of fuzzy set theory have reached impressive levels of sophistication, and ongoing advancements continue to enrich the field today. One of the central challenges in fuzzy topology is the formulation of a suitable notion of a fuzzy metric space. This fundamental problem has been explored extensively from various perspectives. The concept of fuzzy topology plays a significant role in quantum particle physics, particularly in its connections with string theory and E-infinity theory [35, 36]. Notably, George and Veeramani [18] introduced a compelling version of fuzzy metric space using continuous t -norms, a refined and elegant variation of the original concept proposed by Kramosil and Michalek [29]. In a parallel development, Atanassov [2] introduced the framework of intuitionistic fuzzy sets as a natural generalization of fuzzy sets. Since then, the theory of intuitionistic fuzzy sets has witnessed significant advancements through the contributions of numerous researchers [13, 20]. Park [38] introduced the concept of intuitionistic fuzzy metric spaces by extending the framework of fuzzy metric spaces proposed by George and Veeramani [18], utilizing the theory of intuitionistic fuzzy sets. It is based on continuous t -norms and t -conorms, offering a broader and more flexible structure. Furthermore, He established a Hausdorff topology on these spaces

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and demonstrated that every classical metric naturally induces an intuitionistic fuzzy metric. Building upon the concept of \mathcal{L} -fuzzy sets [19], the author in [43] introduced the notion of \mathcal{L} -fuzzy metric spaces using continuous t -norms. This construction serves as a natural generalization of both the fuzzy metric space defined by George and Veeramani [18] and the intuitionistic fuzzy metric space introduced by Park and Saadati [38, 42]. Deschrijver et al. and Saadati extended the framework of intuitionistic fuzzy metric (and normed) spaces by introducing and investigating the concepts of \mathcal{L} -fuzzy metric spaces and \mathcal{L} -fuzzy normed spaces [12, 44], thereby broadening the scope of fuzzy analysis within a more general lattice-valued setting.

The concept of statistical convergence, a significant extension of the traditional convergence structure for sequences of real numbers for single sequences, has its roots in the pioneering work of Zygmund, first mentioned in the 1935 Warsaw edition of his monograph. It was formally introduced soon after by Steinhaus [51] and Fast [16], and later revitalized by Schoenberg [48], laying the groundwork for a rich field of study in summability theory. Since then, it has emerged as a vibrant area of research within summability theory, particularly following the influential contributions of Fridy [17] and Šalát [45]. Building on this foundation, Connor [7] further explored the role of statistical convergence in the context of summability, helping to shape the direction of ongoing studies in this field. Let $\mathcal{M} \subseteq \mathbb{N}$. The natural density of \mathcal{M} , denoted by $\delta(\mathcal{M})$, is defined as:

$$\delta(\mathcal{M}) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in \mathcal{M}\}|,$$

provided the limit exists, where the vertical bars represent the cardinality of the enclosed set. Although statistical convergence was introduced over half a century ago, it has only recently emerged as a vibrant area of mathematical research. Numerous scholars have explored its properties and extended its applications across a wide range of disciplines, including measure theory [32], approximation theory [14], topology [31], probability theory [49], locally convex spaces [30], the structure of subsets within the Stone-Čech compactification of the natural numbers [8], Banach spaces [9], and neutrosophic normed spaces [34].

We now proceed by revisiting some fundamental definitions related to ideals. The concept of \mathcal{I} -convergence, introduced by Kostyrko et al. [28] as a generalization of statistical convergence, has since been further developed and explored in subsequent works [46, 47].

A family \mathcal{I} of subsets of a non-empty set \mathcal{U} is called an ideal [28] if (i) $\emptyset \in \mathcal{I}$; (ii) $\mathcal{C}, \mathcal{B} \in \mathcal{I} \implies \mathcal{C} \cup \mathcal{B} \in \mathcal{I}$; (iii) $\mathcal{C} \in \mathcal{I}$ and $\mathcal{B} \subseteq \mathcal{C} \implies \mathcal{B} \in \mathcal{I}$. An ideal \mathcal{I} is called non-trivial if $\mathcal{U} \notin \mathcal{I}$ and $\mathcal{I} \neq \emptyset$. Moreover, a non-trivial ideal $\mathcal{I} \subseteq 2^{\mathcal{U}}$ is termed admissible if $\{\{x\} : x \in \mathcal{U}\} \subseteq \mathcal{I}$. For instance, the class \mathcal{I}_f of all finite subsets of \mathbb{N} forms an admissible ideal on \mathbb{N} . If $\mathcal{I} \subseteq 2^{\mathcal{U}}$ is a non-trivial ideal, then the class $\mathcal{F}(\mathcal{I}) = \{\mathcal{U} \setminus \mathcal{C} : \mathcal{C} \in \mathcal{I}\}$ forms a filter on \mathcal{U} , known as the filter associated with the ideal \mathcal{I} [28]. A sequence $\{\zeta_s\}$ of a metric space (\mathcal{U}, ρ) is defined as \mathcal{I} -convergent to $v \in \mathcal{U}$ if for every $\omega > 0$ the set $\{s \in \mathbb{N} : \rho(\zeta_s, v) \geq \omega\} \in \mathcal{I}$. Since then, the notion has witnessed significant advancements in various settings, notably within the framework of gradual normed linear spaces [26].

The concept of rough convergence was first introduced by Phu [39] in the setting of finite-dimensional normed spaces. In that work, he demonstrated that the set of rough limit points, denoted by LIM_x^r , is bounded, closed, and convex. He also introduced the notion of rough Cauchy sequences and explored the relationship between rough convergence and other types of convergence, emphasizing the dependence of LIM_x^r on the roughness degree r . In a subsequent study [40], Phu defined the rough continuity of linear operators and proved that any linear operator $\mathcal{G} : \mathcal{X} \rightarrow \mathcal{Y}$ is r -continuous at every point $x \in \mathcal{X}$, provided $\dim \mathcal{Y} < \infty$ and $r > 0$, where \mathcal{X} and \mathcal{Y} are normed spaces. These results were later extended to infinite-dimensional normed spaces in [41]. Expanding on this framework, Aytar [4] introduced the concept of rough statistical convergence, defining the set of rough statistical limit points of a sequence. He established two statistical convergence criteria associated with this set and showed that it is both closed and convex. Furthermore, in [3], Aytar demonstrated that the r -limit set of a sequence equals the intersection of certain related sets, while the r -core of the sequence corresponds to their union. Dündar and Çakan [15], and Pal et al. [37] independently extended the notion of rough convergence by incorporating the concept of ideals, thereby naturally generalizing and unifying the earlier frameworks of rough convergence and rough statistical convergence. Since its inception, this convergence concept has been extensively developed by

numerous researchers, including Demirci et al. [11], Khan et al. [23], and Kişi and Dündar [25]. Significant advancements have also been made in various novel settings such as probabilistic normed spaces [22], 2-normed spaces [21], and intuitionistic fuzzy normed spaces [1, 33], among others.

1.1. Identified research gaps and key insights from the literature review

Statistical convergence, originally introduced by Fast [16] and Schoenberg [48], has played a foundational role in the development of summability theory and the theory of sequence spaces. Building upon this, Kostyrko et al. [28] extended the concept to ideal convergence, thereby enriching its theoretical depth and applicability. Nevertheless, much of the existing research remains confined to specific sequence spaces and particular modes of convergence. While statistical convergence has found applications in diverse fields such as Banach spaces, ergodic theory, Fourier analysis, and topological structures, there remains a compelling need to explore this notion within broader and more abstract frameworks particularly in the context of \mathcal{L} -fuzzy normed spaces.

