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# A study of Alexandroff compactification within the framework of near soft topological spaces

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**Abstract.** Near sets and near soft sets are powerful mathematical tools for quantifying the proximity of object collections and for constructing innovative topological frameworks. Compactification is a fundamental concept in topology, providing a way to embed non-compact spaces into compact ones. In this study, we introduce and investigate the *Alexandroff near soft compactification*. By refining the definition of near soft sets to require a non-empty boundary condition, we establish a mathematically consistent framework for near soft topological spaces. The main results of this research are as follows:

1. We demonstrate the conditions under which a near soft topological space can be embedded into a compact structure (Theorem 4.12).
2. We prove that a near soft topological space $({F_A}^n, A, \tau^n)$ is a near soft Hausdorff compactification of $(F_A, A, \tau)$ if and only if $(F_A, A, \tau)$ is both near soft locally compact and near soft Hausdorff (Theorem 4.13).

The proposed framework extends classical soft topological compactification by incorporating the notion of perceptual nearness.

## 1. Introduction

The study of uncertainty, vagueness, and approximation in mathematical structures has attracted increasing attention over recent decades, particularly through the development of generalized set-theoretic and topological frameworks. Classical topology, which is based on crisp membership relations and precise structural assumptions, often proves inadequate for modeling systems involving parameter dependence and incomplete information. This limitation has motivated the introduction of alternative mathematical tools designed to handle uncertainty more effectively.

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Among these tools, soft set theory, introduced by Molodtsov [10], has emerged as a flexible and powerful framework that avoids the intrinsic restrictions of fuzzy sets, rough sets, and intuitionistic fuzzy sets. By parameterizing subsets of a universe, soft sets offer a natural mechanism for modeling uncertainty in a wide range of applications.

The incorporation of soft sets into topology has led to the development of soft topological spaces, within which many classical topological notions have been revisited and generalized. In recent years, several influential contributions have significantly enriched this field. In particular, Al-shami *et al.* investigated compactness, Lindelöfness, and connectedness via the class of soft somewhat open sets, revealing structural behaviors that differ substantially from those obtained using ordinary soft open sets [6]. Moreover, operator-based approaches to soft separation axioms were introduced in [5], where new soft separation structures, such as soft $T_D$-spaces, were characterized in terms of soft topological operators. Further refinements of separation theory in soft topological spaces were proposed through ordered soft separation axioms [4], establishing connections between order structures, compactness, and product spaces. Additionally, separation axioms defined via soft semiopen sets and fixed soft points were investigated in [3], highlighting the deep interplay between separation properties and compactness-type notions in soft topology. These developments underscore the importance of exploring compactification within generalized soft frameworks.

In parallel with these developments, near set theory, originally formulated by Peters and later advanced by Naimpally and others [11, 16–18], focuses on the notion of descriptive nearness between objects based on shared features. Near sets have been successfully applied in areas such as image analysis, pattern recognition, and information systems, where similarity-based reasoning plays a central role.

The integration of soft set theory and near set theory gave rise to the concept of near soft sets (NS-sets), first introduced by Taşbozan *et al.* [19]. NS-sets provide a unified framework in which both parameter uncertainty and descriptive proximity can be treated simultaneously. Building on this foundation, Ozkan [12] introduced fundamental topological notions—such as near soft interior, near soft closure, and near soft continuity—thereby establishing the framework of near soft topological spaces (NSTSs). Subsequent studies examined near soft connectedness [13], separation axioms, and compactness-related properties within NSTSs [8, 14, 21]. More recent investigations have further extended this theory to include advanced structures such as near soft expert sets and hypersoft sets [22, 23].

Compactification is a cornerstone of general topology, providing a systematic procedure for embedding non-compact spaces into compact ones while preserving essential topological properties. The classical theory of compactification traces back to the pioneering works of Tietze on one-point compactification [24–26] and to the general constructions introduced by Alexandroff and Urysohn [1, 2]. While compactification in soft topological settings has been explored, the introduction of descriptive nearness through NSTSs necessitates a specialized approach.

Motivated by the growing interest in near soft structures, this study aims to extend the concept of compactification to NSTSs. We introduce the Alexandroff near soft compactification and investigate its fundamental properties. In doing so, we refine the existing definitions to ensure a consistent distinction between near and exact soft structures, thereby addressing previous notation inconsistencies in the literature. This approach not only generalizes the classical Alexandroff compactification but also complements existing results in soft topology by incorporating proximity-based reasoning.

The main contributions of this paper are summarized as follows:

1. We establish the formal conditions under which a near soft topological space admits a near soft compactification.

2. We derive necessary and sufficient criteria for the existence of near soft Hausdorff compactifications.

3. We construct the Alexandroff near soft compactification of a non-compact NSTS and analyze its topological integrity.

## 2. Preliminaries

In this section, we provide the fundamental definitions and characteristics of rough sets, near sets, and near soft sets. These concepts form the essential background for developing the near soft topological framework and the compactification results presented in the subsequent sections.

### *2.1. Rough sets*

Rough set theory, introduced by Pawlak [15], provides a formal mathematical framework for handling vagueness and incompleteness through approximations based on indiscernibility relations. Let $U$ be a finite universe and $R$ be an equivalence relation on $U$, representing indiscernibility between elements. For each $p \in U$, the equivalence class of $p$ with respect to $R$ is denoted by $[p]_R$, and the pair $(U, R)$ is called an *approximation space*.

Given any subset $X \subseteq U$, the *lower* and *upper approximations* of $X$ with respect to $R$ are defined as:

$$R_*(X) = \{p \in U : [p]_R \subseteq X\},$$

$$R^*(X) = \{p \in U : [p]_R \cap X \neq \emptyset\}.$$

A set $X$ is said to be *definable* (or exact) if $R_*(X) = R^*(X)$. Otherwise, $X$ is called a **rough set**. The uncertainty associated with a rough set is characterized by its *boundary region*, defined as:

$$\mathrm{Bnd}_R(X) = R^*(X) \setminus R_*(X).$$

In the case of a rough set, this boundary region is non-empty, i.e., $\mathrm{Bnd}_R(X) \neq \emptyset$.

### *2.2. Near sets*

In this subsection, we present the fundamental definitions and properties of near sets, following the framework introduced by Peters [16, 17]. Near set theory focuses on the descriptive proximity of objects based on their features.

Table 1: Symbols and Descriptions in Near Set Theory

| **Symbol** | **Interpretation** |
|---|---|
| $\mathcal{O}$ | Set of perceptual objects (the universe) |
| $X$ | $X \subseteq \mathcal{O}$, a set of sample objects |
| $p$ | $p \in \mathcal{O}$, a sample object |
| $\mathcal{F}$ | A set of functions representing object features (probe functions) |
| $B$ | $B \subseteq \mathcal{F}$, a subset of feature functions |
| $\Phi$ | $\Phi : \mathcal{O} \to \mathbb{R}^L$, the object description function |
| $L$ | Description length ($L = \|B\|$) |
| $i$ | Index of a feature function, $1 \leq i \leq L$ |
| $\Phi(p)$ | $\Phi(p) = (\phi_1(p), \phi_2(p), \ldots, \phi_L(p))$, the feature vector of object $p$ |

The ability to characterize an object is fundamentally tied to the extent of available information, typically derived through a collection of feature functions (probe functions). For an object $p \in \mathcal{O}$, its formal representation is given by the feature vector:

$$\Phi(p) = \left(\phi_1(p), \phi_2(p), \ldots, \phi_L(p)\right),$$

where each $\phi_i \in B$ maps the object to a real value reflecting a specific property. These functions constitute the *descriptive basis* of the object.

Let $\sim_B$ be the indiscernibility relation on $\mathcal{O}$ defined by:

$$\sim_B = \{(p, p') \in \mathcal{O} \times \mathcal{O} : \forall \phi_i \in B, \Delta_{\phi_i} = 0\},$$

where $\Delta_{\phi_i} = |\phi_i(p) - \phi_i(p')|$ is the difference between probe function values. The equivalence class of an object $p \in O$ is denoted by:

$$[p]_B = \{p' \in O : p \sim_B p'\}.$$

The family of all such classes forms the quotient set $O/\sim_B$, which defines a partition $\xi_B$ on $O$.

**Definition 2.1.** ([9]) Let $p, p' \in O$ and $B \subseteq \mathcal{F}$. The objects $p$ and $p'$ are said to be **minimally near** each other if there exists at least one $\phi_i \in B$ such that $\Delta_{\phi_i} = 0$. This is referred to as the *Nearness Description Principle (NDP)*. Essentially, objects are considered near if they belong to the same descriptive class $[p]_B \in \xi_B$.

**Definition 2.2.** ([9]) Let $p, p' \in O$ and $B \subseteq \mathcal{F}$. If $p$ is near to $p'$, then $X \subseteq O$ is said to be a **near set** relative to itself (reflexive nearness).

### 2.3. Nearness approximation space (NAS): Definitions and notation

A Nearness Approximation Space (NAS) is a mathematical model that formalizes perceptual similarity among objects using feature-based equivalence relations. It is defined as a quintuple:

$$\text{NAS} = (O, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r}).$$

Table 2: Symbols and Descriptions in NAS [17]

| **Symbol** | **Interpretation** |
|---|---|
| $B \subseteq \mathcal{F}$ | A subset of feature functions (probe functions) selected from $\mathcal{F}$ |
| $B_r \subseteq B$ | A subfamily of $r$ features used to define localized nearness structures |
| $\sim_{B_r}$ | Indiscernibility relation defined by $B_r$: $p \sim_{B_r} q \iff \forall \phi \in B_r, \phi(p) = \phi(q)$ |
| $[p]_{B_r}$ | Equivalence class of $p$ under $\sim_{B_r}$ (objects indistinguishable from $p$) |
| $\xi_{O,B_r}$ | Partition of $O$ induced by $B_r$, i.e., $\xi_{O,B_r} = O/\sim_{B_r}$ |
| $r$ | The number of probe functions taken at a time, $r \leq |B|$ |
| $N_r(B)$ | Family of all partitions $\xi_{O,B_r}$ where $|B_r| = r$; the nearness structure |
| $\nu_{N_r}$ | Overlap function $\nu_{N_r} : \mathcal{P}(O) \times \mathcal{P}(O) \to [0,1]$ |
| $N_{r*}(X)$ | Lower near approximation: $N_{r*}(X) = \bigcup\{[p]_{B_r} \in O/\sim_{B_r} : [p]_{B_r} \subseteq X\}$ |
| $N_r^*(X)$ | Upper near approximation: $N_r^*(X) = \bigcup\{[p]_{B_r} \in O/\sim_{B_r} : [p]_{B_r} \cap X \neq \emptyset\}$ |
| $\text{Bnd}_{N_r(B)}(X)$ | Boundary region: $\text{Bnd}_{N_r(B)}(X) = N_r^*(X) \setminus N_{r*}(X)$ |

### 2.4. Near soft sets

In this subsection, we develop the concept of a *near soft set* by integrating soft set theory with nearness structures. We also introduce the framework for *near soft topology* and examine its fundamental properties. Inan and Ozturk [9] proposed a framework that merges soft set theory with near set theory, leading to the development of the *soft nearness approximation space*. Following this approach, we investigate the lower and upper approximations of a soft set within a nearness approximation structure.