Despite a few notable contributions such as statistical convergence [54, 55], ideal convergence [56, 59], and lacunary statistical convergence [57, 58]-the study of summability theory and sequence convergence in \mathcal{L} -fuzzy normed spaces is still in its nascent stages. Moreover, the rich potential for developing generalized forms of statistical convergence such as deferred statistical and ideal convergence within this framework remains largely unexplored. This presents a significant gap in understanding the intricate relationships between various types of convergence and boundedness in such spaces. Additionally, there is considerable scope for defining novel sequence spaces based on Orlicz functions and modulus functions in conjunction with ideals, specifically tailored to the \mathcal{L} -fuzzy normed space framework.

1.2. Motivation

The concept of rough convergence in normed linear spaces was initially introduced by H. X. Phu [39–41], and since then, it has been extensively explored and developed by numerous researchers, including Subramanian and Esi [52], Debnath and Subramanian [10], and Kişi and Choudhury [27], across a variety of novel spaces such as probabilistic n -normed spaces [53], intuitionistic fuzzy normed spaces (IFNS) [24], and cone metric spaces [5]. Building on this progression, Çaki et al. [6] introduced the notion of rough statistical convergence in \mathcal{L} -fuzzy normed spaces.

Motivated by the growing applications of ideals in sequence convergence theory, we propose a further extension of the concept of rough convergence through the framework of ideals. In particular, we introduce and investigate the notion of rough ideal convergence within the setting of \mathcal{L} -fuzzy normed spaces. This approach naturally generalizes and unifies earlier notions of rough convergence and rough statistical convergence.

We also define the set of rough ideal cluster points and establish several significant results concerning its structure. Notably, our findings serve as ideal analogues of the corresponding results in [6], and are presented in a more generalized form. These results further enhance the understanding of the intricate relationships among the rough ideal limit set, closed balls, and rough ideal cluster point sets in \mathcal{L} -fuzzy normed spaces.

To proceed, we first revisit essential definitions concerning \mathcal{L} -fuzzy normed spaces. Unless otherwise stated, \mathbb{N} and \mathbb{R} will denote the sets of natural and real numbers, respectively.

Definition 1.1. Let (\mathcal{L}, \leq) be a partially ordered set. If for every pair of elements $\vartheta_1, \vartheta_2 \in \mathcal{L}$, both the infimum $\inf\{\vartheta_1, \vartheta_2\}$ and the supremum $\sup\{\vartheta_1, \vartheta_2\}$ exist and belong to \mathcal{L} , then (\mathcal{L}, \leq) is defined as lattice. Furthermore, the lattice (\mathcal{L}, \leq) is said to be complete if, for every subset $\mathcal{M} \subseteq \mathcal{L}$, both $\inf \mathcal{M}$ and $\sup \mathcal{M}$ exist and are elements of \mathcal{L} .

Definition 1.2. ([50]) Let (\mathcal{L}, \leq) be a complete lattice, and let \mathcal{W} be a nonempty set, referred to as the universe. An \mathcal{L} -fuzzy set on \mathcal{W} is a mapping $\mathcal{G} : \mathcal{W} \rightarrow \mathcal{L}$. For each element $\vartheta \in \mathcal{W}$, the value $\mathcal{G}(\vartheta)$ denotes the degree of ϑ in \mathcal{L} .

The collection of all \mathcal{L} -fuzzy sets defined on a set \mathcal{W} is denoted by $\mathcal{L}^{\mathcal{W}}$. Moreover, the infimum and supremum of \mathcal{L} are denoted by $\mathbf{0}_{\mathcal{L}} = \inf \mathcal{L}$ and $\mathbf{1}_{\mathcal{L}} = \sup \mathcal{L}$ respectively. Henceforth, \mathcal{L} will denote a complete lattice (\mathcal{L}, \leq) . For any given lattice (\mathcal{L}, \leq) , we also use the symbols $\geq, <, >$ with their standard interpretations.

Definition 1.3. ([50]) Let \mathcal{L} be a complete lattice. A t -norm is a function $\mathcal{T} : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ that satisfies the following properties for all $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathcal{L}$:

1. $\mathcal{T}(\vartheta_1, \vartheta_2) = \mathcal{T}(\vartheta_2, \vartheta_1)$;
2. $\mathcal{T}(\mathcal{T}(\vartheta_1, \vartheta_2), \vartheta_3) = \mathcal{T}(\vartheta_1, \mathcal{T}(\vartheta_2, \vartheta_3))$;
3. $\mathcal{T}(\vartheta_1, \mathbf{1}_{\mathcal{L}}) = \mathcal{T}(\mathbf{1}_{\mathcal{L}}, \vartheta_1) = \vartheta_1$;
4. $\mathcal{T}(\vartheta_1, \vartheta_3) \leq \mathcal{T}(\vartheta_2, \vartheta_4)$ whenever $\vartheta_1 \leq \vartheta_2$ and $\vartheta_3 \leq \vartheta_4$.

Definition 1.4. ([50]) Let $\{\zeta_k\}$ and $\{\xi_k\}$ be sequences in a complete lattice \mathcal{L} such that $\zeta_k \rightarrow \vartheta_1 \in \mathcal{L}$ and $\xi_k \rightarrow \vartheta_2 \in \mathcal{L}$. If it follows that $\mathcal{T}(\zeta_k, \xi_k) \rightarrow \mathcal{T}(\vartheta_1, \vartheta_2)$, then the t -norm $\mathcal{T} : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ is referred to as continuous on \mathcal{L} .

Definition 1.5. ([50]) A function $\mathcal{G} : \mathcal{L} \rightarrow \mathcal{L}$ is called a negator on the complete lattice \mathcal{L} if the following conditions are satisfied:

1. $\mathcal{G}(\mathbf{0}_{\mathcal{L}}) = \mathbf{1}_{\mathcal{L}}$;
2. $\mathcal{G}(\mathbf{1}_{\mathcal{L}}) = \mathbf{0}_{\mathcal{L}}$;
3. $\forall \vartheta_1, \vartheta_2 \in \mathcal{L}$, if $\vartheta_1 \leq \vartheta_2$, then $\mathcal{G}(\vartheta_2) \leq \mathcal{G}(\vartheta_1)$.
Furthermore, if
4. $\mathcal{G}(\mathcal{G}(\vartheta)) = \vartheta$ for all $\vartheta \in \mathcal{L}$,
then \mathcal{G} is called an involutive negator on \mathcal{L} .

The mapping $\mathcal{G}_s : [0, 1] \rightarrow [0, 1]$, defined on the lattice $([0, 1], \leq)$ by $\mathcal{G}_s(\vartheta) = 1 - \vartheta$, is a well-known example of an involutive negator. This type of negator plays a central role in the theory of standard fuzzy sets.

Furthermore, consider the lattice $([0, 1]^2, \leq)$ with the order defined as

$$(\vartheta_1, \varrho_1) \leq (\vartheta_2, \varrho_2) \text{ iff } \vartheta_1 \leq \vartheta_2 \text{ and } \varrho_1 \geq \varrho_2$$

for all $(\vartheta_i, \varrho_i) \in [0, 1]^2, i = 1, 2$. In this setting, the mapping $\mathcal{G}_1 : [0, 1]^2 \rightarrow [0, 1]^2$, defined by

$$\mathcal{G}_1(\vartheta, \varrho) = (\varrho, \vartheta)$$

as introduced by Atanassov [2], is another prominent example of an involutive negator. This form is fundamental to the framework of intuitionistic fuzzy sets.

In general, for an arbitrary continuous t -norm \mathcal{T} and a negator \mathcal{G} , it is not always guaranteed that for every $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}, \mathbf{1}_{\mathcal{L}}\}$, there exists an element $\lambda \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}, \mathbf{1}_{\mathcal{L}}\}$ such that $\mathcal{T}(\mathcal{G}(\omega), \mathcal{G}(\omega)) > \mathcal{G}(\lambda)$ [54]. For further discussion and examples where this inequality does hold, the reader is referred to [43]. In the present study, we assume a continuous t -norm \mathcal{T} and an involutive negator \mathcal{G} are given and fixed, satisfying the condition that for each $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}, \mathbf{1}_{\mathcal{L}}\}$, there exists an element $\lambda \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}, \mathbf{1}_{\mathcal{L}}\}$ such that $\mathcal{T}(\mathcal{G}(\omega), \mathcal{G}(\omega)) > \mathcal{G}(\lambda)$.

Definition 1.6. ([50]) Let Q be a vector space and \mathcal{T} a continuous t -norm on a complete lattice \mathcal{L} , Ψ an \mathcal{L} -fuzzy set on $Q \times (0, \infty)$. Then, the triple (Q, Ψ, \mathcal{T}) is called an \mathcal{L} -fuzzy normed space if the following conditions are satisfied for all $\vartheta, \varrho \in Q$ and $\gamma, \lambda > 0$:

1. $\Psi(\vartheta, \gamma) > \mathbf{0}_{\mathcal{L}}$;
2. $\Psi(\vartheta, \gamma) = \mathbf{1}_{\mathcal{L}}$ iff $\vartheta = \theta$;
3. $\Psi(\kappa\vartheta, \gamma) = \Psi\left(\vartheta, \frac{\gamma}{\kappa}\right)$ for all $\kappa \in \mathbb{R} \setminus \{0\}$;
4. $\mathcal{T}(\Psi(\vartheta, \gamma), \Psi(\varrho, \lambda)) \leq \Psi(\vartheta + \varrho, \gamma + \lambda)$;

5. $\lim_{\gamma \rightarrow \infty} \Psi(\vartheta, \gamma) = \mathbf{1}_{\mathcal{L}}$ and $\lim_{\gamma \rightarrow 0} \Psi(\vartheta, \gamma) = \mathbf{0}_{\mathcal{L}}$ for all $\vartheta \in \mathcal{Q} \setminus \{\theta\}$.
6. The functions $f_{\vartheta} : (0, \infty) \rightarrow \mathcal{L}$ defined by $f_{\vartheta}(\gamma) = \Psi(\vartheta, \gamma)$ are continuous.

Henceforth, unless specified otherwise, the \mathcal{L} -fuzzy normed space $(\mathcal{Q}, \Psi, \mathcal{T})$ will simply be denoted by \mathcal{Q} for the sake of brevity.

Example 1.7. ([6]) Let $(\mathbb{R}, |\cdot|)$ be a normed space and let $\mathcal{T} : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ be a continuous t -norm defined by $\mathcal{T}(\xi, \eta) = \xi\eta$ for all $\xi, \eta \in \mathcal{L}$. Define an \mathcal{L} -fuzzy set $\Psi : \mathbb{R} \times (0, \infty) \rightarrow \mathcal{L}$ by

$$\Psi(\vartheta, \gamma) = \frac{\gamma}{\gamma + |\vartheta|}$$

for all $\vartheta \in \mathbb{R}$ and $\gamma > 0$. It can be verified that the function Ψ satisfies the conditions of Definition 1.6. Therefore, the triple $(\mathbb{R}, \Psi, \mathcal{T})$ forms an \mathcal{L} -fuzzy normed space.

Example 1.8. ([55]) Let $\mathcal{Q} = \mathbb{R}^2$, and consider $\mathcal{L} = (P(\mathbb{R}^+), \subseteq)$, the lattice of all subsets of the set of non-negative real numbers. Define the function $\Psi : \mathbb{R}^2 \times (0, \infty) \rightarrow P(\mathbb{R}^+)$ by

$$\Psi((x, y), \gamma) = \{g \in \mathbb{R}^+ : \max\{|gx|, |gy|\} < \gamma\}.$$

It can be verified that the function Ψ satisfies the conditions of Definition 1.6. Hence, the triple $(\mathbb{R}^2, \Psi, \mathcal{T})$ constitutes an \mathcal{L} -fuzzy normed space.

Definition 1.9. ([6]) Let \mathcal{Q} be an \mathcal{L} -fuzzy normed space, and let \mathcal{G} be a negator on \mathcal{L} . Then, for any $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}\}$ and $\gamma > 0$, the set

$$\mathcal{B}(\vartheta, \omega, \gamma) = \{\varrho \in \mathcal{Q} : \Psi(\vartheta - \varrho, \gamma) > \mathcal{G}(\omega)\}$$

is called ω -ball centered at ϑ .

Definition 1.10. ([50]) Let \mathcal{G} be a negator on \mathcal{L} . A sequence $\{\zeta_s\}$ in an \mathcal{L} -fuzzy normed space \mathcal{Q} is referred to as convergent to $\ell \in \mathcal{Q}$ if, for every $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}\}$ and $\gamma > 0$, there exists a natural number s_0 such that

$$s \geq s_0 \implies \Psi(\zeta_s - \ell, \gamma) > \mathcal{G}(\omega).$$

Under this circumstances, we express $\mathcal{L} - \lim_{s \rightarrow \infty} \zeta_s = \ell$, or equivalently, $\zeta_s \xrightarrow{\mathcal{L}} \ell$.

Definition 1.11. ([55]) Let \mathcal{G} be a negator on \mathcal{L} . A sequence $\{\zeta_s\}$ in an \mathcal{L} -fuzzy normed space \mathcal{Q} is referred to as statistically convergent to $\ell \in \mathcal{Q}$ in relation to Ψ if, for every $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}\}$ and $\gamma > 0$, we have

$$\delta(\{s \in \mathbb{N} : \Psi(\zeta_s - \ell, \gamma) \not> \mathcal{G}(\omega)\}) = 0.$$

Under this circumstances, we express $\mathcal{S}_{\mathcal{L}} - \lim_{s \rightarrow \infty} \zeta_s = \ell$, or equivalently, $\zeta_s \xrightarrow{\mathcal{S}_{\mathcal{L}}} \ell$.

Definition 1.12. ([55]) Let \mathcal{G} be a negator on \mathcal{L} . A sequence $\{\zeta_s\}$ in an \mathcal{L} -fuzzy normed space \mathcal{Q} is referred to as statistically bounded in relation to Ψ if, there exists $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}, \mathbf{1}_{\mathcal{L}}\}$ and $\gamma > 0$, such that

$$\delta(\{s \in \mathbb{N} : \Psi(\zeta_s, \gamma) \not> \mathcal{G}(\omega)\}) = 0.$$

Definition 1.13. ([6]) Let \mathcal{G} be a negator on \mathcal{L} and $r \geq 0$. A sequence $\{\zeta_s\}$ in an \mathcal{L} -fuzzy normed space \mathcal{Q} is referred to as rough convergent to $\ell \in \mathcal{Q}$ with respect to Ψ if, for every $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}\}$ and $\gamma > 0$, there exists a natural number s_0 such that

$$s \geq s_0 \implies \Psi(\zeta_s - \ell, r + \gamma) > \mathcal{G}(\omega).$$

Under this circumstances, we express $r_{\mathcal{L}} - \lim_{s \rightarrow \infty} \zeta_s = \ell$, or equivalently, $\zeta_s \xrightarrow{r_{\mathcal{L}}} \ell$. And, ℓ is called an $r_{\mathcal{L}}$ -limit of $\{\zeta_s\}$.

From the above definition, it is clear that $r_{\mathcal{L}}$ -limit of a sequence may not be unique. Hence, the set of $r_{\mathcal{L}}$ -limit of a sequence $\{\zeta_s\}$ is defined as

$$LIM_{\zeta_s}^{r_{\mathcal{L}}} = \{\ell \in \mathcal{Q} : \zeta_s \xrightarrow{r_{\mathcal{L}}} \ell\}.$$

If $LIM_{\zeta_s}^{r_{\mathcal{L}}} \neq \emptyset$, then the sequence $\{\zeta_s\}$ is said to be rough convergent.