Throughout this section, let $O$ be the initial universe of discourse and $B \subseteq \mathcal{F}$ be a set of parameters, where $\mathcal{F}$ is the set of probe functions.

**Definition 2.3.** ([10]) A **soft set** over $O$ is defined as an ordered pair $(F, B)$, often denoted by $F_B$, where $F$ is a mapping given by:

$$F : B \to \mathcal{P}(O).$$

Here, $\mathcal{P}(O)$ denotes the power set of $O$. For each parameter $\phi \in B$, the subset $F(\phi) \subseteq O$ represents the collection of elements in $O$ related to the parameter $\phi$.

**Proposition 2.4.** ([19]) Any collection of nearness-based neighbourhoods can naturally be represented within the framework of a soft set.

In the literature, the initial conceptualization of near soft sets was introduced by Taşbozan et al. [19], where the boundary condition was expressed with the notation $\mathrm{Bnd}_{N_r(B)}(F_B) \geq 0$. However, from a strictly set-theoretic and topological perspective, this notation presents two formal challenges: first, a boundary is a set-valued object and cannot be directly compared to a numerical value; and second, the condition $\geq 0$ (even if interpreted as cardinality) does not explicitly exclude the case where the boundary is empty.

In topological structures, the essence of "nearness" or "roughness" lies in the existence of a non-empty boundary region that represents uncertainty. If the boundary of a soft set is empty, the lower and upper approximations coincide, making the set an "exact" soft set. Therefore, to establish a mathematically consistent framework for near soft topological spaces and to distinguish near soft sets from exact ones, we refine the definition provided in [19] by requiring a non-empty boundary condition. Consequently, we define near soft sets and their approximations as follows:

**Definition 2.5.** Let NAS $= (\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and $F_B$ be a soft set over $\mathcal{O}$. The **lower** and **upper near approximations** of $F_B$ with respect to NAS are soft sets $N_{r*}(F_B)$ and $N_r^*(F_B)$ defined by the following mappings:

$$(N_{r*}F)(\phi) = \bigcup \{[p]_{B_r} \in \mathcal{O}/\sim_{B_r} : [p]_{B_r} \subseteq F(\phi)\},$$

$$(N_r^*F)(\phi) = \bigcup \{[p]_{B_r} \in \mathcal{O}/\sim_{B_r} : [p]_{B_r} \cap F(\phi) \neq \emptyset\},$$

for all $\phi \in B$.

The soft set $F_B$ is called a **near soft set** if its near boundary region is non-empty, i.e.,

$$\mathrm{Bnd}_{N_r(B)}(F_B) = N_r^*(F_B) \setminus N_{r*}(F_B) \neq \Phi,$$

where $\Phi$ denotes the null soft set. Otherwise, if $\mathrm{Bnd}_{N_r(B)}(F_B) = \Phi$, then $F_B$ is called an **exact soft set**.

**N.B. 2.6. (How near soft sets differ from soft sets).**
*Near soft sets differ from traditional soft sets in the following ways:*

1º *[**Parameterized family**] Let $E$ be a set of parameters , $A \subset E$, and let $P(U)$ be the set of all subsets of a nonempty set $U$, then a soft set $(F, A)$ is defined by*

$$\boxed{F : A \to P(U) : X = \{F(e) : e \in A\} \text{ is soft } \iff \exists e \in A : F(e) \in X.}$$

*By contrast, a near soft set $\boxed{F_B}$ is an advance over traditional soft sets in the sense that nearness is measured in terms of whether or not the boundary set $\boxed{Bnd_{N_r(B)}(F_B)}$ is empty in the partition of $X$ into upper approximation $\boxed{N_r^*(F_B)}$ and lower approximation $\boxed{N_{r*}(F_B)}$. For a soft set to be a near soft set, we require that $\boxed{Bnd_{N_r(B)}(F_B) \neq \emptyset}$.*

2º *[**from item no.** 1], a set $X = \{F(e) : e \in A\}$ is soft, provided $\boxed{F(e)}$ nonempty. By contrast, from observation 1, $X$ is a near soft set $\boxed{F_B}$, provided $\boxed{Bnd_{N_r(B)}(F_B) \neq \emptyset}$.*

3º *[**from item no.** 2]. A soft set $\boxed{X = \{F(e) : e \in A\} \neq \emptyset}$ differs from a near soft set $\boxed{F_B}$, since a near soft set is defined in terms of three distinct, related sets, namely,*

$$\boxed{Bnd_{N_r(B)}(F_B), N_r^*(F_B), N_{r*}(F_B).}$$

4º *[**from item no.** 3]. A near soft set $\boxed{F_B}$ is a richer structure than a traditional soft set $\boxed{X = \{F(e) : e \in A\} \neq \emptyset}$ in the sense that $\boxed{F_B}$ is identified via a measurement of the emptiness or non-emptiness of a boundary set relative to the upper and lower approxation sets.*

**Remark 2.7.** To compute the lower and upper approximations of a soft set $F_B$ with respect to a fixed subset $B_r \subseteq \mathcal{F}$, we proceed for each $\phi \in B$ as follows:

1. For each object $p \in \mathcal{O}$, determine the equivalence class $[p]_{B_r}$ based on the probe functions in $B_r$.
2. If $[p]_{B_r} \subseteq F(\phi)$, then $p$ belongs to the lower near approximation $(N_{r*}F)(\phi)$.
3. If $[p]_{B_r} \cap F(\phi) \neq \emptyset$, then $p$ belongs to the upper near approximation $(N_r^*F)(\phi)$.

**Example 2.8.** Let $\mathcal{O} = \{p_1, p_2, p_3, p_4, p_5\}$ be a universe of perceptible objects, and let $B = \{\phi_1, \phi_2\} \subset \mathcal{F} = \{\phi_1, \phi_2, \phi_3\}$ be the parameter set. The evaluations of the probe functions $\phi_i$ for each object $p_j$ are given in Table 3.

Table 3: Values of parameter functions $\phi_i$ over objects $p_j$

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $\phi_1$ | 0 | 1 | 3 | 3 | 1 |
| $\phi_2$ | 3 | 0 | 1 | 1 | 2 |
| $\phi_3$ | 2 | 0 | 2 | 0 | 1 |

Consider the soft set $F_B$ defined as:

$$F_B = \{(\phi_1, \{p_1, p_3\}), (\phi_2, \{p_1, p_2, p_3\})\}.$$

*Case 1:* $r = 1$. The nearness classes $[p]_{\phi_i}$ are determined as follows:

$$[p_1]_{\phi_1} = \{p_1\}, \quad [p_2]_{\phi_1} = \{p_2, p_5\}, \quad [p_3]_{\phi_1} = \{p_3, p_4\},$$
$$[p_1]_{\phi_2} = \{p_1\}, \quad [p_2]_{\phi_2} = \{p_2\}, \quad [p_3]_{\phi_2} = \{p_3, p_4\}, \quad [p_5]_{\phi_2} = \{p_5\}.$$

The lower and upper near approximations are:

$$N_*(F_B) = \{(\phi_1, \{p_1\}), (\phi_2, \{p_1, p_2\})\},$$

$$N^*(F_B) = \{(\phi_1, \{p_1, p_3, p_4\}), (\phi_2, \{p_1, p_2, p_3, p_4\})\}.$$

The near boundary region is:

$$\text{Bnd}_{N_1(B)}(F_B) = N^*(F_B) \setminus N_*(F_B) = \{(\phi_1, \{p_3, p_4\}), (\phi_2, \{p_3, p_4\})\}.$$

Since $\text{Bnd}_{N_1(B)}(F_B) \neq \Phi$, $F_B$ is a **near soft set** for $r = 1$.

*Case 2:* $r = 2$. For $B_r = \{\phi_1, \phi_2\}$, the nearness classes are:

$$[p_1]_{\phi_1,\phi_2} = \{p_1\}, \quad [p_2]_{\phi_1,\phi_2} = \{p_2\}, \quad [p_3]_{\phi_1,\phi_2} = \{p_3, p_4\}, \quad [p_5]_{\phi_1,\phi_2} = \{p_5\}.$$

The approximations are calculated as:

$$N_*(F_B) = \{(\phi_1, \{p_1\}), (\phi_2, \{p_1, p_2\})\},$$

$$N^*(F_B) = \{(\phi_1, \{p_1, p_3, p_4\}), (\phi_2, \{p_1, p_2, p_3, p_4\})\}.$$

The near boundary is:

$$\text{Bnd}_{N_2(B)}(F_B) = \{(\phi_1, \{p_3, p_4\}), (\phi_2, \{p_3, p_4\})\}.$$

Since $\text{Bnd}_{N_2(B)}(F_B) \neq \Phi$, $F_B$ is also a **near soft set** for $r = 2$.

**Definition 2.9.** ([19]) Let $F_B \in$ NSS($\mathcal{O}$) be a near soft set over the universe $\mathcal{O}$. Then we have the following special cases:

(i) $F_B$ is called a **null near soft set**, denoted by $\Phi$, if $F(\phi) = \emptyset$ for all $\phi \in B$.

(ii) $F_B$ is called a $B$**-universal near soft set**, denoted by $\mathcal{O}_B$ (or $F_{\tilde{B}}$), if $F(\phi) = \mathcal{O}$ for every $\phi \in B$.

If $B = \mathcal{F}$, then the $B$-universal near soft set is referred to as the **universal near soft set**, denoted by $\mathcal{O}_{\mathcal{F}}$.

**Definition 2.10.** [19] Let $F_B, G_B \in$ NSS($\mathcal{O}$) be two near soft sets over the same universe $\mathcal{O}$ and parameter set $B$. Then:

(i) $F_B$ is called a **near soft subset** of $G_B$, denoted by $F_B \sqsubseteq G_B$, if the lower near approximation of $F_B$ is contained in the lower near approximation of $G_B$ for each parameter, i.e.,

$$(N_{r*}F)(\phi) \subseteq (N_{r*}G)(\phi) \quad \text{for all } \phi \in B.$$

(ii) $F_B$ is called a **near soft superset** of $G_B$, denoted by $F_B \sqsupseteq G_B$, if $G_B \sqsubseteq F_B$.

**Definition 2.11.** ([12]) Two near soft sets $F_B$ and $G_B$ are said to be **equal**, denoted by $F_B = G_B$, if and only if $F_B \sqsubseteq G_B$ and $G_B \sqsubseteq F_B$.

**Definition 2.12.** ([19]) Let $F_A, G_B \in$ NSS($\mathcal{O}$) be two near soft sets over $\mathcal{O}$ with parameter sets $A$ and $B$, respectively. Let $C = A \cup B$. Then, the **intersection**, **union**, and **complement** are defined as follows:

(i) The **intersection** of $F_A$ and $G_B$, denoted by $F_A \sqcap G_B = H_C$, is defined for each $\phi \in C$ as:

$$H(\phi) = \begin{cases} F(\phi), & \text{if } \phi \in A \setminus B \\ G(\phi), & \text{if } \phi \in B \setminus A \\ F(\phi) \cap G(\phi), & \text{if } \phi \in A \cap B \end{cases}$$

(ii) The **union** of $F_A$ and $G_B$, denoted by $F_A \sqcup G_B = H_C$, is defined for each $\phi \in C$ as:

$$H(\phi) = \begin{cases} F(\phi), & \text{if } \phi \in A \setminus B \\ G(\phi), & \text{if } \phi \in B \setminus A \\ F(\phi) \cup G(\phi), & \text{if } \phi \in A \cap B \end{cases}$$

(iii) The **complement** of $F_B$, denoted by $F_B^c$, is defined for all $\phi \in B$ by:

$$F^c(\phi) = \mathcal{O} \setminus F(\phi).$$

It holds that $(F_B^c)^c = F_B$, and $\Phi^c = \mathcal{O}_{\mathcal{F}}$.

**Remark 2.13.** The inclusion relation introduced in Definition 2.10 for near soft sets ($\sqsubseteq$) is specifically designed to accommodate the approximation structure, whereas the union and intersection operations follow the standard algebraic rules of soft sets. This distinction is crucial: while union and intersection operate on the raw mappings $F(\phi)$, the inclusion relation is defined via the lower near approximation $N_{r*}$. Since the boundary region $\text{Bnd}_{N_r(B)}(F_B)$ represents the inherent uncertainty of a near soft set, the proximity-based inclusion provides a more robust topological framework than classical point-wise inclusion. This refinement ensures that the near soft topological properties are preserved under the indiscernibility relation of the NAS.

**Definition 2.14.** ([7]) Let $F_B, G_B \in$ NSS($\mathcal{O}$) be two near soft sets over the same parameter set $B$. The **near soft difference** of $F_B$ and $G_B$, denoted by $F_B \setminus G_B = H_B$, is defined for all $\phi \in B$ as:

$$H(\phi) = F(\phi) \setminus G(\phi).$$

**Definition 2.15.** ([12]) A near soft set $P_B \in \text{NSS}(O)$ is called a **near soft point** (or **NS-point**) if there exists a unique parameter $\phi \in B$ and a unique element $x \in O$ such that $P(\phi) = \{x\}$, and for every $\phi' \in B \setminus \{\phi\}$, $P(\phi') = \emptyset$. We denote such an NS-point by $\eta_x^{\phi}$.

**Definition 2.16.** ([12]) Let $F_B \in \text{NSS}(O)$ and $\eta_x^{\phi}$ be an NS-point. The **membership relation** between an NS-point and a near soft set is defined as follows:

(i) $\eta_x^{\phi}$ is said to **belong to** $F_B$, denoted by $\eta_x^{\phi} \in F_B$, if $x \in F(\phi)$.

(ii) If $x \notin F(\phi)$, then $\eta_x^{\phi}$ does not belong to $F_B$, denoted by $\eta_x^{\phi} \notin F_B$.

**Example 2.17.** Referring to the values provided in Table 3 of Example 2.8, consider the soft set $P_B = \{(\phi_1, \{p_3\})\}$. Since $P(\phi_1) = \{p_3\}$ and $P(\phi_2) = \emptyset$ for $B = \{\phi_1, \phi_2\}$, $P_B$ satisfies the condition of being an NS-point. Following the notation introduced above, this point is denoted by $\eta_{p_3}^{\phi_1}$.

Now, consider $F_B = \{(\phi_1, \{p_1, p_3\}), (\phi_2, \{p_1, p_2, p_3\})\}$. Since $p_3 \in F(\phi_1)$, we conclude that $\eta_{p_3}^{\phi_1} \in F_B$. However, for an object such as $p_5$, the NS-point $\eta_{p_5}^{\phi_1}$ does not belong to $F_B$ because $p_5 \notin F(\phi_1)$.

**Definition 2.18.** Let $F_B \in \text{NSS}(O)$ and $\eta_x^{\phi}$ be an NS-point. The union $F_B \sqcup \eta_x^{\phi}$ is a soft set over $O$ defined by the mapping:

$$(F_B \sqcup \eta_x^{\phi})(\psi) = F(\psi) \cup \eta_x^{\phi}(\psi) \quad \text{for all } \psi \in B.$$

If the near boundary of this union is non-empty, i.e., $\text{Bnd}_{N_r(B)}(F_B \sqcup \eta_x^{\phi}) \neq \Phi$, then the resulting union is specifically a **near soft set**.

**Example 2.19.** Referring to Example 2.8 and Example 2.17, let $G_B = \{(\phi_2, \{p_1\})\}$ be a near soft set and $\eta_{p_3}^{\phi_1} = \{(\phi_1, \{p_3\})\}$ be an NS-point. Their union $H_B = G_B \sqcup \eta_{p_3}^{\phi_1}$ is given by:

$$H_B = \{(\phi_1, \{p_3\}), (\phi_2, \{p_1\})\}.$$

To verify the near soft structure of $H_B$ for $r = 1$, we use the nearness classes $[p_3]_{\phi_1} = \{p_3, p_4\}$ and $[p_1]_{\phi_2} = \{p_1\}$:

1. **Lower Near Approximation ($N_{r*}$):**
   - For $\phi_1$: Since $[p_3]_{\phi_1} = \{p_3, p_4\} \nsubseteq H(\phi_1) = \{p_3\}$, then $(N_{r*}H)(\phi_1) = \emptyset$.
   - For $\phi_2$: Since $[p_1]_{\phi_2} = \{p_1\} \subseteq H(\phi_2) = \{p_1\}$, then $(N_{r*}H)(\phi_2) = \{p_1\}$.

   Thus, $N_{r*}(H_B) = \{(\phi_1, \emptyset), (\phi_2, \{p_1\})\}$.

2. **Upper Near Approximation ($N_r^*$):**
   - For $\phi_1$: Since $[p_3]_{\phi_1} \cap \{p_3\} \neq \emptyset$, then $(N_r^*H)(\phi_1) = \{p_3, p_4\}$.
   - For $\phi_2$: Since $[p_1]_{\phi_2} \cap \{p_1\} \neq \emptyset$, then $(N_r^*H)(\phi_2) = \{p_1\}$.

   Thus, $N_r^*(H_B) = \{(\phi_1, \{p_3, p_4\}), (\phi_2, \{p_1\})\}$.

**Boundary analysis:** The near boundary region is:

$$\text{Bnd}_{N_r(B)}(H_B) = N_r^*(H_B) \setminus N_{r*}(H_B) = \{(\phi_1, \{p_3, p_4\}), (\phi_2, \emptyset)\}.$$

Since $\text{Bnd}_{N_r(B)}(H_B) \neq \Phi$, the boundary region is non-empty, confirming that $H_B$ is indeed a **near soft set**.

**Definition 2.20.** ([19]) Let $F_B \in \text{NSS}(O)$ and $\tau \subseteq \text{NSS}(O)$ be a collection of near soft subsets of $F_B$. The collection $\tau$ is called a **near soft topology** (NST) on $F_B$ if it satisfies the following axioms:

(T1) $\Phi$ and $F_B$ belong to $\tau$.

(T2) The intersection of any two near soft sets in $\tau$ is also in $\tau$: $G_B^1 \sqcap G_B^2 \in \tau$ for all $G_B^1, G_B^2 \in \tau$.

(T3) The union of any family of near soft sets in $\tau$ belongs to $\tau$: $\bigsqcup_{i\in I} G_B^i \in \tau$ for every $\{G_B^i\}_{i\in I} \subseteq \tau$.

The triple $(F_B, B, \tau)$ is called a **near soft topological space** (NSTS). Elements of $\tau$ are **NS-open** sets, and their complements are **NS-closed** sets.

**Remark 2.21.** Definition 2.20 shows that the axiomatic structure of a near soft topology coincides formally with that of a soft topology. The essential difference lies in the underlying set-theoretic framework: near soft sets are constructed by incorporating descriptive nearness relations through lower and upper near approximations, whereas soft sets do not involve any proximity structure. If the nearness relation is chosen as the identity relation, then the lower and upper near approximations coincide, and every near soft set reduces to a classical soft set. Consequently, every soft topological space can be regarded as a special case of a near soft topological space. However, the converse does not hold in general, since near soft topological spaces allow a richer structure induced by nearness.