Definition 1.14. ([6]) Let \mathcal{G} be a negator on \mathcal{L} and $r \geq 0$. A sequence $\{\zeta_s\}$ in an \mathcal{L} -fuzzy normed space \mathcal{Q} is referred to as rough statistically convergent to $\ell \in \mathcal{Q}$ with respect to Ψ if, for every $\omega \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$ and $\gamma > 0$,

$$\delta(\{s \in \mathbb{N} : \Psi(\zeta_s - \ell, r + \gamma) \not\prec \mathcal{G}(\omega)\}) = 0.$$

Under this circumstances, we express $S_{\mathcal{L}}^r - \lim_{s \rightarrow \infty} \zeta_s = \ell$, or equivalently, $\zeta_s \xrightarrow{S_{\mathcal{L}}^r} \ell$.

Definition 1.15. ([56]) Let \mathcal{I} be a non-trivial ideal on \mathbb{N} . Consider $\{\zeta_s\}$ to be a sequence in an \mathcal{L} -fuzzy normed space \mathcal{Q} . Then, $\{\zeta_s\}$ is defined as \mathcal{I} -convergent to $\ell \in \mathcal{Q}$ with respect to Ψ if, for every $\gamma > 0$ and $\omega \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$, the set

$$\{s \in \mathbb{N} : \Psi(\zeta_s - \ell, \gamma) \not\prec \mathcal{G}(\omega)\} \in \mathcal{I}.$$

Under this circumstances, we express $\mathcal{I}_{\mathcal{L}} - \lim_{s \rightarrow \infty} \zeta_s = \ell$ or $\zeta_s \xrightarrow{\mathcal{I}_{\mathcal{L}}} \ell$.

2. Main results

In what follows, we establish our main results. Throughout this section, \mathcal{I} refers to an admissible ideal, unless stated otherwise.

Definition 2.1. Consider $\{\zeta_s\}$ to be a sequence in an \mathcal{L} -fuzzy normed space \mathcal{Q} . Then, $\{\zeta_s\}$ is defined as rough \mathcal{I} -convergent to $\ell \in \mathcal{Q}$ with respect to Ψ (in short $\mathcal{I}_{\mathcal{L}}^r$ -convergence) if, for every $\gamma > 0$ and $\omega \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$, the set

$$\{s \in \mathbb{N} : \Psi(\zeta_s - \ell, r + \gamma) \not\prec \mathcal{G}(\omega)\} \in \mathcal{I}.$$

Under this circumstances, we express $\mathcal{I}_{\mathcal{L}}^r - \lim_{s \rightarrow \infty} \zeta_s = \ell$ or $\zeta_s \xrightarrow{\mathcal{I}_{\mathcal{L}}^r} \ell$. And, ℓ is called $\mathcal{I}_{\mathcal{L}}^r$ -limit of $\{\zeta_s\}$.

Remark 2.2. If $r = 0$, then the concept of rough \mathcal{I} -convergence for a sequence in an \mathcal{L} -fuzzy normed space \mathcal{Q} reduces to the classical notion of \mathcal{I} -convergence in \mathcal{Q} . Accordingly, the core focus of the discussion lies in the assumption that $r > 0$. According to Definition 2.1, the $\mathcal{I}_{\mathcal{L}}^r$ -limit of a sequence need not be unique. Therefore, we denote the set of all $\mathcal{I}_{\mathcal{L}}^r$ -limits of the sequence $\{\zeta_s\}$ by $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$.

It is worth noting that for an unbounded sequence in \mathcal{Q} , we have $LIM_{\zeta_s}^{r_{\mathcal{L}}} = \emptyset$ [6]. However, in such scenarios, it is possible that $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r \neq \emptyset$ as will be illustrated in the following example.

Example 2.3. Let $\mathcal{Q} = \mathbb{R}$, and consider the mappings Ψ and \mathcal{T} as defined in Example 1.7. With these, the structure $(\mathcal{Q}, \Psi, \mathcal{T})$ forms an \mathcal{L} -fuzzy normed space.

Now, let \mathcal{I} denote the class of all subsets of \mathbb{N} having natural density zero. Clearly, \mathcal{I} is a non-trivial admissible ideal. Define a sequence $\{\zeta_s\}$ in \mathcal{Q} by $\zeta_s = \begin{cases} (-1)^s, & \text{if } s \neq j^2 \text{ for any } j \in \mathbb{N}, \\ s, & \text{otherwise.} \end{cases}$ Then, the set of all

$\mathcal{I}_{\mathcal{L}}^r$ -limits of $\{\zeta_s\}$ is given by $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r = \begin{cases} \emptyset, & \text{if } r < 1, \\ [1 - r, r - 1], & \text{if } r \geq 1. \end{cases}$ Furthermore, since the sequence $\{\zeta_s\}$ is

unbounded, the classical rough limit set is empty, i.e., $LIM_{\zeta_s}^{r_{\mathcal{L}}} = \emptyset$.

From Example 2.3, we observe that

$$\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r \neq \emptyset$$

does not necessarily imply

$$LIM_{\zeta_s}^{r\mathcal{L}} \neq \emptyset.$$

However, if \mathcal{I} is an admissible ideal, then

$$LIM_{\zeta_s}^{r\mathcal{L}} \neq \emptyset \implies \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r \neq \emptyset.$$

Remark 2.4. Let \mathcal{I}_f denote the collection of all finite subsets of \mathbb{N} . Clearly, \mathcal{I}_f is a non-trivial admissible ideal. Hence, the concept of rough \mathcal{I} -convergence with respect to Ψ coincides with the classical notion of rough convergence in the \mathcal{L} -fuzzy normed space \mathcal{Q} .

Consider \mathcal{I}_δ as the family of all subsets of \mathbb{N} having natural density zero. It is evident that \mathcal{I}_δ is also a non-trivial admissible ideal. Therefore, rough \mathcal{I}_δ -convergence with respect to Ψ aligns with the concept of rough statistical convergence in \mathcal{Q} .

We now introduce the concept of \mathcal{I} -boundedness for sequences in an \mathcal{L} -fuzzy normed space \mathcal{Q} and explore its connection with the $\mathcal{I}_{\mathcal{L}}^r$ -limit set of a sequence.

Definition 2.5. Let \mathcal{G} be a negator on \mathcal{L} . A sequence $\{\zeta_s\}$ in an \mathcal{L} -fuzzy normed space \mathcal{Q} is referred to as \mathcal{I} -bounded in relation to Ψ (in short $\mathcal{I}_{\mathcal{L}}$ -bounded) if, there exists $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}, \mathbf{1}_{\mathcal{L}}\}$ and $\gamma > 0$, such that

$$\{s \in \mathbb{N} : \Psi(\zeta_s, \gamma) \not\prec \mathcal{G}(\omega)\} \in \mathcal{I}.$$

Theorem 2.6. A sequence $\{\zeta_s\}$ in an \mathcal{L} -fuzzy normed space \mathcal{Q} is $\mathcal{I}_{\mathcal{L}}$ -bounded if and only if there exists $r > 0$ such that $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r \neq \emptyset$.