**Definition 2.22.** ([14],[21]) Let $(F_B, B, \tau)$ be an NSTS and $G_B \sqsubseteq F_B$. The collection

$$\tau_{G_B} = \{V_B \sqcap G_B : V_B \in \tau\}$$

is called the **NS-relative topology** on $G_B$, and the triple $(G_B, B, \tau_{G_B})$ is called an **NS-topological subspace** of $(F_B, B, \tau)$.

**Definition 2.23.** ([12]) Let $(F_B, B, \tau)$ be an NSTS and $H_B \in \mathrm{NSS}(\mathcal{O})$ such that $H_B \sqsubseteq F_B$. Then:

(i) The **NS-closure** of $H_B$ in $(F_B, B, \tau)$, denoted by $\mathrm{cl}_n(H_B)$, is defined as:

$$\mathrm{cl}_n(H_B) = \sqcap\{G_B \sqsupseteq H_B : G_B \text{ is an NS-closed set in } F_B\}.$$

(ii) The **NS-interior** of $H_B$ in $(F_B, B, \tau)$, denoted by $\mathrm{int}_n(H_B)$, is defined as:

$$\mathrm{int}_n(H_B) = \sqcup\{G_B \sqsubseteq H_B : G_B \text{ is an NS-open set in } F_B\}.$$

**Definition 2.24.** ([12]) Let $(F_B, B, \tau)$ be an NSTS.

(i) A near soft set $N_B \sqsubseteq F_B$ is called an **NS-neighborhood** of an NS-point $\eta_x^\phi \in F_B$ if there exists an NS-open set $H_B$ such that $\eta_x^\phi \in H_B \sqsubseteq N_B$.

(ii) A near soft set $G_B \sqsubseteq F_B$ is called an **NS-neighborhood** of an NS-set $H_B$ if there exists an NS-open set $U_B$ such that $H_B \sqsubseteq U_B \sqsubseteq G_B$.

## 3. Near soft locally compact spaces

In this section, we review the fundamental concept of compactness within the framework of NSTS. Furthermore, we introduce the notions of dense near soft sets and locally compact spaces, along with their characteristic properties, which will serve as the foundation for the Alexandroff compactification.

**Definition 3.1.** ([8]) Let $(F_B, B, \tau)$ be an NSTS and let $C_B \sqsubseteq F_B$. A family of NS-open subsets $\{(U_B)_i\}_{i\in I}$ in $\tau$ is called an **NS-open cover** of the near soft set $C_B$ if:

$$C_B \sqsubseteq \bigsqcup_{i\in I} (U_B)_i.$$

Essentially, $\{(U_B)_i\}_{i\in I}$ is an NS-open cover if the union of the family contains $C_B$ under the near soft inclusion relation.

**Definition 3.2.** ([8]) Let $C_B \sqsubseteq F_B$ and let $\{(U_B)_i\}_{i\in I}$ be an NS-open cover of $C_B$. A subfamily $\{(U_B)_j\}_{j\in J}$ where $J \subseteq I$ is called an **NS-subcover** of $C_B$ if:

$$C_B \sqsubseteq \bigsqcup_{j\in J} (U_B)_j.$$

If the index set $J$ is finite, then $\{(U_B)_j\}_{j\in J}$ is called a **finite NS-subcover** of $C_B$.

**Definition 3.3.** ([8]) Let $(F_B, B, \tau)$ be an NSTS and $K_B \sqsubseteq F_B$. The near soft set $K_B$ is said to be **NS-compact** if every NS-open cover of $K_B$ admits a finite NS-subcover. In particular, the space $(F_B, B, \tau)$ is called NS-compact if $F_B$ is itself an NS-compact set.

**Definition 3.4.** Let $(F_B, B, \tau)$ be an NSTS. A near soft set $G_B \sqsubseteq F_B$ is called **dense** in $(F_B, B, \tau)$ if its NS-closure is equal to the universal near soft set of the space, i.e.,

$$\mathrm{cl}_n(G_B) = F_B.$$

**Example 3.5.** Consider the near soft set $F_B = \{(\phi_1, \{p_1, p_3\}), (\phi_2, \{p_1, p_2, p_3\})\}$ as defined in Example 2.8. Let $\tau = \{\Phi, F_B, D^1_B, D^2_B\}$ be a collection of NS-sets where:

$$D^1_B = \{(\phi_1, \{p_3\}), (\phi_2, \{p_1\})\}, \quad D^2_B = \{(\phi_1, \{p_1\}), (\phi_2, \{p_2, p_3\})\}.$$

The triple $(F_B, B, \tau)$ forms an NSTS. Since the NS-closed sets are $\{F_B, \Phi, (D^1_B)^c, (D^2_B)^c\}$, we observe that:

$$\mathrm{cl}_n(D^1_B) = \sqcap\{H_B \in \tau^c : D^1_B \sqsubseteq H_B\} = (D^2_B)^c \neq F_B.$$

Thus, $D^1_B$ is not NS-dense. However, consider the near soft set $K_B = \{(\phi_1, \{p_3\}), (\phi_2, \{p_1, p_3\})\}$. Since there is no NS-closed set smaller than $F_B$ that contains $K_B$, we have:

$$\mathrm{cl}_n(K_B) = F_B.$$

Therefore, $K_B$ is **NS-dense** in $(F_B, B, \tau)$.

**Proposition 3.6.** *Let $(F_B, B, \tau)$ be an NSTS, and let $D_B$ be an NS-dense set in $F_B$ (i.e., $\mathrm{cl}_n(D_B) = F_B$). For a given parameter $\phi \in B$, the set $D(\phi)$ is not necessarily dense in $F(\phi)$ under the classical subset topology.*

*However, if $D(\phi)$ is dense in $F(\phi)$ for all $\phi \in B$ under the classical topology induced by the nearness structure, then $D_B$ is NS-dense in $F_B$.*

*Proof.* Assume $D_B$ is NS-dense, i.e., $\mathrm{cl}_n(D_B) = F_B$. By the definition of NS-closure, $F_B$ is the smallest NS-closed set containing $D_B$. For any $x \in F(\phi)$, the NS-point $\eta^\phi_x$ belongs to the NS-closure of $D_B$, which implies that every NS-open neighborhood of $\eta^\phi_x$ has a non-empty intersection with $D_B$. However, because the NS-topology is induced by the nearness structure $N_r(B)$, this intersection only guarantees that $x$ is "near" to some element in $D(\phi)$ within the NAS, not necessarily that $x$ is a limit point of $D(\phi)$ in the classical sense. Hence, $D(\phi)$ may not be dense in $F(\phi)$.

Conversely, suppose $D(\phi)$ is dense in $F(\phi)$ for each $\phi \in B$ in the classical sense ($\mathrm{cl}(D(\phi)) = F(\phi)$). This implies that every open set in the universe that contains $x$ also contains at least one element of $D(\phi)$. Since any NS-open set $G_B$ containing $\eta^\phi_x$ must satisfy $x \in G(\phi)$, the classical density ensures $G(\phi) \cap D(\phi) \neq \emptyset$. Consequently, $G_B \sqcap D_B \neq \Phi$, which confirms that $\eta^\phi_x \in \mathrm{cl}_n(D_B)$ for all NS-points. Thus, $\mathrm{cl}_n(D_B) = F_B$. □

**Example 3.7.** Let $O = \{p_1, p_2, p_3\}$ and $B = \{\phi_1, \phi_2\}$. Define the universal near soft set $F_B = \{(\phi_1, \{p_1, p_2\}), (\phi_2, \{p_2, p_3\})\}$. We define an NSTS $(F_B, B, \tau)$ with:

$$\tau = \{\Phi, F_B, G_B\}, \quad \text{where } G_B = \{(\phi_1, \{p_2\}), (\phi_2, \{p_3\})\}.$$

The set of NS-closed sets is $\tau^c = \{F_B, \Phi, G^c_B\}$, where $G^c_B = \{(\phi_1, \{p_1\}), (\phi_2, \{p_2\})\}$.

Now, consider $D_B = \{(\phi_1, \{p_1\}), (\phi_2, \{p_3\})\}$. Let us find $\mathrm{cl}_n(D_B)$:

- $D_B \not\sqsubseteq G^c_B$ because at parameter $\phi_2$, $D(\phi_2) = \{p_3\} \not\subseteq G^c(\phi_2) = \{p_2\}$.
- The only NS-closed set containing $D_B$ is $F_B$.

Thus, $\mathrm{cl}_n(D_B) = F_B$, making $D_B$ **NS-dense**. However, at $\phi_1$, $D(\phi_1) = \{p_1\}$ is not dense in $F(\phi_1) = \{p_1, p_2\}$ in the classical sense if we consider a topology where $\{p_2\}$ is open (like the discrete topology), because the closure of $\{p_1\}$ would just be $\{p_1\}$. This demonstrates that NS-denseness is a global property of the soft mapping and the nearness structure.

**Definition 3.8.** ([8]) Let $(F_B, B, \tau)$ be an NSTS. It is called an **NS-Hausdorff space** or **NS-$T_2$-space** ($nT_2$) if for any two distinct NS-points $\eta^\phi_x$ and $\beta^\psi_y$ in $F_B$ (where $\eta^\phi_x \neq \beta^\psi_y$), there exist NS-open sets $U_B, V_B \in \tau$ such that:

$$\eta^\phi_x \in U_B, \quad \beta^\psi_y \in V_B, \quad \text{and} \quad U_B \sqcap V_B = \Phi.$$

**Definition 3.9.** ([8]) An NSTS $(F_B, B, \tau)$ is called **NS-regular** if for every NS-point $\eta^\phi_x \in F_B$ and every NS-closed set $K_B \sqsubseteq F_B$ such that $\eta^\phi_x \notin K_B$, there exist disjoint NS-open sets $U_B$ and $V_B$ such that:

$$\eta^\phi_x \in U_B, \quad K_B \sqsubseteq V_B, \quad \text{and} \quad U_B \sqcap V_B = \Phi.$$

**Theorem 3.10.** *Every NS-compact $nT_2$-space $(F_B, B, \tau)$ is NS-regular.*

*Proof.* Let $(F_B, B, \tau)$ be an NS-compact $nT_2$-space. Consider an NS-closed set $K_B \sqsubseteq F_B$ and an NS-point $\eta^\phi_x \in F_B$ such that $\eta^\phi_x \notin K_B$.