Proof. First, suppose that $\{\zeta_s\}$ is $\mathcal{I}_{\mathcal{L}}$ -bounded in \mathcal{Q} . Then, there exist $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}, \mathbf{1}_{\mathcal{L}}\}$ and $\gamma > 0$ such that the set

$$\mathcal{M} = \{s \in \mathbb{N} : \Psi(\zeta_s, \gamma) \not\prec \mathcal{G}(\omega)\}$$

belongs to the ideal \mathcal{I} . Consequently, for all $s \in \mathcal{M}^c$ (the complement of \mathcal{M}), we have

$$\Psi(\zeta_s, \gamma) \succ \mathcal{G}(\omega).$$

Using the properties of the \mathcal{L} -fuzzy norm Ψ and the t-norm \mathcal{T} , we get:

$$\begin{aligned} \Psi(\zeta_s, r + \gamma) &\geq \mathcal{T}(\Psi(\theta, r), \Psi(\zeta_s, \gamma)) \\ &= \mathcal{T}(\mathbf{1}_{\mathcal{L}}, \Psi(\zeta_s, \gamma)) \\ &= \Psi(\zeta_s, \gamma) \\ &> \mathcal{G}(\omega), \end{aligned}$$

for all $s \in \mathcal{M}^c$. Thus, we have

$$\mathcal{M}^c \subseteq \{s \in \mathbb{N} : \Psi(\zeta_s, r + \gamma) \succ \mathcal{G}(\omega)\}.$$

Equivalently,

$$\{s \in \mathbb{N} : \Psi(\zeta_s, r + \gamma) \not\prec \mathcal{G}(\omega)\} \in \mathcal{I}.$$

Hence, the origin θ is an $\mathcal{I}_{\mathcal{L}}^r$ -limit of the sequence $\{\zeta_s\}$, i.e., $\theta \in \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$, and so $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r \neq \emptyset$.

Conversely, assume that $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r \neq \emptyset$ for some $r > 0$. Then there exists an element $\ell \in \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$. By the definition of $\mathcal{I}_{\mathcal{L}}^r$ -limit, for every $\gamma > 0$ and $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}\}$, the set

$$\{s \in \mathbb{N} : \Psi(\zeta_s - \ell, r + \gamma) \not\prec \mathcal{G}(\omega)\} \in \mathcal{I}.$$

This implies that for almost all $s \in \mathbb{N}$ (i.e., for all s outside a set in \mathcal{I}), the elements ζ_s lie within a ball centered at ℓ with radius $r + \gamma$. Therefore, the sequence $\{\zeta_s\}$ is $\mathcal{I}_{\mathcal{L}}$ -bounded in \mathcal{Q} . \square

Next, we present an algebraic characterization of rough \mathcal{I} -convergent sequences in an \mathcal{L} -fuzzy normed space \mathcal{Q} .

Theorem 2.7. Let $\{\zeta_s\}$ and $\{\sigma_s\}$ be two sequences in \mathcal{Q} , and let $r_1, r_2 > 0$. Then the following algebraic properties of $\mathcal{I}_{\mathcal{L}}^r$ -convergence hold:

1. If $\zeta_s \xrightarrow{\mathcal{I}_{\mathcal{L}}^{r_1}} \ell$ and $\sigma_s \xrightarrow{\mathcal{I}_{\mathcal{L}}^{r_2}} \alpha$ then $\zeta_s + \sigma_s \xrightarrow{\mathcal{I}_{\mathcal{L}}^{r_1+r_2}} \ell + \alpha$.
2. If $\zeta_s \xrightarrow{\mathcal{I}_{\mathcal{L}}^{r_1}} \ell$ then $c\zeta_s \xrightarrow{\mathcal{I}_{\mathcal{L}}^{|c|r_1}} c\ell, c \in \mathbb{R} \setminus \{0\}$.

Proof. The result is straightforward to verify and thus the proof is omitted. \square

In the following, we examine certain topological and geometrical aspects of the $\mathcal{I}_{\mathcal{L}}^r$ -limit set associated with a sequence in an \mathcal{L} -fuzzy normed space \mathcal{Q} .

Theorem 2.8. The $\mathcal{I}_{\mathcal{L}}^r$ -limit set $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ of a sequence $\{\zeta_s\}$ in the space \mathcal{Q} is a closed subset of \mathcal{Q} .

Proof. If $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r = \emptyset$, then there is nothing to prove. So, let us assume that $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r \neq \emptyset$. Then, there exists a sequence $\{\sigma_s\} \subseteq \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ such that $\sigma_s \xrightarrow{\mathcal{L}} \sigma_0 \in \mathcal{Q}$. Now, for every $\omega \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$ and $\gamma > 0$, there exists a natural number s_0 such that

$$s \geq s_0 \implies \Psi\left(\sigma_s - \sigma_0, \frac{\gamma}{2}\right) > \mathcal{G}(\omega).$$

Since each $\sigma_t \in \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ for $t > s_0$, by the definition of $\mathcal{I}_{\mathcal{L}}^r$ -limit set, we have

$$\mathcal{M} = \left\{s \in \mathbb{N} : \Psi\left(\zeta_s - \sigma_t, r + \frac{\gamma}{2}\right) \not> \mathcal{G}(\omega)\right\} \in \mathcal{I}.$$

Thus, $\mathbb{N} \setminus \mathcal{M} \in \mathcal{F}(\mathcal{I})$, and for any $k \in \mathbb{N} \setminus \mathcal{M}$, we obtain

$$\Psi\left(\zeta_k - \sigma_t, r + \frac{\gamma}{2}\right) > \mathcal{G}(\omega). \tag{1}$$

Also, for $t > s_0$ we have

$$\Psi\left(\sigma_t - \sigma_0, \frac{\gamma}{2}\right) > \mathcal{G}(\omega). \tag{2}$$

Combining inequalities (1) and (2), we obtain

$$\begin{aligned} \Psi(\zeta_k - \sigma_0, r + \gamma) &\geq \mathcal{T}\left(\Psi\left(\zeta_k - \sigma_t, r + \frac{\gamma}{2}\right), \Psi\left(\sigma_t - \sigma_0, \frac{\gamma}{2}\right)\right) \\ &> \mathcal{T}(\mathcal{G}(\omega), \mathcal{G}(\omega)) \\ &> \mathcal{G}(\lambda). \end{aligned}$$

Therefore,

$$\mathbb{N} \setminus \mathcal{M} \subseteq \{k \in \mathbb{N} : \Psi(\zeta_k - \sigma_0, r + \gamma) > \mathcal{G}(\lambda)\} \in \mathcal{F}(\mathcal{I}),$$

which implies

$$\{k \in \mathbb{N} : \Psi(\zeta_k - \sigma_0, r + \gamma) \not> \mathcal{G}(\lambda)\} \in \mathcal{I}.$$

Hence, $\sigma_0 \in \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ and therefore, $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ is a closed set. \square

Theorem 2.9. The $\mathcal{I}_{\mathcal{L}}^r$ -limit set $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ of a sequence $\{\zeta_s\}$ in the space \mathcal{Q} is a convex set.

Proof. Let $\ell_1, \ell_2 \in \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ and let $\varrho \in (0, 1)$. Then, for every $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}\}$ and $\gamma > 0$, consider the sets

$$\begin{aligned} \mathcal{C}_1 &= \left\{s \in \mathbb{N} : \Psi\left(\zeta_s - \ell_1, r + \frac{\gamma}{2(1-\varrho)}\right) \not> \mathcal{G}(\omega)\right\}, \\ \mathcal{C}_2 &= \left\{s \in \mathbb{N} : \Psi\left(\zeta_s - \ell_2, r + \frac{\gamma}{2\varrho}\right) \not> \mathcal{G}(\omega)\right\}. \end{aligned}$$

Since $\ell_1, \ell_2 \in \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$, we have $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{I}$ which implies

$$\mathcal{C}_1^c \cap \mathcal{C}_2^c \in \mathcal{F}(\mathcal{I}).$$

Now, for each $s \in \mathcal{C}_1^c \cap \mathcal{C}_2^c$ we obtain,

$$\begin{aligned} &\Psi(\zeta_s - [(1-\varrho)\ell_1 + \varrho\ell_2], r + \gamma) \\ &\geq \mathcal{T}\left(\Psi\left((1-\varrho)(\zeta_s - \ell_1), (1-\varrho)r + \frac{\gamma}{2}\right), \Psi\left(\varrho(\zeta_s - \ell_2), \varrho r + \frac{\gamma}{2}\right)\right) \\ &= \mathcal{T}\left(\Psi\left(\zeta_s - \ell_1, r + \frac{\gamma}{2(1-\varrho)}\right), \Psi\left(\zeta_s - \ell_2, \varrho r + \frac{\gamma}{2\varrho}\right)\right) \\ &> \mathcal{T}(\mathcal{G}(\omega), \mathcal{G}(\omega)) \\ &> \mathcal{G}(\lambda). \end{aligned}$$