Since $K_B$ is an NS-closed subset of an NS-compact space, $K_B$ is itself NS-compact (by Theorem 4.2.5 in [8]). Because the space is $nT_2$, for each NS-point $\zeta \in K_B$, there exist disjoint NS-open neighborhoods $U_\zeta$ and $V_\zeta$ such that:

$$\eta^\phi_x \in U_\zeta, \quad \zeta \in V_\zeta, \quad \text{and} \quad U_\zeta \sqcap V_\zeta = \Phi.$$

The family $\{V_\zeta : \zeta \in K_B\}$ forms an NS-open cover of $K_B$. By the NS-compactness of $K_B$, there exists a finite subcover $\{V_{\zeta_1}, V_{\zeta_2}, \ldots, V_{\zeta_n}\}$ such that:

$$K_B \sqsubseteq \bigsqcup_{i=1}^{n} V_{\zeta_i}.$$

Now, let $U_B = \sqcap_{i=1}^n U_{\zeta_i}$. Since $U_B$ is a finite intersection of NS-open sets, $U_B \in \tau$. Clearly, $\eta^\phi_x \in U_B$. Furthermore, for the union $V_B = \sqcup_{i=1}^n V_{\zeta_i}$, we have $K_B \sqsubseteq V_B$.

To show $U_B \sqcap V_B = \Phi$, note that:

$$U_B \sqcap \left( \bigsqcup_{i=1}^{n} V_{\zeta_i} \right) = \bigsqcup_{i=1}^{n} (U_B \sqcap V_{\zeta_i}) \sqsubseteq \bigsqcup_{i=1}^{n} (U_{\zeta_i} \sqcap V_{\zeta_i}) = \bigsqcup_{i=1}^{n} \Phi = \Phi.$$

Thus, $\eta^\phi_x$ and $K_B$ are separated by disjoint NS-open sets $U_B$ and $V_B$, proving the space is NS-regular. □

**Definition 3.11.** Let $(F_B, B, \tau)$ be an NSTS. The space $(F_B, B, \tau)$ is said to be **NS-locally compact** if every NS-point $\eta^\phi_x \in F_B$ has an NS-neighborhood $N_B$ such that $N_B \sqsubseteq K_B$ for some NS-compact subset $K_B \sqsubseteq F_B$.

**Corollary 3.12.** *Every NS-compact space $(F_B, B, \tau)$ is NS-locally compact.*

*Proof.* Let $(F_B, B, \tau)$ be an NS-compact space. For any NS-point $\eta^\phi_x \in F_B$, the absolute near soft set $F_B$ is itself an NS-compact set. Since $F_B$ is an NS-neighborhood of every NS-point it contains ($\eta^\phi_x \in F_B \sqsubseteq F_B$), the condition for NS-local compactness is trivially satisfied. □

**Theorem 3.13.** *Let $(F_B, B, \tau)$ be an $nT_2$-space. Then $(F_B, B, \tau)$ is NS-locally compact if and only if for every NS-point $\eta_x^\phi \in F_B$, there exists an NS-compact set $K_B$ such that:*

$$\eta_x^\phi \in \text{int}_n(K_B).$$

*Proof.* The necessity follows directly from Definition 3.11, as the existence of an NS-neighborhood within a compact set implies that the point belongs to the interior of that compact set.

Conversely, assume that for each $\eta_x^\phi \in F_B$, there exists an NS-compact set $K_B$ such that $\eta_x^\phi \in \text{int}_n(K_B)$. Let $U_B$ be any NS-neighborhood of $\eta_x^\phi$. Then, the intersection $W_B = \text{int}_n(K_B) \sqcap U_B$ is an NS-neighborhood of $\eta_x^\phi$ contained in $U_B$.

Since $(F_B, B, \tau)$ is $nT_2$ and $K_B$ is an NS-compact subset, by Theorem 3.10, the subspace $K_B$ inherits the NS-regularity property. In a regular space, for the point $\eta_x^\phi$ and its NS-open neighborhood $W_B$, there exists an NS-open set $V_B$ such that:

$$\eta_x^\phi \in V_B \sqsubseteq \text{cl}_n(V_B) \sqsubseteq W_B \sqsubseteq \text{int}_n(K_B) \sqsubseteq K_B.$$

Since $\text{cl}_n(V_B)$ is an NS-closed subset of the NS-compact set $K_B$, it is also NS-compact (Theorem 4.2.5 in [8]). Thus, we have found an NS-neighborhood $V_B$ of $\eta_x^\phi$ whose closure is NS-compact and contained in $U_B$. This confirms that the space is NS-locally compact. □

**Corollary 3.14.** *Every NS-Hausdorff NS-compact space is NS-locally compact.*

*Proof.* The result is immediate from Definition 3.11 and the properties established in Theorem 3.13, as compactness implies that the space itself provides the required compact neighborhood for every NS-point. □

**Corollary 3.15.** *Let $(F_B, B, \tau)$ be an $nT_2$-space. If $(F_B, B, \tau)$ is NS-locally compact, then $F_B$ is NS-open in its NS-closure $\text{cl}_n(F_B)$.*

*Proof.* This follows directly from the local structure provided by Theorem 3.13, which ensures that each point is contained within the NS-interior of a compact set relative to the closure. □

**Corollary 3.16.** *A dense, NS-locally compact subspace of an $nT_2$-space $(F_B, B, \tau)$ is NS-open in $(F_B, B, \tau)$.*

*Proof.* If the subspace is dense, its NS-closure is the absolute set $F_B$. By Corollary 3.15, an NS-locally compact subspace is NS-open in its closure, which in this case is the entire space. □

**Theorem 3.17.** *Every NS-closed or NS-open subspace of an NS-Hausdorff locally compact space is itself NS-locally compact.*

*Proof.* Let $(F_B, B, \tau)$ be an NS-locally compact $nT_2$-space.

*Case 1: NS-open subspace.* Let $U_B$ be an NS-open subspace of $F_B$ and $\eta_x^\phi \in U_B$. Every NS-neighborhood $V_B$ of $\eta_x^\phi$ relative to $U_B$ is also an NS-neighborhood in the absolute space $F_B$. By Theorem 3.13, there exists an NS-compact set $K_B$ such that:

$$\eta_x^\phi \in \text{int}_n(K_B) \sqsubseteq K_B \sqsubseteq V_B \sqsubseteq U_B.$$

This confirms that $U_B$ is NS-locally compact. Note that this part of the argument holds regardless of the $nT_2$ property.

*Case 2: NS-closed subspace.* Let $G_B \sqsubseteq F_B$ be an NS-closed subspace and $\eta_x^\phi \in G_B$. Since the master space is locally compact, there exists an NS-compact set $H_B$ such that $\eta_x^\phi \in \text{int}_n(H_B) \sqsubseteq H_B$. Since $F_B$ is $nT_2$, the NS-compact set $H_B$ is NS-closed. The intersection $G_B \sqcap H_B$ is an NS-closed subset of the NS-compact set $H_B$, and is therefore NS-compact itself. Consequently,

$$\eta_x^\phi \in \text{int}_n(G_B \sqcap H_B) \sqsubseteq G_B \sqcap H_B \sqsubseteq G_B,$$

which proves that $G_B$ is NS-locally compact. □

**Proposition 3.18.** *Let* $(F_B, B, \tau)$ *be an* $nT_2$*-space and* $\eta_x^\phi \in F_B$*. If for every NS-open set* $U_B$ *containing* $\eta_x^\phi$*, there exists an NS-open set* $V_B$ *such that:*

$$\eta_x^\phi \in V_B \sqsubseteq \mathrm{cl}_n(V_B) \sqsubseteq U_B,$$

*where* $\mathrm{cl}_n(V_B)$ *is NS-compact, then* $(F_B, B, \tau)$ *is NS-locally compact.*

*Proof.* The statement follows immediately from Definition 3.11. The existence of such a $V_B$ ensures that every NS-point has an NS-neighborhood whose closure is NS-compact, which is exactly the requirement for NS-local compactness. □

**Theorem 3.19.** *Let* $(F_B, B, \tau)$ *be an* $nT_2$*-space and* $G_B \sqsubseteq F_B$ *be a non-empty subset. Then the subspace* $(G_B, B, \tau_{G_B})$ *is NS-locally compact if:*

*(i)* $(F_B, B, \tau)$ *is NS-locally compact,*

*(ii)* $G_B$ *is NS-open in* $F_B$*.*

*Proof.* Let $\eta_x^\phi \in G_B$. Since $G_B$ is NS-open in $F_B$ and the master space $(F_B, B, \tau)$ is NS-locally compact, there exists an NS-open set $V_B$ containing $\eta_x^\phi$ such that $\mathrm{cl}_n(V_B)$ is NS-compact and:

$$\eta_x^\phi \in V_B \sqsubseteq \mathrm{cl}_n(V_B) \sqsubseteq G_B.$$

Because $G_B$ is NS-open, $V_B$ is also NS-open in the subspace topology $\tau_{G_B}$. Furthermore, since $\mathrm{cl}_n(V_B)$ is an NS-compact subset of $F_B$ contained within $G_B$, it remains NS-compact in the relative topology of $G_B$.

Thus, every NS-point in $G_B$ has an NS-compact neighborhood within the subspace. By Proposition 3.18, the subspace $(G_B, B, \tau_{G_B})$ is NS-locally compact. □

## 4. Alexandroff compactification of near soft topological spaces

Compactification is a fundamental construction in classical topology, allowing non-compact spaces to be embedded into compact ones by adding minimal structure. This process is crucial for extending topological analysis to boundary behavior, convergence properties, and structural completeness.

In this section, we investigate the Alexandroff (one-point) compactification within the framework of NSTSs. By extending the classical notion, we construct a near soft topological space that contains the original NSTS as a dense subspace and gains compactness via a single additional NS-point, denoted by $\infty_B$.

Such compactifications typically rely on topological embeddings that preserve structure through homeomorphisms. In the near soft setting, these homeomorphisms are defined via bijective mappings that are near soft continuous in both directions. To formalize this framework, we first refine the notions of near soft continuity and injective mappings.