This implies

$$\mathcal{C}_1^c \cap \mathcal{C}_2^c \subseteq \{s \in \mathbb{N} : \Psi(\zeta_s - [(1-\varrho)\ell_1 + \varrho\ell_2], r + \gamma) > \mathcal{G}(\lambda)\}.$$

Hence, the set

$$\{s \in \mathbb{N} : \Psi(\zeta_s - [(1-\varrho)\ell_1 + \varrho\ell_2], r + \gamma) \not> \mathcal{G}(\lambda)\} \in \mathcal{I},$$

which shows that $(1-\varrho)\ell_1 + \varrho\ell_2 \in \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$. Therefore, the set $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ is convex. \square

In the following theorem, we establish a sufficient condition under which a sequence $\{\zeta_s\}$ becomes $\mathcal{I}_{\mathcal{L}}^r$ -convergent, based on the existence of an $\mathcal{I}_{\mathcal{L}}$ -convergent sequence $\{\sigma_s\}$.

Theorem 2.10. *A sequence $\{\zeta_s\} \subseteq \mathcal{Q}$ is $\mathcal{I}_{\mathcal{L}}^r$ -convergent to $\ell \in \mathcal{Q}$ with respect to Ψ for some $r > 0$, if there exists a sequence $\{\sigma_s\} \subseteq \mathcal{Q}$ such that $\sigma_s \xrightarrow{\mathcal{I}_{\mathcal{L}}} \ell$, and for every $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}\}$ and $\gamma > 0$, the inequality $\Psi(\zeta_s - \sigma_s, r) > \mathcal{G}(\omega)$ hold for all $s \in \mathbb{N}$.*

Proof. Let $\sigma_s \xrightarrow{\mathcal{I}_{\mathcal{L}}} \ell$ and suppose that $\Psi(\zeta_s - \sigma_s, r) > \mathcal{G}(\omega)$ hold for all $s \in \mathbb{N}$. Then, for every $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}\}$ and $\gamma > 0$, consider the set

$$\mathcal{C} = \{s \in \mathbb{N} : \Psi(\sigma_s - \ell, \gamma) \not> \mathcal{G}(\omega)\}.$$

Since $\sigma_s \xrightarrow{\mathcal{I}_{\mathcal{L}}} \ell$, it follows that $\mathcal{C} \in \mathcal{I}$, and hence its complement $\mathcal{C}^c \in \mathcal{F}(\mathcal{I})$, which implies $\mathcal{C}^c \neq \emptyset$. Now, for each $s \in \mathcal{C}^c$, we have

$$\begin{aligned} \Psi(\zeta_s - \ell, r + \gamma) &\geq \mathcal{T}(\Psi(\zeta_s - \sigma_s, r), \Psi(\sigma_s - \ell, \gamma)) \\ &> \mathcal{T}(\mathcal{G}(\omega), \mathcal{G}(\omega)) \\ &> \mathcal{G}(\lambda). \end{aligned}$$

This implies:

$$\mathcal{C}^c \subseteq \{s \in \mathbb{N} : \Psi(\zeta_s - \ell, r + \gamma) > \mathcal{G}(\lambda)\}.$$

Therefore, the set

$$\{s \in \mathbb{N} : \Psi(\zeta_s - \ell, r + \gamma) \not> \mathcal{G}(\lambda)\} \in \mathcal{I},$$

which shows that the sequence $\{\zeta_s\} \subseteq \mathcal{Q}$ is $\mathcal{I}_{\mathcal{L}}^r$ -convergent to $\ell \in \mathcal{Q}$. This completes the proof. \square

If ℓ_2 is any point in the $\mathcal{I}_{\mathcal{L}}$ -limit set $\mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ with roughness degree r , then every point $\ell_1 \in \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ lies within an ω -ball centered at ℓ_2 and roughness degree $\hbar r$ for some $\hbar > 2$. This relationship is formally established in the following theorem.

Theorem 2.11. *Let $\{\zeta_s\}$ be a sequence in an \mathcal{L} -fuzzy normed space \mathcal{Q} . Then, for some $r > 0$ and for every $\lambda \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$, there do not exist two points $\ell_1, \ell_2 \in \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ such that $\Psi(\ell_1 - \ell_2, \hbar r) \not\prec \mathcal{G}(\lambda)$ for any $\hbar > 2$.*

Proof. Suppose, to the contrary, that there exist two elements $\ell_1, \ell_2 \in \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ such that

$$\Psi(\ell_1 - \ell_2, \hbar r) \not\prec \mathcal{G}(\lambda) \quad \text{for any } \hbar > 2. \tag{3}$$

Then, for every $\omega \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$ and $\gamma > 0$, consider the sets

$$\begin{aligned} \mathcal{B}_1 &= \left\{s \in \mathbb{N} : \Psi\left(\zeta_s - \ell_1, r + \frac{\gamma}{2}\right) \not\prec \mathcal{G}(\omega)\right\}, \\ \mathcal{B}_2 &= \left\{s \in \mathbb{N} : \Psi\left(\zeta_s - \ell_2, r + \frac{\gamma}{2}\right) \not\prec \mathcal{G}(\omega)\right\}. \end{aligned}$$

Since $\ell_1, \ell_2 \in \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$, it follows that $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{I}$, and hence their complements intersect in a non-empty set: $\mathcal{B}_1^c \cap \mathcal{B}_2^c \in \mathcal{F}(\mathcal{I})$, so, $\mathcal{B}_1^c \cap \mathcal{B}_2^c \neq \emptyset$. For any $s \in \mathcal{B}_1^c \cap \mathcal{B}_2^c$, we then have

$$\begin{aligned} \Psi(\ell_1 - \ell_2, 2r + \gamma) &\geq \mathcal{T}\left(\Psi\left(\zeta_s - \ell_1, r + \frac{\gamma}{2}\right), \Psi\left(\zeta_s - \ell_2, r + \frac{\gamma}{2}\right)\right) \\ &> \mathcal{T}(\mathcal{G}(\omega), \mathcal{G}(\omega)) \\ &> \mathcal{G}(\lambda), \end{aligned}$$

which implies

$$\Psi(\ell_1 - \ell_2, 2r + \gamma) > \mathcal{G}(\lambda). \tag{4}$$

Now, choosing $\gamma = \hbar r - 2r$ for any $\hbar > 2$, from (4) we obtain

$$\Psi(\ell_1 - \ell_2, \hbar r) > \mathcal{G}(\lambda)$$

which contradicts (3). Hence, no such pair $\ell_1, \ell_2 \in \mathcal{I}_{\mathcal{L}} - LIM_{\zeta_s}^r$ can exist for which $\Psi(\ell_1 - \ell_2, \hbar r) \not\prec \mathcal{G}(\lambda)$ holds for all $\hbar > 2$. \square

We now introduce the notions of \mathcal{I} -cluster points and rough \mathcal{I} -cluster points for sequences in an \mathcal{L} -fuzzy normed space.

Definition 2.12. Let $\{\zeta_s\}$ be a sequence in an \mathcal{L} -fuzzy normed space \mathcal{Q} and $\ell \in \mathcal{Q}$. Then, ℓ is referred to as an \mathcal{I} -cluster point of the sequence $\{\zeta_s\}$ with respect to Ψ , if for every $\omega \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$ and $\gamma > 0$ the set $\{s \in \mathbb{N} : \Psi(\zeta_s - \ell, \gamma) > \mathcal{G}(\omega)\} \notin \mathcal{I}$.