**Definition 4.1.** ([20]) Let $(F_B, B, \tau)$ and $(G_B, B, \tau')$ be two NSTSs over the same parameter set $B$. A mapping

$$f : (F_B, B, \tau) \to (G_B, B, \tau')$$

is said to be **near soft continuous** if for every NS-open set $V_B \in \tau'$, the preimage $f^{-1}(V_B)$ is an NS-open set in $(F_B, B, \tau)$, i.e., $f^{-1}(V_B) \in \tau$.

**Definition 4.2.** [Near Soft Injective Mapping] Let $(F_B, B, \tau)$ and $(G_B, B, \tau')$ be two near soft topological spaces over the same parameter set $B$. A mapping $f : F_B \to G_B$ between the underlying near soft universes is called **near soft injective** if the following conditions are satisfied:

(i) For any two distinct NS-points $\eta_x^\phi, \beta_y^\psi \in F_B$:

$$\eta_x^\phi \neq \beta_y^\psi \implies f(\eta_x^\phi) \neq f(\beta_y^\psi),$$

i.e., $f$ is injective at the level of NS-points.

(ii) For any two near soft sets $H_B, K_B \sqsubseteq F_B$, the condition:

$$f(H_B)(\phi) = f(K_B)(\phi) \quad \forall \phi \in B,$$

implies $H(\phi) = K(\phi)$ for all $\phi \in B$, ensuring that $f$ preserves the structural distinctness of the parameter-wise mappings.

**Definition 4.3.** [Near Soft Surjective Mapping] Let $(F_B, B, \tau)$ and $(G_B, B, \tau')$ be two NSTSs over the same parameter set $B$. A mapping $f : F_B \to G_B$ between the near soft universes is called **near soft surjective** if for every NS-point $\beta_y^\phi \in G_B$, there exists at least one NS-point $\eta_x^\phi \in F_B$ such that:

$$f(\eta_x^\phi) = \beta_y^\phi.$$

In other words, every point in the codomain's mapping is an image of at least one point from the domain's mapping under the parameter-preserving function.

**Example 4.4.** Let $U = \{a, b, c\}$ and $V = \{1, 2, 3\}$ be two universes, with $B = \{\phi_1, \phi_2\}$ as the parameter set. We define two near soft sets $F_B \in \text{NSS}(U)$ and $G_B \in \text{NSS}(V)$ as follows:

$$F(\phi_1) = \{a, b\}, \quad F(\phi_2) = \{b, c\}$$

$$G(\phi_1) = \{1, 2\}, \quad G(\phi_2) = \{2, 3\}$$

Consider a mapping $f : U \to V$ defined by $f(a) = 1, f(b) = 2, f(c) = 3$. The induced near soft mapping $f_{pu} : \text{NSS}(U) \to \text{NSS}(V)$ is defined for any NS-point $\eta_x^\phi$ as $f_{pu}(\eta_x^\phi) = \eta_{f(x)}^\phi$.

**Claim 1: The mapping $f_{pu}$ is near soft injective.**

*Proof.* Let $\eta_{x_1}^\phi, \eta_{x_2}^\phi \in F_B$ be two distinct NS-points.

- If the parameters are the same but $x_1 \neq x_2$, the injectivity of $f$ on the universes ($f(x_1) \neq f(x_2)$) ensures $\eta_{f(x_1)}^\phi \neq \eta_{f(x_2)}^\phi$.
- If the parameters are different, the images will naturally have different parameters, thus remaining distinct.

Hence, $f_{pu}$ satisfies the near soft injective condition.

**Claim 2: The mapping $f_{pu}$ is near soft surjective.**

*Proof.* We verify that every NS-point in $G_B$ has a preimage in $F_B$:

- For $\eta_1^{\phi_1} \in G_B$: Since $f(a) = 1$ and $a \in F(\phi_1)$, the preimage is $\eta_a^{\phi_1} \in F_B$.
- For $\eta_2^{\phi_1} \in G_B$: Since $f(b) = 2$ and $b \in F(\phi_1)$, the preimage is $\eta_b^{\phi_1} \in F_B$.
- For $\eta_2^{\phi_2} \in G_B$: Since $f(b) = 2$ and $b \in F(\phi_2)$, the preimage is $\eta_b^{\phi_2} \in F_B$.
- For $\eta_3^{\phi_2} \in G_B$: Since $f(c) = 3$ and $c \in F(\phi_2)$, the preimage is $\eta_c^{\phi_2} \in F_B$.

Since every near soft point of the codomain is covered, $f_{pu}$ is near soft surjective.

**Definition 4.5.** [Near Soft Homeomorphism] Let $(F_B, B, \tau)$ and $(G_B, B, \tau')$ be two near soft topological spaces over the universes $U$ and $V$, respectively, with the same parameter set $B$. A near soft mapping $f_{pu} : \text{NSS}(U) \to \text{NSS}(V)$ is called a **near soft homeomorphism** if it satisfies the following three conditions:

(i) $f_{pu}$ is a **near soft bijection** (it is both near soft injective and near soft surjective as per Definitions 4.2 and 4.3),

(ii) $f_{pu}$ is **near soft continuous**, i.e., $f_{pu}^{-1}(H_B) \in \tau$ for every NS-open set $H_B \in \tau'$,

(iii) The inverse mapping $f_{pu}^{-1}$ is **near soft continuous**, i.e., $f_{pu}(K_B) \in \tau'$ for every NS-open set $K_B \in \tau$.

If such a mapping exists, the spaces $(F_B, B, \tau)$ and $(G_B, B, \tau')$ are said to be **near soft homeomorphic**, denoted by $(F_B, B, \tau) \cong (G_B, B, \tau')$.

**Example 4.6.** Let the universes be $O_F = \{p_1, p_2, p_3, p_4\}$ and $O_G = \{q_1, q_2, q_3, q_4\}$, with the parameter set $B = \{\phi_1, \phi_2\}$. Define the near soft sets:

$$F_B = \{(\phi_1, \{p_1, p_2\}), (\phi_2, \{p_3, p_4\})\},$$

$$G_B = \{(\phi_1, \{q_1, q_2\}), (\phi_2, \{q_3, q_4\})\}.$$

Consider the near soft topologies $\tau$ on $F_B$ and $\tau'$ on $G_B$ defined as:

$$\tau = \{\Phi, F_B, U_B\}, \quad \text{where } U_B = \{(\phi_1, \{p_1\}), (\phi_2, \{p_3, p_4\})\},$$

$$\tau' = \{\Phi, G_B, V_B\}, \quad \text{where } V_B = \{(\phi_1, \{q_1, q_2\}), (\phi_2, \{q_3\})\}.$$

Define the universe bijection $f : O_F \to O_G$ by:

$$f(p_1) = q_1, \quad f(p_2) = q_2, \quad f(p_3) = q_3, \quad f(p_4) = q_4.$$

Let $f_{pu}$ be the induced near soft mapping. We verify that $f_{pu}$ is a near soft homeomorphism:

- **Bijectivity:** Since $f$ is a bijection between the underlying universes and preserves the parameter set $B$, $f_{pu}$ is a near soft bijection between the sets of NS-points.
- **Near Soft Continuity of $f_{pu}$:** We check the preimages of NS-open sets in $\tau'$:
  - $f_{pu}^{-1}(\Phi) = \Phi \in \tau$.
  - $f_{pu}^{-1}(G_B) = F_B \in \tau$.
  - $f_{pu}^{-1}(V_B) = \{(\phi_1, f^{-1}\{q_1, q_2\}), (\phi_2, f^{-1}\{q_3\})\} = \{(\phi_1, \{p_1, p_2\}), (\phi_2, \{p_3\})\}$.

  *Note:* In this specific mapping, if we adjust $\tau$ such that $U_B$ matches the structure of $f_{pu}^{-1}(V_B)$, the continuity is strictly satisfied.
- **Near Soft Continuity of $f_{pu}^{-1}$:** Similarly, the image of every NS-open set in $\tau$ under $f_{pu}$ must be NS-open in $\tau'$.

**Remarks:**

- The NS-open sets are explicitly constructed with distinct parameter-wise subsets to highlight the influence of the near soft structure $N_r(B)$.
- The mapping $f_{pu}$ preserves the structural integrity of the near soft sets across different parameters.
- This demonstrates that a near soft homeomorphism is not merely a set-theoretic bijection, but a structure-preserving map that respects the nearness approximations.

**Definition 4.7.** [Compactification of NSTS] A **compactification** of an NSTS $(F_B, B, \tau)$ is an embedding of $F_B$ as a dense subspace of an NS-compact space $(G_B, B, \sigma)$ via a near soft mapping:

$$j : (F_B, B, \tau) \to (G_B, B, \sigma)$$

such that $j : F_B \to j(F_B)$ is a near soft homeomorphism and:

$$\text{cl}_n(j(F_B)) = G_B.$$

**Definition 4.8.** [NS-Alexandroff Compactification] Let $(F_B, B, \tau)$ be a non-NS-compact near soft topological space. We define the extended near soft universe as:

$$F_B^* = F_B \sqcup \{\infty_B\},$$

where $\infty_B = \{(\phi, \{\infty\}) : \phi \in B\}$ is a new NS-point not belonging to $F_B$. The collection $\tau^*$ of NS-open sets on $F_B^*$ is defined as:

$$\tau^* = \tau \cup \{(F_B \setminus C_B) \sqcup \{\infty_B\} : C_B \text{ is NS-closed and NS-compact in } (F_B, B, \tau)\}.$$

The triple $(F_B^*, B, \tau^*)$ is called the **NS-Alexandroff compactification** (or one-point compactification) of $(F_B, B, \tau)$.

**Lemma 4.9.** *Let $(F_B, B, \tau)$ be an NSTS over the universe $O$. Let $F_B^*$ be the absolute near soft set over the extended universe $O^* = O \cup \{\infty\}$. The collection*

$$\tau^* := \tau \cup \{(F_B \setminus K_B) \cup \{\infty_B\} : K_B \text{ is NS-compact and NS-closed in } (F_B, B, \tau)\}$$

*defines a near soft topology on $F_B^*$, where $\infty_B$ denotes the NS-point at infinity.*

*Proof.* To prove that $\tau^*$ is a near soft topology on $F_B^*$, we verify the three NSTS axioms using the near soft difference defined in Definition 2.14:

**(T1) Null and Absolute NS-sets:** Clearly, $\Phi \in \tau \subseteq \tau^*$. For the absolute set $F_B^*$, notice that $F_B^* = (F_B \setminus \Phi) \sqcup \{\infty_B\}$. Since the null set $\Phi$ is trivially NS-compact and NS-closed in any NSTS, it follows that $F_B^* \in \tau^*$.

**(T2) Finite Intersections:** Let $U_B, V_B \in \tau^*$. We examine three possible cases:

- If $U_B, V_B \in \tau$, then $U_B \sqcap V_B \in \tau \subseteq \tau^*$ by the definition of $\tau$.
- If $U_B \in \tau$ and $V_B = (F_B \setminus K_B) \sqcup \{\infty_B\} \in \tau^* \setminus \tau$, then

  $$U_B \sqcap V_B = U_B \sqcap (F_B \setminus K_B)$$

  since $\infty_B \notin U_B$. Because $U_B$ and $(F_B \setminus K_B)$ are both in $\tau$, their near soft intersection is in $\tau \subseteq \tau^*$.
- If both $U_B, V_B \in \tau^* \setminus \tau$, let $U_B = (F_B \setminus K_1) \sqcup \{\infty_B\}$ and $V_B = (F_B \setminus K_2) \sqcup \{\infty_B\}$. Then:

  $$U_B \sqcap V_B = (F_B \setminus (K_1 \sqcup K_2)) \sqcup \{\infty_B\}.$$

  Since the finite near soft union of NS-compact and NS-closed sets $(K_1 \sqcup K_2)$ is NS-compact and NS-closed, the intersection $U_B \sqcap V_B$ satisfies the condition to be in $\tau^*$.

**(T3) Arbitrary Unions:** Let $\{U_i\}_{i \in I}$ be a collection of NS-sets in $\tau^*$.

- If $U_i \in \tau$ for all $i \in I$, then $\bigsqcup_{i \in I} U_i \in \tau \subseteq \tau^*$.
- If there exists at least one $i_0 \in I$ such that $U_{i_0} = (F_B \setminus K_{i_0}) \sqcup \{\infty_B\}$, then $\infty_B \in \bigsqcup U_i$. Let the union be expressed as:

  $$\bigsqcup_{i \in I} U_i = G_B \sqcup \{\infty_B\}, \quad \text{where } G_B \in \tau.$$

  The complement of the base part in $F_B$ is $F_B \setminus G_B = \bigsqcap_{i \in I}(F_B \setminus U_i)$. For $i = i_0$, $(F_B \setminus U_{i_0}) = K_{i_0}$ is NS-compact. Since an arbitrary near soft intersection of NS-closed sets where at least one is NS-compact is itself NS-compact, the set $F_B \setminus G_B$ is NS-compact and NS-closed. Thus, $\bigsqcup U_i \in \tau^*$.

Hence, $\tau^*$ is a valid near soft topology on $F_B^*$. □

**Example 4.10.** Let the initial near soft universe be defined by $F_B$ over $\mathcal{O} = \{p_1, p_2, p_3, p_4, p_5\}$ with parameters $B = \{\phi_1, \phi_2\}$. We construct the Alexandroff extension by defining:

- The extended universe $\mathcal{O}^* = \mathcal{O} \cup \{b_1\}$.
- The extended parameter set $B^* = B \cup \{\alpha\}$.

The **NS-point at infinity** is defined as $\infty_{B^*} = \eta^{\alpha}_{b_1}$, where the mapping satisfies $F^*_{B^*}(\alpha) = \{b_1\}$.

Let $(F_B, B, \tau)$ be the NSTS from Example 3.5. We define the extended near soft topology $\tau^*$ on $(F^*_{B^*}, B^*)$ as:

$$\tau^* = \tau \cup \{(F_B \setminus K_B) \sqcup \{\eta^{\alpha}_{b_1}\} : K_B \text{ is NS-compact and NS-closed in } \tau\}.$$

Consider a specific NS-compact and NS-closed set $K_B \sqsubseteq F_B$. An element $G_B \in \tau^*$ containing the point at infinity would take the form:

$$G_B = (F_B \setminus K_B) \sqcup \{\eta^{\alpha}_{b_1}\} = \{(\phi_1, \{p_3, b_1\}), (\phi_2, \{p_1, b_1\}), (\alpha, \{b_1\})\}.$$

By analyzing the restrictions to each parameter in $B^*$, the topology $\tau^*$ exhibits the following structure:

$$\begin{cases} \phi_1 : & \{\emptyset, \{p_1, \ldots, p_5\}, \{p_1, \ldots, p_5, b_1\}, \{p_3\}, \{p_1\}, \{p_1, p_3\}, \{p_3, b_1\}, \{p_1, p_3, b_1\}\} \\ \phi_2 : & \{\emptyset, \{p_1, \ldots, p_5\}, \{p_1, \ldots, p_5, b_1\}, \{p_1\}, \{p_2, p_3\}, \{p_1, p_2, p_3\}, \{p_1, b_1\}, \{p_1, p_2, p_3, b_1\}\} \\ \alpha : & \{\emptyset, \{b_1\}\} \end{cases}$$

The resulting space $(F^*_{B^*}, B^*, \tau^*)$ is the **NS-Alexandroff compactification** of the original space. Note that the restriction to the new parameter $\alpha$ yields a Sierpinski-like structure $\{\emptyset, \{b_1\}\}$, which is the minimal structure required to host the point at infinity while maintaining the compactness of the global near soft union.

**Theorem 4.11.** *Let $(F_B, B, \tau)$ be a non-NS-compact near soft topological space. The following statements are equivalent:*

1. *$(F_B, B, \tau)$ is NS-locally compact and satisfies the $nT_2$ separation axiom.*
2. *There exists a one-point NS-Alexandroff compactification $(F^*_B, B, \tau^*)$ of $(F_B, B, \tau)$.*

*Proof.* **(1) $\implies$ (2):**
Assume $(F_B, B, \tau)$ is NS-locally compact and $nT_2$. Let $F^*_B = F_B \sqcup \{\infty_B\}$ be the extended near soft universe, where $\infty_B$ is the NS-point at infinity as defined in Definition 4.8. We define the collection $\tau^*$ as in Lemma 4.9.

*Density:* Since $(F_B, B, \tau)$ is non-NS-compact, for any NS-compact subset $K_B \sqsubseteq F_B$, the set $F_B \setminus K_B$ is non-empty. Consequently, every NS-open neighborhood of $\infty_B$ in $\tau^*$, which necessarily takes the form $U_B = (F_B \setminus K_B) \sqcup \{\infty_B\}$, has a non-empty intersection with $F_B$. This implies:

$$\text{cl}_n(F_B) = F^*_B,$$

confirming that $(F_B, B, \tau)$ is NS-dense in $(F^*_B, B, \tau^*)$.

*NS-Compactness:* Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an NS-open cover of $F^*_B$. By the covering property, there exists some $i_0 \in I$ such that $\infty_B \in U_{i_0}$. According to the definition of $\tau^*$, the complement $F^*_B \setminus U_{i_0} = K_B$ is an NS-compact subset of $F_B$. Since $\mathcal{U}$ covers $F^*_B$, it also provides an NS-open cover for the set $K_B$. By the NS-compactness of $K_B$, there exists a finite sub-collection $\{U_{i_1}, U_{i_2}, \ldots, U_{i_n}\}$ such that:

$$K_B \sqsubseteq \bigsqcup_{k=1}^{n} U_{i_k}.$$

Thus, the finite collection $\{U_{i_0}, U_{i_1}, \ldots, U_{i_n}\}$ covers $F^*_B$, establishing the NS-compactness of the extended space.

*$nT_2$ Property:* To show $F^*_B$ is $nT_2$, let $\eta^{\phi}_x$ and $\beta^{\psi}_y$ be two distinct NS-points in $F^*_B$.

- If both points belong to $F_B$, they are separated by disjoint NS-open sets due to the $nT_2$ property of $\tau$.
- If one point is $\infty_B$ and the other is $\eta_x^\phi \in F_B$, we utilize the NS-local compactness of $F_B$. There exists an NS-compact set $K_B \sqsubseteq F_B$ such that $\eta_x^\phi \in \text{int}_n(K_B)$. Then, the sets $V_B = \text{int}_n(K_B)$ and $W_B = (F_B \setminus K_B) \sqcup \{\infty_B\}$ are disjoint NS-open sets in $\tau^*$ that separate $\eta_x^\phi$ and $\infty_B$.

**(2) $\Longrightarrow$ (1):**
Suppose $j : (F_B, B, \tau) \to (F_B^*, B, \tau^*)$ is a one-point NS-compactification.

*$nT_2$ Property:* Since $F_B^*$ is an $nT_2$-space and $F_B$ is a subspace of $F_B^*$, the $nT_2$ separation axiom is inherited by $(F_B, B, \tau)$.

*NS-Local Compactness:* Let $\eta_x^\phi \in F_B$. Since $F_B^*$ is $nT_2$, we can separate the distinct NS-points $\eta_x^\phi$ and $\infty_B$ by disjoint NS-open sets $U_B, V_B \in \tau^*$. Since $\infty_B \in V_B$ and $U_B \sqcap V_B = \Phi$, it follows that $\infty_B \notin U_B$, and thus $U_B \sqsubseteq F_B$. The NS-closure $\text{cl}_n(U_B)$ is an NS-closed subset of the NS-compact space $F_B^*$, making $\text{cl}_n(U_B)$ NS-compact. Furthermore, because $U_B \sqcap V_B = \Phi$ and $V_B$ is an NS-neighborhood of $\infty_B$, the point at infinity cannot be a limit point of $U_B$. Therefore, $\text{cl}_n(U_B) \sqsubseteq F_B$. This shows that $\eta_x^\phi$ has an NS-compact neighborhood in $F_B$, proving the space is NS-locally compact. □

**Theorem 4.12.** *Let $(F_B^*, B, \tau^*)$ be the NS-Alexandroff extension of the near soft topological space $(F_B, B, \tau)$. Then $(F_B^*, B, \tau^*)$ is an NS-compact space.*

*Proof.* Let $\widetilde{\mathcal{U}} = \{U_{B_i} : i \in I\}$ be an arbitrary NS-open cover of the extended space $F_B^*$. Our objective is to demonstrate that there exists a finite subfamily of $\widetilde{\mathcal{U}}$ that still covers $F_B^*$.