The collection of all \mathcal{I} -cluster points of the sequence $\{\zeta_s\}$ is denoted by $\Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)$.

Definition 2.13. Let $\{\zeta_s\}$ be a sequence in an \mathcal{L} -fuzzy normed space \mathcal{Q} and $\ell \in \mathcal{Q}$. Then, ℓ is referred to as rough \mathcal{I} -cluster point of the sequence $\{\zeta_s\}$ with respect to Ψ for some non-negative number r , if for every $\omega \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$ and $\gamma > 0$ the set $\{s \in \mathbb{N} : \Psi(\zeta_s - \ell, r + \gamma) > \mathcal{G}(\omega)\} \notin \mathcal{I}$.

The collection of all rough \mathcal{I} -cluster points of the sequence $\{\zeta_s\}$ is denoted by $\Gamma_{\mathcal{I}}^{r, \mathcal{L}}(\zeta_s)$.

Remark 2.14. It is evident that for $r = 0$, Definition 2.13 reduces to Definition 2.12.

Theorem 2.15. *Let $\{\zeta_s\}$ be a sequence in an \mathcal{L} -fuzzy normed space \mathcal{Q} . Then, the set $\Gamma_{\mathcal{I}}^{r, \mathcal{L}}(\zeta_s)$ is a closed subset of \mathcal{Q} .*

Proof. The proof follows along similar lines as Theorem 2.8, and is therefore omitted for brevity. \square

In the following theorem, we establish a condition under which a point ℓ qualifies as a rough \mathcal{I} -cluster point of a sequence in an \mathcal{L} -fuzzy normed space.

Theorem 2.16. *Let $\{\zeta_s\}$ be a sequence in an \mathcal{L} -fuzzy normed space \mathcal{Q} and let $\ell \in \mathcal{Q}$. If, for every $\omega \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$, the inequality $\Psi(\ell - \varrho, r) > \mathcal{G}(\omega)$ holds for every $\varrho \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)$, then $\ell \in \Gamma_{\mathcal{I}}^{r, \mathcal{L}}(\zeta_s)$.*

Proof. Let $\omega \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$ be arbitrary. Suppose that for every such ω , the following inequality holds:

$$\Psi(\ell - \varrho, r) > \mathcal{G}(\omega), \tag{5}$$

for some $\varrho \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)$. Then, for any $\gamma > 0$, consider the set

$$\mathcal{C} = \{s \in \mathbb{N} : \Psi(\zeta_s - \varrho, \gamma) > \mathcal{G}(\omega)\}. \tag{6}$$

By Definition 2.12, it follows that $\mathcal{C} \notin \mathcal{I}$, and evidently, $\mathcal{C} \neq \emptyset$. Now, for each $s \in \mathcal{C}$, using (5) and (6), we derive:

$$\begin{aligned} \Psi(\zeta_s - \ell, r + \gamma) &\geq \mathcal{T}(\Psi(\zeta_s - \varrho, \gamma), \Psi(\ell - \varrho, r)) \\ &> \mathcal{T}(\mathcal{G}(\omega), \mathcal{G}(\omega)) \\ &> \mathcal{G}(\lambda), \end{aligned}$$

for some $\lambda \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$. Hence,

$$\mathcal{C} \subseteq \{s \in \mathbb{N} : \Psi(\zeta_s - \ell, r + \gamma) > \mathcal{G}(\lambda)\},$$

which implies that the set

$$\{s \in \mathbb{N} : \Psi(\zeta_s - \ell, r + \gamma) > \mathcal{G}(\lambda)\} \notin \mathcal{I},$$

since otherwise, \mathcal{C} would also belong to \mathcal{I} , a contradiction. Therefore, we conclude that $\ell \in \Gamma_{\mathcal{I}}^{r, \mathcal{L}}(\zeta_s)$. \square

We explore the connection between the set $\Gamma_{\mathcal{I}}^{r, \mathcal{L}}(\zeta_s)$ and closed balls in the space \mathcal{Q} . In particular, we demonstrate that $\Gamma_{\mathcal{I}}^{r, \mathcal{L}}(\zeta_s)$ encompasses the union of closed balls centered at each point belonging to $\Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)$.

Theorem 2.17. *Consider a sequence $\{\zeta_s\}$ in the space \mathcal{Q} . Then, for a fixed $\ell \in \mathcal{Q}$ and some $r > 0$, it holds that*

$$\bigcup_{\ell \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)} \overline{\mathcal{B}(\ell, \omega, r)} \subseteq \Gamma_{\mathcal{I}}^{r, \mathcal{L}}(\zeta_s),$$

where $\overline{\mathcal{B}(\ell, \omega, r)}$ denotes the closed ball centered at ℓ with radius r .

Proof. Let $\vartheta \in \bigcup_{\ell \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)} \overline{\mathcal{B}(\ell, \omega, r)}$. Then there exists some $\ell \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)$ such that $\vartheta \in \overline{\mathcal{B}(\ell, \omega, r)}$. This implies that, for any $\omega \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$, we have

$$\Psi(\ell - \vartheta, r) \geq \mathcal{G}(\omega).$$

Since ℓ belongs to $\Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)$, it follows that for every $\gamma > 0$, the set

$$\mathcal{C} = \{s \in \mathbb{N} : \Psi(\zeta_s - \ell, \gamma) > \mathcal{G}(\omega)\} \notin \mathcal{I}.$$

Clearly, \mathcal{C} is nonempty. Then for each $s \in \mathcal{C}$, we obtain the following estimate:

$$\begin{aligned} \Psi(\zeta_s - \vartheta, r + \gamma) &\geq \mathcal{T}(\Psi(\zeta_s - \ell, \gamma), \Psi(\ell - \vartheta, r)) \\ &> \mathcal{T}(\mathcal{G}(\omega), \mathcal{G}(\omega)) \\ &> \mathcal{G}(\lambda). \end{aligned}$$

Consequently,

$$\mathcal{C} \subseteq \{s \in \mathbb{N} : \Psi(\zeta_s - \vartheta, r + \gamma) > \mathcal{G}(\lambda)\}.$$

which implies that the set on the right-hand side does not belong to \mathcal{I} . Therefore,

$$\vartheta \in \Gamma_{\mathcal{I}}^{r\mathcal{L}}(\zeta_s),$$

and hence,

$$\bigcup_{\ell \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)} \overline{\mathcal{B}(\ell, \omega, r)} \subseteq \Gamma_{\mathcal{I}}^{r\mathcal{L}}(\zeta_s).$$

□

The structure of rough \mathcal{I} -limit sets can be elegantly described using closed balls, as demonstrated in the following two theorems.

Theorem 2.18. Consider a sequence $\{\zeta_s\}$ in the space \mathcal{Q} . Then, for all $\omega \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}\}$, the subsequent statements are true:

1. If $\ell \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)$, then $\mathcal{I}_{\mathcal{L}} - \text{LIM}_{\zeta_s}^r \subseteq \overline{\mathcal{B}(\ell, \omega, r)}$.
2. $\mathcal{I}_{\mathcal{L}} - \text{LIM}_{\zeta_s}^r \subseteq \bigcap_{\ell \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)} \overline{\mathcal{B}(\ell, \omega, r)} \subseteq \{\vartheta \in \mathcal{Q} : \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s) \subseteq \overline{\mathcal{B}(\vartheta, \omega, r)}\}$.

Proof. 1. Assume that $\ell \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)$ and let $z \in \mathcal{I}_{\mathcal{L}} - \text{LIM}_{\zeta_s}^r$. For any $\gamma > 0$ and $\lambda \in \mathcal{L} \setminus \{\mathbf{0}_{\mathcal{L}}\}$, consider the sets

$$\begin{aligned} \mathcal{M} &= \{s \in \mathbb{N} : \Psi(\zeta_s - z, r + \gamma) \not> \mathcal{G}(\lambda)\} \\ \mathcal{H} &= \{s \in \mathbb{N} : \Psi(\zeta_s - \ell, \gamma) > \mathcal{G}(\lambda)\}. \end{aligned}$$

By assumption, we have $\mathcal{M} \in \mathcal{I}$ and $\mathcal{H} \notin \mathcal{I}$. Define $\mathcal{M}^c = \mathcal{B}$. Clearly $\mathcal{B} \cap \mathcal{H} \neq \emptyset$.