By the definition of the extended universe, the NS-point at infinity $\infty_B$ is a member of $F_B^*$. Since $\widetilde{\mathcal{U}}$ covers the entire space, there must exist at least one index $i_0 \in I$ such that:

$$\infty_B \in U_{B_{i_0}} \in \widetilde{\mathcal{U}}.$$

According to the construction of the NS-topology $\tau^*$, the presence of the point at infinity implies that $U_{B_{i_0}}$ takes the form:

$$U_{B_{i_0}} = (F_B \setminus K_B) \sqcup \{\infty_B\},$$

where $K_B$ is an NS-compact subset of the original space $(F_B, B, \tau)$.

Since $K_B \sqsubseteq F_B \sqsubseteq F_B^*$, the collection $\widetilde{\mathcal{U}}$ also serves as an NS-open cover for the set $K_B$. By the definition of NS-compactness for $K_B$, there exists a finite subcollection $\{U_{B_1}, U_{B_2}, \dots, U_{B_n}\} \subseteq \widetilde{\mathcal{U}}$ such that:

$$K_B \sqsubseteq \bigsqcup_{k=1}^{n} U_{B_k}.$$

Now, we observe that the extended space $F_B^*$ can be partitioned into the compact core $K_B$ and the neighborhood of infinity $U_{B_{i_0}}$. Specifically:

$$F_B^* = K_B \sqcup U_{B_{i_0}}.$$

Substituting the finite cover of $K_B$ into this expression, we obtain:

$$F_B^* \sqsubseteq \left( \bigsqcup_{k=1}^{n} U_{B_k} \right) \sqcup U_{B_{i_0}}.$$

This confirms that the finite subcollection $\{U_{B_1}, U_{B_2}, \dots, U_{B_n}, U_{B_{i_0}}\}$ forms an NS-open subcover of $F_B^*$. Therefore, the space $(F_B^*, B, \tau^*)$ is NS-compact. □

**Theorem 4.13.** *Let $(F_B, B, \tau)$ be a near soft topological space, and suppose $(F^n_B, B, \tau^n)$ is an NSTS over the extended universe. Then $(F^n_B, B, \tau^n)$ is an NS-Hausdorff compactification of $(F_B, B, \tau)$ if and only if the following conditions hold:*

1. *$(F^n_B, B, \tau^n)$ is an NS-compactification of $(F_B, B, \tau)$.*
2. *$(F_B, B, \tau)$ is NS-locally compact and satisfies the $nT_2$ separation axiom.*
3. *There exists an NS-compact set $C_B \sqsubseteq F_B$ such that for every $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$, there exist NS-open neighborhoods $U_{B_i}$ of $\eta^B_i$ and $U_{B_j}$ of $\eta^B_j$ such that:*

$$(U_{B_i} \sqcap U_{B_j}) \sqsubseteq C_B.$$

*Proof.* **( $\Longrightarrow$ ):** Assume $(F^n_B, B, \tau^n)$ is an NS-Hausdorff compactification of $(F_B, B, \tau)$.

1. By definition, $(F^n_B, B, \tau^n)$ is an NS-compact space and $(F_B, B, \tau)$ is an NS-dense open subspace.

2. Since the $nT_2$ property is hereditary, $(F_B, B, \tau)$ inherits it from $(F^n_B, B, \tau^n)$. Furthermore, $F_B = F^n_B \setminus \{\eta^B_1, \dots, \eta^B_n\}$. As a finite set of points in an $nT_2$ space is NS-closed, $F_B$ is NS-open. By Theorem 3.19, an NS-open subspace of an NS-locally compact $nT_2$ space is itself NS-locally compact.

3. For $i \neq j$, since $(F^n_B, B, \tau^n)$ is $nT_2$, there exist disjoint NS-open sets $V_{B_i}$ and $V_{B_j}$ in $F^n_B$ separating $\eta^B_i$ and $\eta^B_j$. The condition follows naturally as their intersection in the absolute space is empty ($\Phi$), which is trivially contained in any NS-compact set $C_B$.

**( $\Longleftarrow$ ):** Conversely, suppose conditions (1), (2), and (3) are satisfied. We prove $(F^n_B, B, \tau^n)$ is $nT_2$. Let $\alpha, \beta \in F^n_B$ be distinct NS-points:

1. *Case 1:* $\alpha, \beta \in F_B$. Disjoint NS-open neighborhoods exist in $\tau$ due to the $nT_2$ property of $F_B$, and these remain NS-open in $\tau^n$.

2. *Case 2:* $\alpha = \eta^B_i$ and $\beta = \eta^B_j$ $(i \neq j)$. By condition (3), there exist NS-open neighborhoods $U_i, U_j$ such that their intersection is contained in an NS-compact set $C_B$. Since $F_B$ is $nT_2$, we can refine these neighborhoods to exclude the compact core $C_B$, effectively separating the two points at infinity.

3. *Case 3:* $\alpha \in F_B$ and $\beta = \eta^B_i$. By NS-local compactness of $F_B$, there exists an NS-open set $W_B$ containing $\alpha$ such that $\mathrm{cl}_n(W_B)$ is NS-compact and does not contain $\eta^B_i$. Then $W_B$ and $F^n_B \setminus \mathrm{cl}_n(W_B)$ are disjoint NS-open neighborhoods of $\alpha$ and $\eta^B_i$.

Thus, $(F^n_B, B, \tau^n)$ is an NS-Hausdorff compactification. □

**Corollary 4.14.** *Let $(F_B, B, \tau)$ be an NS-locally compact but non-compact $nT_2$-space. Then, $(F_B, B, \tau)$ is NS-open in every NS-compactification $(G_B, B, \sigma)$ containing $F_B$ as a subspace.*

*Proof.* The result follows directly from the fundamental property of NS-locally compact $nT_2$ spaces, which states that such a space is NS-open in any $nT_2$ space that contains it as a dense subspace. Since every NS-compactification $(G_B, B, \sigma)$ is an $nT_2$ space by definition, and $F_B$ is NS-locally compact, the original space $(F_B, B, \tau)$ must remain NS-open in $(G_B, B, \sigma)$. □

**Theorem 4.15.** *The near soft set $F_B$ is NS-dense in the Alexandroff space $(F^*_B, B, \tau^*)$.*

*Proof.* By the definition of NS-closure, $\mathrm{cl}_n(F_B)$ is the intersection of all NS-closed supersets of $F_B$ in $(F^*_B, B, \tau^*)$. According to the construction of $\tau^*$, any NS-closed set in $F^*_B$ that is smaller than $F^*_B$ must be an NS-compact subset of $F_B$. However, since $(F_B, B, \tau)$ is assumed to be non-compact, it cannot be contained within any such NS-compact subset. Therefore, the only NS-closed set containing $F_B$ is the absolute set $F^*_B$ itself. Hence, $\mathrm{cl}_n(F_B) = F^*_B$, establishing that $F_B$ is NS-dense in $F^*_B$. □

**Theorem 4.16.** *Let $U_B$ be an NS-open subset of $(F^*_B, B, \tau^*)$. Then the intersection $U_B \sqcap F_B$ is NS-open in the subspace $(F_B, B, \tau)$.*

*Proof.* Let $U_B \in \tau^*$ be an NS-open subset. We consider the two cases based on the definition of $\tau^*$:

1. If the NS-point at infinity $\infty_B \notin U_B$, then by the construction of the extended topology, $U_B \in \tau$. In this case, the intersection is $U_B \sqcap F_B = U_B$, which is naturally NS-open in $(F_B, B, \tau)$.

2. If $\infty_B \in U_B$, then $U_B = (F_B \setminus K_B) \sqcup \{\infty_B\}$, where $K_B$ is an NS-compact and NS-closed subset of $(F_B, B, \tau)$. The intersection with $F_B$ yields:

$$U_B \sqcap F_B = ((F_B \setminus K_B) \sqcup \{\infty_B\}) \sqcap F_B = F_B \setminus K_B.$$

Since $K_B$ is NS-closed in the original topology $\tau$, its complement $F_B \setminus K_B$ is NS-open in $(F_B, B, \tau)$.
In both scenarios, $U_B \sqcap F_B$ belongs to $\tau$, confirming that the mapping preserves the NS-open structure of the subspace. □

## 5. Conclusion

The primary objective of this study was to contribute to the advancement of topological structures specifically designed to effectively model and manage the uncertainty inherent in object states and parametric descriptions. The theory of compact spaces occupies a central role in general topology due to its fundamental influence on continuity, convergence, and separation properties.

In this research, the concept of **NS-Alexandroff compactification** (one-point compactification) has been successfully introduced and characterized within the framework of NSTS. By extending classical topological compactification to the near soft universe, we have demonstrated that an NSTS admits a one-point Hausdorff compactification if and only if it is NS-locally compact and satisfies the $nT_2$ separation axiom. This extension not only reinforces the theoretical foundations of soft topology but also provides a more general structure for handling unstable, incomplete, or inconsistent information.

Beyond the Alexandroff method, this study paves the way for adapting other well-established compactification techniques—such as **Stone–Čech, Wallman, and Fan–Gottesman**—into the near soft context. The NSTS framework allows for the systematic extension of non-compact structures, enriching the analytical toolkit available for researchers in decision theory, information science, and applied mathematics.

Given the increasing importance of uncertainty modeling in artificial intelligence, big data analysis, and engineering, the theoretical progress made here offers promising tools for tackling real-world data inconsistencies. Future research may focus on the practical application of these compactification techniques in fields such as:

- Optimization under uncertainty and vague constraints,
- Fuzzy control systems and near soft approximation operators,
- Knowledge representation in ambiguous and dynamic environments.

In conclusion, this study aligns with and expands upon the existing literature in soft set theory, rough logic, and fuzzy topology. It provides a more comprehensive framework for quantifying and reasoning about uncertainty, bridging the gap between abstract near soft structures and their compact representations.

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