Suppose, on the contrary, that $\mathcal{B} \cap \mathcal{H} = \emptyset$. Then it follows that $\mathcal{H} \subseteq \mathbb{N} \setminus \mathcal{B} = \mathcal{M}$, which implies $\mathcal{H} \in \mathcal{I}$, contradicting the assumption that $\mathcal{H} \notin \mathcal{I}$. Hence, $\mathcal{B} \cap \mathcal{H} \neq \emptyset$.

Now, for any $s \in \mathcal{B} \cap \mathcal{H}$, we observe that

$$\begin{aligned} \Psi(z - \ell, r) &\geq \mathcal{T}(\Psi(\zeta_s - \ell, \gamma), \Psi(\zeta_s - z, r + \gamma)) \\ &> \mathcal{T}(\mathcal{G}(\lambda), \mathcal{G}(\lambda)) \\ &> \mathcal{G}(\omega). \end{aligned}$$

Therefore, $z \in \mathcal{B}(\ell, \omega, r) \subseteq \overline{\mathcal{B}(\ell, \omega, r)}$, and consequently,

$$\mathcal{I}_{\mathcal{L}} - \text{LIM}_{\zeta_s}^r \subseteq \overline{\mathcal{B}(\ell, \omega, r)}.$$

2. From 1, it follows that

$$\mathcal{I}_{\mathcal{L}} - \text{LIM}_{\zeta_s}^r \subseteq \bigcap_{\ell \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)} \overline{\mathcal{B}(\ell, \omega, r)}. \tag{7}$$

Now, let $\vartheta \in \bigcap_{\ell \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)} \overline{\mathcal{B}(\ell, \omega, r)}$. Then, for every $\ell \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)$, we have

$$\begin{aligned} \Psi(\vartheta - \ell, r) &\geq \mathcal{G}(\omega) \\ \implies \Psi(\ell - \vartheta, r) &\geq \mathcal{G}(\omega). \end{aligned}$$

This implies that $\Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s) \subseteq \overline{\mathcal{B}(\vartheta, \omega, r)}$, and hence,

$$\bigcap_{\ell \in \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s)} \overline{\mathcal{B}(\ell, \omega, r)} \subseteq \{\vartheta \in \mathcal{Q} : \Gamma_{\mathcal{I}}^{\mathcal{L}}(\zeta_s) \subseteq \overline{\mathcal{B}(\vartheta, \omega, r)}\}. \tag{8}$$

Combining (7) and (8), we arrive at the desired conclusion.

□

Theorem 2.19. Consider a sequence $\{\zeta_s\}$ in the space \mathcal{Q} such that $\zeta_s \xrightarrow{\mathcal{I}_L} \ell$. Then, for every $\omega \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$,

$$\mathcal{I}_L - LIM_{\zeta_s}^r = \overline{\mathcal{B}(\ell, \omega, r)}.$$

Proof. Assume that $\zeta_s \xrightarrow{\mathcal{I}_L} \ell$. Then, for every $\omega \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$ and any $\gamma > 0$, the set

$$\mathcal{C} = \{s \in \mathbb{N} : \Psi(\zeta_s - \ell, \gamma) \not\prec \mathcal{G}(\omega)\} \in \mathcal{I}.$$

Now, let $\vartheta \in \overline{\mathcal{B}(\ell, \omega, r)}$, i.e.,

$$\Psi(\vartheta - \ell, r) \geq \mathcal{G}(\omega).$$

Since $\mathcal{C}^c \in \mathcal{F}(\mathcal{I})$, we have $\mathcal{C}^c \neq \emptyset$. For every $s \in \mathcal{C}^c$, it follows that

$$\begin{aligned} \Psi(\zeta_s - \vartheta, r + \gamma) &\geq \mathcal{T}(\Psi(\zeta_s - \ell, \gamma), \Psi(\vartheta - \ell, r)) \\ &> \mathcal{T}(\mathcal{G}(\omega), \mathcal{G}(\omega)) \\ &> \mathcal{G}(\lambda). \end{aligned}$$

This implies that

$$\{s \in \mathbb{N} : \Psi(\zeta_s - \vartheta, r + \gamma) \not\prec \mathcal{G}(\lambda)\} \subseteq \mathcal{C} \in \mathcal{I}.$$

Hence, $\vartheta \in \mathcal{I}_L - LIM_{\zeta_s}^r$, and therefore,

$$\overline{\mathcal{B}(\ell, \omega, r)} \subseteq \mathcal{I}_L - LIM_{\zeta_s}^r.$$

Moreover, since $\zeta_s \xrightarrow{\mathcal{I}_L} \ell$, it follows that $\Gamma_{\mathcal{I}_L}^{\mathcal{L}}(\zeta_s) = \{\ell\}$. Then, by item 1 of Theorem 2.18, we have

$$\mathcal{I}_L - LIM_{\zeta_s}^r \subseteq \overline{\mathcal{B}(\ell, \omega, r)}.$$

Combining both inclusions yields

$$\mathcal{I}_L - LIM_{\zeta_s}^r = \overline{\mathcal{B}(\ell, \omega, r)}$$

which completes the proof. \square

3. Conclusion and future scope

In this article, we introduced the concept of rough ideal convergence for sequences in an \mathcal{L} -fuzzy normed space \mathcal{Q} . We defined the notion of \mathcal{I} -boundedness and investigated its relationship with the rough \mathcal{I} -limit set (\mathcal{I}_L^r) of a sequence, exploring various topological and geometric properties of this set. Additionally, we presented several algebraic characterizations of rough \mathcal{I} -convergent sequences with respect to an \mathcal{L} -fuzzy norm Ψ . A sufficient condition has been established under which a sequence $\{\zeta_s\}$ is \mathcal{I}_L^r -convergent, based on the existence of an \mathcal{I}_L -convergent sequence $\{\sigma_s\}$. Furthermore, we introduced the concept of a rough \mathcal{I} -cluster point and derived key results involving closed balls and their connection to the rough \mathcal{I} -limit set in the setting of \mathcal{L} -fuzzy normed spaces.

\mathcal{L} -fuzzy normed spaces, natural generalizations of normed spaces, fuzzy normed spaces and intuitionistic fuzzy normed spaces based on some logical algebraic structures, which also enriches the notion of a \mathcal{L} -fuzzy metric space. The introduction of \mathcal{L} -fuzzy sets by Goguen [19], in which membership degrees are taken from a complete lattice \mathcal{L} , has provided flexibility and generality. In \mathcal{L} -fuzzy normed spaces, a complete lattice \mathcal{L} is considered in place of the unit interval $[0, 1]$, resulting in a profound generalization of classical fuzzy normed spaces. This generalization offers a more flexible and comprehensive framework for the investigation of problems in which uncertainty and ordered structures coexist. It has wide-ranging applications in fuzzy functional analysis and summability theory, including, for instance, fixed point theory, the stability of functional equations, cubic functional equations in non-Archimedean \mathcal{L} -fuzzy normed

spaces, and fuzzy topological spaces induced by implicators, among others. Owing to its broad applicability, the present work can be effectively employed in fuzzy functional analysis and summability theory, particularly in the study of convergence of sequences. Furthermore, several promising directions remain open for future investigation. One possible extension is to explore rough convergence by employing tools such as λ -density and θ -density, thereby generalizing statistical convergence to double and triple sequences within the framework of the \mathcal{L} -fuzzy norm Ψ .

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